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Research article

Attractors for a Navier–Stokes–Allen–Cahn system with unmatched densities

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Abstract: This paper investigates the long-time behavior for a Navier–Stokes–Allen–Cahn system, a diffuse interface model for two-phase incompressible flows with unmatched densities, non-constant viscosities, and a singular Flory–Huggins potential. First, we establish the dissipativity of strong solutions via some a priori estimates. Then, we demonstrate the regular-continuity of the semigroup, which allows us to prove the existence of the global attractor in the strong solutions space.

Keywords: two phase flow; global attractor; Navier–Stokes–Allen–Cahn system; dissipativity; singular Flory–Huggins potential
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1. Introduction

This paper considers the following Navier–Stokes–Allen–Cahn (NSAC) system modeling for two-phase flow with unmatched densities and viscosities, reading as follows:

$$\begin{cases} \rho(\phi)\partial_t \boldsymbol{u} - \operatorname{div}(\nu(\phi)D\boldsymbol{u}) + \rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla P = -\operatorname{div}(\nabla\phi\otimes\nabla\phi), \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\phi + \boldsymbol{u}\cdot\nabla\phi = -\mu - \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2} + \overline{\mu + \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}}, \end{cases} \quad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary. Considering the no-slip boundary condition for $\boldsymbol{u}$, the homogeneous Neumann boundary condition for ϕ

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \\ \partial_n\phi = 0, \end{cases} \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.2}$$

and the initial conditions

$$\begin{cases} \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \\ \phi(\cdot,0) = \phi_0, \end{cases} \quad \text{in } \Omega. \tag{1.3}$$

Here, $\boldsymbol{u} = \boldsymbol{u}(x, t)$ represents the volume-averaged fluid velocity field, while $P = P(x, t)$ denotes the pressure. The viscosity of the mixture, ν, is not constant, and ρ denotes the density of the mixture, which depends on the phase function ϕ. D is the symmetric gradient, which has the following form: $D = \frac{1}{2}(\nabla + \nabla^T)$. The chemical potential is defined by

$$\mu = \Psi'(\phi) - \Delta\phi, \tag{1.4}$$

and $\overline{X} = \frac{1}{|\Omega|}\int_\Omega X \mathrm{d}x$ denotes the spatial average of the term X. As an example, though not the only possibility, we can consider the averaged density

$$\rho(\phi) = \rho_1 \frac{1+\phi}{2} + \rho_2 \frac{1-\phi}{2},$$

and the averaged viscosity of the binary fluids

$$\nu(\phi) = \nu_1 \frac{1+\phi}{2} + \nu_2 \frac{1-\phi}{2},$$

where ρ_1 and ρ_2 denote the densities of the two fluids, and ν_1 and ν_2 represent their respective viscosities. The function Ψ is the double-well free energy density, also known as the Flory–Huggins potential, which is given by

$$\begin{aligned} \Psi(s) &= \frac{\theta}{2}\Big((1+s)\ln(1+s) + (1-s)\ln(1-s)\Big) - \frac{\theta_0}{2}s^2 \\ &= F(s) - \frac{\theta_0}{2}s^2, \end{aligned} \tag{1.5}$$

for every $s \in [-1, 1]$, where θ and θ_0 are two positive constants representing the absolute temperature of the mixture and the critical temperature, respectively, and they satisfy $0 < \theta < \theta_0$.

Investigating the dynamics of two-phase flows is one of the most attractive and important problems within the hydrodynamic theory of fluids, with the Allen–Cahn equation playing a fundamental role (see [1, 2]). The interface between two fluids is a $(d-1)$-dimensional manifold, posing great challenges both to the theoretical analysis and to the computational applications. Recently, a method called the diffuse-interface approach has emerged as a powerful technique for the study of interface theory (see [3–6]). The diffuse-interface method introduces a labeling function to replace the sharp interfaces with transition layers of width $\varepsilon > 0$, where ε is a small parameter. Under this framework, the dynamics of interfaces between two fluids recognized as level sets of the order parameter can be naturally described (see [7]). Within the diffuse-interface framework, the phase function ϕ represents the contrast between local concentrations of the two fluids.

Two commonly used model equations in the study of the evolution of binary fluid systems with mass conservation are the following.

(1) Mass-conserving Allen–Cahn equation (see [8])

$$\begin{cases} \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi + m\,(\mu - \overline{\mu}) = 0, & \text{in } \Omega\times(0,T), \\ \partial_{\boldsymbol{n}}\varphi = 0, & \text{on } \partial\Omega\times(0,T); \end{cases} \tag{1.6}$$

(2) Cahn–Hilliard equation (see [9])

$$\begin{cases} \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi - \mathrm{div}(m\nabla\mu) = 0, & \text{in } \Omega\times(0,T), \\ \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.7}$$

where μ represents the chemical potential, and m is a physically relevant constant.

The transport equation, in contrast to the Allen–Cahn and Cahn–Hilliard equations, does not include a diffusive term, and thus fails to maintain the proper shape of the diffuse interface along the normal direction, which motivates us to study the incompressible Navier–Stokes equations coupled with either the Allen–Cahn equation (1.6) or the Cahn–Hilliard equation (1.7), without dropping the crucial conservation laws (see (4.1)).

Recent research on incompressible binary fluid mixtures has led to significant findings. For the phase-field model of binary fluids, including the case of equal densities or small density contrasts, which can be approximated by the Boussinesq equations, we refer readers to [3, 10–13] and references therein. We also refer interested readers to [14] for a nice reference about the compressible Navier–Stokes equations with Onsager's regularity. Nevertheless, in most physical models, the density differences between two fluids are non-negligible. Significant contributions to the Navier–Stokes–Allen–Cahn models with constant density have been made by the authors in [15–17]. However, the authors chose the potential as the classical Landau double-well form as well as the lack of mass conservation (see e.g. [18] and [19] and references therein).

The model (1.1) was derived by Onsager (see [20, 21]), and we also refer interested readers to [22] for the Navier–Stokes–Cahn–Hilliard system. We also mention that there are several works for coupled nonlinear parabolic systems (see [23–25]) by introducing the so-called potential well method for the global existence of weak solutions (see [26–28]). Such systems are particularly relevant in the modeling of two-phase fluid systems, where the complex interplay between the phases often results in nonlinear coupling. The authors in [21] demonstrated both the well-posedness and the existence of the global attractor associated with system (1.1) in the 2-dimensional case. Their analysis focused on a specific case when the potential is smooth. The system they considered lacks a mass-conserving law and has constant viscosity. As a result, they ensured that the phase function ϕ remained confined within the physical range $[-1, 1]$, which is essential to their analysis. The well-posedness, regularity, and existence of the global attractor for the Navier–Stokes–Cahn–Hilliard system were established by the authors in [29] and [30]. The system they considered has matched density, and consequently, the uniqueness of the weak solution was easily obtained. This, together with the higher-order regularity of the phase function ϕ in (1.7), ensured that the dynamical system they constructed was on a lower-order regularity space $\mathbf{H}_\sigma \times H^1(\Omega)$, and they obtained the compact absorbing set by dissipativity estimates in $\mathbf{V}_\sigma \times H^2(\Omega)$ (see definition in section 2). Nevertheless, compared to the NSCH system, the NSAC system (1.1) contains only second-order diffusion terms. As a result, the regularity of ϕ is lower and we need more delicate estimates for ϕ (for more details, we refer to Proposition 4.1 in this paper and the argument of absorbing set in Theorem 4.1 in [30]). Moreover, since there is currently no theoretical proof of the uniqueness of weak solutions for the NSAC systems with unmatched densities, we can only consider strong solutions and construct the global attractor in a higher-regularity space $\mathbf{V}_\sigma \times H^2(\Omega)$, and therefore we need a much higher estimate of solutions in $\mathbf{H}^2_\sigma \times H^3(\Omega)$ to get the existence of a compact absorbing set (see Proposition 4.4 and Proposition 4.5 for more details).

The authors in [31] established the existence of a global weak solution of (1.1) in both 2-dimensional and 3-dimensional cases, together with the uniqueness of weak solutions with matched densities in the 2-dimensional case. Additionally, they proved the existence and uniqueness of strong solutions in the 2-dimensional case and derived several entropy estimates. However, there is no successful method that gives the uniqueness of weak solutions to system (1.1) with unmatched densities in the 2-dimensional case.

Before concluding this introduction, we give some additional remarks about our work. This study investigates the long-time behavior of solutions to the NSAC system (1.1). The system we considered here is more closely related to the actual physical model since the differences between densities and viscosities are not dropped. We also consider the system added a nonlinear term $(1/2)\rho'(\phi)|\boldsymbol{u}|^2$ representing the force which effectively models the impact of macroscopic fluid effects on the microscopic description arising from density differences (see [31]). Building on the framework established in [31], we demonstrate the dissipativity in the complete metric space $\mathcal{H}_m$. Due to Theorem 3.2, the existence of strong solutions provided in [31], we focus our analysis on strong solutions and construct an absorbing set on a suitable phase space $\mathbb{Y}_m$ (refer to Section 3 for details). Moreover, because the chosen Flory–Huggins potential has singular derivatives, the uniform bound of $F''(\phi)$ as time t away from zero is obtained by a corollary of Theorem 3.3, and this result enables us to derive compactness of the trajectories by proving dissipativity in a higher-regularity function space. Finally, applying the interpolation techniques, we demonstrate the continuity of the semigroup on the phase space $\mathbb{Y}_m$ and obtain the existence of the global attractor. For further research, one may get a higher regularity of the global attractor by the framework in [32]. This analysis lies beyond the framework of the present study and will be investigated in future work.

The plan of this paper reads as follows: In section 2, we present the function spaces, several inequalities in analysis, the theory of elliptic and the Stokes problems, as well as some Gronwall-type lemmas. In section 3, we recall the well-posedness results shown in [31], and we introduce the dynamical system in a suitable phase space generated by (1.1)-(1.3). Section 4 gives the existence of the global attractor, demonstrating the existence of a bounded and compact absorbing set in the phase space together with the continuity of the semigroup.

2. Preliminaries

2.1. Function spaces

Throughout this paper, the notation $C = C(a_1, a_2, ..., a_N)$ indicates that the constant C is a positive constant depending on the quantities $a_1, a_2, ..., a_N$. The boldface letter (e.g., $\mathbf{L}$) denotes the space of vector fields. If X is a metric space, $B_X(R)$ denotes the closed ball in X with radius R, centered at the origin. In this paper, $A : B$ is defined as the inner product of two matrices A and B, given by $A : B = tr(A^T B)$, and we denote norms $\|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{H^1(\Omega)}...$ by $\|\cdot\|_{L^p}, \|\cdot\|_{H^1}...$ $(1 \leqslant p \leqslant +\infty)$ unless otherwise specified.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\Omega$. We denote by $\mathbf{H}_\sigma$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $\mathbf{L}^2(\Omega)$, $\mathbf{V}_\sigma$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $\mathbf{H}^1(\Omega)$, and $\mathbf{H}^2_\sigma$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $\mathbf{H}^2(\Omega)$, where

$$C^\infty_{0,\sigma}(\Omega) = \{\boldsymbol{u} \in C^\infty_0(\Omega) : \operatorname{div} \boldsymbol{u} = 0\}.$$

Then they are Hilbert spaces, and for convenience, we may still use $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^2}$ for the norms in those spaces.

The Korn inequality related to the symmetric gradient reads as follows:

$$\|\nabla \boldsymbol{u}\|_{L^2} \leq \sqrt{2}\|D\boldsymbol{u}\|_{L^2} \leq \sqrt{2}\|\nabla \boldsymbol{u}\|_{L^2}, \quad \text{for all } \boldsymbol{u} \in \mathbf{V}_\sigma. \tag{2.1}$$

We also recall the following inequalities in the 2D case (see [31]):

$$\|f\|_{L^4} \leq C\|f\|^{\frac{1}{2}}_{L^2}\|f\|^{\frac{1}{2}}_{H^1}, \qquad \text{for all } f \in H^1(\Omega), \tag{2.2}$$

$$\|f\|_{L^\infty} \le C\|f\|_{L^2}^{\frac{1}{2}}\|f\|_{H^2}^{\frac{1}{2}}, \qquad \text{for all } f \in H^2(\Omega), \tag{2.3}$$

$$\|\nabla f\|_{L^4} \le C\|f\|_{H^2}^{\frac{1}{2}}\|f\|_{L^\infty}^{\frac{1}{2}}, \qquad \text{for all } f \in H^2(\Omega), \tag{2.4}$$

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}\ln^{\frac{1}{2}}(\mathrm{e}\frac{\|f\|_{H^2}}{\|f\|_{H^1}}), \qquad \text{for all } f \in H^2(\Omega), \tag{2.5}$$

$$\|f\|_{L^\infty} \le C(p)\|f\|_{H^1}\ln^{\frac{1}{2}}(C(p)\frac{\|f\|_{W^{1,p}}}{\|f\|_{H^1}}), \qquad \text{for all } f \in W^{1,p}(\Omega), p > 2. \tag{2.6}$$

We recall the following lemma and refer interested readers to [31] for a detailed proof.

Lemma 2.1 ([31]). *Let $f \in H^1(\Omega), g \in L^p(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary and $p > 2$. Then*

$$\|fg\|_{L^2} \le C(\frac{p}{p-2})^{\frac{1}{2}}\|f\|_{H^1}\|g\|_{L^2}\ln^{\frac{1}{2}}(\mathrm{e}|\Omega|^{\frac{p-2}{2p}}\frac{\|g\|_{L^p}}{\|g\|_{L^2}}),$$

for some $C = C(\Omega)$.

In the following, we recall an important differential inequality in order to obtain the dissipativities later (see [29, 32] for more details).

Lemma 2.2 (Uniform Gronwall lemma in logarithm)**.** *Assume $f > 0$ is absolutely continuous on $[0,\infty)$ and $g, h > 0$ are both locally integrable on $[0,\infty)$, satisfying*

$$f'(t) \le g(t)f(t)\ln(\mathrm{e} + f(t)) + h(t), \ \text{a.e. } t \ge 0,$$

and in addition the uniform bounds: for every $t \ge 0$,

$$\int_t^{t+r} f(\tau)\mathrm{d}\tau \le a_1, \quad \int_t^{t+r} g(\tau)\mathrm{d}\tau \le a_2, \quad \int_t^{t+r} h(\tau)\mathrm{d}\tau \le a_3,$$

for some r, $a_1, a_2, a_3 > 0$. Then for every $t \ge r$,

$$f(t) \le \mathrm{e}^{(\frac{a_1+r}{r}+a_3)\mathrm{e}^{a_2}}.$$

2.1.1. Stokes problems and Neumann problems

Now we recall two lemmas for the Stokes problem and the elliptic estimate of Neumann problems (see [3] and [31]).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. $\nu \in W^{2,\infty}(\mathbb{R})$, and satisfies $0 < \nu_* \le \nu(s) \le \nu^*$ for all $s \in \mathbb{R}$. $\varphi \in W^{1,r}(\Omega)$, with $r > 2$. The force $\boldsymbol{g} \in \mathbf{L}^p(\Omega)$, with $p \in (1,\infty)$. Assume that $\boldsymbol{u} \in \mathbf{V}_\sigma$ is a weak solution of*

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\boldsymbol{u}) + \nabla P = \boldsymbol{g}, & \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases}$$

in the following sense

$$(\nu(\varphi)D\boldsymbol{u}, \nabla\boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_\sigma.$$

Then,

$$\|\boldsymbol{u}\|_{W^{2,p}} \le C\left(\|\boldsymbol{g}\|_{L^p} + \|\nabla\varphi\|_{L^r}\|D\boldsymbol{u}\|_{L^2}\right),$$

for some positive constant $C = C(p,\Omega)$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Assume that φ is the solution to the Neumann problem:*

$$\begin{cases} -\Delta\varphi + F'(\varphi) = g, & \text{in } \Omega, \\ \partial_{\boldsymbol{n}}\varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

Then we have:

(a) If $g \in L^p(\Omega)$, $p \in [2, \infty]$, then

$$\|F'(\varphi)\|_{L^p} \leq \|g\|_{L^p}.$$

(b) If $g \in H^1(\Omega)$, then

$$\|\Delta\varphi\|_{L^2} \leq \|\nabla\varphi\|_{L^2}^{\frac{1}{2}}\|\nabla g\|_{L^2}^{\frac{1}{2}},$$

and for every $p \geq 2$, there exists a positive constant $C = C(p, \Omega)$, such that

$$\|\varphi\|_{W^{2,p}} + \|F'(\varphi)\|_{L^p} \leq C(1 + \|g\|_{H^1} + \|\varphi\|_{L^2}).$$

3. Well-posedness and dynamical system

First we assume that the density and the viscosity $\rho, \nu \in C^2([-1, 1])$ satisfy

$$\begin{aligned} &0 < \rho_* \leq \rho(s) \leq \rho^*, \\ &0 < \nu_* \leq \nu(s) \leq \nu^*, \end{aligned} \tag{3.1}$$

for every $s \in [-1, 1]$.

Next, we recall the well-posedness and regularity theorems given in [31].

Theorem 3.1 ([31]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary, $(\boldsymbol{u}_0, \phi_0) \in \mathbf{H}_\sigma \times (H^1(\Omega) \cap L^\infty(\Omega))$ with $\|\phi_0\|_{L^\infty} \leq 1$ and $|\overline{\phi}_0| < 1$. Then there exists a weak solution $(\boldsymbol{u}, \phi)$ to problem* (1.1)-(1.3) *on the interval* $[0, \infty)$*, satisfying:*

(a) For every $T > 0$,

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(0, T; \mathbf{H}_\sigma) \cap L^2(0, T; \mathbf{V}_\sigma), \\ &\phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad \partial_t\phi \in L^2(0, T; L^2(\Omega)), \\ &\phi \in L^\infty(\Omega \times (0, T)) : |\phi(x, t)| < 1 \quad a.e. \quad in \quad \Omega \times (0, T), \\ &\mu \in L^2(0, T; L^2(\Omega)), \quad F'(\phi) \in L^2(0, T; L^2(\Omega)). \end{aligned}$$

(b) The pair $(\boldsymbol{u}, \phi)$ solves the problem in the following sense:

$$\begin{aligned} -\int_0^T \int_\Omega (\rho'(\phi)\partial_t\phi\eta(t) + \rho(\phi)\eta'(t))\, \boldsymbol{u} \cdot \boldsymbol{w} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega (\rho(\phi)\boldsymbol{u} \cdot \nabla\boldsymbol{u}) \cdot \boldsymbol{w}\eta(t)\mathrm{d}x\mathrm{d}t \\ + \int_0^T \int_\Omega \nu(\phi)(D\boldsymbol{u} : D\boldsymbol{w})\eta(t)\mathrm{d}x\mathrm{d}t \\ = \int_\Omega \rho(\phi_0)\boldsymbol{u}_0 \cdot \boldsymbol{w}\eta(0)\mathrm{d}x + \int_0^T \int_\Omega ((\nabla\phi \otimes \nabla\phi) : \nabla\boldsymbol{w})\eta(t)\mathrm{d}x\mathrm{d}t, \end{aligned}$$

for every $T > 0$, $\boldsymbol{w} \in \mathbf{V}_\sigma$, $\eta \in C^1([0,T])$ *with* $\eta(T) = 0$, *and*

$$\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi = \Delta\phi - \Psi'(\phi) - \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2} + \overline{\Psi'(\phi) + \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}},\ a.e.\ (x,t) \in \Omega\times(0,T).$$

$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$, $\phi(\cdot,0) = \phi_0$ *in* Ω, *and* $\partial_{\boldsymbol{n}}\phi = 0$ *almost everywhere on* $\partial\Omega\times(0,T)$.

(c) Set the total energy of the system by

$$E(\boldsymbol{u},\phi) = \int_\Omega \frac{1}{2}\rho(\phi)|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\mathrm{d}x, \tag{3.2}$$

then the weak solutions satisfy the energy inequality as follows:

$$E(\boldsymbol{u}(t),\phi(t)) + \int_0^t\int_\Omega \nu(\phi(\tau))|D\boldsymbol{u}(\tau)|^2\mathrm{d}x + \int_0^t \|(\partial_t\phi(\tau) + \boldsymbol{u}(\tau)\cdot\nabla\phi(\tau))\|_{L^2}^2 \le E(\boldsymbol{u}_0,\phi_0), \tag{3.3}$$

for all $t > 0$.

Theorem 3.2 ([31])**.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with a smooth boundary. Assume that* $(\boldsymbol{u}_0,\phi_0) \in \mathbf{V}_\sigma(\Omega)\times H^2(\Omega)$ *such that* $\|\phi_0\|_{L^\infty} \le 1$, $|\overline{\phi}_0| < 1$, $\mu_0 = \Psi'(\phi_0) - \Delta\phi_0 \in H^1(\Omega)$ *and* $\partial_{\boldsymbol{n}}\phi_0 = 0$ *on* $\partial\Omega$. *Then there is a strong solution* $(\boldsymbol{u},\phi)$ *to problem* (1.1)-(1.3) *on the interval* $[0,\infty)$, *satisfying:*

(a) For every $T > 0$ *and for every* $p \in (2,\infty)$,

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{H}_\sigma),\\
&\phi \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;W^{2,p}(\Omega)),\\
&\partial_t\phi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&F'(\phi) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^p(\Omega)).
\end{aligned}$$

The solution $(\boldsymbol{u},\phi)$ *solves the system* (1.1) *almost everywhere in* $\Omega\times(0,\infty)$. *Moreover,* $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$, $\phi(\cdot,0) = \phi_0$ *in* Ω, $\partial_{\boldsymbol{n}}\phi = 0$ *a.e. on* $\partial\Omega\times(0,T)$.

(b) If in addition there exists $\eta_1 = \eta_1(E(\boldsymbol{u}_0,\phi_0), \|\boldsymbol{u}_0\|_{\mathbf{V}_\sigma}, \|\phi_0\|_{H^2}, \|F'(\phi_0)\|_{L^2}, \theta, \theta_0)$, *such that* $\|\rho'\|_{L^\infty(-1,1)} \le \eta_1$ *and* $F''(\phi_0) \in L^1(\Omega)$, *then for every* $T > 0$,

$$(F''(\phi))^2\ln(1 + F''(\phi)) \in L^1(\Omega\times(0,T)), \tag{3.4}$$

and furthermore, the strong solution that satisfies (3.4) *is unique.*

Theorem 3.3 ([31])**.** *Let the assumptions of Theorem 3.2 be satisfied. Assume in addition that* $\|\rho'\|_{L^\infty(-1,1)} \le \eta_1$. *If* $(\boldsymbol{u},\phi)$ *is the strong solution of system* (1.1), *then for every* $\xi > 0$, *there exists a positive constant* $\delta(\xi)$, *such that the absolute value of* ϕ *is away from one:*

$$-1 + \delta(\xi) \le \phi(x,t) \le 1 - \delta(\xi),$$

for every $x \in \overline{\Omega}$ *and* $t \ge \xi$.

Remark 3.4. *In contrast to the Navier–Stokes–Cahn–Hilliard system, the phase function* ϕ *can approach* ± 1 *arbitrarily closely as t goes to zero (see [31] and [30] for a detailed discussion).*

For any $m \in (-1, 1)$, we define the following spaces:

$$\mathcal{H}_m = \mathbf{H}_\sigma \times V_m,$$

$$\mathbb{Y}_m = \{(\boldsymbol{u}, \phi) \in \mathbf{V}_\sigma \times H^2(\Omega) : |\phi| \leq 1, \text{ a.e. }, \overline{\phi} = m, \partial_{\boldsymbol{n}}\phi = 0 \text{ on } \partial\Omega\},$$

where

$$V_m = \{\phi \in H^1(\Omega) \cap L^\infty(\Omega) : \|\phi\|_{L^\infty} \leq 1, \overline{\phi} = m\}.$$

Then $\mathcal{H}_m$ and $\mathbb{Y}_m$ are two complete metric spaces.

According to Theorems 3.1-3.2, the problem (1.1)-(1.3) generates a dynamical system: for each $t \geq 0$,

$$S(t) : \mathbb{Y}_m \to \mathbb{Y}_m,$$

in the following sense

$$S(t)(\boldsymbol{u}_0, \phi_0) = (\boldsymbol{u}(t), \phi(t)),$$

where $(\boldsymbol{u}(t), \phi(t))$ is the unique solution of problem (1.1)-(1.3). The dynamical system is a semigroup $S(t)$ on $\mathbb{Y}_m$ satisfying:

(a) $S(0) = \mathrm{Id}_{\mathbb{Y}_m}$;

(b) $S(t + \tau) = S(t)S(\tau)$, for every $t, \tau \geq 0$;

(c) $t \to S(t)(\boldsymbol{u}_0, \phi_0) \in C([0, \infty), \mathbb{Y}_m)$, for every $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}_m$.

4. The Global Attractor

In this section, we will prove the existence of the global attractor $\mathscr{A}_m$ of the semigroup $S(t)$ on the phase space $\mathbb{Y}_m$.

4.1. Absorbing Set

Proposition 4.1. *There is a bounded set $B_0 \subset \mathcal{H}_m$, such that for any bounded subset B of $\mathcal{H}_m$, there exists $t_0(B) > 0$, which depends only on the $\mathcal{H}_m$-bounds of B, satisfying*

$$(\boldsymbol{u}(t), \phi(t)) \in B_0, \quad \text{for all } t \geq t_0(B),$$

where $(\boldsymbol{u}, \phi)$ is the weak solution of (1.1) *subject to the initial value $(\boldsymbol{u}_0, \phi_0) \in B$.*

Proof. Let us fix $R > 0$. We consider $(\boldsymbol{u}_0, \phi_0) \in B_{\mathcal{H}_m}(R) \subset \mathcal{H}_m$. First we integrate the equation $(1.1)_3$ over Ω to obtain the mass conservation: for every $t \geq 0$,

$$\int_\Omega \phi(t)\mathrm{d}x = \int_\Omega \phi_0 \mathrm{d}x, \tag{4.1}$$

and we define

$$m = \overline{\phi}(t) = \frac{1}{|\Omega|}\int_\Omega \phi(t)\mathrm{d}x.$$

By Theorem 3.1, we recall the energy identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_\Omega \nu(\phi)|D\boldsymbol{u}|^2\mathrm{d}x + \|\partial_t\phi + \boldsymbol{u} \cdot \nabla\phi\|_{L^2}^2 = 0, \tag{4.2}$$

for every $t > 0$. For $(1.1)_3$, we take the L^2-inner-product with $\phi - \overline{\phi} = \phi - m$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2 + \int_\Omega F'(\phi)(\phi - m)\mathrm{d}x - \theta_0\int_\Omega \phi(\phi - m)\mathrm{d}x + \int_\Omega \rho'(\phi)(\phi - m)\frac{|\boldsymbol{u}|^2}{2}\mathrm{d}x = 0.$$

By multiplying the above equation by ε and summing with (4.2), we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E(t) + \frac{\varepsilon}{2}\|\phi\|_{L^2}^2) + \varepsilon\|\nabla\phi\|_{L^2}^2 + \varepsilon\int_\Omega F'(\phi)(\phi - m)\mathrm{d}x + \int_\Omega \nu(\phi)|D\boldsymbol{u}|^2\mathrm{d}x&\\
+\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 \le C - \varepsilon\int_\Omega \rho'(\phi)(\phi - m)\frac{|u|^2}{2}\mathrm{d}x&\\
\le C + C_1\varepsilon\|\boldsymbol{u}\|_{L^2}^2&\\
\le C + C_1\varepsilon\|\nabla\boldsymbol{u}\|_{L^2}^2,&\end{aligned}$$

where $C = C(\theta_0, m, \Omega, \varepsilon)$ and $C_1 = C_1(\rho, m, \Omega)$. Then by the Korn inequality and (3.1), we obatin

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E(t)+\frac{\varepsilon}{2}\|\phi\|_{L^2}^2) + (\frac{1}{2}\nu_* - \varepsilon)\|\nabla\boldsymbol{u}\|_{L^2}^2 + \varepsilon\int_\Omega F'(\phi)(\phi - m)\mathrm{d}x&\\
+\varepsilon\|\nabla\phi\|_{L^2} + \|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 \le C.&\end{aligned}$$

Next we need an inequality, which can be found in [33]:

$$\beta\int_\Omega |F'(\phi)|\mathrm{d}x \le \int_\Omega F'(\phi)(\phi - m)\mathrm{d}x + C_0, \tag{4.3}$$

for some β, $C_0 > 0$, depending only on F and m. Then we obatin

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E(t)+\frac{\varepsilon}{2}\|\phi\|_{L^2}^2) + (\frac{1}{2}\nu_* - \varepsilon)\|\nabla\boldsymbol{u}\|_{L^2}^2 + \varepsilon\beta\|F'(\phi)\|_{L^1}^2 +&\\
\varepsilon\|\nabla\phi\|_{L^2} + \|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 \le C.&\end{aligned}$$

Taking $\varepsilon = \frac{1}{4C_1}\nu_*$, and $\Psi^* := \max_{s\in[-1,1]}|\Psi(s)|, \Psi_* := \max_{s\in[-1,1]}|\Psi(s)|$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(E(t) + \frac{\varepsilon}{2}\|\phi\|_{L^2}^2) + \frac{1}{4}\nu_*\|\nabla\boldsymbol{u}\|_{L^2}^2 + \frac{1}{4}\nu_*\|\nabla\phi\|_{L^2}^2 + \int_\Omega \Psi(\phi)\mathrm{d}x + \varepsilon\|\phi\|_{L^2}^2 \le \tilde{C}_1,$$

where $\tilde{C_1} = \tilde{C}_1(F, \theta_0, m, \nu, \Omega)$. By the definition of $E(t)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(E(t) + \frac{\varepsilon}{2}\|\phi\|_{L^2}^2) + \alpha(E(t) + \frac{\varepsilon}{2}\|\phi\|_{L^2}^2) \le K_0^2, \tag{4.4}$$

where $\alpha = \alpha(\lambda_1, \Psi_*)$, depends on parameters of system (1.1), and λ_1 is the first eigenvalue of the Stokes operator $\mathbf{A}$, while $K_0^2 := \tilde{C}_1 + |\Omega|\Psi^*$. By the Gronwall lemma, for each $t \ge 0$,

$$E(t) \le (E(0) + \frac{\varepsilon}{2}\|\phi_0\|_{L^2}^2)\mathrm{e}^{-\alpha t} + (K_1')^2,$$

where $(K_1')^2 = \frac{K_0^2}{\alpha}$. Thus by the definition of the energy $E(t)$ again, we obtain the crucial inequality:

$$\|\boldsymbol{u}\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2 \le C(\rho)(\|\boldsymbol{u}_0\|_{L^2}^2 + \|\nabla\phi_0\|_{L^2}^2)\mathrm{e}^{-\alpha t} + K^2.$$

As $(u_0, \phi_0) \in B_{\mathcal{H}_m}(R)$, when $t \geq t_e(R)$, for some $t_e(R) = t_e(\rho, R)$,

$$C(\rho)(\|\boldsymbol{u}_0\|_{L^2}^2 + \|\nabla\phi_0\|_{L^2}^2)\mathrm{e}^{-\alpha t} \leq 1,$$

and

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2 \leq K_1^2, \tag{4.5}$$

where $K_1^2 := K^2 + 1$, depends only on parameters of system (1.1). Then we can finish the proof by taking $B_0 = B_{\mathcal{H}_m}(K_1)$. □

Remark 4.2. *As a direct consequence of the above proposition, we indeed obtain the dissipativity in the weak solution space. This may allow us to construct the so-called trajectory attractor for the weak solutions of the NSAC system* (1.1) *without uniqueness. On the other hand,* (4.5) *together with* (4.4) *yields, for all $t \geq 0$,*

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2 + \int_t^{t+1} (\|\nabla\boldsymbol{u}(s)\|_{L^2}^2 + \|\partial_t\phi(s) + \boldsymbol{u}\cdot\nabla\phi(s)\|_{L^2}^2 + \|F'(\phi)\|_{L^1})\mathrm{d}s \leq M_0, \tag{4.6}$$

for some M_0 that depends only on the parameters of the system and K_1.

Lemma 4.3. *The following estimates hold for all $t \geq t_e(R)$:*

$$\begin{aligned} \|\phi\|_{H^2}^2 \leq& C(1 + \|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2) \\ \leq& C_1(1 + \|\partial_t\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2), \end{aligned} \tag{4.7}$$

and

$$\|\partial_t\phi\|_{L^2}^2 \leq C_2(\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2), \tag{4.8}$$

for some C_1, C_2 dependent on parameters of (1.1) *and K_1.*

Proof. Multiply (1.4) by $-\Delta\phi$ and integrate over Ω to obtain

$$\|\Delta\phi\|_{L^2}^2 + \int_\Omega F''(\phi)|\nabla\phi|^2\mathrm{d}x = \theta_0\|\nabla\phi\|_{L^2}^2 - \int_\Omega (\mu - \overline{\mu})\Delta\phi\mathrm{d}x.$$

As $t \geq t_e(R)$, and by (4.5),

$$\|\Delta\phi\|_{L^2}^2 \leq \theta_0 K_1^2 + \|\Delta\phi\|_{L^2}\|\mu - \overline{\mu}\|_{L^2} \leq \theta_0 K_1^2 + \frac{1}{2}\|\Delta\phi\|_{L^2}^2 + \frac{1}{2}\|\mu - \overline{\mu}\|_{L^2}^2.$$

Thus,

$$\|\Delta\phi\|_{L^2}^2 \leq C(1 + \|\mu - \overline{\mu}\|_{L^2}^2).$$

Then by (4.5), we obatin

$$\|\phi\|_{H^2}^2 \leq C(1 + \|\mu - \overline{\mu}\|_{L^2}^2),$$

for any $t \geq t_e(R)$. Also by (2.4) and (4.5),

$$\begin{aligned} \|\boldsymbol{u}\cdot\nabla\phi\|_{L^2} \leq \|\boldsymbol{u}\|_{L^4}\|\nabla\phi\|_{L^4} &\leq C\|\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}} \\ &\leq CK_1^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}K_1^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}} \\ &\leq C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}}. \end{aligned}$$

We can infer from $(1.1)_3$ that

$$\int_\Omega (\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi)(\mu-\overline{\mu})\mathrm{d}x + \|\mu-\overline{\mu}\|_{L^2}^2 + \int_\Omega \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}(\mu-\overline{\mu})\mathrm{d}x$$
$$= \int_\Omega (\mu + \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2} - \overline{\mu + \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}})(\mu-\overline{\mu})\mathrm{d}x = 0.$$

Denoting $(\rho')^* = \max_{s\in[-1,1]}|\rho'(s)|$, we obtain that

$$\|\mu-\overline{\mu}\|_{L^2}^2 = \left|-\int_\Omega (\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi)(\mu-\overline{\mu})\mathrm{d}x - \int_\Omega \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}(\mu-\overline{\mu})\mathrm{d}x\right|$$
$$\le \|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}\|\mu-\overline{\mu}\|_{L^2} + \|\rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}\|_{L^2}\|\mu-\overline{\mu}\|_{L^2}.$$

Thus, we obtain

$$\|\mu-\overline{\mu}\|_{L^2} \le C\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2} + (\rho')^*\|\boldsymbol{u}\|_{L^4}^2$$
$$\le C(\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2} + \|\nabla\boldsymbol{u}\|_{L^2}),$$

where $C = C(K_1)$. Consequently, we obtain

$$\begin{aligned}\|\phi\|_{H^2}^2 &\le C(1 + \|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2)\\ &\le C(1 + \|\partial_t\phi\|_{L^2}^2 + \|\boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2)\\ &\le C(1 + \|\partial_t\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}\|\phi\|_{H^2} + \|D\boldsymbol{u}\|_{L^2}^2)\\ &\le C(1 + \|\partial_t\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2) + \frac{1}{2}\|\phi\|_{H^2}^2,\end{aligned}$$

which implies

$$\|\phi\|_{H^2}^2 \le C_1(1 + \|\partial_t\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2)$$

for all $t \ge t_e(R)$.

Since

$$\begin{aligned}\|\partial_t\phi\|_{L^2}^2 &\le C(\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|\boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2)\\ &\le C(\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}\|\phi\|_{H^2})\\ &\le C(\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2) + \frac{1}{2}\|\partial_t\phi\|_{L^2}^2,\end{aligned}$$

we obtain that for each $t \ge t_e(R)$,

$$\|\partial_t\phi\|_{L^2}^2 \le C_2(\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2), \tag{4.9}$$

where the parameters C_1, C_2 depend only on parameters of the system (1.1) and K_1. □

Proposition 4.4. *The dynamical system $(\mathbb{Y}_m, S(t))$ possesses a bounded absorbing set B_1, i.e., for any bounded set $B \subset \mathbb{Y}_m$, there exists $t_1(B) > 0$, depending only on the $\mathbb{Y}_m$-bound of B, such that for any $t \ge t_1(B)$, $S(t)B \subset B_1$.*

Proof. This part of the proof is similar to the higher regular estimates of the NSAC system in [31], whereas keep in mind that (4.6) is valid for any $t > t_e(R) + 1$.

Multiplying (1.1) by $\partial_t \boldsymbol{u}$, integrating over Ω, we get

$$\int_\Omega \rho(\phi)|\partial_t \boldsymbol{u}|^2 \mathrm{d}x + (\rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \partial_t\boldsymbol{u}) + \int_\Omega \nu(\phi) D\boldsymbol{u}\cdot D\partial_t\boldsymbol{u}\mathrm{d}x = -\int_\Omega \Delta\phi\nabla\phi\cdot\partial_t\boldsymbol{u}\mathrm{d}x,$$

where

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu(\phi)|D\boldsymbol{u}|^2\mathrm{d}x = \int_\Omega \nu(\phi)D\boldsymbol{u} : D\partial_t\boldsymbol{u}\mathrm{d}x + \frac{1}{2}\int_\Omega \nu'(\phi)\partial_t\phi|D\boldsymbol{u}|^2\mathrm{d}x,$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu(\phi)|D\boldsymbol{u}|^2\mathrm{d}x + \int_\Omega \rho(\phi)|\partial_t\boldsymbol{u}|^2\mathrm{d}x \\
&= -(\rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \partial_t\boldsymbol{u}) + \frac{1}{2}\int_\Omega \nu'(\phi)\partial_t\phi|D\boldsymbol{u}|^2\mathrm{d}x - \int_\Omega \Delta\phi\nabla\phi\cdot\partial_t\boldsymbol{u}\mathrm{d}x.
\end{aligned} \tag{4.10}$$

Differentiating $(1.1)_3$ with respect to t, multiplying by $\partial_t\phi$ and integrating over Ω, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\phi\|_{L^2}^2 + \|\nabla\partial_t\phi\|_{L^2}^2 + \int_\Omega F''(\phi)|\partial_t\phi|^2\mathrm{d}x \\
&= -\int_\Omega \partial_t\boldsymbol{u}\cdot\nabla\phi\partial_t\phi\mathrm{d}x + \theta_0\|\partial_t\phi\|_{L^2}^2 - \int_\Omega \rho''(\phi)|\partial_t\phi|^2\frac{|\boldsymbol{u}|^2}{2}\mathrm{d}x \\
&\quad - \int_\Omega \rho'(\phi)\boldsymbol{u}\cdot\partial_t\boldsymbol{u}\partial_t\phi\mathrm{d}x + \partial_t\overline{(\mu + \rho'(\phi)\frac{|\boldsymbol{u}|^2}{2})}\int_\Omega \partial_t\phi\mathrm{d}x.
\end{aligned} \tag{4.11}$$

By summing up (4.10) and (4.11), we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\int_\Omega \nu(\phi)|D\boldsymbol{u}|^2\mathrm{d}x + \|\partial_t\phi\|_{L^2}^2) + \|\nabla\partial_t\phi\|_{L^2}^2 + \rho_*\|\partial_t\boldsymbol{u}\|_{L^2}^2 + \int_\Omega F''(\phi)|\partial_t\phi|^2\mathrm{d}x \\
&\le -\int_\Omega \Delta\phi\nabla\phi\cdot\partial_t\boldsymbol{u}\mathrm{d}x + \frac{1}{2}\int_\Omega \nu'(\phi)\partial_t\phi|D\boldsymbol{u}|^2\mathrm{d}x + \theta_0\|\partial_t\phi\|_{L^2}^2 - \int_\Omega \partial_t\boldsymbol{u}\cdot\nabla\phi\partial_t\phi\mathrm{d}x \\
&\quad -(\rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \partial_t\boldsymbol{u}) - \int_\Omega \rho''(\phi)|\partial_t\phi|^2\frac{|\boldsymbol{u}|^2}{2}\mathrm{d}x - \int_\Omega \rho'(\phi)\boldsymbol{u}\cdot\partial_t\boldsymbol{u}\partial_t\phi\mathrm{d}x.
\end{aligned} \tag{4.12}$$

Then by (2.2)-(2.6) and Lemma 2.1, as well as a fundamental inequality: for each $x, y > 0$,

$$x^2\ln\left(\frac{y}{x}\right) \le x^2\ln(y) + 1, \tag{4.13}$$

we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}G(t) + \frac{\rho_*}{2}\|\partial_t\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla\partial_t\phi\|_{L^2}^2 \\
&\le C\left(G^2(t)\ln(C\|\boldsymbol{u}\|_{W^{1,p}}) + (1+G^2(t))\ln(C\|\phi\|_{W^{2,p}}) + G^2(t) + 1\right),
\end{aligned} \tag{4.14}$$

where

$$G(t) = \frac{1}{2}\int_\Omega \nu(\phi(t))|D\boldsymbol{u}(t)|^2\mathrm{d}x + \frac{1}{2}\|\partial_t\phi(t)\|_{L^2}^2, \tag{4.15}$$

and

$$\frac{\nu_*}{2}\|D\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\partial_t\phi\|_{L^2}^2 \le G(t) \le C\left(\|D\boldsymbol{u}\|_{L^2}^2 + \|\partial_t\phi\|_{L^2}^2\right). \tag{4.16}$$

From Lemma 2.3, for the Stokes problem with the force term $\boldsymbol{g} = -\rho(\phi)\partial_t \boldsymbol{u} - \rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \Delta\phi\nabla\phi$, we have, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}\|\boldsymbol{u}\|_{W^{2,1+\varepsilon}} &\le C\left(\|\partial_t \boldsymbol{u}\|_{L^{1+\varepsilon}} + \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^{1+\varepsilon}} + \|\Delta\phi\nabla\phi\|_{L^{1+\varepsilon}} + \|D\boldsymbol{u}\|_{L^2}\|\nabla\phi\|_{L^r}\right)\\ &\le C(\|\partial_t\boldsymbol{u}\|_{L^2} + G(t) + 1).\end{aligned} \tag{4.17}$$

By the Sobolev embedding $W^{2,1+\varepsilon}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for every $p \in (2,\infty)$ and $\frac{1}{p} = \frac{1}{1+\varepsilon} - \frac{1}{2}$,

$$\|\boldsymbol{u}\|_{W^{1,p}} \le C(\|\partial_t\boldsymbol{u}\|_{L^2} + G(t) + 1). \tag{4.18}$$

Consider the elliptic problem

$$\begin{cases}-\Delta\phi + F'(\phi) = \mu + \theta_0\phi, & \text{a.e. in } \Omega\times(0,T),\\ \partial_n\phi = 0, & \text{a.e. on } \partial\Omega\times(0,T),\end{cases} \tag{4.19}$$

and Lemma 2.4,

$$\begin{aligned}\|\phi\|_{W^{2,p}} + \|F'(\phi)\|_{L^p} &\le C(1 + \|\phi\|_{L^2} + \|\mu + \theta_0\phi\|_{L^p})\\ &\le C(1 + \|\phi\|_{L^p} + \|\mu\|_{L^p}),\end{aligned} \tag{4.20}$$

for any $p \in [2,\infty)$. By $(1.1)_3$, we have

$$\begin{aligned}\|\mu - \overline{\mu}\|_{L^p} &\le \|\partial_t\phi\|_{L^p} + \|\boldsymbol{u}\cdot\nabla\phi\|_{L^p} + \left\|\rho'(\phi)\frac{|\boldsymbol{u}|^2}{2} - \overline{\rho'(\phi)\frac{|\boldsymbol{u}|^2}{2}}\right\|_{L^p}\\ &\le C(\|\nabla\partial_t\phi\|_{L^2} + \|\boldsymbol{u}\|_{H^1}\|\phi\|_{H^2} + \|\nabla\boldsymbol{u}\|_{L^2}^2).\end{aligned}$$

Note that $\overline{\mu} = \overline{F'(\phi) - \theta_0\phi}$, and thus $|\overline{\mu}| \le C(1 + \|F'(\phi)\|_{L^1})$. Taking the L^2-inner product of (1.1) with $\phi - m$, we obtain

$$\|\nabla\phi\|_{L^2}^2 + \int_\Omega F'(\phi)(\phi - m)\mathrm{d}x = \int_\Omega (\mu - \overline{\mu})(\phi - m)\mathrm{d}x + \int_\Omega \theta_0\phi(\phi - m)\mathrm{d}x.$$

By (4.3),

$$\|F'(\phi)\|_{L^1} \le C(1 + \|\mu - \overline{\mu}\|_{L^2}). \tag{4.21}$$

Thus,

$$\|\mu\|_{L^p} \le C\|\mu - \overline{\mu}\|_{L^p} + C|\overline{\mu}| \le C(G(t) + \|\nabla\partial_t\phi\|_{L^2} + 1).$$

Then by (4.20)

$$\|\phi\|_{W^{2,p}} \le C(G(t) + \|\nabla\partial_t\phi\|_{L^2} + 1). \tag{4.22}$$

Consequentially, applying the following inequality: for each $x, y > 0$,

$$xy \le (x\ln x - x + 1) + (\mathrm{e}^y - 1),$$

we obtain that

$$G^2(t)\ln(C\|\boldsymbol{u}\|_{W^{1,p}}) \le \frac{\rho_*}{4}\|\partial_t\boldsymbol{u}\|_{L^2}^2 + C(G^2(t)\ln(\mathrm{e} + G(t)) + G^2(t) + 1).$$

Similarly, we can obtain that

$$(1+G^2(t))\ln\left(C\|\phi\|_{W^{2,p}}\right) \le \frac{1}{8}\|\nabla\partial_t\phi\|_{L^2}^2 + C\left(\left(G(t)+G^2(t)\right)\ln(\mathrm{e}+G(t)) + G^2(t)+1\right).$$

Then by (4.14),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}+G(t)) + \frac{\rho_*}{4}\|\partial_t \boldsymbol{u}\|_{L^2}^2 + \frac{1}{4}\|\nabla\partial_t\phi\|_{L^2}^2 \le C\left(1+G(t)(\mathrm{e}+G(t))\ln(\mathrm{e}+G(t))\right). \tag{4.23}$$

Due to (4.16), we have

$$\int_t^{t+1} G(s)\mathrm{d}s \le C(1+\int_t^{t+1}\|D\boldsymbol{u}\|_{L^2}^2 + \|\partial_t\phi\|_{L^2}^2\mathrm{d}s) \le C,$$

where using (4.5) and (4.6), we obtain

$$\int_t^{t+1}\|D\boldsymbol{u}\|_{L^2}^2 \le CE(t) \le K_1^2,$$

and

$$\int_t^{t+1}\|\partial_t\phi\|_{L^2}^2 \le C\int_t^{t+1}\|\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi\|_{L^2}^2 + \|D\boldsymbol{u}\|_{L^2}^2\mathrm{d}s \le 2CE(t) \le CK_1^2.$$

Then by the uniform Gronwall lemma in logarithm, we get that, for all $t \ge t_e(R)+1 := t_1(R)$,

$$G(t) \le (K_2')^2.$$

On the other hand, from (4.7) and (4.16), for every $t \ge t_e(R)+1 := t_1(R)$,

$$\|\boldsymbol{u}\|_{H^1}^2 + \|\phi\|_{H^2}^2 \le K_2^2, \tag{4.24}$$

where K_2 depends only on the parameters of the system. Therefore, we can finish the proof by taking $B_1 = B_{\mathbb{Y}_m}(K_2)$. □

4.2. Further Dissipativity

Proposition 4.5. *For every $R > 0$ and for any $(\boldsymbol{u}_0,\phi_0) \in B_{\mathbb{Y}_m}(R)$ there exists $t_2 > 0$ and a bounded subset B_2 of $\mathbf{H}_\sigma^2 \times H^3(\Omega)$, such that*

$$S(t)(\boldsymbol{u}_0,\phi_0) \in B_1 \cap B_2,$$

for all $t \ge t_2$, and therefore $B_1 \cap B_2$ is the compact absorbing set for $S(t)$ on phase space $\mathbb{Y}_m$.

Recall that the set B_1 was obtained in Proposition 4.4.

Proof. Recall that $\mathbf{A}$ is the Stokes operator, and there exists $\Pi \in L^2(0,T;L^2(\Omega))$. Then $-\Delta\boldsymbol{u} + \nabla\Pi = \mathbf{A}\boldsymbol{u}$ a.e. in $\Omega\times(0,\infty)$ with the following estimates:

$$\|\Pi\|_{L^2} \le C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \text{ and } \|\Pi\|_{H^1} \le C\|\mathbf{A}\boldsymbol{u}\|_{L^2}. \tag{4.25}$$

Multiplying $(1.1)_1$ by $\mathbf{A}\boldsymbol{u}$ and taking integration over Ω, we obtain

$$\begin{aligned}&\int_\Omega \rho(\phi)\,\partial_t \boldsymbol{u}\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x-\int_\Omega \frac{\nu(\phi)}{2}\Delta \boldsymbol{u}\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\\ &=-\int_\Omega \rho(\phi)\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x+\int_\Omega \nu'(\phi)\,(Du\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x-\int_\Omega \operatorname{div}(\nabla\phi\otimes\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x,\end{aligned}$$

where

$$-\int_\Omega \frac{\nu(\phi)}{2}\Delta \boldsymbol{u}\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x=\int_\Omega \frac{\nu(\phi)}{2}|\mathbf{A}\boldsymbol{u}|^2\mathrm{d}x+\int_\Omega \frac{\nu'(\phi)}{2}\Pi\nabla\phi\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x,$$

and

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}|^2\mathrm{d}x=&\frac12\int_\Omega \rho'(\phi)\phi_t|\nabla\boldsymbol{u}|^2\mathrm{d}x+\int_\Omega \rho(\phi)u_t\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\\ &-\int_\Omega \rho'(\phi)\nabla\phi\nabla\boldsymbol{u}\cdot\boldsymbol{u}_t\mathrm{d}x+\int_\Omega \rho'(\phi)\nabla\phi\cdot\boldsymbol{u}_t\Pi\mathrm{d}x.\end{aligned}$$

Then we obtain

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}|^2\mathrm{d}x+\int_\Omega \frac{\nu(\phi)}{2}|\mathbf{A}\boldsymbol{u}|^2\mathrm{d}x=-\int_\Omega \rho(\phi)\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\\ &+\int_\Omega \nu'(\phi)\,(\boldsymbol{D}\boldsymbol{u}\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x-\int_\Omega \operatorname{div}(\nabla\phi\otimes\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x-\int_\Omega \frac{\nu'(\phi)}{2}\Pi\nabla\phi\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\\ &+\frac12\int_\Omega \rho'(\phi)\phi_t|\nabla\boldsymbol{u}|^2\mathrm{d}x\int_\Omega \rho'(\phi)\nabla\phi\nabla\boldsymbol{u}\boldsymbol{u}_t\mathrm{d}x+\int_\Omega \rho'(\phi)\nabla\phi\cdot\boldsymbol{u}_t\Pi\mathrm{d}x.\end{aligned}\tag{4.26}$$

By (2.2), (4.5), (4.24), and the Young inequality, we obtain

$$\begin{aligned}\left|-\int_\Omega \rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\right|&\le \rho^*\|\boldsymbol{u}\|_{L^4}\|\nabla\boldsymbol{u}\|_{L^4}\|\mathbf{A}\boldsymbol{u}\|_{L^2}\\ &\le C\rho^*K_1\|\nabla\boldsymbol{u}\|_{L^2}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^{\frac32}\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2+C\|\nabla\boldsymbol{u}\|_{L^2}^4,\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega \nu'(\phi)\,(\boldsymbol{D}\boldsymbol{u}\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\right|&\le \|\nu'(\phi)\|_{L^\infty}\|\boldsymbol{D}\boldsymbol{u}\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\mathbf{A}\boldsymbol{u}\|_{L^2}\\ &\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}^2.\end{aligned}$$

By (2.2), (2.4), (4.5), (4.24), (4.25), and the global bound of ϕ in $L^\infty(\Omega)$, we obtain

$$\begin{aligned}\left|\int_\Omega \operatorname{div}(\nabla\phi\otimes\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\right|&=\left|\int_\Omega (\Delta\phi\nabla\phi)\cdot \mathbf{A}\boldsymbol{u}\mathrm{d}x\right|\\ &\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}^2\|\Delta\phi\|_{L^2}^2\\ &\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}^2,\end{aligned}$$

$$\left|-\int_\Omega \frac{\nu'(\phi)}{2}\Pi\nabla\phi\cdot\mathbf{A}\boldsymbol{u}\mathrm{d}x\right| \le \frac{1}{2}\|\nu'(\phi)\|_{L^\infty}\|\Pi\|_{L^4}\|\nabla\phi\|_{L^4}\|\mathbf{A}\boldsymbol{u}\|_{L^2}$$
$$\le C_{\Omega,\nu}\|\Pi\|_{L^2}^{\frac{1}{2}}\|\Pi\|_{H^1}^{\frac{1}{2}}\|\phi\|_{L^\infty}^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}}\|\mathbf{A}\boldsymbol{u}\|_{L^2}$$
$$\le C_{\Omega,\nu}\|\phi\|_{H^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{4}}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^{\frac{7}{4}}$$
$$\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 + C\|\phi\|_{H^2}^4\|\nabla\boldsymbol{u}\|_{L^2}^2$$
$$\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 + C,$$

$$\left|\frac{1}{2}\int_\Omega \rho'(\phi)\phi_t|\nabla\boldsymbol{u}|^2\mathrm{d}x\right| \le C\|\phi_t\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^4}^2$$
$$\le C\|\phi_t\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}\|\mathbf{A}\boldsymbol{u}\|_{L^2}$$
$$\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 + C\|\phi_t\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}^2$$
$$\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 + C,$$

$$\left|\int_\Omega \rho'(\phi)\nabla\phi\nabla\boldsymbol{u}\boldsymbol{u}_t\mathrm{d}x\right| \le C\,\|\nabla\phi\|_{L^\infty}\,\|\nabla\boldsymbol{u}\|_{L^2}\,\|\boldsymbol{u}_t\|_{L^2}$$
$$\le \frac{\rho_*}{16}\|\boldsymbol{u}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{L^2}^2\|\nabla\phi\|_{L^\infty}^2$$
$$\le \frac{\rho_*}{16}\|\boldsymbol{u}_t\|_{L^2}^2 + C\|\nabla\phi\|_{L^\infty}^2,$$

and

$$\left|\int_\Omega \rho'(\phi)\nabla\phi\cdot\boldsymbol{u}_t\Pi dx\right| \le C\|\nabla\phi\|_{L^4}\|\boldsymbol{u}_t\|_{L^2}\|\Pi\|_{L^4}$$
$$\le C\|\nabla\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}}\|\boldsymbol{u}_t\|_{L^2}\|\Pi\|_{L^2}^{\frac{1}{2}}\|\Pi\|_{H^1}^{\frac{1}{2}}$$
$$\le C\|\boldsymbol{u}_t\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{4}}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^{\frac{3}{4}}$$
$$\le C\|\boldsymbol{u}_t\|_{L^2}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^{\frac{3}{4}}$$
$$\le \frac{\rho_*}{16}\|\boldsymbol{u}_t\|_{L^2}^2 + C\|\mathbf{A}\boldsymbol{u}\|_2^{\frac{3}{2}}$$
$$\le \frac{\rho_*}{16}\|\boldsymbol{u}_t\|_{L^2}^2 + \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 + C.$$

Thus, by the Sobolev embeddings, (2.6) and (4.20), we arrive at

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}|^2\mathrm{d}x + \frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 &\le C(1+\|\nabla\phi\|_{L^\infty}^2) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\|\nabla\phi\|_{H^1}^2 \ln(C\frac{\|\nabla\phi\|_{W^{1,p}}}{\|\nabla\phi\|_{H^1}})) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\|\phi\|_{H^2}^2 + \|\phi\|_{H^2}^2 \ln\|\phi\|_{W^{2,p}}) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\ln\|\phi\|_{W^{2,p}}) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\ln(1+\|\phi\|_{L^p}+\|\mu\|_{L^p}) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\ln(1+\|\mu\|_{L^p}) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\ln(1+\|\nabla\phi_t\|_{L^2}+G(t)) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\ln(1+\|\nabla\phi_t\|_{L^2}) + \ln(1+G(t))) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+\|\nabla\phi_t\|_{L^2}) + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 \\
&\le \frac{1}{8}\|\nabla\phi_t\|_{L^2}^2 + \frac{\rho_*}{8}\|\partial_t \boldsymbol{u}\|_{L^2}^2 + C.
\end{aligned}
\tag{4.27}
$$

Adding (4.27) and (4.23), we obtain that as $t \ge t_2(R)$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e} + G(t) + \frac{1}{2}\int_\Omega \rho(\phi)|\nabla\boldsymbol{u}|^2\mathrm{d}x\Big) + \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}\|_{L^2}^2 + \frac{1}{8}\|\nabla\partial_t\phi\|_{L^2}^2 + \frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}\|_{L^2}^2 \\
&\le C(1+G(t)(\mathrm{e}+G(t))\ln(\mathrm{e}+G(t))) \le C,
\end{aligned}
\tag{4.28}
$$

where $C = C(K_1, K_2)$ and depends on the parameters of system (1.1). Integrating (4.28), and by (4.24), we get there exists M_1 depending on K_1, K_2, such that

$$
\mathrm{e} + G(t) + \frac{\rho_*}{2}\|\nabla\boldsymbol{u}\|_{L^2}^2 + C\int_t^{t+1}(\|\partial_t\boldsymbol{u}\|_{L^2}^2 + \|\nabla\partial_t\phi\|_{L^2}^2 + \|\Delta\boldsymbol{u}\|_{L^2}^2)\mathrm{d}s \le M_1,
\tag{4.29}
$$

for every $t \ge t_2$.

Multiplying $(1.1)_1$ with $-\Delta\boldsymbol{u}_t$ and taking integration over Ω, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{\nu(\phi)}{2}\Delta\boldsymbol{u}\cdot\Delta\boldsymbol{u}_t\mathrm{d}x + \int_\Omega \nu'(\phi)\nabla\phi D\boldsymbol{u}\cdot\Delta\boldsymbol{u}_t\mathrm{d}x + \int_\Omega \rho(\phi)|\nabla\boldsymbol{u}_t|^2\mathrm{d}x + \int_\Omega \rho'(\phi)\nabla\phi\nabla\boldsymbol{u}_t\boldsymbol{u}_t\mathrm{d}x \\
&\quad + \int_\Omega \nabla\boldsymbol{u}_t : \nabla(\rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u})\mathrm{d}x = -\int_\Omega \nabla\,\mathrm{div}(\nabla\phi\otimes\nabla\phi) : \nabla\boldsymbol{u}_t\mathrm{d}x,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\nu(\phi)}{2}|\Delta\boldsymbol{u}|^2\mathrm{d}x + \int_\Omega \rho(\phi)|\nabla\boldsymbol{u}_t|^2\mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega \frac{\nu'(\phi)}{2}\phi_t|\Delta\boldsymbol{u}|^2\mathrm{d}x - \int_\Omega \nabla(\nu'(\phi)D\boldsymbol{u}\nabla\phi) : \nabla\boldsymbol{u}_t\mathrm{d}x - \int_\Omega \rho'(\phi)\nabla\phi\nabla\boldsymbol{u}_t\boldsymbol{u}_t\mathrm{d}x \\
&\quad - \int_\Omega \nabla\boldsymbol{u}_t : \nabla(\rho(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u})\mathrm{d}x - \int_\Omega \nabla\,\mathrm{div}(\nabla\phi\otimes\nabla\phi) : \nabla\boldsymbol{u}_t\mathrm{d}x.
\end{aligned}
\tag{4.30}
$$

By (2.5) and (4.24), and the Sobolev embeddings, we obtain

$$
\begin{aligned}
\Big|\int_\Omega \frac{\nu'(\phi)}{4}\phi_t|\Delta \boldsymbol{u}|^2 \mathrm{d}x\Big| &\le C\|\phi_t\|_{L^\infty}\|\Delta \boldsymbol{u}\|_{L^2}^2 \\
&\le C\|\phi_t\|_{H^1}\ln^{\frac{1}{2}}\Big(C\frac{\|\phi_t\|_{H^2}}{\|\phi_t\|_{H^1}}\Big)\|\Delta \boldsymbol{u}\|_{L^2}^2 \\
&\le C(\|\phi_t\|_{H^1} + \|\phi_t\|_{H^1}^{\frac{1}{2}}\|\phi_t\|_{H^2}^{\frac{1}{2}})\|\Delta \boldsymbol{u}\|_{L^2}^2 \\
&\le C(\|\nabla\phi_t\|_{L^2} + \|\Delta\phi_t\|_{L^2})\|\Delta \boldsymbol{u}\|_{L^2}^2 \\
&\le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + C(\|\Delta \boldsymbol{u}\|_{L^2}^4 + \|\nabla\phi_t\|_{L^2}^2 + 1),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_\Omega \nabla(\nu'(\phi)D\boldsymbol{u}\nabla\phi) : \nabla \boldsymbol{u}_t \mathrm{d}x\Big| \\
&\le C\int_\Omega (|\nabla\phi|^2|D\boldsymbol{u}||\nabla \boldsymbol{u}_t| + |D^2\boldsymbol{u}||\nabla\phi||\nabla u_t| + |D\boldsymbol{u}||D^2\phi||\nabla \boldsymbol{u}_t|)\mathrm{d}x \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C\int_\Omega (|\nabla\phi|^4|D\boldsymbol{u}|^2 + |D^2\boldsymbol{u}|^2|\nabla\phi|^2 + |D\boldsymbol{u}|^2|D^2\phi|^2)\mathrm{d}x \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C(\|\nabla\phi\|_{L^8}^4\|D\boldsymbol{u}\|_{L^4}^2 + \|\nabla\phi\|_{L^\infty}^2\|\Delta \boldsymbol{u}\|_{L^2}^2 + \|D^2\phi\|_{L^4}^2\|D\boldsymbol{u}\|_{L^4}^2 \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C(\|\Delta \boldsymbol{u}\|_{L^2} + (\|\nabla\phi_t\|_{L^2} + 1)\|\Delta \boldsymbol{u}\|_{L^2}^2 + \|\nabla\Delta\phi\|_{L^2}\|\Delta \boldsymbol{u}\|_{L^2}).
\end{aligned}
$$

By the same argument in (4.27), the Agmon inequality together with the Sobolev embeddings, we obtain

$$
\begin{aligned}
\Big|\int_\Omega \rho'(\phi)\nabla\phi\nabla \boldsymbol{u}_t\boldsymbol{u}_t\mathrm{d}x\Big| &\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C\int_\Omega |\nabla\phi|^2|\boldsymbol{u}_t|^2\mathrm{d}x \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C(\|\nabla\phi\|_{L^\infty}^2\|\boldsymbol{u}_t\|_{L^2}^2) \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C(1 + \|\nabla\phi_t\|_{L^2}^2)\|\boldsymbol{u}_t\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_\Omega \nabla \boldsymbol{u}_t : \nabla(\rho(\phi)\boldsymbol{u}\cdot\nabla \boldsymbol{u})\mathrm{d}x\Big| \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C\int_\Omega (|\nabla\phi|^2|\boldsymbol{u}|^2|\nabla \boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^4 + |\boldsymbol{u}|^2|D^2\boldsymbol{u}|^2)\mathrm{d}x \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C(\|\nabla\phi\|_{L^\infty}^2\|\boldsymbol{u}\|_{L^4}^2\|\nabla \boldsymbol{u}\|_{L^4}^2 + \|\nabla \boldsymbol{u}\|_{L^4}^4 + \|\boldsymbol{u}\|_{L^\infty}^2\|D^2\boldsymbol{u}\|_{L^2}^2)\mathrm{d}x \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x + C((1 + \|\nabla\phi_t\|_{L^2})\|\Delta \boldsymbol{u}\|_{L^2} + \|\Delta \boldsymbol{u}\|_{L^2}^2 + \|\Delta \boldsymbol{u}\|_{L^2}^4),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_\Omega \nabla\,\mathrm{div}(\nabla\phi\otimes\nabla\phi) : \nabla \boldsymbol{u}_t \mathrm{d}x\Big| \\
&= \Big|\int_\Omega \nabla(\frac{1}{2}\nabla\,|\nabla\phi|^2 + \Delta\phi\nabla\phi) : \nabla \boldsymbol{u}_t \mathrm{d}x\Big| \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2 \mathrm{d}x + C\int_\Omega (|D^3\phi|^2|\nabla\phi|^2 + |\Delta\phi|^2|D^2\phi|^2)\mathrm{d}x \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2 \mathrm{d}x + C(\|\nabla\phi\|_{L^\infty}^2\|\nabla\Delta\phi\|_{L^2}^2 + \|\Delta\phi\|_{L^4}^2\|D^2\phi\|_{L^4}^2) \\
&\le \frac{1}{8}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2 \mathrm{d}x + C((1+\|\nabla\phi_t\|_{L^2})\|\nabla\Delta\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{L^2}^2).
\end{aligned}
$$

From the above estimates, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\nu(\phi)}{2}|\Delta \boldsymbol{u}|^2\mathrm{d}x + \frac{1}{2}\int_\Omega \rho(\phi)|\nabla \boldsymbol{u}_t|^2\mathrm{d}x \le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + C(\|\nabla\Delta\phi\|_{L^2}^4 \\
+ \|\Delta \boldsymbol{u}\|_{L^2}^4 + \|\nabla\phi_t\|_{L^2}^2 + \|\boldsymbol{u}_t\|_{L^2}^2(1+\|\nabla\phi_t\|_{L^2}^2) + \|\Delta \boldsymbol{u}\|_{L^2}^2 + 1).
\end{aligned}
\tag{4.31}
$$

On the other hand, differentiating $(1.1)_3$ with respect to t, multiplying $-\Delta\phi_t$, integrating over Ω, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi_t\|_{L^2}^2 + \|\Delta\phi_t\|_{L^2}^2 = &\int_\Omega (\boldsymbol{u}_t\cdot\nabla\phi + \boldsymbol{u}\cdot\nabla\phi_t + F''(\phi)\phi_t)\Delta\phi_t\mathrm{d}x \\
&+ \theta_0\|\nabla\phi_t\|_{L^2}^2 + \int_\Omega (\rho''(\phi)\phi_t\frac{|\boldsymbol{u}|^2}{2} + \rho'(\phi)\boldsymbol{u}\cdot\boldsymbol{u}_t)\Delta\phi_t\mathrm{d}x.
\end{aligned}
\tag{4.32}
$$

By Theorem 3.3, there exists $\delta = \delta(1) > 0$, such that for every $x\in\overline{\Omega}$ and $t\ge 1$,

$$
-1+\delta \le \phi(x,t) \le 1-\delta.
$$

Then $\|F''(\phi)\|_{L^\infty} \le C(\delta)$. By (4.5), (4.24), and (4.27), we obtain

$$
\begin{aligned}
&\Big|\int_\Omega (\boldsymbol{u}_t\cdot\nabla\phi + \boldsymbol{u}\cdot\nabla\phi_t + F''(\phi)\phi_t)\Delta\phi_t\mathrm{d}x\Big| \\
&\le \frac{1}{8}\|\Delta\phi_t\|_{L^2}^2 + \int_\Omega (|\boldsymbol{u}_t|^2|\nabla\phi|^2 + |\boldsymbol{u}|^2|\nabla\phi_t|^2 + \|F''(\phi)\|_{L^\infty}^2\|\phi_t\|_{L^2}^2\mathrm{d}x \\
&\le \frac{1}{8}\|\Delta\phi_t\|_{L^2}^2 + C(\|\nabla\phi\|_{L^\infty}^2\|\boldsymbol{u}_t\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^4}^2\|\nabla\phi_t\|_{L^4}^2 + 1) \\
&\le \frac{1}{8}\|\Delta\phi_t\|_{L^2}^2 + C((1+\|\nabla\phi_t\|_{L^2}^2)\|\boldsymbol{u}_t\|_{L^2}^2 + \|\Delta\phi_t\|_{L^2} + 1) \\
&\le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + C(1+\|\boldsymbol{u}_t\|_{L^2}^2(1+\|\nabla\phi_t\|_{L^2}^2)),
\end{aligned}
$$

and by the Agmon inequality (2.3), we have that

$$\begin{aligned}\Big|\int_\Omega (\rho''(\phi)\phi_t\frac{|\boldsymbol{u}|^2}{2} + \rho'(\phi)\boldsymbol{u}\cdot\boldsymbol{u}_t)\Delta\phi_t \mathrm{d}x\Big| &\le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + C\int_\Omega (|\phi_t|^2|\boldsymbol{u}|^4 + |\boldsymbol{u}|^2|\boldsymbol{u}_t|^2)\mathrm{d}x \\ &\le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + C(\|\boldsymbol{u}\|_{L^\infty}^4\|\phi_t\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^4}^2\|\boldsymbol{u}_t\|_{L^4}^2) \\ &\le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + C(\|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{u}_t\|_{L^2}\|\nabla\boldsymbol{u}_t\|_{L^2}) \\ &\le \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 + \frac{1}{4}\int_\Omega \rho(\phi)|\nabla\boldsymbol{u}_t|^2\mathrm{d}x + C(\|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{u}_t\|_{L^2}^2).\end{aligned}$$

From the above estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi_t\|_{L^2}^2 + \frac{1}{2}\|\Delta\phi_t\|_{L^2}^2 \\ &\quad\le \frac{1}{4}\int_\Omega \rho(\phi)|\nabla\boldsymbol{u}_t|^2\mathrm{d}x + C((1 + \|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\nabla\phi_t\|_{L^2}^2)\|\boldsymbol{u}_t\|_{L^2}^2 + \|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\nabla\phi_t\|_{L^2}^2).\end{aligned}$$

Adding (4.31) into the above inequality, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\int_\Omega \frac{\nu(\phi)}{2}|\Delta\boldsymbol{u}|^2\mathrm{d}x + \|\nabla\phi_t\|_{L^2}^2) + \frac{1}{4}\int_\Omega \rho(\phi)|\nabla\boldsymbol{u}_t|^2\mathrm{d}x + \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 \\ &\le C(\|\nabla\Delta\phi\|_{L^2}^4 + \|\Delta\boldsymbol{u}\|_{L^2}^4 + (1 + \|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\nabla\phi_t\|_{L^2}^2)\|\boldsymbol{u}_t\|_{L^2}^2 + 1).\end{aligned} \tag{4.33}$$

Taking the gradient of $(1.1)_3$, we obtain

$$\nabla\phi_t + \nabla\boldsymbol{u}\cdot\nabla\phi + \boldsymbol{u}\cdot D^2\phi - \nabla\Delta\phi + F''(\phi)\nabla\phi - \theta_0\nabla\phi + \rho''(\phi)\nabla\phi\frac{|\boldsymbol{u}|^2}{2} + \rho'(\phi)\boldsymbol{u}\cdot\nabla\boldsymbol{u} = 0,$$

and consequently,

$$\begin{aligned}\|\nabla\Delta\phi\|_{L^2}^2 &\le \|\nabla\phi_t\|_{L^2}^2 + \|\nabla\boldsymbol{u}\|_{L^4}^2\|\nabla\phi\|_{L^4}^2 + \|\boldsymbol{u}\|_{L^\infty}^2\|\Delta\phi\|_{L^2}^2 + \|F''(\phi)\|_{L^\infty}^2\|\nabla\phi\|_{L^2}^2 \\ &\quad + \theta_0\|\nabla\phi\|_{L^2}^2 + C(\|\nabla\phi\|_{L^\infty}^2\|\boldsymbol{u}\|_{L^4}^4 + \|\boldsymbol{u}\|_{L^4}^2\|\nabla\boldsymbol{u}\|_{L^4}^2) \\ &\le C(1 + \|\nabla\phi_t\|_{L^2}^2 + \|\Delta\boldsymbol{u}\|_{L^2}^2).\end{aligned} \tag{4.34}$$

Then by (4.33), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + \frac{1}{4}\int_\Omega \rho(\phi)|\nabla\boldsymbol{u}_t|^2\mathrm{d}x + \frac{1}{4}\|\Delta\phi_t\|_{L^2}^2 \le C\Lambda(t)(1 + \Lambda(t) + \|\boldsymbol{u}_t\|_{L^2}^2), \tag{4.35}$$

where $\Lambda(t) = \frac{1}{2}\int_\Omega \frac{\nu(\phi)}{2}|\Delta\boldsymbol{u}|^2\mathrm{d}x + \|\nabla\phi_t\|_{L^2}^2$, and

$$C(\|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\nabla\phi_t\|_{L^2}^2) \le \Lambda(t) \le C'(\|\Delta\boldsymbol{u}\|_{L^2}^2 + \|\nabla\phi_t\|_{L^2}^2). \tag{4.36}$$

By (4.29), we obtain

$$\int_t^{t+1} \Lambda(s)\,\mathrm{d}s \le C, \qquad \int_t^{t+1} \|\boldsymbol{u}_t\|_{L^2}^2\,\mathrm{d}s \le C.$$

Then by the uniform Gronwall lemma (e.g., see [32]), we can obtain that

$$\|\Delta \boldsymbol{u}\|_{L^2}^2 + \|\nabla \phi_t\|_{L^2}^2 \leq (K_3')^2,$$

and by (4.34),

$$\|\Delta \boldsymbol{u}\|_{L^2}^2 + \|\phi\|_{H^3}^2 \leq (K_3'')^2,$$

for every $t \geq t_2 + 1 = t_e(R) + 2$, and

$$\|\boldsymbol{u}\|_{H^2}^2 + \|\phi\|_{H^3}^2 \leq K_3^2,$$

for each $t \geq t_3 := t_2 + 1$. Thus, we obtain that $B_1 \cap B_2$ is a compact absorbing set by the compact Sobolev embedding theorem, where $B_1 = B_{\mathbb{Y}_m}(K_2)$ and $B_2 = B_{\mathbf{H}_\sigma^2 \times H^3(\Omega)}(K_3)$. □

The proposition above directly leads to the following corollary.

Corollary 4.6. *For each bounded set $B \subset \mathbb{Y}_m$, there exists $t_3 = t_3(B) > t_e(R) + 2$, such that $\bigcup_{t \geq t_2} S(t)B$ is relatively compact in $\mathbb{Y}_m$.*

4.3. Continuity of the semigroup

In this section, we will prove that for every $t > 0$, the semigroup $S(t)$ is continuous on the phase space $\mathbb{Y}_m$. Let $\{(\boldsymbol{u}_0^n, \phi_0^n)\}_{n\geq 1}$ be a sequence in $\mathbb{Y}_m$ and $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}_m$, such that $(\boldsymbol{u}_0^n, \phi_0^n) \to (\boldsymbol{u}_0, \phi_0)$ in the strong topology of $\mathbb{Y}_m$. Let $(\boldsymbol{u}^n(t), \phi^n(t)) = S(t)(\boldsymbol{u}_0^n, \phi_0^n)$, $(\boldsymbol{u}(t), \phi(t)) = S(t)(\boldsymbol{u}_0, \phi_0)$. Then by assumption, there exists $\tilde{M}_1 > 0$, such that, for all $n = 1, 2, \cdots$,

$$\|\boldsymbol{u}_0^n\|_{\mathbf{V}_\sigma}^2 + \|\phi_0^n\|_{H^2}^2 \leq \tilde{M}_1^2 \quad \text{and} \quad \|\boldsymbol{u}_0\|_{\mathbf{V}_\sigma}^2 + \|\phi_0\|_{H^2}^2 \leq \tilde{M}_1^2.$$

Proposition 4.7. *For any $t \geqslant 0$, there is a constant $C = C(\tilde{M}_1, t)$, $\tilde{M} = \tilde{M}(\tilde{M}_1, t)$, such that*

$$\|\boldsymbol{u}^n(t) - \boldsymbol{u}(t)\|_{\mathbf{V}_\sigma} + \|\phi^n(t) - \phi(t)\|_{H^2} \leq C(\|\boldsymbol{u}_0^n - \boldsymbol{u}_0\|_{\mathbf{H}_\sigma}^{\frac{1}{2}} + \|\phi_0^n - \phi_0\|_{H^1}^{\frac{1}{2}})\mathrm{e}^{-\tilde{M}}.$$

Proof. Let $(\boldsymbol{u}_1, \phi_1), (\boldsymbol{u}_2, \phi_2)$ be the corresponding strong solutions initiated from $(\boldsymbol{u}_{01}, \phi_{01}), (\boldsymbol{u}_{02}, \phi_{02})$ and let $(\boldsymbol{u}, \phi) = (\boldsymbol{u}_1, \phi_1) - (\boldsymbol{u}_2, \phi_2)$.

Following the proof of Theorem 4.1 in [31], and the uniqueness of the strong solution, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{\rho(\phi_1)}{2}|\boldsymbol{u}|^2\mathrm{d}x + \int_\Omega \frac{1}{2}|\nabla\phi|^2\mathrm{d}x\Big) + \frac{\nu_*}{2}\int_\Omega |D\boldsymbol{u}|^2\mathrm{d}x + \frac{1}{2}\|\Delta\phi\|_{L^2}^2 \\
&\leq A(t)\int_\Omega \frac{\rho(\phi_1)}{2}|\boldsymbol{u}|^2\mathrm{d}x + B(t)\|\nabla\phi\|_{L^2}^2,
\end{aligned}$$

where

$$A(t) = C\left(\|\partial_t \boldsymbol{u}_2 + \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u}_2\|_{L^2}^2 + \|\boldsymbol{u}_1\|_{L^\infty}^2 + \|\boldsymbol{u}_2\|_{L^\infty}^2 + 1\right)$$

and

$$\begin{aligned}
B(t) = C(&\|\partial_t \boldsymbol{u}_2 + \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u}_2\|_{L^2}^2 + \|\Delta\phi_1\|_{L^4}^2 + \|D\boldsymbol{u}_2\|_{L^4}^2 + \|\boldsymbol{u}_1\|_{L^\infty}^2 \\
&+ \|F''(\phi_1)\|_{L^2}^2 \ln(\|F''(\phi_1)\|_{L^2}) + \|F''(\phi_2)\|_{L^2}^2 \ln(\|F''(\phi_2)\|_{L^2}) + 1),
\end{aligned}$$

moreover, A and B are both in $L^1(0, T)$ for any $T > 0$. Let $\tilde{M} = \max\{\int_0^t A(s)\mathrm{d}s, \int_0^t B(s)\mathrm{d}s\}$ which depends on $\tilde{M}_1$ following the proof in [31], and use the Gronwall lemma to obtain

$$\int_\Omega \rho(\phi_1)\frac{|\boldsymbol{u}|^2}{2}\mathrm{d}x + \int_\Omega \frac{1}{2}|\nabla\phi|^2\mathrm{d}x \leq \Big(\int_\Omega \rho(\phi_{01})\frac{|\boldsymbol{u}_{01} - \boldsymbol{u}_{02}|^2}{2}\mathrm{d}x + \|\nabla\phi_{01} - \nabla\phi_{02}\|_{L^2}^2\Big)\mathrm{e}^{-\tilde{M}}.$$

Then

$$\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^2}^2+\|\phi_1-\phi_2\|_{H^1}^2\le C(t)(\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\|_{L^2}^2+\|\phi_{01}-\phi_{02}\|_{H^1}^2)\mathrm{e}^{-\tilde{M}}.$$

Following the proof of Proposition 4.5, we obtain that for each $\tau, t : 0<\tau<t\le T$,

$$\begin{aligned}\mathrm{e}+G(t)+\rho_*\|\nabla\boldsymbol{u}\|_{L^2}^2+\int_\tau^t(\|\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\phi_t\|_{L^2}^2+\|\Delta\boldsymbol{u}\|_{L^2}^2)\mathrm{d}s\\ \le C(\tilde{M}_1)\left(\mathrm{e}+G(\tau)+\rho_*\|\nabla\boldsymbol{u}(\tau)\|_{L^2}^2\right).\end{aligned}\tag{4.37}$$

Note that in the proof of Proposition 4.5, M_1 in (4.29) is independent of the initial value $(\boldsymbol{u}_0,\phi_0)$ and (4.29) is valid for $t\ge t_2+1$. However, here in (4.37), $C(\tilde{M}_1)$ depends on the initial data, but we need the inequality for $t>0$. Then there is a $\tau_0\in(0,\tau)$ such that $(\boldsymbol{u}(\tau_0),\phi(\tau_0))$ satisfies $\boldsymbol{u}(\tau_0)\in\mathbb{V}_\sigma,\phi(\tau_0)\in H^2(\Omega),\partial_n\phi(\tau_0)=0$ on $\partial\Omega$, $F''(\phi(\tau_0))\in L^2(\Omega)$ and

$$G(\tau_0)+\rho_*\|\nabla\boldsymbol{u}(\tau_0)\|_{L^2}^2\le\tilde{M}_1^2.$$

Taking $(\boldsymbol{u}(\tau_0),\phi(\tau_0))$ as the initial datum, there exists a global strong solution $(\tilde{\boldsymbol{u}}(t),\tilde{\phi}(t))$ on $[\tau_0,\infty)$, for each $t\ge\tau_0$, there exists a constant $C_0=C_0(\tilde{M}_1)$,

$$G(\tilde{\boldsymbol{u}}(t),\tilde{\phi}(t))+\rho_*\|\nabla\tilde{\boldsymbol{u}}\|_{L^2}^2+\int_\tau^t(\|\tilde{\boldsymbol{u}}_t\|_{L^2}^2+\|\nabla\tilde{\phi}_t\|_{L^2}^2+\|\Delta\tilde{\boldsymbol{u}}\|_{L^2}^2)\mathrm{d}s\le C_0,$$

where

$$G(\tilde{\boldsymbol{u}}(t),\tilde{\phi}(t)):=\frac{1}{2}\int_\Omega\nu(\tilde{\phi}(t))|D\tilde{\boldsymbol{u}}(t)|^2\mathrm{d}x+\frac{1}{2}\|\tilde{\phi}_t(t)\|_{L^2}^2.$$

Also by (4.35), there is a constant $C_1=C_1(\tilde{M}_1)$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(\tilde{\boldsymbol{u}}(t),\tilde{\phi}(t))\le C_1\Lambda(\tilde{\boldsymbol{u}}(t),\tilde{\phi}(t))(\Lambda(\tilde{\boldsymbol{u}}(t),\tilde{\phi}(t))+\|\tilde{\boldsymbol{u}}_t(t)\|_{L^2}^2+1).$$

Using the uniform Gronwall lemma again, we obtain

$$\|\Delta\tilde{\boldsymbol{u}}(t)\|_{L^2}^2+\|\nabla\tilde{\phi}_t(t)\|_{L^2}^2\le\tilde{M}_2'^2,$$

and

$$\|\Delta\tilde{\boldsymbol{u}}(t)\|_{L^2}^2+\|\widetilde{\phi}(t)\|_{H^3}^2\le\tilde{M}_2^2,$$

for every $t\ge\tau$, where $\tilde{M}_2'=\tilde{M}_2'(\tau,\tilde{M}_1)$ and $\tilde{M}_2=\tilde{M}_2(\tau,\tilde{M}_1)$. Then we obtain that for every $\tau>0$, $i=1,2$,

$$\sup_{t\ge\tau}(\|\Delta\boldsymbol{u}_i(t)\|_{L^2}^2+\|\phi_i(t)\|_{H^3}^2)\le\tilde{M}_2^2,$$

by viewing $(\boldsymbol{u}_{01},\phi_{01}),(\boldsymbol{u}_{02},\phi_{02})$ as initial values of the trajectories. Now let $t^*>0$, $\tau=\frac{1}{2}t^*$, $t^*>\tau>0$. By interpolation,

$$\begin{aligned}&\|\boldsymbol{u}_1(t^*)-\boldsymbol{u}_2(t^*)\|_{\mathbb{V}_\sigma}+\|\phi_1(t^*)-\phi_2(t^*)\|_{H^2}\\ &\le C(\|\boldsymbol{u}_1(t^*)-\boldsymbol{u}_2(t^*)\|_{\mathbf{H}_\sigma}^{\frac{1}{2}}+\|\phi_1(t^*)-\phi_2(t^*)\|_{H^1}^{\frac{1}{2}})(\|\boldsymbol{u}_1(t^*)-\boldsymbol{u}_1(t^*)\|_{\mathbf{H}_\sigma^2}^{\frac{1}{2}}+\|\phi_1(t^*)-\phi_2(t^*)\|_{H^3}^{\frac{1}{2}})\\ &\le C\tilde{M}_2^{\frac{1}{2}}(\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\|_{\mathbf{H}_\sigma}^{\frac{1}{2}}+\|\phi_{01}-\phi_{02}\|_{H^1}^{\frac{1}{2}})\mathrm{e}^{-\tilde{M}}.\end{aligned}$$

Replacing $(\boldsymbol{u}_1, \phi_1)$ by $(\boldsymbol{u}_n, \phi_n)$, $(\boldsymbol{u}_2, \phi_2)$ by $(\boldsymbol{u}, \phi)$, we obtain

$$\|\boldsymbol{u}^n(t^*) - \boldsymbol{u}(t^*)\|_{\mathbf{V}_\sigma} + \|\phi^n(t^*) - \phi(t^*)\|_{H^2} \leq C\tilde{M}_2^{\frac{1}{2}}(\|\boldsymbol{u}_0^n - \boldsymbol{u}_0\|_{\mathbf{H}_\sigma}^{\frac{1}{2}} + \|\phi_0^n - \phi_0\|_{H^1}^{\frac{1}{2}})\mathrm{e}^{-\tilde{M}},$$

where $C = C(t^*, \tilde{M}_1)$ and $\tilde{M} = \tilde{M}(t^*, \tilde{M}_1)$. □

Thanks to these propositions above, we obtain the following theorem by Theorem 1.1 in [32].

Theorem 4.8. *Let the assumptions of Theorem 3.2 and Theorem 3.3 hold. The dynamical system* $(\mathbb{Y}_m, S(t))$ *possesses a unique global attractor* $\mathcal{A}_m \subset \mathbb{Y}_m$*, which is a connected compact set and has the following properties:*

(a) $\mathcal{A}_m$ *is strictly invariant in* $\mathbb{Y}_m$*, i.e.,* $S(t)\mathcal{A}_m = \mathcal{A}_m$*, for every* $t \geq 0$*;*

(b) $\mathcal{A}_m$ *is an attracting set for* $S(t)$ *on* $\mathbb{Y}_m$*, i.e., for every bounded ball* $B_{\mathbb{Y}_m}$ *in the phase space* $\mathbb{Y}_m$

$$\lim_{t\to\infty} \operatorname{dist}(S(t)B_{\mathbb{Y}_m}(R), \mathcal{A}_m) = 0,$$

where the Hausdorff semi-distance between sets is defined by $\operatorname{dist}(A, B) := \sup_{a\in A} \inf_{b\in B} \operatorname{dist}_{\mathbb{Y}_m}(a, b)$.

Note that from Proposition 4.5, we know that the obtained global attractor $\mathcal{A}_m$ is bounded in $\mathbf{H}_\sigma^2 \times H^3(\Omega)$.

Author contributions

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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