Normalized solutions for pseudo-relativistic Schrödinger equations

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Abstract: In this paper, we consider the existence and multiplicity of normalized solutions to the following pseudo-relativistic Schrödinger equations

\begin{align*}
\begin{cases}
\sqrt{-\Delta + m^2} u + \lambda u = \theta |u|^{p-2} v + |u|^{2s-2} v, & x \in \mathbb{R}^N, \\
u > 0, & \int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}
\end{align*}

where $N \geq 2$, $a, \theta, m > 0$, $\lambda$ is a real Lagrange parameter, $2 < p < 2s = \frac{2N}{N-1}$ and $2s$ is the critical Sobolev exponent. The operator $\sqrt{-\Delta + m^2}$ is the fractional relativistic Schrödinger operator. Under appropriate assumptions, with the aid of truncation technique, concentration-compactness principle and genus theory, we show the existence and the multiplicity of normalized solutions for the above problem.

Keywords: pseudo-relativistic Schrödinger operator; normalized solutions; Sobolev critical exponent

Mathematics Subject Classification: 35J20, 35R03, 58E05

1. Introduction

This paper deals with the following pseudo-relativistic equation of the form:

\begin{align*}
\begin{cases}
\sqrt{-\Delta + m^2} u + \lambda u = \theta |u|^{p-2} u + |u|^{2s-2} u, & x \in \mathbb{R}^N, \\
u > 0, & \int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}
\end{align*}

(1.1)

where the frequency $\lambda$ as a real Lagrange parameter and is part of the unknowns, $2 < p < 2s$. For $s \in (0, 1)$, the operator $(-\Delta + m^2)^s$ is defined in Fourier space as multiplication by the symbol $(|\xi|^2 + m^2)^s$ (see [1, 2]) i.e., for any $u : \mathbb{R}^N \to \mathbb{R}$ belonging to the Schwartz space $S(\mathbb{R}^N)$ of rapidly decreasing functions,

$$\mathcal{F}((-\Delta + m^2)^s u)(\xi) := (|\xi|^2 + m^2)^s \mathcal{F} u(\xi), \quad \forall \xi \in \mathbb{R}^N,$$
where we denote by
\[ \mathcal{F}u(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{i\xi \cdot x} u(x) dx, \quad \xi \in \mathbb{R}^N, \]
the Fourier transform of \( u \). Also, we show an alternative definition of \((-\Delta + m^2)^s\) (see [2, 3]):
\[ (-\Delta + m^2)^s u(x) := m^{2s} u(x) + C(N, s) m^{\frac{N+2s}{2}} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} K_{\frac{N+2s}{2}}(m|x-y|) dy, \quad x \in \mathbb{R}^N, \quad (1.2) \]
where \( P.V. \) is the Cauchy principal value, \( K_\iota \) is the modified Bessel function of the third kind of index \( \iota \) (see [4, 5]) and
\[ C(N, s) := 2^{-\frac{N+2s}{2}} \Gamma^{\frac{N}{2}} 2^{2s} s(1-s) \frac{\Gamma(2-s)}{\Gamma(2-\iota s)}. \]

Once \( m \to 0 \), then \((-\Delta + m^2)^s\) reduces to the classical fractional Laplacian \((-\Delta)^s\) defined via Fourier transform by
\[ \mathcal{F}((-\Delta)^s u)(\xi) := |\xi|^{2s} \mathcal{F}u(\xi), \quad \xi \in \mathbb{R}^N. \]

At the same time, by singular integrals, we also get
\[ (-\Delta)^s u(x) := C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N, \quad C_{N,s} := \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} s(1-2s) \quad (1.3) \]
for \( s \in (0, 1) \). We observe that the most important difference between operators \((-\Delta)^s\) and \((-\Delta + m^2)^s\) is showed in scaling: the first one is homogeneous in scaling, while the second one is inhomogeneous, which is evident from the Bessel function \( K_\iota \) in (1.2). There are many scholars devoted to the exploration of fractional Schrödinger equation
\[ (-\Delta)^s u + V(x) u = f(u), \quad x \in \Omega, \]
where \((-\Delta)^s\) as the fractional Laplacian, \( f(u) \) represents the nonlinearity, the function \( V(x) : \mathbb{R}^N \to \mathbb{R} \) is an external potential function, and \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) or \( \Omega = \mathbb{R}^N \). It was first introduced in the work of Laskin [6, 7] and originated from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. Note that the Feynman path integral produces the classical Schrödinger equation, however, the fractional Schrödinger equation is obtained by the path integral over Lévy trajectories.

When \( s = \frac{1}{2} \), the operator \( \sqrt{-\Delta + m^2} \) associates with the free Hamiltonian of a free relativistic particle of mass \( m \). It is worth noting that works of Lieb and Yau [8, 9] on the stability of relativistic matter bring great inspiration to the exploration of \( \sqrt{-\Delta + m^2} \). There are some results for this topic, here we just quote a few, please refer to [10–12]. In particularly, it is interesting to consider results for fractional equations involving the operator \( \sqrt{-\Delta u + m^2} \) with \( m > 0 \). From the perspective of mathematics, many scholars focused on finding a solution to the following pseudo-relativistic equation
\[ \sqrt{-\Delta u + m^2} u + \lambda u = \vartheta|u|^{p-2} u + g(u) \text{ in } \mathbb{R}^n, \quad (1.4) \]
with \( g(u) = |u|^{p^*-2} u \). Now, there are two different approaches to consider problem (1.4) according to the characteristics of the frequency \( \lambda \):

(i) the frequency \( \lambda \) is a fixed given constant,
(ii) the frequency $\lambda$ is part of the unknown in problem (1.4).

In case (i), we use a variant of extension method [13] to consider problem (1.4) due to the presence of the nonlocal operator $\sqrt{-\Delta u + m^2 u}$ and we shall introduce this tool in detail in Section 2. Therefore, it can be seen that the solution of problem (1.1) is a critical point connected with the energy functional $I_\lambda(v)$ defined in $H^1(\mathbb{R}^{N+1}_+)$ by

$$I_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dx dy + \frac{\lambda}{2} \int_{\mathbb{R}^N} |v(x,0)|^2 dx - \frac{\theta}{p} \int_{\mathbb{R}^N} |v(x,0)|^p dx - \frac{1}{2\rho} \int_{\mathbb{R}^N} |v(x,0)|^{2\rho} dx.$$  

In this case, we are devoted to looking for the ground state solutions because they possess many more properties, such as positivity, symmetry and stability. In particularly, the ground state solutions are regarded as minimizers of $I_\lambda$ on the Nehari manifold

$$M_\lambda := \{ v \in H^1(\mathbb{R}^{N+1}_+) \setminus \{0\} : \langle I_\lambda'(v), v \rangle = 0 \},$$

(see [14]). In addition, by building a nonempty closed subset of the sign-changing Nehari manifold, Yang and Tang [15] obtained the existence of least energy sign-changing solutions for Schrödinger-Poisson system involving concave-convex nonlinearities in $\mathbb{R}^3$.

Alternatively, in case (ii) other papers are devoted to looking for nontrivial solutions of problem (1.4) when the frequency $\lambda$ is unknown. In this situation, $\lambda$ is regarded as a Lagrange multiplier. Moreover, this method from the perspective of physics seems particularly interesting because of the conservation of mass and the mass has a clear physical meaning. On the other hand, such solutions help us to better understand the dynamical properties, such as orbital stability or instability, where $\theta > 0$ represents the strength of attractive interactions between cold atoms. In general, the solutions with prescribed $L^2$-norms of solutions is called normalized solutions, i.e., the solutions satisfy $|u_2| = c > 0$ for a priori given $c$. Here, in order to look for normalized solutions of problem (1.1), we shall take advantage of a variant of extension method [13] and transform problem (1.1) into a local problem in a upper half-space $\mathbb{R}^{N+1}_+$ with Neumann boundary condition. In addition, we look for the critical point of the functional on the constraint manifold $S(a)$. We shall introduce $S(a)$ and the upper half-space $\mathbb{R}^{N+1}_+$ in detail in Section 2.

In recent years, many scholars have paid great attention to exploration of normalization solutions to various classes of local and non-local problems, and have obtained many results, which are not only of special significance in physics, but also closely related to nonlinear optics and Bose-Einstein condensation. In addition, more and more mathematical scholars begin to explore also solutions with prescribed $L^2$-norms. This kind of problems was first proposed by Jeanjean in [16], who considered the existence of normalized solutions for the Schrödinger equations

$$\begin{cases}  
-\Delta u = \lambda u + g(u) & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}$$

(1.5)

where $N \geq 1$, $\lambda \in \mathbb{R}$ and $g$ satisfies suitable assumptions. Inspired by pioneering work of Jeanjean [16], with the help of variational methods, Alves et al. [17] considered the existence of normalized solutions to the nonlinear Schrödinger equation with critical growth both when $N \geq 3$ and $N = 2$. The author in [18] established existence and several properties of ground states for the following critical equation

$$\begin{cases}  
-\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}$$

Communications in Analysis and Mechanics
Later, Soave [19] also was interested in existence and qualitative properties of normalized solutions of the nonlinear Schrödinger equation with combined power nonlinearities driven by two different Laplacian operators. With the aid of an approximation method, Deng and Wu [20] obtained the existence of normalized solutions for the Schrödinger equation, and the positive solution is mountain-pass type for \( p = 2^* \). Li and Zou [21] were interested in the exploration of fractional Schrödinger equation, they obtained the existence of multiple normalized solutions in both the \( L^2 \)-subcritical and \( L^2 \)-supercritical cases by truncation technique, concentration-compactness principle, genus theory and a fiber map. Wang et al. [22] explored the existence results of normalized solutions for \( p \)-Laplacian equations in the case \((N+2)/N, p^\ast\) by a mountain-pass argument and constrained variational methods. Yao et al. [23] considered several nonexistence and existence results of normalized solutions for the Choquard equations involving lower critical exponent by variational methods. With the aid of a perturbation method, Jeanjean et al. [24] verified the existence of two solutions involving a prescribed \( L^2 \)-norm for a quasi-linear Schrödinger equation. We point out that, in [19, 25–27], several applications are discussed. However, results about the pseudo-relativistic equation are relatively few, as far as we know.

Inspired by the works above, we treat existence of the multiple normalized solutions for problem (1.1). Undoubtedly, we shall encounter some difficulties in proving the existence of the normalized solutions of problem (1.1). One is that Sobolev critical exponent \( 2^d = \frac{2N}{N-1} \) which makes the lack of compactness occur. On the other hand, since the embedding \( X_{rad}(\mathbb{R}_N^N) \hookrightarrow L^2(\mathbb{R}_N^N) \) is not compact, we observe that the weak limit of \( (PS) \) sequence can not be established in the constraint manifold \( S(a) \). Therefore, we have to prove that the Lagrange multipliers \( \lambda \) are non-negative in case \( 2 < p < 2 + 2/N < 2^d \), which is crucial for us to be able to obtain the compactness. Using the compactness principle, the difficulty is solved.

In the following, in case \( 2 < p < 2 + 2/N < 2^d \), the energy functional \( J \) is unbounded from below on \( S(a) \), which results in the failure to get the existence of the solution to problem (1.1) via minimizing problem. In the case \( 2 < p < 2 + \frac{2}{N} \), inspired by [17, 28], we use a truncation technique that allows the truncation function to be bounded from below and coercive.

Finally, problem (1.1) is nonlocal, we shall encounter new difficulties and the study of this kind of equations becomes very meaningful. Therefore, by the extension method in [13], we transform problem (1.1) into a local problem in a upper half-space with a nonlinear Neumann boundary condition.

Our main result is stated in the following theorem:

**Theorem 1.1.** Let \( 2 < p < 2 + \frac{2}{N} < 2^d \) be satisfied. Then for given \( k \in \mathbb{N} \), there exists \( \beta > 0 \) independent of \( k \) and \( \theta_k := \theta(k) \) such that problem (1.1) has at least \( k \) couples \((u_j, \lambda_j) \in H^1(\mathbb{R}_N^N) \times \mathbb{R} \) of weak solutions for \( \theta \geq \theta_k \) and \( a \in (0, (\beta/\theta)^1/\beta) \), with \( \int_{\mathbb{R}_N^N} |u_j|^2 dx = a^2, \lambda_j < 0 \) for all \( j \in [1, k] \) and \( \theta = (p-2)/(2(N-1)) \).

The organizational structure of present paper in what follows. In Section 2, we give some necessary preliminaries and outline the variational framework. In Section 3, we are devoted to the proof of Theorem 1.1.

**2. Preliminaries**

Let \( H^1(\mathbb{R}_N^N) \) be the fractional Sobolev space defined as the completion of \( C_c^\infty(\mathbb{R}_N^N) \) with the following norm

\[
|u|_{H^1(\mathbb{R}_N^N)} := \left( \int_{\mathbb{R}_N^N} \sqrt{|\xi|^2 + m^2}|u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

*Communications in Analysis and Mechanics* Volume 16, Issue 1, 217–236.
Therefore, $H^\frac{1}{2}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^\ast]$ and $H^\frac{1}{2}(\mathbb{R}^N)$ is compactly embedded in $W_{loc}^{1,p}(\mathbb{R}^N)$ for all $p \in [1, 2^\ast)$, please refer to [2,4,29,30]. Let $H^1(\mathbb{R}^{N+1}_+)$ denote the completion of $C^\infty_c(\mathbb{R}^{N+1}_+)$ in the norm:

$$||v|| := ||v||_{H^1(\mathbb{R}^{N+1}_+)} = \left(\int\int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy\right)^{\frac{1}{2}}.$$ 

According to Lemma 3.1 in [3], for $s \in (0, 1)$, we have that the continuous imbedding $H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})$, this fact means

$$\|\nabla v\|_{L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})} \leq \hat{S} \|v\| \quad \text{for all } v \in H^1(\mathbb{R}^{N+1}_+),$$

(2.1)

for some $\hat{S} > 0$, where $\gamma := 1 + \frac{2}{N-2s}$, and $L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})$ is the weighted Lebesgue space for $r \in (1, \infty)$, equipped with the norm

$$\|v\|_{L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})} := \left(\int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |v|^r \, dx \, dy\right)^{\frac{1}{2}}.$$

Using Lemma 3.1.2 in [31], it follows that $H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})$ for all $R > 0$. In the light of Proposition 5 in [3], there exists a (unique) linear trace operator $\text{Tr} : H^1(\mathbb{R}^{N+1}_+) \rightarrow H^\frac{1}{2}(\mathbb{R}^N)$ such that

$$\sqrt{\sigma_s} |\text{Tr}(v)|_{H^\frac{1}{2}(\mathbb{R}^N)} \leq \|v\|_{H^1(\mathbb{R}^{N+1}_+)} \quad \text{for all } v \in H^1(\mathbb{R}^{N+1}_+),$$

(2.2)

where $\sigma_s := 2^{1-2s} \Gamma(1-s)/\Gamma(s)$, please refer to [32,33]. For the sake of simplicity, we will show $\text{Tr}(v)$ by $v(\cdot, 0)$. It is worth noting that (2.2) implies

$$\sigma_s m^{2s} \int_{\mathbb{R}^N} v^2(x, 0) \, dx \leq \int\int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy,$$

(2.3)

for all $v \in H^1(\mathbb{R}^{N+1}_+)$, which is equivalent to

$$\sigma_s \int_{\mathbb{R}^N} v^2(x, 0) \, dx \leq m^{-2s} \int\int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy + m^{2-s} \int\int_{\mathbb{R}^{N+1}_+} v^2 \, dx \, dy.$$  (2.4)

To simplify the notation, we can get rid of the constant $\sigma_s$ in (2.4).

In the following, we define the work space

$$\mathcal{X} := \{v \in H^1(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} |v(x, 0)|^2 \, dx < \infty\}$$

equipped with the norm

$$||v||_{\mathcal{X}} := (||v||^2 + \int_{\mathbb{R}^N} |v(x, 0)|^2 \, dx)^{\frac{1}{2}}.$$

Clearly, $\mathcal{X} \subset H^1(\mathbb{R}^{N+1}_+)$ and using (2.3), we see that

$$||v|| \leq ||v||_{\mathcal{X}} \quad \text{for all } v \in \mathcal{X}.$$
Moreover, $\mathcal{X}$ is a Hilbert space equipped with the inner product
\[ \langle v, w \rangle = \int_{\mathbb{R}^{N+1}} (\nabla v \cdot \nabla w + m^2 vw) dx dy + \int_{\mathbb{R}^N} v(x, 0)w(x, 0) dx. \]

At the same time, $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Now, we recall some results in the case $s \in (0, 1)$. Since $H^1(\mathbb{R}^{N+1}_+)$ is a Hilbert space equipped with the inner product $\langle u, w \rangle$, and $H^1(\mathbb{R}^{N+1}_+)$ is the completion of $C_c(\mathbb{R}^{N+1}_+)$ in the norm
\[ \left( \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 dx dy \right)^{\frac{1}{2}} \]
and the best constant is given by
\[ S_s = \frac{2\pi^{\frac{1}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{N+1}{2})\Gamma(\frac{N}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})\Gamma(N)^{\frac{1}{2}}}. \]

This constant is obtained on the family of functions $\omega_\epsilon = E_{1/2}(u_\epsilon)$, where $E_{1/2}$ denotes the $\frac{1}{2}$-harmonic extension [13], and
\[ u_\epsilon(x) := \frac{\epsilon^{\frac{N+1}{2}}}{(|x|^2 + \epsilon^2)^{\frac{N+1}{2}}}, \quad \epsilon > 0, \]
see [29, 32]. Therefore,
\[ \omega_\epsilon(x, y) := \langle P_{1/2}(\cdot, y) * u_\epsilon \rangle(x) = p_{N,1/2} y \int_{\mathbb{R}^N} \frac{u_\epsilon(\xi)}{|x - \xi|^2 + y^2} d\xi, \]
where
\[ P_{1/2}(x, y) := \frac{p_{N,1/2} y}{(|x|^2 + y^2)^{\frac{N+1}{2}}}. \]
as the Poisson kernel for the extension problem in \( \mathbb{R}^{N+1}_+ \). We observe that \( \omega_{\varepsilon}(x,y) = e^{\frac{1+\varepsilon}{\varepsilon}} \omega_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) \).

We are devoted to studying the existence and multiplicity of normalized solutions of problem (1.1) in present paper. To consider problem (1.1) by variational methods, we make full use of a variant of the extension method [13] given in [3, 29, 33]. To be more precise, the nonlocal operator \( \sqrt{-\Delta + m^2} \) in \( \mathbb{R}^N \) can be achieved by a local problem in \( \mathbb{R}^N \times (0, \infty) \). In the following, we shall describe this construction in detail. For any function \( u \in H^2(\mathbb{R}^N) \), there exists a unique function \( v \in H^1(\mathbb{R}^{N+1}_+) \) (here, \( \mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\} \) such that

\[
\begin{aligned}
-\Delta v + m^2 v &= 0 & & \text{in } \mathbb{R}^{N+1}_+ , \\
v(x,0) &= u(x) & & \text{for } x \in \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+. 
\end{aligned}
\]  

Set

\[
Tu(x) = -\frac{\partial v}{\partial y}(x,0),
\]

we have the following equation

\[
\begin{aligned}
-\Delta w + m^2 w &= 0 & & \text{in } \mathbb{R}^{N+1}_+ , \\
w(x,0) &= Tu(x) & & \text{for } x \in \partial \mathbb{R}^{N+1}_+ = \{0\} \times \mathbb{R}^N \approx \mathbb{R}^N
\end{aligned}
\]

with the solution \( w(x,y) = \frac{\partial w}{\partial y}(x,y) \). By (2.6), we have

\[
T(Tu)(x) = -\frac{\partial w}{\partial y}(x,0) = \frac{\partial^2 v}{\partial y^2}(x,0) = (-\Delta_x v + m^2 v)(x,0)
\]

and hence \( T^2 = (-\Delta + m^2) \). Thus, the operator \( T \) that maps the Dirichlet-type data \( u \) to the Neumann-type data \( \frac{\partial v}{\partial y}(x,0) \) is actually \( \sqrt{-\Delta + m^2} \). Therefore, for problem (1.1), we shall consider the following nonlinear boundary value problem:

\[
\begin{aligned}
-\Delta v + m^2 v &= 0 & & \text{in } \mathbb{R}^{N+1}_+ , \\
\frac{\partial v}{\partial y} &= \theta |v(x,0)|^{p-2} u + |v(x,0)|^{2^* - 2} u - \lambda v(x,0) & & \text{on } \mathbb{R}^N , \\
v > 0, & & \int_{\mathbb{R}^N} |v(x,0)|^2 dx = a^2.
\end{aligned}
\]

Furthermore, we shall look for the critical points of the energy functional \( J : \mathcal{X}_{rad}(\mathbb{R}^{N+1}_+) \to \mathbb{R} \) associated with problem (2.7):

\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dx dy - \frac{\theta}{p} \int_{\mathbb{R}^N} |v(x,0)|^p dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v(x,0)|^{2^*} dx
\]

on the constraint

\[
S(a) := \{ v \in \mathcal{X}_{rad} : |v(x,0)|_2^2 = a^2 \}.
\]

3. Proof of Theorem 1.1

Let us start the section by recalling the definition of genus. Let \( X \) be a Banach space and \( D \) be a subset of \( X \). The set \( D \) is called to be symmetric if \( -u \in D \) for all \( u \in D \). Denote by \( \Sigma \) the family of closed symmetric subsets \( D \) of \( X \) such that \( 0 \not\in D \), that is

\[
\Sigma = \{ D \subset X \setminus \{ 0 \} : D \text{ is closed and symmetric with respect to the origin} \}.
\]
For $D \in \Sigma$, we define
\[
\gamma(A) = \begin{cases} 
0, & \text{if } D = 0, \\
\inf\{k \in \mathbb{N} : \exists \text{ an odd map } \phi \in C(D, \mathbb{R}^k \setminus \{0\})\}, & \text{if such an odd map does not exist,}
\end{cases}
\]
and $\Sigma_k = \{D \in \Sigma : \gamma(D) \geq k\}$. Now, we are ready to give some lemmas that play important roles in proving Theorem 1.1.

**Lemma 3.1.** Let $v \in H^1(\mathbb{R}_+^{N+1})$ and $2 < t < 2^\sharp$, then
\[
\int_{\mathbb{R}^N} |v(x, 0)|^t dx \leq S_\star^{-\frac{2\theta}{t}} \left( \int_{\mathbb{R}^N} |v(x, 0)|^2 dx \right)^{1-\theta} \left( \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) dx dy \right)^{\frac{2\theta}{t}} ,
\]
where $\theta = \frac{(t-2)(N-1)}{2}$. Then we have completed the proof of Lemma 3.1.

**Proof.** Since $v \in H^1(\mathbb{R}_+^{N+1})$ and $2 < t < 2^\sharp$, by Hölder inequality and (2.5), we obtain
\[
\int_{\mathbb{R}^N} |v(x, 0)|^t dx = \int_{\mathbb{R}^N} |v(x, 0)|^{2(1-\theta)} \cdot |v(x, 0)|^{2\theta} dx \\
\leq \left( \int_{\mathbb{R}^N} |v(x, 0)|^2 dx \right)^{1-\theta} \left( \int_{\mathbb{R}^N} |v|^2 dx \right)\theta \\
\leq \left( \int_{\mathbb{R}^N} |v(x, 0)|^2 dx \right)^{1-\theta} \left( \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) dx dy \right)^{\frac{2\theta}{t}} \\
= S_\star^{-\frac{2\theta}{t}} \left( \int_{\mathbb{R}^N} |v(x, 0)|^2 dx \right)^{1-\theta} \left( \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) dx dy \right)^{\frac{2\theta}{t}} ,
\]
where $\theta = \frac{(t-2)(N-1)}{2}$. Then we have completed the proof of Lemma 3.1.

We state the concentration-compactness principle for $s = \frac{1}{2}$ in what follows.

**Lemma 3.2** (Proposition 3.1 in [35]). Let $\{v_k\}$ be a bounded tight sequence in $H^1(\mathbb{R}_+^{N+1})$, such that $v_k$ converges weakly to $v$ in $H^1(\mathbb{R}_+^{N+1})$. Let $\mu, \nu$ be two non-negative measures on $\mathbb{R}_+^{N+1}$ and $\mathbb{R}^N$ respectively and such that
\[
\lim_{n \to \infty} (|\nabla v_k|^2 + m^2 u_k^2) =: \mu
\]
and
\[
\lim_{n \to \infty} |v_k(x, 0)|^{2^\sharp} =: \nu,
\]
in the sense of measures. Then, there exist an at most countable set $I$ and three families $\{x_i\}_{i \in I}, \{\mu_i\}_{i \in I}, \{\nu_i\}_{i \in I}$, with $\mu_i, \nu_i \geq 0$ for all $i \in I$, such that
\[
\nu = |v(., 0)|^{2^\sharp} + \sum_{i \in I} v_i \delta_{x_i},
\]
\[
\mu \geq (|\nabla v|^2 + m^2 v^2) + \sum_{i \in I} \mu_i \delta_{(x_i, 0)},
\]
\[
\mu_i \geq S \nu_i^{\frac{2^\sharp}{2}} \text{ for all } i \in I.
\]
Lemma 3.3. Let \( \{v_k\} \) be a sequence in \( H^1(\mathbb{R}^{N+1}_+) \) as in Lemma 3.2 and define

\[
\mu_\infty = \lim_{n \to \infty} \lim_{k \to \infty} \int_{B_n} (|\nabla v_k|^2 + m^2 v_k^2) dx dy, \quad \nu_\infty = \lim_{n \to \infty} \lim_{k \to \infty} \int_{B_n} |v_k(\cdot, 0)|^2 dx.
\]  
(3.1)

Then

\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2) dx dy = \mu(\mathbb{R}^{N+1}_+) + \mu_\infty,
\]  
(3.2)

\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_{\mathbb{R}^N} |v_k(\cdot, 0)|^2 dx = \nu(\mathbb{R}^N) + \nu_\infty, \quad \mu_\infty \geq S. \nu_\infty^{\frac{2}{N}},
\]  
(3.3)

where \( \mu, \nu \) are the finite non-negative measures in Lemma 3.2.

Proof. Fix a sequence \( \{v_k\} \) in \( H^1(\mathbb{R}^{N+1}_+) \), as in the statement of Lemma 3.2. Let \( \eta \in C_0^\infty(\mathbb{R}^{N+1}_+) \) such that \( 0 \leq \eta \leq 1, \eta = 0 \) in \( B^+_1 \) and \( \eta = 1 \) in \( B^+_2 \). Take \( R > 0 \) and put \( \eta_R(x, y) = \eta(\frac{x}{R}, \frac{y}{R}) \). We write

\[
\int_{B^+_R} (|\nabla v_k|^2 + m^2 v_k^2) dx dy = \int_{B^+_R} (|\nabla v_k|^2 + m^2 v_k^2) \eta_R^2 dx dy + \int_{B^+_R} (|\nabla v_k|^2 + m^2 v_k^2)(1 - \eta_R^2) dx dy.
\]  
(3.4)

We first observe that

\[
\int_{(B^+_R)^+} (|\nabla v_k|^2 + m^2 v_k^2) dx dy \leq \int_{B^+_R} (|\nabla v_k|^2 + m^2 v_k^2) \eta_R^2 dx dy
\]

\[
\leq \int_{(B^+_R)^+} (|\nabla v_k|^2 + m^2 v_k^2)(1 - \eta_R^2) dx dy.
\]

So by (3.1),

\[
\mu_\infty = \lim_{R \to \infty} \lim_{k \to \infty} \int_{B^+_R} (|\nabla v_k|^2 + m^2 v_k^2) \eta_R^2 dx dy.
\]  
(3.5)

On the other hand, since \( \mu \) is finite, \( 1 - \eta_R^2 \) has compact support and \( \eta_R \to 0 \) a.e. in \( \mathbb{R}^{N+1}_+ \), by the definition of \( \mu \) and the Dominated convergence theorem, we have

\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B^+_R} (|\nabla v_k|^2 + m^2 v_k^2)(1 - \eta_R^2) dx dy
\]

\[
= \lim_{R \to \infty} \int_{B^+_R} (1 - \eta_R^2) d\mu = \mu(\mathbb{R}^{N+1}_+).
\]  
(3.6)

Using (3.5)-(3.6) in (3.4), we can obtain (3.2). Arguing similarly for \( \nu \), we see that

\[
\lim_{R \to \infty} \lim_{K \to \infty} \int_{\mathbb{R}^N} (1 - \eta_R^2)|v_k(\cdot, 0)|^2 dx = \nu(\mathbb{R}^N).
\]

Thus, the first part of (3.3) is proved.

In order to verify the last part of (3.3), we consider again the function \( \eta_R \). Let \( K := \text{supp}(\eta_R) \). By the fact that

\[
S_\ast(\int_{\mathbb{R}^N} |v_k(\cdot, 0)|^{\frac{2}{N+1}} dx)^{\frac{N+1}{2}} \leq \int_{\mathbb{R}^{N+1}_+} |\nabla v_k|^2 dx dy
\]
Therefore, together with (3.10) and taking $R$ by the definition of $\eta$, using the Hölder inequality, the boundedness of $\eta$ for all $k$ and applying this to $\eta$, we obtain the desired conclusion.

Putting (3.10)-(3.12) into (3.8), we obtain

$$S \left( \int_{\mathbb{R}^{N+1}} |v_k(\cdot, 0)|^2 R^2 dx \right)^{\frac{2}{3}} \leq \int_{\mathbb{R}^{N+1}} (|\nabla(v_k R)|^2 + m^2(v_k R)^2) dx dy \quad (3.8)$$

for all $k$. On the other hand,

$$\int_{\mathbb{R}^{N+1}} |\nabla(\eta R v_k)|^2 + m^2(\eta R v_k)^2 \, dx dy = \int_{\mathbb{R}^{N+1}} \eta^2 v_k^2 |\nabla v_k|^2 \, dx dy + \int_{\mathbb{R}^{N+1}} v_k^2 |\nabla \eta R|^2 \, dx dy + 2 \int_{\mathbb{R}^{N+1}} v_k \eta R \nabla \eta R \cdot \nabla v_k \, dx dy. \quad (3.9)$$

By the definition of $\eta$, we know

$$\lim_{R \to \infty} \lim_{k \to \infty} \int_{\mathbb{R}^{N+1}} v_k^2 |\nabla \eta R|^2 \, dx dy \to 0. \quad (3.10)$$

Using the Hölder inequality, the boundedness of $\{v_k\}$ in $H^1(\mathbb{R}^{N+1})$ and (3.10), we get

$$\left| \int_{\mathbb{R}^{N+1}} v_k \eta R \nabla \eta R \cdot \nabla v_k \, dx dy \right| \leq \left( \int_{\mathbb{R}^{N+1}} v_k^2 |\nabla \eta R|^2 \, dx dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{N+1}} \eta R^2 |\nabla v_k|^2 \, dx dy \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^{N+1}} v_k^2 |\nabla \eta R|^2 \, dx dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{N+1}} |\nabla v_k|^2 \, dx dy \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^{N+1}} v_k^2 |\nabla \eta R|^2 \, dx dy \right)^{\frac{1}{2}}. \quad (3.11)$$

Therefore, together with (3.10) and taking $R \to \infty$, $k \to \infty$ in (3.11), we obtain

$$\lim_{R \to \infty} \lim_{k \to \infty} \int_{\mathbb{R}^{N+1}} v_k \eta R \nabla \eta R \cdot \nabla v_k \, dx dy = 0. \quad (3.12)$$

Putting (3.10)-(3.12) into (3.8), we obtain the desired conclusion.

For $v \in \mathcal{S}(a)$, by Lemma 3.1 and (3.7), we have

$$\mathcal{J}(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) dx dy - \frac{\theta}{p} \int_{\mathbb{R}^N} |v(x, 0)|^p dx - \frac{1}{2^p} \int_{\mathbb{R}^N} |v(x, 0)|^{2^*} dx \geq \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) dx dy - \frac{\theta}{p} S^{\frac{a}{2}} \, a^{1-\theta} ||v||^{2\theta} - \frac{1}{2^p} S^{\frac{a}{2}} ||v||^{2^*}$$

Communications in Analysis and Mechanics

Volume 16, Issue 1, 217–236.
\[= \frac{1}{2}||v||^2 - \frac{\theta}{p} S^{-\frac{\beta}{\theta}} a^{1-\theta} ||v||^2 - \frac{1}{2^a} S^{-\frac{\beta}{\theta}} ||v||^2 \]

\[:= \mathcal{K}(||v||), \]

where

\[\mathcal{K}(t) = \frac{1}{2} t^2 - \frac{\theta}{p} S^{-\frac{\beta}{\theta}} a^{1-\theta} t^{2\theta} - \frac{1}{2^a} S^{-\frac{\beta}{\theta}} t^{2a} \]

and \(\theta = \frac{(p-2)(N-1)}{2}\). By \(2 < p < 2 + \frac{2}{N}\), we get that \(0 < \theta < 1\) and there exists \(\beta > 0\) such that \(\theta a^{1-\theta} \leq \beta\). Thus, the function \(\mathcal{K}\) has a positive local maximum. To be more precisely, there exist two numbers \(0 \leq W_1 < W_2 < \infty\) such that \(\mathcal{K} < 0\) in the intervals \((0, W_1)\) and \((W_2, \infty)\), while \(\mathcal{K} > 0\) in \((W_1, W_2)\).

Suppose that \(\sigma \in C^\infty(\mathbb{R}^+, [0, 1])\) is a nonincreasing function such that \(\sigma(t) = 1\) for \(t \leq W_1\) and \(\sigma(t) = 0\) for \(t \geq W_2\).

We define the truncated functional by

\[\mathcal{J}_\sigma(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + m^2 v^2) dx dy - \frac{\theta}{p} \int_{\mathbb{R}^N} |v(x, 0)|^p dx - \frac{\sigma(||v||)}{2^a} \int_{\mathbb{R}^N} |v(x, 0)|^2 dx.\]

For \(v \in S(a)\), by Lemma 3.1 and (3.7), we get

\[\mathcal{J}_\sigma(v) \geq \frac{1}{2} ||v||^2 - \frac{\theta}{p} S^{-\frac{\beta}{\theta}} a^{1-\theta} ||v||^2 - \frac{\sigma(||v||)}{2^a} \]

\[:= \widetilde{\mathcal{K}}(||v||), \]

where

\[\widetilde{\mathcal{K}}(t) = \frac{1}{2} t^2 - \frac{\theta}{p} S^{-\frac{\beta}{\theta}} a^{1-\theta} t^{2\theta} - \frac{\sigma(t)}{2^a} t^{2a}.\]

Therefore, with the help of the definition of \(\sigma\), we obtain \(\widetilde{\mathcal{K}} < 0\) in \((0, W_1)\) and \(\widetilde{\mathcal{K}} > 0\) in \((W_2, \infty)\) when \(a \in \left(0, \left(\frac{\beta}{\theta}\right)^{\frac{\beta}{\theta}}\right]\). From now on, we assume that

\[a \in \left(0, \left(\frac{\beta}{\theta}\right)^{\frac{\beta}{\theta}}\right].\]

Without loss of generality, taking \(W_1 > 0\) small enough if necessary, we also assume

\[0 < W_1^2 < S_N^N, \quad \text{so that} \quad \frac{r^2}{2} - \frac{1}{2^a S_N^{2a}} r^{2a} \geq 0 \quad \text{for all} \quad r \in [0, W_1].\]  \(\text{(3.13)}\)

**Lemma 3.4.**  
(a) \(\mathcal{J}_\sigma \in C^1(\mathcal{K}_{rad}(\mathbb{R}^N), \mathbb{R})\).

(b) \(\mathcal{J}_\sigma\) is coercive and bounded from below on \(S(a)\). Furthermore, if \(\mathcal{J}_\sigma \leq 0\), then \(||v|| \leq W_1\) and \(\mathcal{J}_\sigma(v) = \mathcal{J}(v)\).

(c) \(\mathcal{J}_\sigma|_{S(a)}\) satisfies the \((PS)_c\) condition for all \(c < 0\).

**Proof.** (a) and (b) hold true with the aid of a standard argument.

For (a). As the proof of the Proposition B.10 in the book [36], conclusion (a) is satisfied.
For (b). Let \( v \in S(a) \), by the definition of \( \sigma \), we obtain \( \sigma(||v||^2) = 0 \) when \( ||v|| \to \infty \). Thus,
\[
\mathcal{F}_\sigma(v) \geq \frac{1}{2}||v||^2 - \frac{\theta}{p} S_\star \frac{2^p}{a^1-\theta} ||v||^{2\theta} \to +\infty,
\]
since \( N(p - 2) \leq 2 \) and \( \theta = \frac{(p-2)(N-1)}{2} \), that is \( \mathcal{F}_\sigma \) is coercive. On the other hand, it follows from the definition of \( K(t) \) that \( K \) has a maximum value, and then \( \mathcal{F}_\sigma(v) \) is bounded from below on \( S(a) \). Furthermore, if \( \mathcal{F}_\sigma(v) \leq 0 \), so \( \mathcal{K} < 0 \). Also, by the definition of \( \mathcal{K} \), we obtain \( ||v|| \leq W_1 \). Therefore, from the definition of \( \sigma \), we get \( \sigma = 1 \). This fact implies \( \mathcal{F}_\sigma(v) = \mathcal{F}(v) \).

For (c). Assume that \( \{v_k\}_k \) is a \((PS)_\epsilon\) sequence of \( \mathcal{F}_\sigma \) restricted to \( S(a) \) with \( \epsilon < 0 \), that is,
\[
\mathcal{F}_\sigma(v_k) \to c < 0 \quad \text{and} \quad ||\mathcal{F}_\sigma'(S(a))(v_k)|| \to 0 \quad \text{as} \quad k \to \infty.
\]
By (b), \( ||v_k|| \leq W_1 \) for \( k \) large enough. Therefore, \( \{v_k\}_k \) is bounded in \( X_{rad}(\mathbb{R}^{N+1}_+) \). Then, up to subsequence, there exists \( X_{rad}(\mathbb{R}^{N+1}_+) \) such that \( v_k \rightharpoonup v \) in \( X_{rad}(\mathbb{R}^{N+1}_+) \) and \( v_k \to v \) in \( L^p(\mathbb{R}^N) \) for all \( p \in (2,2^*) \) and \( \nu_k \to \nu \) a.e. in \( \mathbb{R}^N \). Due to the fact that \( 2 < p < 2 + \frac{2}{N} < 2^* \), we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n(x,0)|^p dx = \int_{\mathbb{R}^N} |v(x,0)|^p dx.
\]
Moreover, we claim \( v \neq 0 \). Otherwise, \( \lim_{k \to \infty} \int_{\mathbb{R}^N} |v_k|^p dx = 0 \). Combining this and (3.13), we see that
\[
0 > c = \lim_{k \to \infty} \mathcal{F}(v_k) = \lim_{k \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2) dx dy - \frac{\theta}{p} \int_{\mathbb{R}^N} |v_k(x,0)|^p dx - \frac{1}{2^s} \int_{\mathbb{R}^N} |v_k(x,0)|^{2s} dx \right]
\geq \lim_{k \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dx dy - \frac{\theta}{p} \int_{\mathbb{R}^N} |v(x,0)|^p dx - \frac{1}{2^s} \int_{\mathbb{R}^N} |v(x,0)|^{2s} dx \right]
\geq \lim_{k \to \infty} - \frac{\theta}{p} \int_{\mathbb{R}^N} |v(x,0)|^p dx = 0
\]
which is impossible and proves the claim.

Let
\[
\Psi(v) := \frac{1}{2} \int_{\mathbb{R}^N} |v(x,0)|^2 dx, \quad \forall v \in X(\mathbb{R}^{N+1}_+).
\]
Thus, \( S(a) = \Psi^{-1}((\frac{a^2}{2})) \). By the Lagrange multiplier, there exists \( \lambda_a \in \mathbb{R} \) such that
\[
\mathcal{F}'(v) = \lambda_a \Psi'(v)
\]
in \((H^1(\mathbb{R}^{N+1}_+))^*\). Therefore, using this fact, we have
\[
\begin{aligned}
-\Delta v + m^2 v &= 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \\
-\frac{\partial v}{\partial n} &= \theta |v(x,0)|^{p-2} v + |v(x,0)|^{2^*-2} v - \lambda_a v(x,0) \quad \text{on} \quad \mathbb{R}^N, \\
v &> 0, \quad \int_{\mathbb{R}^N} |v(x,0)|^2 dx = a^2.
\end{aligned}
\]
(3.14)

With the help of Proposition 5.12 in [14], there exists \( \lambda_k \in \mathbb{R} \) such that
\[
||\mathcal{F}'(v_k) - \lambda_k \Psi'(v_k)|| \to 0
\]
as $k \to \infty$. Hence, for $\varphi \in X_{rad}(\mathbb{R}^{N+1})$, 
\[
\int_{\mathbb{R}^{N+1}} (\nabla v_k \cdot \nabla \varphi + m^2 v_k \varphi) dxdy - \theta \int_{\mathbb{R}^N} |v_k(x, 0)|^{p-2} v_k(x, 0) \varphi dx - \int_{\mathbb{R}^N} |v_k(x, 0)|^{2^*-1} v_k(x, 0) \varphi dx = \lambda_k \int_{\mathbb{R}^N} v_k \varphi dx + o(1)\|\varphi\|. \tag{3.15}
\]
In particular, 
\[
\|v_k\|^2 - \theta \int_{\mathbb{R}^N} |v_k(x, 0)|^p dx - \int_{\mathbb{R}^N} |v_k(x, 0)|^{2^*} dx = \lambda_k a^2 + o(1). \tag{3.16}
\]
The boundedness of $\{\|v_k\|\}$ implies that $\{\lambda_k\}$ is also bounded in $\mathbb{R}$. Therefore, up to a subsequence, there exists $\lambda_a \in \mathbb{R}$ such that $\lambda_k \to \lambda_a$ as $k \to \infty$. Therefore, by (3.15) and a standard argument, we obtain that $v$ satisfies problem (3.14). In fact, for any $\varphi \in X_{rad}(\mathbb{R}^{N+1})$, it follows from the definition of weak convergence that
\[
\int_{\mathbb{R}^{N+1}} (\nabla v_k \nabla \varphi + m^2 v_k \varphi) dxdy \to \int_{\mathbb{R}^{N+1}} (\nabla \varphi + m^2 v \varphi) dxdy
\]
as $k \to \infty$. Since $\lambda_k \to \lambda_a$ as $k \to \infty$, we also obtain that
\[
\lambda_k \int_{\mathbb{R}^N} v_k \varphi dx \to \lambda_a \int_{\mathbb{R}^N} v \varphi dx \tag{3.17}
\]
as $k \to \infty$. Moreover, since $\{|v_k|^{2^*-2} v_k\}$ is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$ and
\[
|v_k(x, 0)|^{2^*-2} u_k(x, 0) \to |v(x, 0)|^{2^*-2} v(x, 0) \quad \text{a.e. in } \mathbb{R}^N, \tag{3.18}
\]
then
\[
|v_k(x, 0)|^{2^*-2} v_k(x, 0) \to |v(x, 0)|^{2^*-2} v(x, 0) \quad \text{in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N).
\]
This implies that
\[
\int_{\mathbb{R}^N} |v_k(x, 0)|^{2^*-2} v_k \varphi dx \to \int_{\mathbb{R}^N} |v(x, 0)|^{2^*-2} v \varphi dx
\]
as $k \to \infty$. Next, we show that $\lambda_a < 0$. Indeed, thanks to $2 < p < 2 + \frac{2}{N} < 2^*$, we have
\[
0 > c = \liminf_{k \to \infty} \mathcal{J}(v_k) = \liminf_{k \to \infty} \left(\mathcal{J}(v_k) - \frac{1}{2}\|\mathcal{J}'(v_k) - \lambda_k \mathcal{J}'(v_k)\|\right)
= \left(\frac{1}{2} - \frac{1}{p}\right) \theta \int_{\mathbb{R}^N} |v(x, 0)|^p dx + \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |v(x, 0)|^{2^*} dx + \frac{1}{2} \lambda_a \int_{\mathbb{R}^N} |v(x, 0)|^2 dx.
\]
Therefore,
\[
\frac{1}{2} \lambda_a \int_{\mathbb{R}^N} |v(x, 0)|^2 dx < -\left(\frac{1}{2} - \frac{1}{p}\right) \theta \int_{\mathbb{R}^N} |v(x, 0)|^p dx - \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |v(x, 0)|^{2^*} dx < 0
\]
which shows $\lambda_a < 0$. 

Communications in Analysis and Mechanics

Volume 16, Issue 1, 217–236.
In the following, we shall recover the compactness with an application of the concentration-compactness principle [35]. Indeed, since $\|v_k\| \leq W_1$ for $k$ enough large, using the Prokhorov theorem [37, Theorem 8.6.2], there exist two positive measures $\mu, \nu \in$ $\mathcal{M}(\mathbb{R}^{N+1})$ such that

$$\lim_{k \to \infty} (\|\nabla v_k\|^2 + m^2 v_k^2) =: \mu$$

and

$$\lim_{k \to \infty} |v_k(x, 0)|^{2l} =: \nu$$

in $\mathcal{M}(\mathbb{R}_+^{N+1})$. (3.19)

Hence, Lemma 3.2-Lemma 3.3 hold. Together with Lemma 3.2, either $v_k \to v$ in $L^{2l}(\mathbb{R}^N)$ or there exists a (at most countable) set of distinct points $\{x_i\}_i \subset \mathbb{R}^N$ and positive numbers $\{\nu_i\}$ such that

$$\nu = |v^+(x, 0)|^{2l} + \sum_i \nu_i \delta_{x_i}. $$

If the latter holds, we can also verify $v_k \to v$ in $L^{2l}(\mathbb{R}^N)$. We shall verify the following three claims hold.

**Claim 1.** We verify that $\mu(x_i) \leq \nu_i$ for any $i \in I$.

Assume that $x_i \in \mathbb{R}^N$ for some $i \in I$. For any $\rho > 0$, we define, $\varphi_\rho(x, y) = \varphi(x - \frac{x_i}{\rho}, y)$, where $\varphi \in C_c(\mathbb{R}^{N+1})$ such that $\varphi = 1$ in $B_1$ and $\varphi = 0$ in $(B_{1})^c$, $\varphi \in [0, 1]$ and $\|\nabla \varphi\|_{L^\infty(\mathbb{R}^{N+1})} \leq 2$. We suppose that $\rho > 0$ such that $\text{supp}(\varphi_\rho(., 0)) \subset \mathbb{R}_+^{N+1}$. By the boundedness of $\{v_k\}$ in $\mathcal{X}_\text{rad}(\mathbb{R}_+^{N+1})$, we know that $\{\varphi_\rho v_k\}$ is also bounded in $\mathcal{X}_\text{rad}(\mathbb{R}_+^{N+1})$. Therefore,

$$o(1) = (J'(v_k), v_k \varphi_\rho) = \int_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \varphi_\rho \, dx dy + \int_{\mathbb{R}_+^{N+1}} \nabla v_k \cdot \nabla \varphi_\rho \, dx dy - \theta \int_{\mathbb{R}^N} \varphi_\rho |v_k(x, 0)|^p \, dx - \int_{\mathbb{R}^N} \varphi_\rho |v_k(x, 0)|^{2l} \, dx.$$  \hspace{1cm} (3.20)

That means

$$\int_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \varphi_\rho \, dx dy = \theta \int_{\mathbb{R}^N} \varphi_\rho |v_k(x, 0)|^p \, dx - \int_{\mathbb{R}_+^{N+1}} \nabla v_k \cdot \nabla \varphi_\rho \, dx dy + \int_{\mathbb{R}^N} \varphi_\rho |v_k(x, 0)|^{2l} \, dx + o(1).$$

Consequently,

$$\lim_{\rho \to 0^+} \lim_{k \to \infty} \int_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \varphi_\rho \, dx dy = \lim_{\rho \to 0^+} \lim_{k \to \infty} \int_{\mathbb{R}_+^{N+1}} \varphi_\rho \, d\mu \geq \mu_j.$$  \hspace{1cm} (3.21)

Together with the definition of $\varphi_\rho$, we obtain

$$\lim_{\rho \to 0^+} \lim_{k \to \infty} \int_{\mathbb{R}^N} \varphi_\rho |v_k(x, 0)|^p \, dx = \lim_{\rho \to 0^+} \int_{\mathbb{R}^N} \varphi_\rho |v(x, 0)|^p \, dx = \lim_{\rho \to 0^+} \int_{B_{\rho}^c} \varphi_\rho |v(x, 0)|^p \, dx = 0.$$  \hspace{1cm} (3.22)

Moreover, (3.19) implies

$$\lim_{\rho \to 0^+} \lim_{k \to \infty} \int_{\mathbb{R}^N} \varphi_\rho |v_k(x, 0)|^{2l} \, dx = \lim_{\rho \to 0^+} \int_{\mathbb{R}^N} \varphi_\rho \, dv = \nu_i.$$  \hspace{1cm} (3.23)

In the following, we show that

$$\lim_{\rho \to 0^+} \limsup_{k \to \infty} \int_{\mathbb{R}_+^{N+1}} v_k \nabla v_k \cdot \nabla \varphi_\rho \, dx dy = 0.$$  \hspace{1cm} (3.24)

Communications in Analysis and Mechanics

Volume 16, Issue 1, 217–236.
In fact, by the Hölder inequality, the boundedness of $\{v_k\}_k$ in $X_{rad}(\mathbb{R}_+^{N+1})$, the fact that $\|\nabla \varphi_i\|_{L^\infty(\mathbb{R}_+^{N+1})} \leq \frac{C}{\rho}$ and $X(\mathbb{R}_+^{N+1})$ is compactly embedded into $L^2(B^+_R(x_i,0),y^{-2s})$ with $s = \frac{1}{2}$, we obtain

$$
\limsup_{k \to \infty} \left| \int_{\mathbb{R}_+^{N+1}} v_k \nabla v_k \cdot \nabla \varphi_i dxdy \right| 
\leq \limsup_{k \to \infty} \left( \int_{\mathbb{R}_+^{N+1}} |\nabla v_k|^2 dxdy \right)^{\frac{1}{2}} \left( \int_{B^+_R(x_i,0)} |v_k|^2 |\nabla \varphi_i|^2 dxdy \right)^{\frac{1}{2}} 
\leq \frac{C}{\rho} \left( \int_{B^+_R(x_i,0)} |v_k|^2 dxdy \right)^{\frac{1}{2}}.
$$

By Hölder inequality with $\frac{1}{p} + \frac{1}{r} = 1$ and (2.1), we have

$$
\frac{C}{\rho} \left( \int_{B^+_R(x_i,0)} |v_k|^2 dxdy \right)^{\frac{1}{2}} 
\leq \frac{C}{\rho} \left( \int_{B^+_R(x_i,0)} |v_k|^2 r^2 dxdy \right)^{\frac{1}{2}} \left( \int_{B^+_R(x_i,0)} dxdy \right)^{\frac{1}{2}} 
\leq C \left( \int_{B^+_R(x_i,0)} |v_k|^2 dxdy \right)^{\frac{1}{2}} \to 0 \text{ as } \rho \to 0^+
$$

which shows that (3.24) holds. Therefore, inserting (3.21)-(3.24) into (3.20), taking $k \to \infty$ and $\rho \to 0^+$, we obtain

$$
\mu(x_i) \leq \nu_i
$$

and the claim holds.

**Claim 2.** We claim that $\mu_{\infty} \leq \nu_{\infty}$.

Let $\phi \in C^\infty_c(\mathbb{R}_+^{N+1})$ such that $0 \leq \phi \leq 1$, $\phi = 0$ in $B^+_R$ and $\phi = 1$ in $(B^+_R)^c$. Take $R > 0$ and put $\phi_R(x) = \phi(\frac{x-x_i}{R})$. Again, by the boundedness of $\{v_k\}_k$ in $X_{rad}(\mathbb{R}_+^{N+1})$, we know that $\{v_k\}_k$ is also bounded in $X_{rad}(\mathbb{R}_+^{N+1})$. Hence,

$$
\phi(0) = \langle T'(v_k), \phi_R \rangle = \int_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \phi_R dxdy + \int_{\mathbb{R}_+^{N+1}} v_k \nabla v_k \nabla \phi_R dxdy 
- \theta \int_{\mathbb{R}^N} \phi_R |v_k(x,0)|^p dx - \int_{\mathbb{R}^N} \phi_R |v_k(x,0)|^{2s} dx.
$$

From the aforementioned proof, we obtain

$$
\lim_{R \to \infty} \limsup_{k \to \infty} \int_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \phi_R dxdy = \int_{\mathbb{R}_+^{N+1}} \phi_R d\mu \geq \mu_{\infty}.
$$

By Hölder’s inequality, $0 \leq \phi_R \leq 1$ and $\{v_k\}_k$ is bounded in $X_{rad}(\mathbb{R}_+^{N+1})$, we have

$$
| \int_{\mathbb{R}_+^{N+1}} v_k \nabla v_k \nabla \phi_R dxdy | 
\leq \frac{C}{R} \int_{\mathbb{R}_+^{N+1}} v_k |\nabla v_k| dxdy 
\leq \frac{C}{R} \left( \int_{\mathbb{R}_+^{N+1}} |v_k|^2 dxdy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}_+^{N+1}} |\nabla v_k|^2 dxdy \right)^{\frac{1}{2}}.
$$
\[
\leq \frac{C}{R} \to 0
\]
as \( R \to \infty \). Therefore,
\[
\lim_{R \to \infty} \limsup_{k \to \infty} \int_{R^{N+1}} v_k \nabla v_k \nabla \phi_R \, dxdy \to 0.
\]
By the proof of Lemma 3.3 in [38], we obtain
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{R^{N}} \phi_R |v_k(x, 0)|^p \, dx = \lim_{R \to \infty} \int_{R^{N}} \phi_R |v(x, 0)|^p \, dx = \lim_{R \to \infty} \int_{|x| \geq R} \phi_R |v(x, 0)|^p \, dx = 0
\]
and
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{R^{N}} \phi_R |v_k(x, 0)|^{2s} \, dx = \nu_\infty.
\]
Therefore, it follows from (3.25) that
\[
\mu_\infty \leq \nu_\infty
\]
and this proves Claim 2.

**Claim 3.** We shall verify that \( v_i = 0 \) for any \( i \in I \) and \( \nu_\infty = 0 \).

By contradiction, we suppose that there exists \( i \in I \) such that \( \nu_i > 0 \). Steps 1 implies that
\[
v_i \leq (S x^{-1} \mu(x_i))^{\frac{2s}{p}} \leq (S x^{-1} v_i)^{\frac{2s}{p}}.
\]
It implies that \( \nu_i \geq S_N^N \). If this case is valid, we get
\[
W^2_1 \geq \lim_{\rho \to 0^+} \lim_{k \to \infty} \|v_k\|^2 \geq S \lim_{\rho \to 0^+} \lim_{k \to \infty} \|v_k(x, 0)\|_{2s}^2
\]
\[
\geq \lim_{\rho \to 0^+} \lim_{k \to \infty} S \left( \int_{R^N} \varphi_\rho |v_k(x, 0)|^{2s} \, dx \right)^{\frac{1}{2}} = S \lim_{k \to \infty} \left( \int_{R^N} \varphi_\rho dy \right)^{\frac{1}{2}}
\]
\[
= S \cdot v_i^{\frac{2s}{p}} \geq S_N^N
\]
which is impossible by (3.13). If the latter holds, by the same discussion above, we get
\[
W^2_1 \geq \lim_{R \to \infty} \lim_{k \to \infty} \|v_k\|^2 \geq \mu_\infty \geq S \cdot v_\infty^{\frac{2s}{p}} \geq S_N^N
\]
which contradicts with (3.13), and together with Lemma 3.2 implies \( v_k \to v \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). Moreover, combining with Lemma 3.3, we obtain \( v_k \to v \) in \( L^2(\mathbb{R}^N) \). Taking into account (3.15)–(3.17), we obtain
\[
\lim_{k \to \infty} \left[ \|v_k\|^2 - \lambda_d |v_k(x, 0)|^2 \right]
\]
\[
= \lim_{k \to \infty} \left[ \theta |v_k(x, 0)|^p + |v_k(x, 0)|^{2s} + o(1) \right]
\]
\[
= \theta |v(x, 0)|^p + |v(x, 0)|^{2s} = ||v||^2 - \lambda_d |v(x, 0)|^2.
\]

*Communications in Analysis and Mechanics* Volume 16, Issue 1, 217–236.
Therefore, we obtain that the following result holds.

\[-\lambda_\nu |v(x, 0)|_2^2 \leq \liminf_{k \to \infty} -\lambda_\nu |v_k(x, 0)|_2^2 \leq \limsup_{k \to \infty} -\lambda_\nu |v_k(x, 0)|_2^2 \]

\[\leq \limsup_{k \to \infty} -\lambda_\nu |v_k(x, 0)|_2^2 + \liminf_{k \to \infty} ||v_k||^2 - ||v||^2 \]

\[\leq \limsup_{k \to \infty} \left[ ||v_k||^2 - \lambda_\nu |v_k(x, 0)|_2^2 \right] - ||v||^2 \]

\[= -\lambda_\nu |v(x, 0)|_2^2. \]

Hence,

\[\lim_{k \to \infty} -\lambda_\nu |v_k(x, 0)|_2^2 = -\lambda_\nu |v(x, 0)|_2^2. \]

Moreover, we obtain

\[\lim_{k \to \infty} |v_k(x, 0)|_2^2 = |v(x, 0)|_2^2. \]

By (3.26), we get

\[\lim_{k \to \infty} ||v_k||^2 = ||v||^2. \]

Then \(v_k \to v\) in \(X_{rad}(\mathbb{R}_+^{N+1})\) and \(|v_k(x, 0)|_2 = a\). The proof of Lemma 3.4 is completed.

Set

\[\mathcal{J}_\nu = \{ v \in X_{rad}(\mathbb{R}_+^{N+1}) \cap S(a) : \mathcal{J}_\nu(v) \leq -\nu \} \subset X_{rad}(\mathbb{R}_+^{N+1}) \]

for \(\nu > 0\). By the fact that \(\mathcal{J}_\nu\) is even and continuous on \(X_{rad}(\mathbb{R}_+^{N+1})\), gives that \(\mathcal{J}_\nu\) is closed and symmetric. Consequently, the following lemma is true and its proof is the same as that of Lemma 3.2 in [28].

**Lemma 3.5.** Given \(k \in \mathbb{N}\), there exist \(\epsilon_k := \epsilon(k)\) and \(\vartheta_k := \vartheta(k)\) such that whenever \(0 < \epsilon \leq \epsilon_k\) and \(\vartheta \geq \vartheta_k, \gamma(\mathcal{J}_\nu^{-\epsilon}) \geq k\).

Set

\[\Sigma_k := \{ E \subset X_{rad}(\mathbb{R}_+^{N+1}) \cap S(a) : E \text{ is closed and symmetric, } \gamma(E) \geq k \} \]

and

\[c_k := \inf_{E \in \Sigma_k} \sup_{v \in E} \mathcal{J}_\nu(v) > -\infty \]

for all \(k \in E\) by Lemma 3.4 (b). In order to verify Theorem 1.1, we given by

\[\mathcal{K}_c = \{ v \in X_{rad}(\Omega) \cap S(a) : \mathcal{J}_\nu(v) = 0, \mathcal{J}_\nu(c) = c \} \]

Therefore, we obtain that the following result holds.

**Lemma 3.6.** If \(c = c_k = c_{k+1} = \cdots = c_{k+m}\), then \(\gamma(\mathcal{K}_c) \geq m + 1\). Especially, \(\mathcal{J}_\nu\) has at least \(m + 1\) nontrival critical points.

**Proof.** For \(\epsilon > 0\), we know that \(\mathcal{J}_\nu^{-\epsilon} \in \Sigma\). With the help of Lemma 3.5, for any \(k \in \mathbb{N}\), there exists \(\epsilon_k := \epsilon(k) > 0\) and \(\vartheta_k = \vartheta(k)\) such that if \(0 < \epsilon \leq \epsilon_k\) and \(\vartheta \geq \vartheta_k\), we have \(\gamma(\mathcal{J}_\nu^{-\epsilon}) \geq k\). Therefore, \(\mathcal{J}_\nu^{-\epsilon_k} \in \Sigma_k\), and

\[c_k \leq \sup_{v \in \mathcal{J}_\nu^{-\epsilon_k}} \mathcal{J}_\nu(v) = -\epsilon_k < 0. \]
Let $0 > c = c_k = c_{k+1} = \cdots = c_{k+m}$ are satisfied. Therefore, Lemma 3.4 (c) shows that $\mathcal{J}_c$ satisfies the $(PS)_c$ condition. Consequently, $\mathcal{K}_c$ is a compact set. Theorem 2.1 in [39] yields that $\mathcal{J}_c|_{S(a)}$ has at least $m + 1$ critical points.

**Proof of Theorem 1.1.** By Lemma 3.4 (b) the critical points of $\mathcal{J}_c$ obtained in Lemma 3.6 are the critical points of $\mathcal{J}$. Hence, we complete the proof.

**Use of AI tools declaration**

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

**Acknowledgments**

Sihua Liang is supported by the Science and Technology Development Plan Project of Jilin Province, China (Grant No. YDZJ202201ZYTS582), the Research Foundation of Department of Education of Jilin Province (Grant No. JJKH20230902KJ) and Innovation and Entrepreneurship Talent Funding Project of Jilin Province (No.2023QN21).

**Conflict of interest**

The authors declare there is no conflict of interest.

**References**


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