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Research article

A boundary integral equation method for the fluid-solid interaction problem

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Abstract: In this paper, a boundary integral equation method is proposed for the fluid-solid interaction scattering problem, and a high-precision numerical method is developed. More specifically, by introducing the Helmholtz decomposition, the corresponding problem is transformed into a coupled boundary value problem for the Helmholtz equation. Based on the integral equation method, the coupled value problem is reduced to a system of three coupled hypersingular integral equations. Semi-discrete and fully-discrete collocation methods are proposed for the singular integral equations. The presented method is based on trigonometric interpolation and discretized singular operators applied to differentiated interpolation. The convergence of the method is verified by a numerical experiment.

Keywords: Fluid-solid interaction problem; singular operator; boundary integral equation method; collocation method
Mathematics Subject Classification: 35P25, 45A05, 74F10

1. Introduction

The fluid-solid interaction scattering problem has received more and more attention due to its wide application in seismology, oceanography, biomedicine and other fields [1–3]. When an incident acoustic plane wave encounters an elastic solid, which is immersed in a homogeneous fluid, the elastic solid will have small displacements. We call such a problem the fluid-solid interaction problem. This physical phenomenon is generally described by a transmission problem with acoustic scattering and displacements in the elastic solid. The fluid-solid interaction problem has many applications, such as underwater nondestructive testing (see [4] for details). There are many numerical methods to solve such scattering problems, such as the variational methods [5, 6], the finite element method [7–10], mixed finite element method [11–13], T-matrix method [14, 15], immersed boundary method [16, 17] and pressure-correction schemes [18]. Some other related methods can be found [19–22] for inverse problems and [23–27] for the direct problems.

The fluid-solid interaction scattering problem is mathematically expressed as a class of boundary value transport problems. Gatica et al. [8, 9, 12] give some numerical methods for the fluid-solid interaction problem based on the finite element method. For the scattering transmission problems, the boundary integral equation method is effective [28–32]. The main idea is to obtain the boundary integral expression of the unknown function by using Green's formula or potential theory, and then to obtain the boundary integral equation equivalent to the original scattering problem by using the limiting idea to restrict the solution to the boundary of the domain. In [33], Luke and Martin gave several kinds of boundary integral equations for solving fluid-solid interaction direct scattering problems of bounded structures, as well as the analysis of existence and uniqueness of solutions. Atkinson [34] proposed that the most efficient method for solving boundary integral equations on smooth boundaries is based on trigonometric polynomial approximation. In addition, due to the singularity of the integral equation, the solution of the equation requires special handling of the singularity of the integral kernel; see [35] for details. Kress [36] studied the quadrature method of logarithmic singular integral equations, which discretized the principal part of the singular operator based on triangular interpolation. The quadrature method of hypersingular integral equation was studied based on triangular interpolation and differentiation in [37], a fully discrete collocation method was proposed, and the convergence was analyzed in [38].

In this paper, we study a transmission problem with acoustic scattering and displacements in the elastic solid. When there are not Jones frequencies [39, 40], the corresponding problem is always uniquely solvable. The Fredholm theory combined with the variational method can give a theoretical analysis about this problem. The boundary element methods can get the accurate numerical solution of this problem [33,41]. Inspired by [42–45] singular integral operators can be decomposed into isomorphic operators and compact operators, and the fluid-solid interaction scattering problem is reduced to the coupled singular integral equations by the Helmholtz decomposition. Then, the convergence analysis of integral equations can be carried out by using the collocation method.

The organization of this paper is as follows. In section 2, we introduce the fluid-solid interaction scattering problem. In section 3, we give the boundary integral equation of the model and decompose the singular integral operator. In section 4, the semi-discrete and fully discrete forms of the boundary integral equation are given, and then the convergence is analyzed using the collocation method based on triangular interpolation and differentiation. Section 5 presents a benchmark example to demonstrate the effectiveness of the proposed method.

2. Problem formulation

In this paper, the model is that there is a sufficiently long elastic cylinder immersed in homogeneous compressible inviscid fluid. We consider the corresponding mathematical problem between the cross section of the elastic cylinder and the fluid, that is, the two dimensional fluid-solid interaction problem. We denoted $\Omega \subset \mathbb{R}^2$ be an isotropic elastic solid obstacle, and the boundary $\partial\Omega$ is analytic. Outside the solution domain Ω, there is full filled with the compressible inviscid fluid in $\mathbb{R}^2 \setminus \overline{\Omega}$. The densities of the elastic solid obstacle and the fluid are denoted by ρ_e and ρ_f. $\nu = (\nu_1, \nu_2)^\top$ is the unit normal vector, and $\tau = (\tau_1, \tau_2)^\top$ is the tangential vector on $\partial\Omega$. In general, the components of ν and τ satisfy $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$.

Given an incident field $u^{inc}(x) = e^{i\kappa_a x\cdot \boldsymbol{d}}$, find the elastic displacement $\mathbf{u} \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ and the acoustic scattered field $u^s \in C^2(\mathbb{R}^2 \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^2 \backslash \Omega)$. Here, the elastic displacement $\mathbf{u}$ satisfies the following Navier equation

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho_e\omega^2\mathbf{u} = 0, \quad \text{in } \Omega. \tag{2.1}$$

λ and μ, usually called Lamé constants, satisfy $\mu > 0$ and $\mu + \lambda > 0$. $\omega > 0$ is the frequency. The acoustic scattered field u^s satisfies the Helmholtz equation

$$\Delta u^s + \kappa_a^2 u^s = 0, \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \tag{2.2}$$

and the Sommerfeld radiation condition gives

$$\lim_{r\to\infty} r^{1/2}(\partial_r u^s - i\kappa_a u^s) = 0, \quad r = |x|. \tag{2.3}$$

Here, $\kappa_a = \omega/c$ is the wavenumber, with c being the sound speed in the fluid. In addition, the elastic displacement $\mathbf{u}$ and the total field $u = u^s + u^{inc}$ satisfy the transmission conditions on the interface $\partial\Omega$,

$$T(\mathbf{u}) = -u\nu, \quad \mathbf{u}\cdot\nu = \frac{1}{w^2\rho_f}\partial_\nu u, \tag{2.4}$$

where the traction operator T is given by

$$T(\mathbf{u}) := \mu\partial_\nu\mathbf{u} + (\lambda+\mu)(\nabla\cdot\mathbf{u})\nu.$$

It has been shown (see [39]) that for certain geometries and some frequencies ω, which are called Jones frequencies, the solution of the corresponding transmission problem (2.1)−(2.3) is not unique. In this paper, we assume that the frequency ω is not one of the Jones frequencies.

The vector operator curl for a scalar function w is given by

$$\nabla\times w = \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1}\right)^\top.$$

The Helmholtz decomposition for the solution $\mathbf{u}$ of (2.1) gives the following form:

$$\mathbf{u} = \nabla u_p + \nabla\times u_s, \tag{2.5}$$

where u_p and u_s, respectively, are the solutions of the Helmholtz equations

$$\Delta u_p + \kappa_p^2 u_p = 0, \quad \Delta u_s + \kappa_s^2 u_s = 0,$$

with the compressional wave number $\kappa_p = \sqrt{\frac{\rho_e\omega^2}{\lambda+2\mu}}$ and the shear wave number $\kappa_s = \sqrt{\frac{\rho_e\omega^2}{\mu}}$, respectively. Combining (2.4) and (2.5), we get

$$\mu\partial_\nu(\nabla u_p + \nabla\times u_s) + (\lambda+\mu)(\nabla\cdot(\nabla u_p + \nabla\times u_s))\nu = -u\nu, \tag{2.6}$$

$$(\nabla u_p + \nabla\times u_s)\cdot\nu = \partial_\nu u/(\omega^2\rho_f). \tag{2.7}$$

We can rewrite equation (2.6) by the normal vector direction and the tangential vector direction, respectively. Together with (2.7), we will have

$$\begin{cases}\Delta u_p + \kappa_p^2 u_p = 0, & \text{in } \Omega,\\ \Delta u_s + \kappa_s^2 u_s = 0, & \text{in } \Omega,\\ \Delta u^s + \kappa_a^2 u^s = 0, & \text{in } \mathbb{R}^2 \setminus \overline{\Omega},\\ \mu\nu \cdot \partial_\nu \nabla u_p + \mu\nu \cdot \partial_\nu \nabla \times u_s - (\lambda+\mu)\kappa_p^2 u_p + u^s = f_1, & \text{on } \partial\Omega,\\ \tau \cdot \partial_\nu \nabla u_p + \tau \cdot \partial_\nu \nabla \times u_s = f_2, & \text{on } \partial\Omega,\\ \partial_\nu u_p + \partial_\tau u_s - \partial_\nu u^s/(\omega^2 \rho_f) = f_3, & \text{on } \partial\Omega,\\ \lim_{r\to\infty} r^{\frac{1}{2}}(\partial_r u^s - i\kappa_a u^s) = 0, & r = |x|,\end{cases} \tag{2.8}$$

with $f_1 = -u^{inc}, f_2 = 0, f_3 = \partial_\nu u^{inc}/(\rho_f \omega^2)$.

For the above problem, we are interested in the case $\kappa_p > 0$, $\kappa_s > 0$ and $\kappa_a > 0$, since the problem we considered is always a practice problem, such as the copper alloy in the water or the rock in the magma.

3. Boundary integral equations

3.1. Coupled integral equations.

From [44], the solution of the BVPs for the Helmholtz equation can be given by the form of single-layer potentials, and thus the solution of (2.8) will be given as follows:

$$u_p(x) = \int_{\partial\Omega} \Phi(\kappa_p|x-y|)g_1(y)ds(y), \quad x \in \Omega, \tag{3.1}$$

$$u_s(x) = \int_{\partial\Omega} \Phi(\kappa_s|x-y|)g_2(y)ds(y), \quad x \in \Omega, \tag{3.2}$$

$$u^s(x) = \int_{\partial\Omega} \Phi(\kappa_a|x-y|)g_3(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{3.3}$$

with unknown densities $g_i \in C^{1,\alpha}(\partial\Omega), i = 1,2,3$. $\Phi(\kappa|x-y|) = \frac{i}{4}H_0^{(1)}(\kappa|x-y|), x \neq y$, is the fundamental solution of the two-dimensional Helmholtz equation with $H_0^{(1)}$ being the Hankel function of the first kind of order zero.

If we let the point x tend to boundary $\partial\Omega$ in (3.1)–(3.3), together with the jump relations of the single-layer potentials (see e.g. [43, 44]), we can get

$$
\begin{aligned}
f_1(x) =& -\mu\kappa_p^2\nu^\top(x)\int_{\partial\Omega}\Phi(\kappa_p|x-y|)[\nu(y)\nu^\top(y)]g_1(y)\nu(x)ds(y) + \frac{1}{2}\mu\nu^\top(x)\frac{\partial(g_1\tau)}{\partial\tau}(x) \\
& -\mu\nu^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_p|x-y|)}{\partial\tau(x)}\frac{\partial(g_1\nu)}{\partial\tau}(y)ds(y) + \mu\nu^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_p|x-y|)}{\partial\nu(x)}\frac{\partial(g_1\tau)}{\partial\tau}(y)ds(y) \\
& +\mu\kappa_s^2\nu^\top(x)\int_{\partial\Omega}\Phi(\kappa_s|x-y|)[\tau(y)\nu^\top(y)]g_2(y)\nu(x)ds(y) + \frac{1}{2}\mu\nu^\top(x)\frac{\partial(g_2\nu)}{\partial\tau}(x) \\
& +\mu\nu^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_s|x-y|)}{\partial\tau(x)}\frac{\partial(g_2\tau)}{\partial\tau}(y)ds(y) + \mu\nu^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_s|x-y|)}{\partial\nu(x)}\frac{\partial(g_2\nu)}{\partial\tau}(y)ds(y) \\
& -(\lambda+\mu)\kappa_p^2\int_{\partial\Omega}\Phi(\kappa_p|x-y|)g_1(y)ds(y) + \int_{\partial\Omega}\Phi(\kappa_a|x-y|)g_3(y)ds(y), \\
f_2(x) =& -\kappa_p^2\tau^\top(x)\int_{\partial\Omega}\Phi(\kappa_p|x-y|)[\nu(y)\nu^\top(y)]g_1(y)\nu(x)ds(y) + \frac{1}{2}\tau^\top(x)\frac{\partial(g_1\tau)}{\partial\tau}(x) \\
& -\tau^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_p|x-y|)}{\partial\tau(x)}\frac{\partial(g_1\nu)}{\partial\tau}(y)ds(y) + \tau^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_p|x-y|)}{\partial\nu(x)}\frac{\partial(g_1\tau)}{\partial\tau}(y)ds(y) \\
& +\kappa_s^2\tau^\top(x)\int_{\partial\Omega}\Phi(\kappa_s|x-y|)[\tau(y)\nu^\top(y)]g_2(y)\nu(x)ds(y) + \frac{1}{2}\tau^\top(x)\frac{\partial(g_2\nu)}{\partial\tau}(x) \\
& +\tau^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_s|x-y|)}{\partial\tau(x)}\frac{\partial(g_2\tau)}{\partial\tau}(y)ds(y) + \tau^\top(x)\int_{\partial\Omega}\frac{\partial\Phi(\kappa_s|x-y|)}{\partial\nu(x)}\frac{\partial(g_2\nu)}{\partial\tau}(y)ds(y), \\
f_3(x) =& \int_{\partial\Omega}\frac{\partial\Phi(\kappa_p|x-y|)}{\partial\nu(x)}g_1(y)ds(y) + \frac{1}{2}g_1(x) + \int_{\partial\Omega}\frac{\partial\Phi(\kappa_s|x-y|)}{\partial\tau(x)}g_2(y)ds(y) \\
& -\frac{1}{\rho_f\omega^2}\int_{\partial\Omega}\frac{\partial\Phi(\kappa_a|x-y|)}{\partial\nu(x)}g_3(y)ds(y) + \frac{g_3(x)}{2\rho_f\omega^2}.
\end{aligned}
\tag{3.4}
$$

Through using the single-layer operator

$$(S_\sigma g)(x) = 2\int_{\partial\Omega}\Phi(\kappa_\sigma|x-y|)g(y)ds(y), \qquad x\in\partial\Omega,$$

and its normal and tangential derivative operators

$$\begin{cases}(K_\sigma g)(x) = 2\int_{\partial\Omega}\frac{\partial\Phi(\kappa_\sigma|x-y|)}{\partial\nu(x)}g(y)ds(y), \\ (H_\sigma g)(x) = 2\int_{\partial\Omega}\frac{\partial\Phi(\kappa_\sigma|x-y|)}{\partial\tau(x)}g(y)ds(y),\end{cases} \qquad x\in\partial\Omega,$$

the corresponding coupled equations (3.4) can be rewritten as the form

$$
\begin{aligned}
2f_1(x) =& -\mu\kappa_p^2\nu^\top S_p[\nu\nu^\top g_1]\nu + \mu\nu^\top K_p[\tau\partial_\tau g_1 + g_1\partial_\tau\tau] - \mu\nu^\top H_p[\nu\partial_\tau g_1 + g_1\partial_\tau\nu] \\
& +\mu\kappa_s^2\nu^\top S_s[\tau\nu^\top g_2]\nu + \mu\nu^\top K_s[\nu\partial_\tau g_2 + g_2\partial_\tau\nu] + \mu\nu^\top H_s[\tau\partial_\tau g_2 + g_2\partial_\tau\tau] \\
& -(\lambda+\mu)\kappa_p^2 S_p[g_1] + S_a[g_3] + \mu(\nu\cdot\partial_\tau\tau)g_1 + \mu(\nu\cdot\partial_\tau\nu)g_2 + \mu\partial_\tau g_2, \\
2f_2(x) =& -\kappa_p^2\tau^\top S_p[\nu\nu^\top g_1]\nu + \tau^\top K_p[\tau\partial_\tau g_1 + g_1\partial_\tau\tau] - \tau^\top H_p[\nu\partial_\tau g_1 + g_1\partial_\tau\nu] \\
& +\kappa_s^2\tau^\top S_s[\tau\nu^\top g_2]\nu + \tau^\top K_s[\nu\partial_\tau g_2 + g_2\partial_\tau\nu] + \tau^\top H_s[\tau\partial_\tau g_2 + g_2\partial_\tau\tau] \\
& +(\tau\cdot\partial_\tau\tau)g_1 + (\tau\cdot\partial_\tau\nu)g_2 + \partial_\tau g_1, \\
2f_3(x) =& K_p[g_1] + H_s[g_2] - \frac{K_a[g_3]}{\rho_f\omega^2} + g_1 + \frac{g_3}{\rho_f\omega^2}.
\end{aligned}
\tag{3.5}
$$

We will get the density functions g_1, g_2 and g_3 by solving the system (3.5).

3.2. Decomposition of the operators

Suppose that ∂D is given by

$$z(t) = (z_1(t), z_2(t)), \quad 0 \le t < 2\pi,$$

with $\sqrt{(z_1'(t))^2 + (z_2'(t))^2} > 0$, and $z(t)$ is a $2\pi-$periodic function. To simplify the corresponding coupled system, we introduce the parameterization of the integral operators S_σ, $S_{\sigma ij}$, K_σ, $K_{\sigma i}$, $H_{\sigma j}$ as follows:

$$\begin{aligned}
(S_\sigma\varphi)(t) &= |z'(t)| \int_0^{2\pi} s_\sigma(t,\eta)\varphi(\eta)d\eta,\\
(S_{\sigma ij}\varphi)(t) &= |z'(t)| \int_0^{2\pi} m_i(t,\eta)m_j(t,\eta)s_\sigma(t,\eta)\varphi(\eta)d\eta,\\
(K_\sigma\varphi)(t) &= \int_0^{2\pi} k_\sigma(t,\eta)\varphi(\eta)d\eta,\\
(K_{\sigma i}\varphi)(t) &= \int_0^{2\pi} m_i(t,\eta)k_\sigma(t,\eta)\varphi(\eta)d\eta,\\
(H_\sigma\varphi)(t) &= \int_0^{2\pi} h_\sigma(t,\eta)\varphi(\eta)d\eta,\\
(H_{\sigma i}\varphi)(t) &= \int_0^{2\pi} m_i(t,\eta)h_\sigma(t,\eta)\varphi(\eta)d\eta,
\end{aligned}$$

with

$$\begin{aligned}
s_\sigma &= \frac{i}{2}H_0^{(1)}(\kappa_\sigma|z(t)-z(\eta)|),\\
k_\sigma &= \frac{i\kappa_\sigma H_1^{(1)}(\kappa_\sigma|z(t)-z(\eta)|)}{2|z(t)-z(\eta)|}[z(\eta)-z(t)]\cdot \boldsymbol{n}(t),\\
h_\sigma &= \frac{i\kappa_\sigma H_1^{(1)}(\kappa_\sigma|z(t)-z(\eta)|)}{2|z(t)-z(\eta)|}[z(\eta)-z(t)]\cdot \boldsymbol{n}(t)^\perp,
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{n}(t) := (z_2'(t), -z_1'(t))^\top, \quad \tilde{\nu} = \nu \circ z,\\
&\boldsymbol{n}^\perp(t) := (z_1'(t), z_2'(t))^\top, \quad \tilde{\tau} = \tau \circ z,\\
&\tilde{\tau}' = (\tilde{\tau}_1', \tilde{\tau}_2')^\top, \qquad \tilde{\nu}' = (\tilde{\nu}_1', \tilde{\nu}_2')^\top, \qquad m_0(t,\eta) = |z'(t)|,\\
&m_1(t,\eta) = \tilde{\nu}^\top(t)\tilde{\nu}(\eta) = \tilde{\tau}^\top(t)\tilde{\tau}(\eta), \qquad m_2(t,\eta) = \tilde{\nu}^\top(t)\tilde{\nu}'(\eta) = \tilde{\tau}^\top(t)\tilde{\tau}'(\eta),\\
&m_3(t,\eta) = \tilde{\nu}^\top(t)\tilde{\tau}(\eta) = -\tilde{\tau}^\top(t)\tilde{\nu}(\eta), \qquad m_4(t,\eta) = \tilde{\nu}^\top(t)\tilde{\tau}'(\eta) = -\tilde{\tau}^\top(t)\tilde{\nu}'(\eta).
\end{aligned}$$

Obviously, the functions $m_i(t,\eta), i = 0, 1, 2, 3, 4$, are analytic.

For the $2\pi-$periodic scalar function $w : \mathbb{R} \to \mathbb{C}$, we define $H^p[0, 2\pi]$, $p \ge 0$ by the corresponding space, which is equipped with the norm

$$\| w \|_p^2 := \sum_{m=-\infty}^{\infty} (1+m^2)^p |\hat{w}_m|^2 < \infty.$$

Here,

$$\hat{w}_m = \frac{1}{2\pi}\int_0^{2\pi} w(t)e^{-imt}dt$$

denotes the Fourier coefficients of w. We introduce the Sobolev space

$$H^p[0,2\pi]^3 = \{\boldsymbol{v} = (v_1, v_2, v_3)^\top : v_i(t) \in H^p[0,2\pi], i = 1,2,3\},$$

and equip the norm

$$\| \boldsymbol{v} \|_p = \| v_1 \|_p + \| v_2 \|_p + \| v_3 \|_p \ .$$

Introducing the operators $E_i : H^p[0,2\pi] \to H^p[0,2\pi]$,

$$(E_i\varphi)(t) = m_i(t,t)\varphi(t), \quad i = 0,2,4,$$

and the differentiation operator $D : H^p[0,2\pi] \to H^{p-1}[0,2\pi]$,

$$(D\varphi)(t) = \varphi'(t).$$

By equation (3.5), multiplying $|z'(t)|$, we can get the system

$$\mathcal{A}\varphi = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \tag{3.6}$$

where

$$\begin{aligned}
A_{11} &= -\mu\kappa_p^2 S_{p11}E_0 + \mu K_{p3}D + \mu K_{p4} - \mu H_{p1}D - \mu H_{p2} - (\lambda+\mu)\kappa_p^2 S_p E_0 + \mu E_4, \\
A_{12} &= \mu\kappa_s^2 S_{s31}E_0 + \mu K_{s1}D + \mu K_{s2} + \mu H_{s3}D + \mu H_{s4} + \mu E_2 + \mu D, \\
A_{13} &= S_a E_0, \\
A_{21} &= \kappa_p^2 S_{p31}E_0 + K_{p1}D + K_{p2} + H_{p3}D + H_{p4} + E_2 + D, \\
A_{22} &= \kappa_s^2 S_{s11}E_0 - K_{s3}D - K_{s4} + H_{s1}D + H_{s2} - E_4, \\
A_{23} &= 0, \\
A_{31} &= K_p E_0 + D, \qquad A_{32} = H_s E_0, \qquad A_{33} = (E_0 - K_a E_0)/\omega^2\rho_f.
\end{aligned}$$

and $\varphi_j = g_j \circ z$, $w_j = 2|z'(t)|(f_j \circ z)$, $j = 1,2,3$.

Since the kernel $s_\sigma(t,\eta)$ of the single-layer has a weak singularity at $t = \eta$, the kernel $s_\sigma(t,\eta)$ can be rewritten by

$$s_\sigma(t,\eta) = s_\sigma^1(t,\eta)\ln\left(4\sin^2\frac{t-\eta}{2}\right) + s_\sigma^2(t,\eta), \tag{3.7}$$

with

$$\begin{aligned}
s_\sigma^1(s,\eta) &= -\frac{1}{2\pi}J_0(\kappa_\sigma|z(t)-z(\eta)|), \\
s_\sigma^2(t,\eta) &= s_\sigma(t,\eta) - s_\sigma^1(t,\eta)\ln\left(4\sin^2\frac{t-\eta}{2}\right).
\end{aligned}$$

The above two parts are analytic, and the values at $t = \eta$ are given by

$$s_\sigma^1(t,t) = -\frac{1}{2\pi}, \quad s_\sigma^2(t,t) = \frac{i}{2} - \frac{E_c}{\pi} - \frac{1}{\pi}\ln\Big(\frac{\kappa_\sigma}{2}|z'(t)|\Big).$$

Based on the above equations, especially by equation (3.7), the singular integral operators S_σ, $S_{\sigma ij}$ will be split into

$$S_\sigma = \widetilde{S}_\sigma^1 + \widetilde{S}_\sigma^2 + S^3, \tag{3.8}$$

$$S_{\sigma ij} = \widetilde{S}_{\sigma ij}^1 + \widetilde{S}_{\sigma,ij}^2 + S_{ij}^3, \tag{3.9}$$

with

$$\begin{aligned}
(\widetilde{S}_\sigma^1\varphi)(t) &= \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\tilde{s}_\sigma^1(t,\eta)\varphi(\eta)d\eta,\\
(\widetilde{S}_\sigma^2\varphi)(t) &= \int_0^{2\pi} \tilde{s}_\sigma^2(t,\eta)\varphi(\eta)d\eta,\\
(S^3\varphi)(t) &= \frac{1}{\pi}\int_0^{2\pi}\varphi(\eta)d\eta - \frac{1}{2\pi}\int_0^{2\pi}\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\varphi(\eta)d\eta,\\
(\widetilde{S}_{\sigma ij}^1\varphi)(t) &= \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\tilde{s}_\sigma^1(t,\eta)m_i(t,\eta)m_j(t,\eta)\varphi(\eta)d\eta,\\
(\widetilde{S}_{\sigma ij}^2\varphi)(t) &= \int_0^{2\pi} \tilde{s}_\sigma^2(t,\eta)m_i(t,\eta)m_j(t,\eta)\varphi(\eta)d\eta,\\
(S_{ij}^3\varphi)(t) &= \frac{1}{2\pi}\int_0^{2\pi}\Big(2 - \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)\varphi(\eta)m_i(t,\eta)m_j(t,\eta)d\eta.
\end{aligned}$$

The kernels

$$\tilde{s}_\sigma^1(t,\eta) = s_\sigma^1(t,\eta) + \frac{1}{2\pi}, \qquad \tilde{s}_\sigma^2(t,\eta) = s_\sigma^2(t,\eta) - \frac{1}{\pi}$$

are also analytic with $\tilde{s}_\sigma^1(t,t) = 0$ at $t = \eta$.

As in [46], the kernel $k_\sigma(t,\eta)$ has two parts as follows:

$$k_\sigma(t,\eta) = k_\sigma^2(t,\eta) + k_\sigma^1(t,\eta)\ln\Big(4\sin^2\frac{t-\eta}{2}\Big), \tag{3.10}$$

with

$$\begin{aligned}
k_\sigma^1(t,\eta) &= \frac{\kappa_\sigma J_1(\kappa_\sigma|z(t)-z(\eta)|)}{2\pi|z(t)-z(\eta)|}[z(t)-z(\eta)]\cdot\boldsymbol{n}(t),\\
k_\sigma^2(t,\eta) &= k_\sigma(t,\eta) - k_\sigma^1(t,\eta)\ln\Big(4\sin^2\frac{t-\eta}{2}\Big).
\end{aligned}$$

These two parts are analytic, and the values at $t = \eta$ are

$$k_\sigma^2(t,t) = \frac{z''(t)\cdot\boldsymbol{n}(t)}{2\pi|z'(t)|^2}, \qquad k_\sigma^1(t,t) = 0.$$

Therefore, $K_\sigma\varphi$, $K_{\sigma i}\varphi$ will be in the following form

$$\begin{aligned}
(K_\sigma\varphi)(t) =& (K_\sigma^1\varphi)(t) + (K_\sigma^2\varphi)(t)\\
\stackrel{def}{=}& \int_0^{2\pi} k_\sigma^2(t,\eta)\varphi(\eta)d\eta + \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)k_\sigma^1(t,\eta)\varphi(\eta)d\eta,\\
(K_{\sigma i}\varphi)(t) =& (K_{\sigma i}^1\varphi)(t) + (K_{\sigma i}^2\varphi)(t)\\
\stackrel{def}{=}& \int_0^{2\pi} k_\sigma^2(t,\eta)m_i(t,\eta)\varphi(\eta)d\eta + \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)k_\sigma^1(t,\eta)m_i(t,\eta)\varphi(\eta)d\eta.
\end{aligned}$$

Following the idea in [42], the kernel $h_\sigma(t,\eta)$ will be split into

$$h_\sigma(t,\eta) = h_\sigma^3(t,\eta) + h_\sigma^2(t,\eta)\ln\Big(4\sin^2\frac{t-\eta}{2}\Big) + h^1(t,\eta)\cot\frac{\eta-t}{2} \tag{3.11}$$

with

$$\begin{aligned}
h^1(t,\eta) &= \frac{\tan\frac{\eta-t}{2}}{\pi|z(t)-z(\eta)|^2}[z(\eta)-z(t)]\cdot n^\perp(t),\\
h_\sigma^2(t,\eta) &= \frac{\kappa_\sigma J_1(\kappa_\sigma|z(t)-z(\eta)|)}{2\pi|z(t)-z(\eta)|}[z(t)-z(\eta)]\cdot n^\perp(t),\\
h_\sigma^3(t,\eta) &= h_\sigma(t,\eta) - h_\sigma^1(t,\eta)\cot\frac{\eta-t}{2} - h_\sigma^2(t,\eta)\ln(4\sin^2\frac{t-\eta}{2}).
\end{aligned}$$

When $\eta = t$, the corresponding form will be

$$h_\sigma^3(t,t) = 0, \qquad h_\sigma^2(t,t) = 0, \qquad h^1(t,t) = \frac{1}{2\pi}.$$

Based on the above equations, especially by the equation (3.11), the singular integral operators H_σ, $H_{\sigma i}$ will be given by

$$H_\sigma = H^1 + H^2 + \tilde{H}^1 + \tilde{H}_\sigma^2 + \tilde{H}_\sigma^3, \tag{3.12}$$

$$H_{\sigma i} = H_i^1 + H_i^2 + \tilde{H}_i^1 + \tilde{H}_{\sigma i}^2 + \tilde{H}_{\sigma i}^3, \tag{3.13}$$

with

$$
\begin{aligned}
(H^1\varphi)(t) =& \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\eta-t}{2}\varphi(\eta)d\eta + \frac{i}{2\pi}\int_0^{2\pi}\varphi(\eta)d\eta,\\
(H^2\varphi)(t) =& \frac{1}{2\pi}\int_0^{2\pi}\Big(2-\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)\sin(t-\eta)\varphi(\eta)d\eta,\\
(\widetilde{H}^1\varphi)(t) =& \int_0^{2\pi}\tilde{h}^1(t,\eta)\varphi(s)d\eta,\\
(\widetilde{H}^2_\sigma\varphi)(t) =& \int_0^{2\pi}\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\tilde{h}^2_\sigma(t,\eta)\varphi(s)d\eta,\\
(\widetilde{H}^3_\sigma\varphi)(t) =& \int_0^{2\pi}\tilde{h}^3_\sigma(t,\eta)\varphi(s)d\eta,\\
(H^1_i\varphi)(t) =& \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\eta-t}{2}m_i(t,\eta)\varphi(\eta)d\eta + \frac{i}{2\pi}\int_0^{2\pi} m_i(t,\eta)\varphi(\eta)d\eta,\\
(H^2_i\varphi)(t) =& \frac{1}{2\pi}\int_0^{2\pi}\Big(2-\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)\sin(t-\eta)m_i(t,\eta)\varphi(\eta)d\eta\\
(\widetilde{H}^1_i\varphi)(t) =& \int_0^{2\pi}\tilde{h}^1(t,\eta)m_i(t,\eta)\varphi(s)d\eta,\\
(\widetilde{H}^2_{\sigma i}\varphi)(t) =& \int_0^{2\pi}\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\tilde{h}^2_\sigma(t,\eta)m_i(t,\eta)\varphi(s)d\eta,\\
(\widetilde{H}^3_{\sigma i}\varphi)(t) =& \int_0^{2\pi}\tilde{h}^3_\sigma(t,\eta)m_i(t,\eta)\varphi(s)d\eta.
\end{aligned}
$$

The kernels

$$\tilde{h}^1(t,\eta) = \Big(h^1(t,\eta) - \frac{1}{2\pi}\Big)\cot\frac{\eta-t}{2},$$

$$\tilde{h}^2_\sigma(t,\eta) = h^2_\sigma(t,\eta) + \frac{1}{2\pi}\sin(t-s),$$

$$\tilde{h}^3_\sigma(t,\eta) = h^3_\sigma(t,\eta) - \frac{i+2\sin(t-s)}{2\pi}$$

are analytic. The values at $\eta = t$ are given by $\tilde{h}^1(t,t) = \tilde{h}^2_\sigma(t,t) = 0$.

3.3. Operators equation

By the decomposition of operators, we rewrite integral equations (3.6) by the following from:

$$\mathcal{A}\varphi = (\mathcal{T}+\mathcal{H}+\mathcal{B}+C)\varphi = w, \tag{3.14}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^\top$, $w = (w_1, w_2, w_3)^\top$, and $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, $C = C_1 + C_2 + C_3 + C_4$, and

$$\mathcal{T} \stackrel{def}{=} \begin{bmatrix} -\mu H_1^1 D & \mu H_3^1 D + \mu D & 0 \\ H_3^1 D + D & H_1^1 D & 0 \\ 0 & 0 & H_3^1 D \end{bmatrix}, \quad \mathcal{H} \stackrel{def}{=} \begin{bmatrix} \mu E_4 - \mu H_2^1 & \mu E_2 + \mu H_4^1 & 0 \\ E_2 + H_4^1 & H_2^1 - E_4 & 0 \\ E_0 & H^1 E_0 & \frac{E_0}{\omega^2 \rho_f} - H_3^1 D \end{bmatrix},$$

$$\mathcal{B}_1 \stackrel{def}{=} \begin{bmatrix} -\mu\kappa_p^2 S_{11}^3 E_0 - (\lambda+\mu)\kappa_p^2 S^3 E_0 - \mu H_2^2 & \mu\kappa_s^2 S_{31}^3 E_0 + \mu H_4^2 & S^3 E_0 \\ \kappa_p^2 S_{31}^3 + H_4^2 & \kappa_s^2 S_{11}^3 E_0 + H_2^2 & 0 \\ 0 & H^2 E_0 & 0 \end{bmatrix},$$

$$\mathcal{B}_2 \stackrel{def}{=} \widetilde{\mathcal{B}}_2 \mathcal{D} \begin{bmatrix} -\mu H_1^2 & \mu H_3^2 & 0 \\ H_3^2 & H_1^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D & D & 0 \\ D & D & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_1 \stackrel{def}{=} \begin{bmatrix} \mu K_{p4}^1 - \mu \widetilde{H}_{p2}^2 - \mu\kappa_p^2 \widetilde{S}_{p11}^1 E_0 - (\lambda+\mu)\kappa_p^2 \widetilde{S}_p^1 E_0 & \mu K_{s2}^1 + \mu \widetilde{H}_{s4}^2 + \mu\kappa_s^2 \widetilde{S}_{s13}^1 E_0 & \widetilde{S}_a^1 E_0 \\ K_{p2}^1 + \widetilde{H}_{p4}^2 + \mu\kappa_p^2 \widetilde{S}_{p13}^1 E_0 & -K_{s4}^1 + \widetilde{H}_{s2}^2 + \kappa_s^2 \widetilde{S}_{s11}^1 E_0 & 0 \\ K_p^1 E_0 & \widetilde{H}_s^2 E_0 & \frac{-K_a^1 E_0}{\omega^2 \rho_f} \end{bmatrix},$$

$$C_2 \stackrel{def}{=} \widetilde{C}_2 \mathcal{D} \begin{bmatrix} \mu K_{p3}^1 - \mu \widetilde{H}_{p1}^2 & \mu K_{s1}^1 + \mu \widetilde{H}_{s3}^2 & 0 \\ K_{p1}^1 + \widetilde{H}_{p3}^2 & -K_{s3}^1 + \widetilde{H}_{s1}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D & D & 0 \\ D & D & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_3 \stackrel{def}{=} \begin{bmatrix} \mu(K_{p4}^2 - \widetilde{H}_{p2}^1 - \widetilde{H}_{p2}^3 - \kappa_p^2 \widetilde{S}_{p11}^2 E_0) - (\lambda+\mu)\kappa_p^2 \widetilde{S}_p^2 E_0 & \mu(K_{s2}^2 + \widetilde{H}_{s4}^1 + \widetilde{H}_{s4}^3 + \kappa_s^2 \widetilde{S}_{s13}^2 E_0) & \widetilde{S}_a^2 E_0 \\ K_{p2}^2 + \widetilde{H}_{p4}^1 + \widetilde{H}_{p4}^3 + \mu\kappa_p^2 \widetilde{S}_{p13}^2 E_0 & -K_{s4}^2 + \widetilde{H}_{s2}^1 + \widetilde{H}_{s2}^3 + \kappa_s^2 \widetilde{S}_{s11}^2 E_0 & 0 \\ K_p^2 E_0 & \widetilde{H}_s^1 E_0 + \widetilde{H}_s^3 E_0 & \frac{-K_a^2 E_0}{\omega^2 \rho_f} \end{bmatrix},$$

$$C_4 \stackrel{def}{=} \widetilde{C}_4 \mathcal{D} \begin{bmatrix} \mu K_{p3}^2 - \mu \widetilde{H}_{p1}^1 - \mu \widetilde{H}_{p1}^3 & \mu K_{s1}^2 + \mu \widetilde{H}_{s3}^1 + \mu \widetilde{H}_{s3}^3 & 0 \\ K_{p1}^2 + \widetilde{H}_{p3}^1 + \widetilde{H}_{p3}^3 & -K_{s3}^2 + \widetilde{H}_{s1}^1 + \widetilde{H}_{s1}^3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D & D & 0 \\ D & D & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It should be noted that, for all $p \geq 0$, the differential operator $D : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ is bounded for the nullspace containing only the constant functions.

Theorem 1. *The integral operators $\mathcal{H}$ and $\mathcal{B}$ are compact operators from $H^p[0, 2\pi]^3$ to $H^{p-1}[0, 2\pi]^3$.*

Proof. Noting that $E_i\varphi = m_i(t,t)\varphi$ for $i = 0, 2, 4$, where $m_i(t,t)$ are analytic, using [43, Theorem 3.1], we know that E_i are bounded operators from $H^p[0, 2\pi]$ to $H^p[0, 2\pi]$. The operators $H_2^1, H_4^1 : H^p[0, 2\pi] \to H^p[0, 2\pi]$ are bounded. Noting $m_3(t,t) = 0$ and using [44, Theorem 12.15,13.20], we get that H_3^1 is a bounded operator from $H^p[0, 2\pi]$ to $H^{p+1}[0, 2\pi]$ for $p \geq 0$, then $H_3^1 D$ is also a bounded operator from $H^p[0, 2\pi]$ to $H^p[0, 2\pi]$ for $p \geq 0$.

Therefore, $\mathcal{H} : H^p[0, 2\pi]^3 \to H^p[0, 2\pi]^3$ is bounded and consequently is compact from $H^p[0, 2\pi]^3$ into $H^{p-1}[0, 2\pi]^3$.

In fact, it is sufficient to prove that the operators $\mathcal{B}_1, \mathcal{B}_2$ are compact by $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_1$. From [44, Theorem 8.24], we have that H^2, S^3 are bounded operators from $H^p[0, 2\pi]$ to $H^{p+1}[0, 2\pi]$, then the integral operators $S_{11}^3, S_{31}^3, H_2^2$ are bounded operators from $H^p[0, 2\pi]$ to $H^{p+1}[0, 2\pi]$ for $p \geq 0$, hence, $\mathcal{B}_1, \widetilde{\mathcal{B}}_2 : H^p[0, 2\pi]^3 \to H^{p+1}[0, 2\pi]^3$ are bounded. Note that $\mathcal{D}$ is bounded from $H^p[0.2\pi]$ to $H^{p-1}[0, 2\pi]$, then $\mathcal{B}_2$ is bounded from $H^p[0, 2\pi]^3$ to $H^p[0, 2\pi]^3$.

Therefore, the operators $\mathcal{B}_1, \mathcal{B}_2$ consequently are compact from $H^p[0, 2\pi]^3$ into $H^{p-1}[0, 2\pi]^3$.

Theorem 2. *The integral operator C is a compact operator from $H^p[0,2\pi]^3$ to $H^p[0,2\pi]^3$.*

Proof. First, it should be noted that

$$k^1_\sigma(t,t) = \tilde{h}^2_\sigma(t,t) = \tilde{s}^1_\sigma(t,t) = 0, \sigma = p, s, a. \tag{3.15}$$

Together with [44, Theorems 13.20], we can get that K^1_σ, $\tilde{H}^2_\sigma$, $\tilde{S}^{\,1}_\sigma$ are bounded from $H^p[0,2\pi]$ to $H^{p+2}[0,2\pi]$. Thus $C_1, \widetilde{C_2}$ are bounded from $H^p[0,2\pi]^3$ into $H^{p+2}[0,2\pi]^3$. Further, $C_2 = \widetilde{C}_2\mathcal{D}$. Then, C_2 is bounded from $H^p[0,2\pi]^3$ to $H^{p+1}[0,2\pi]^3$.

Second, the goal is to show the boundedness of C_3, C_4. Noting that kernel functions $\widetilde{h}^3_\sigma$, k^2_σ and $\widetilde{s}^2_\sigma$ $\widetilde{h}^1$ are analytic. From [47, Theorem 3.3], we can see that the kernel function $\widetilde{h}^1$ is also analytic. Together with [48, Theorem A.45] and [44, Theorems 8.13], we know that the operators K^2_σ, $S^{\,2}_\sigma$, $\widetilde{H}^3_\sigma$, $\widetilde{H}^3_\sigma$: $H^p[0,2\pi] \to H^{p+r}[0,2\pi]$ are boundness for $p \geq 0$ and all integers $r \geq 0$. Specifically, for $r \geq 0, p \geq 0$, we can get the boundness of the operators K^2_σ, $S^{\,2}_\sigma$, $\widetilde{H}^3_\sigma$, $\widetilde{H}^3_\sigma$: $H^p[0,2\pi] \to H^{p+2}[0,2\pi]$. Thus, the operators $C_3, \widetilde{C_4}$ are bounded from $H^p[0,2\pi]^3$ into $H^{p+2}[0,2\pi]^3$. C_4 is bounded from $H^p[0,2\pi]^3$ to $H^{p+1}[0,2\pi]^3$.

Therefore, the operator C is a bounded operator from $H^p[0,2\pi]^3$ into $H^{p+1}[0,2\pi]^3$, and thus a compact operator from $H^p[0,2\pi]^3$ to $H^p[0,2\pi]^3$.

4. Collocation method

Consider the operator equation (3.14), $\mathcal{H}$, $\mathcal{B}$ and C are compact operators from $H^p[0,2\pi]^3$ into $H^{p-1}[0,2\pi]^3$. In this section, we use the collocation method to give the convergence of the numerical method.

4.1. Semi-discrete collocation

We describe a semi-discrete method by collocation via trigonometric interpolation. Let X_n be an n-dimensional space of trigonometric polynomials of the form

$$\varphi(t) = \sum_{m=0}^{n} \alpha_m \cos mt + \sum_{m=1}^{n-1} \beta_m \sin mt.$$

Let P_n denote the interpolation operator. If there are $2n$ points $\eta^n_j := j\pi/n, j = 0, \cdots, 2n-1$, uniformly distributed on $[0,2\pi]$, the operator P_n, for a function g, will give a trigonometric polynomial $P_n g$ satisfying $(P_n g)(\eta^n_j) = g(\eta^n_j)$.

Let $X^3_n = \{\varphi = (\varphi_1, \varphi_2, \varphi_3)^\top : \varphi_i \in X_n\}$ and define the interpolation operator $\mathcal{P}_n : H^p[0,2\pi]^3 \to X^3_n$ by $\mathcal{P}_n g = (P_n g_1, P_n g_2, P_n g_3)^\top, \forall g = (g_1, g_2, g_3) \in H^p[0,2\pi]^3$ For the interpolation error, we note that

$$\| \mathcal{P}_n g - g \|_q \leq \frac{C}{n^{p-q}} \| g \|_p, \ 0 \leq q \leq p, \ p > \frac{1}{2}, \tag{4.1}$$

for all $g \in H^p[0,2\pi]$ and some constant C depending p and q.

We denote the numerical solution of $\varphi = (\varphi_1, \varphi_2, \varphi_3)^\top$ corresponding to the equation (3.14) by $\varphi^n = (\varphi^n_1, \varphi^n_2, \varphi^n_3)^\top \in X^3_n$, which is the solution of the following equation:

$$\mathcal{P}_n \mathcal{T} \varphi^n + \mathcal{P}_n(\mathcal{H} + \mathcal{B} + C)\varphi^n = \mathcal{P}_n w. \tag{4.2}$$

Remark 4.1. *For the semi-discrete collocation method, it is expected that the following estimate holds under certain conditions:*

$$\| \varphi^n - \varphi \|_p \leq M \| \mathcal{P}_n(\mathcal{T}\varphi) - \mathcal{T}\varphi \|_{p-1}$$

for each $p \geq 1$, where M is a positive constant depending on $\mathcal{T}, \mathcal{H}, \mathcal{B}$ and C.

4.2. Fully discrete collocation

For the fully discrete method, we need to approximate all the integral operators $S_\sigma, S_{\sigma ij}, K_\sigma, K_{\sigma i}, H_\sigma, H_{\sigma i}$ and the differentiation operator D. For the differentiation operator D, we have a description of trigonometric differentiation which approximates D by $D_n := DP_n$, i.e., the derivative Dg of a $2\pi-$periodic function g by the derivative $D_n g$ of the trigonometric interpolation polynomial $P_n g \in X_n$. Denote the Lagrange basis by

$$\mathfrak{L}_j(t) = \frac{1}{2n}\{1 + 2\sum_{k=1}^{n-1} \cos k(t - \eta_j^{(n)}) + \cos n(t - \eta_j^{(n)})\}, \quad j = 0, 1, \cdots, 2n-1.$$

For $g \in H^p[0, 2\pi]$, from the boundness of $D : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$, we have the error estimate

$$\| D_n g - Dg \|_{q-1} \leq \frac{C}{n^{p-q}} \| g \|_p, \quad 0 \leq q \leq p, \frac{1}{2} < p, \tag{4.3}$$

whith the constant C depending on p and q.

The trigonometric polynomial numerical solution $\widetilde{\varphi}^n(t) = (\widetilde{\varphi}_1^n(t), \widetilde{\varphi}_2^n(t), \widetilde{\varphi}_3^n(t))^\top \in X_n^3$ of $\varphi = (\varphi_1, \varphi_2, \varphi_3)^\top$ satisfies the following projected equation:

$$\mathcal{P}_n \mathcal{T}_n \widetilde{\varphi}^n + \mathcal{P}_n(\mathcal{H}_n + \mathcal{B}_n + C_n)\widetilde{\varphi}^n = \mathcal{P}_n w. \tag{4.4}$$

Here, $\mathcal{B}_n = \mathcal{B}_{1,n} + \mathcal{B}_{2,n}$, $C_n = C_{1,n} + C_{2,n} + C_{3,n} + C_{4,n}$, and the quadrature operators are described by $\mathcal{T}_n = \mathcal{T}\mathcal{P}_n$, $\mathcal{H}_n = \mathcal{H}\mathcal{P}_n$, $\mathcal{D}_n = \mathcal{D}\mathcal{P}_n$,

$$\mathcal{T}_n = \begin{bmatrix} -\mu H_{1,n}^1 D_n & \mu H_{3,n}^1 D_n + \mu D_n & 0 \\ H_{3,n}^1 D_n + D_n & H_{1,n}^1 D_n & 0 \\ 0 & 0 & H_{3,n}^1 D_n \end{bmatrix}, \quad \mathcal{H}_n = \begin{bmatrix} \mu E_{4,n} - \mu H_{2,n}^1 & \mu E_{2,n} + \mu H_{4,n}^1 & 0 \\ E_{2,n} + H_{4,n}^1 & H_{2,n}^1 - E_{4,n} & 0 \\ E_{0,n} & H_n^1 E_{0,n} & \frac{E_{0,n}}{\omega^2 \rho_a} - H_{3,n}^1 D_n \end{bmatrix},$$

$$\mathcal{B}_{1,n} = \begin{bmatrix} -\mu\kappa_p^2 S_{11,n}^3 E_{0,n} - (\lambda + \mu)\kappa_p^2 S_n^3 E_{0,n} - \mu H_{2,n}^2 & \mu\kappa_s^2 S_{31,n}^3 E_{0,n} + \mu H_{4,n}^2 & S_n^3 E_{0,n} \\ \kappa_p^2 S_{31,n}^3 + H_{4,n}^2 & \kappa_s^2 S_{11,n}^3 E_{0,n} + H_{2,n}^2 & 0 \\ 0 & H_n^2 E_{0,n} & 0 \end{bmatrix},$$

$$\mathcal{B}_{2,n} = \widetilde{\mathcal{B}}_{2,n}\mathcal{D}_n \overset{def}{=} \begin{bmatrix} -\mu H_{1,n}^2 & \mu H_{3,n}^2 & 0 \\ H_{3,n}^2 & H_{1,n}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_n & D_n & 0 \\ D_n & D_n & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_{1,n} = \begin{bmatrix} \mu K_{p4,n}^1 - \mu \widetilde{H}_{p2,n}^2 - \mu\kappa_p^2 \widetilde{S}_{p11,n}^1 E_{0,n} - (\lambda + \mu)\kappa_p^2 \widetilde{S}_{p,n}^1 E_{0,n} & \mu K_{s2,n}^1 + \mu \widetilde{H}_{s4,n}^2 + \mu\kappa_s^2 \widetilde{S}_{s13,n}^1 E_{0,n} & \widetilde{S}_{a,n}^1 E_{0,n} \\ K_{p2,n}^1 + \widetilde{H}_{p4,n}^2 + \mu\kappa_p^2 \widetilde{S}_{p13,n}^1 E_{0,n} & -K_{s4,n}^1 + \widetilde{H}_{s2,n}^2 + \kappa_s^2 \widetilde{S}_{s11,n}^1 E_{0,n} & 0 \\ K_{p,n}^1 E_{0,n} & \widetilde{H}_{s,n}^2 E_{0,n} & \frac{-K_{a,n}^1 E_{0,n}}{\omega^2 \rho_f} \end{bmatrix},$$

$$C_{2,n} = \widetilde{C}_{2,n}\mathcal{D}_n \overset{def}{=} \begin{bmatrix} \mu K_{p3,n}^1 - \mu \tilde{H}_{p1,n}^2 & \mu K_{s1,n}^1 + \mu \tilde{H}_{s3,n}^2 & 0 \\ K_{p1,n}^1 + \tilde{H}_{p3,n}^2 & -K_{s3,n}^1 + \tilde{H}_{s1,n}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_n & D_n & 0 \\ D_n & D_n & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_{3,n} = \begin{bmatrix} \mu(K_{p4,n}^2 - \widetilde{H}_{p2,n}^1 - \widetilde{H}_{p2,n}^3 - \kappa_p^2 \widetilde{S}_{p11,n}^2 E_{0,n}) - (\lambda + \mu)\kappa_p^2 \widetilde{S}_{p,n}^2 E_{0,n} & \mu(K_{s2,n}^2 + \widetilde{H}_{s4,n}^1 + \widetilde{H}_{s4,n}^3 + \kappa_s^2 \widetilde{S}_{s13,n}^2 E_{0,n}) & \widetilde{S}_{a,n}^2 E_{0,n} \\ K_{p2,n}^2 + \tilde{H}_{p4,n}^1 + \widetilde{H}_{p4,n}^3 + \mu\kappa_p^2 \widetilde{S}_{p13,n}^2 E_{0,n} & -K_{s4,n}^2 + \widetilde{H}_{s2,n}^1 + \widetilde{H}_{s2,n}^3 + \kappa_s^2 \widetilde{S}_{s11,n}^2 E_{0,n} & 0 \\ K_{p,n}^2 E_{0,n} & \widetilde{H}_{s,n}^1 E_{0,n} + \widetilde{H}_{s,n}^3 E_0 & \frac{-K_{a,n}^2 E_{0,n}}{\omega^2 \rho_f} \end{bmatrix},$$

$$C_{4,n} = \widetilde{C}_{4,n}\mathcal{D}_n \overset{def}{=} \begin{bmatrix} \mu K_{p3,n}^2 - \mu \tilde{H}_{p1,n}^1 - \mu \tilde{H}_{p1,n}^3 & \mu K_{s1,n}^2 + \mu \tilde{H}_{s3,n}^1 + \mu \tilde{H}_{s3,n}^3 D_n & 0 \\ K_{p1,n}^2 + \tilde{H}_{p3,n}^1 + \tilde{H}_{p3,n}^3 & -K_{s3,n}^2 + \tilde{H}_{s1,n}^1 + \tilde{H}_{s1,n}^3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_n & D_n & 0 \\ D_n & D_n & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Define $E_{i,n}\psi = m_i(t,t)\psi = E_i\psi, i = 0,1,2,3,4$, $S^3_n = S^3P_n$, $H^1_n = H^1P_n$, $D_n = DP_n$ with P_n being the interpolation with respect to the variable $(\cdot)$, where

$$
\begin{aligned}
(\widetilde{S}^1_{\sigma ij,n}\psi)(t) &= \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)P_n\Big\{\tilde{s}^1_\sigma(t,\cdot)m_i(t,\cdot)m_j(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(\widetilde{S}^2_{\sigma ij,n}\psi)(t) &= \int_0^{2\pi} P_n\Big\{\tilde{s}^2_\sigma(t,\cdot)m_i(t,\cdot)m_j(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(S^3_{ij,n}\psi)(t) &= \frac{1}{2\pi}\int_0^{2\pi} \Big(2-\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)P_n\Big\{m_i(t,\cdot)m_j(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(K^1_{\sigma i,n}\psi)(t) &= \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)P_n\Big\{k^1_\sigma(t,\cdot)m_i(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(K^2_{\sigma i,n}\psi)(t) &= \int_0^{2\pi} P_n\Big\{m_i(t,\cdot)k^2_\sigma(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(H^1_{i,n}\psi)(t) &= \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\eta-t}{2}P_n\Big\{m_i(t,\cdot)\psi\Big\}(\eta)d\eta + \frac{i}{2\pi}\int_0^{2\pi} P_n\Big\{m_i(t,\cdot)\psi\Big\}(\eta)d\eta,
\end{aligned}
$$

$$
\begin{aligned}
(H^2_{i,n}\psi)(t) &= \frac{1}{2\pi}\int_0^{2\pi} \Big(2-\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)P_n\Big\{\sin(t-\cdot)m_i(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(\widetilde{H}^1_{\sigma i,n}\psi)(t) &= \int_0^{2\pi} P_n\Big\{\widetilde{h}^1(t,\cdot)m_i(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(\widetilde{H}^2_{\sigma i,n}\psi)(t) &= \int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)P_n\Big\{\tilde{h}^2_\sigma(t,\cdot)m_i(t,\cdot)\psi\Big\}(\eta)d\eta,\\
(\widetilde{H}^3_{\sigma i,n}\psi)(t) &= \int_0^{2\pi} P_n\Big\{\widetilde{h}^3_\sigma(t,\cdot)m_i(t,\cdot)\psi\Big\}(\eta)d\eta.
\end{aligned}
$$

Theorem 3. *Assume that* $0 \le q \le p$ *and* $p > \frac{1}{2}$. *Then, for the quadrature operators* $\mathcal{B}_{1,n}$, $C_{1,n}$, $C_{3,n}$, *the following estimates hold:*

$$\|\, \mathcal{B}_{1,n}\mathcal{X} - \mathcal{B}_1\mathcal{X}\,\|_{q+1} \le \frac{M_1}{n^{p-q}}\,\|\,\mathcal{X}\,\|_p, \tag{4.5}$$

$$\|\,(C_{1,n}+C_{3,n})\mathcal{X} - (C_1+C_3)\mathcal{X}\,\|_{q+1} \le \frac{M_2}{n^{p-q}}\,\|\,\mathcal{X}\,\|_p, \tag{4.6}$$

$$\|\,\mathcal{P}_n(C_{1,n}+C_{3,n})\varphi - \mathcal{P}_n(C_1+C_3)\varphi\,\|_{q+1} \le \frac{\widetilde{M}_2}{n^{p+1-q}}\,\|\,\varphi\,\|_p, \tag{4.7}$$

for all $\mathcal{X} \in H^p[0,2\pi]^3$ *and all* $\varphi \in X^3_n$, *where* M_1, M_2, $\widetilde{M}_2$ *are positive constants depending on p and q.*

Proof. We rewrite the functions $\mathcal{B}_1\mathcal{X}$, $(C_1+C_3)\mathcal{X}$ in the form of

$$(\mathcal{B}_1\mathcal{X})(t) = \frac{1}{2\pi}\int_0^{2\pi}\Big(2-\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)N(t,\eta)\mathcal{X}(\eta)d\eta,$$

$$(C_1\mathcal{X}+C_3\mathcal{X})(t) = \int_0^{2\pi}\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)P(t,\eta)\mathcal{X}(\eta)d\eta + \int_0^{2\pi}Q(t,\eta)\mathcal{X}(\eta)d\eta,$$

where

$$N(t,\eta)=\begin{bmatrix} n_1(t,\eta) & n_2(t,\eta) & n_3(t,\eta)\\ n_4(t,\eta) & n_5(t,\eta) & n_6(t,\eta)\\ n_7(t,\eta) & n_8(t,\eta) & n_9(t,\eta)\end{bmatrix},$$

$$P(t,\eta)=\begin{bmatrix} p_1(t,\eta) & p_2(t,\eta) & p_3(t,\eta)\\ p_4(t,\eta) & p_5(t,\eta) & p_6(t,\eta)\\ p_7(t,\eta) & p_8(t,\eta) & p_9(t,\eta)\end{bmatrix},\quad Q(t,\eta)=\begin{bmatrix} q_1(t,\eta) & q_2(t,\eta) & q_3(t,\eta)\\ q_4(t,\eta) & q_5(t,\eta) & q_6(t,\eta)\\ q_7(t,\eta) & q_8(t,\eta) & q_9(t,\eta)\end{bmatrix},$$

$$\begin{aligned}
n_1(t,\eta) &= -\mu\kappa_p^2 m_1^2(t,\eta)|z'(\eta)| - (\lambda+\mu)\kappa_p^2|z'(\eta)| - \mu m_2(t,\eta)h^2(t,\eta),\\
n_2(t,\eta) &= \mu\kappa_s^2 m_1(t,\eta)m_3(t,\eta)|z'(\eta)| + \mu m_4(t,\eta)h^2(t,\eta),\\
n_3(t,\eta) &= |z'(\eta)|,\\
n_4(t,\eta) &= \kappa_p^2 m_1(t,\eta)m_3(t,\eta)|z'(\eta)| + m_4(t,\eta)h^2(t,\eta),\\
n_5(t,\eta) &= \kappa_s^2 m_1^2(t,\eta)|z'(\eta)| + m_2(t,\eta)h^2(t,\eta),\\
n_8(t,\eta) &= h^2(t,\eta)|z'(\eta)|,\\
n_6(t,\eta) &= n_7(t,\eta) = n_9(t,\eta) = 0.\\
p_1(t,\eta) &= -[\mu m_1^2(t,\eta) + (\lambda+\nu)]\kappa_p^2\widetilde{s}_p^1(t,\eta)|z'(\eta)| - \mu m_2(t,\eta)\widetilde{h}_p^2(t,\eta) + \mu m_4(t,\eta)k_p^1(t,s),\\
p_2(t,\eta) &= \mu\kappa_s^2 m_1(t,\eta)m_3(t,\eta)\widetilde{s}_s^1(t,\eta)|z'(\eta)| + \mu m_4(t,\eta)\widetilde{h}_s^2(t,\eta) + \mu m_2(t,\eta)k_s^1(t,s),\\
p_3(t,\eta) &= \widetilde{s}_a^1(t,\eta)|z'(\eta)|,\\
p_4(t,\eta) &= \kappa_p^2 m_1(t,\eta)m_3(t,\eta)\widetilde{s}_p^1(t,\eta)|z'(\eta)| + m_4(t,\eta)\widetilde{h}_p^2(t,\eta) + m_2(t,\eta)k_p^1(t,\eta),\\
p_5(t,\eta) &= \kappa_s^2 m_1^2(t,\eta)\widetilde{s}_s^1(t,\eta)|z'(\eta)| + m_2(t,\eta)\widetilde{h}_s^2(t,\eta),\\
p_6(t,\eta) &= 0,\\
p_7(t,\eta) &= k_p^1(t,\eta)|z'(\eta)|,\\
p_8(t,\eta) &= \widetilde{h}_s^2(t,\eta)|z'(\eta)|,\\
p_9(t,\eta) &= -k_a^1(t,\eta)|z'(\eta)|/\omega^2\rho_a.\\
q_1(t,\eta) &= -[\mu m_1^2(t,\eta) + (\lambda+\nu)]\kappa_p^2\widetilde{s}_p^2(t,\eta)|z'(\eta)| - \mu m_2[(t,\eta)\widetilde{h}_p^1(t,\eta) + \widetilde{h}_p^3(t,\eta)] + \mu m_4(t,\eta)k_p^2(t,s),\\
q_2(t,\eta) &= \mu\kappa_s^2 m_1(t,\eta)m_3(t,\eta)\widetilde{s}_s^2(t,\eta)|z'(\eta)| + \mu m_4(t,\eta)[(t,\eta)\widetilde{h}_p^1(t,\eta) + \widetilde{h}_p^3(t,\eta)] + \mu m_2(t,\eta)k_s^2(t,s),\\
q_3(t,\eta) &= \widetilde{s}_a^2(t,\eta)|z'(\eta)|,\\
q_4(t,\eta) &= \kappa_p^2 m_1(t,\eta)m_3(t,\eta)\widetilde{s}_p^2(t,\eta)|z'(\eta)| + m_4(t,\eta)[\widetilde{h}_p^1(t,\eta) + \widetilde{h}_p^3(t,\eta)] + m_2(t,\eta)k_p^2(t,\eta),\\
q_5(t,\eta) &= \kappa_s^2 m_1^2(t,\eta)\widetilde{s}_s^2(t,\eta)|z'(\eta)| + m_2(t,\eta)[\widetilde{h}_s^1(t,\eta) + \widetilde{h}_s^3(t,\eta)],\\
q_6(t,\eta) &= 0,\\
q_7(t,\eta) &= k_p^2(t,\eta)|z'(\eta)|,\\
q_8(t,\eta) &= [\widetilde{h}_s^1(t,\eta) + \widetilde{h}_s^3(t,\eta)]|z'(\eta)|,\\
q_9(t,\eta) &= -k_a^2(t,\eta)|z'(\eta)|/\omega^2\rho_a.
\end{aligned}$$

Denote the full-discretization of $\mathcal{B}_1$, $C_1 + C_3$ via interpolatory quadrature

$$(B_{1,n}\mathcal{X})(t) = \int_0^{2\pi}\Big(2 - \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\Big)\mathcal{P}_n\Big\{N(t,\eta)\mathcal{X}\Big\}(\eta)d\eta,$$

$$(C_{1,n}\mathcal{X} + C_{3,n}\mathcal{X})(t) = \int_0^{2,\pi}\ln\Big(4\sin^2\frac{t-\eta}{2}\Big)\mathcal{P}_n\Big\{P(t,\eta)\mathcal{X}\Big\}(\eta)d\eta + \int_0^{2,\pi}\mathcal{P}_n\Big\{Q(t,\eta)\mathcal{X}\Big\}(\eta)d\eta.$$

Since the kernel functions $h^2(t,\eta)$ and $m_i(t,\eta)$, i=1,2,3,4, are analytic, we get that $n_i(t,\eta)$, $i = 1,\cdots,9$, are analytic. Following from [44, theorem 12.15, 12.18], for all $\mathcal{X} \in H^p[0,2\pi]^3$, we deduce

$$\| \mathcal{B}_{1,n}\mathcal{X} - \mathcal{B}_1\mathcal{X} \|_{q+1} \le \frac{M_1}{n^{p-q}} \| \mathcal{X} \|_p,$$

for $p > \frac{1}{2}$ and $0 \le q \le p$. For $C_{1,n}$, $C_{3,n}$, we have the analogous estimate

$$\| (C_{1,n} + C_{3,n})\mathcal{X} - (C_1 + C_3)\mathcal{X} \|_{q+1} \le \frac{M_2}{n^{p-q}} \| \mathcal{X} \|_p,$$

for some constants M_1 and M_2 depending on p and q. Further, due to $k^1_\sigma(t,t) = \widetilde{s}^1_\sigma(t,t) = \widetilde{h}^2_\sigma(t,t) = 0$, $P(t,t) = 0$, using [44, Lemma 13.21], we get

$$\| \mathcal{P}_n(C_{1,n} + C_{3,n})\varphi - \mathcal{P}_n(C_1 + C_3)\varphi \|_{q+1} \le \frac{\widetilde{M}_2}{n^{p+1-q}} \| \varphi \|_p,$$

for all trigonometric polynomials $\varphi \in X_n^3$ and some constant $\widetilde{M}_2$ depending on p and q.

Theorem 4. *Assume that $p > \frac{3}{2}$. Then, the operators $\mathcal{H}_n$, $\mathcal{B}_{1,n}$, $C_{1,n}$ and $C_{3,n}$ have estimate*

$$\| \mathcal{P}_n[(\mathcal{H}_n + \mathcal{B}_{1,n} + C_{1,n} + C_{3,n}) - (\mathcal{H} + \mathcal{B}_1 + C_1 + C_3)]\varphi \|_{p-1} \lesssim \frac{1}{n} \| \varphi \|_p$$

for all trigonometric polynomials $\varphi \in X_n^3$.

Proof. From the boundness of the operator $\mathcal{H} : H^p[0,2\pi]^3 \to H^p[0,2\pi]^3$ and the estimate (4.1), for $0 \le q \le p$, $p > \frac{1}{2}$, we obtain

$$\| \mathcal{H}_n\mathcal{X} - \mathcal{H}\mathcal{X} \|_q = \| \mathcal{H}(\mathcal{P}_n\mathcal{X} - \mathcal{X}) \|_q \le L_1 \| (\mathcal{P}_n\mathcal{X} - \mathcal{X}) \|_q \le \frac{L_2}{n^{p-q}} \| \mathcal{X} \|_p, \tag{4.8}$$

for all $\mathcal{X} \in H^p[0,2\pi]^3$, where L_1 is a positive constant depending on q, and L_2 is a positive constant depending on q and p. Using the boundness of the operator $\mathcal{P}_n : H^{p-1}[0,2\pi]^3 \to H^{p-1}[0,2\pi]^3$, combining the estimate (4.1) and (4.8), for the trigonometric polynomials $\varphi \in X_n^3$, we obtain

$$\| \mathcal{P}_n(\mathcal{H}_n - \mathcal{H})\varphi \|_{p-1} \le \frac{c_1}{n} \| \varphi \|_p, \tag{4.9}$$

for $p > \frac{3}{2}$ and some constant c_1. Similarly, combining estimate (4.1) and (4.5), we have

$$\| \mathcal{P}_n(\mathcal{B}_{1,n} - \mathcal{B}_1)\varphi \|_{p-1} \le \frac{c_2}{n} \| \varphi \|_p, \tag{4.10}$$

for $p > \frac{3}{2}$ and some constant c_2. Recalling (4.3) implies

$$\| \mathcal{P}_n(C_{1,n} + C_{3,n})\varphi - \mathcal{P}_n(C_1 + C_3)\varphi \|_{p-1} \le \frac{c_3}{n} \| \varphi \|_p, \tag{4.11}$$

for $p > \frac{3}{2}$ and some constant c_3. Therefore, combining the estimates (4.9), (4.10), (4.11), the proof is completed.

Theorem 5. *Assume that $p \geq \frac{3}{2}$. Then, the operators $\mathcal{B}_{2,n}$, $C_{2,n}$ and $C_{4,n}$ have estimate*

$$\| \mathcal{P}_n[(\mathcal{B}_{2,n} + C_{2,n} + C_{4,n}) - (\mathcal{B}_2 + C_2 + C_4)]\varphi \|_{p-1} \lesssim \frac{1}{n} \| \varphi \|_p$$

for all trigonometric polynomials $\varphi \in X_n^3$.

Proof. Noting $\mathcal{D}_n\varphi = \mathcal{D}\varphi$, for all trigonometric polynomials $\varphi \in X_n^3$, we can transform

$$\mathcal{P}_n(\mathcal{B}_{2,n} - \mathcal{B}_2)\varphi = \mathcal{P}_n(\widetilde{\mathcal{B}}_{2,n}\mathcal{D}_n - \widetilde{\mathcal{B}}_2\mathcal{D})\varphi = \mathcal{P}_n(\widetilde{\mathcal{B}}_{2,n} - \widetilde{\mathcal{B}}_2)\mathcal{D}\varphi. \tag{4.12}$$

$$\begin{aligned}\mathcal{P}_n(C_{2,n} + C_{4,n} - C_2 - C_4)\varphi &= \mathcal{P}_n(\widetilde{C}_{2,n}\mathcal{D}_n + \widetilde{C}_{4,n}\mathcal{D}_n - \widetilde{C}_2\mathcal{D} - \widetilde{C}_4\mathcal{D})\varphi \\ &= \mathcal{P}_n(\widetilde{C}_{2,n} + \widetilde{C}_{4,n} - \widetilde{C}_2 - \widetilde{C}_4)\mathcal{D}\varphi.\end{aligned} \tag{4.13}$$

By the analogous discussion in Theorem 3, for all $\mathcal{X} \in H^p[0, 2\pi]^3$ and all $\varphi \in X_n^3$, we get

$$\| \widetilde{\mathcal{B}}_{2,n}\mathcal{X} - \widetilde{\mathcal{B}}_2\mathcal{X} \|_{q+1} \leq \frac{M_3}{n^{p-q}} \| \mathcal{X} \|_p,$$

$$\| (\widetilde{C}_{2,n} + \widetilde{C}_{4,n})\mathcal{X} - (\widetilde{C}_2 + \widetilde{C}_4)\mathcal{X} \|_{q+1} \leq \frac{M_4}{n^{p-q}} \| \mathcal{X} \|_p,$$

for $p > \frac{1}{2}$ and $0 \leq q \leq p$, where some constants M_3 and M_4 depend on p and q. Thus, in particular, $\widetilde{\mathcal{B}}_{2,n}$, $\widetilde{\mathcal{B}}_{2,n} - \widetilde{\mathcal{B}}_2$, $(\widetilde{C}_{2,n} + \widetilde{C}_{4,n}) - (\widetilde{C}_2 + \widetilde{C}_4)$ $\widetilde{C}_{2,n} + \widetilde{C}_{4,n}$ are bounded operators from $H^p[0, 2\pi]$ to $H^p[0, 2\pi]$, for $p > \frac{1}{2}$.

From the boundedness of $\mathcal{D} : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ and the uniform boundedness of $\mathcal{P}_n : H^{p-1}[0, 2\pi] \to H^{p-1}[0, 2\pi]$ for $p > \frac{3}{2}$, together with (4.12) and (4.13), we obtain

$$\| \mathcal{P}_n(\mathcal{B}_{2,n} - \mathcal{B}_2)\varphi \|_{p-1} = \| \mathcal{P}_n(\widetilde{\mathcal{B}}_{2,n} - \widetilde{\mathcal{B}}_2)\mathcal{D}\varphi \|_{p-1} \leq \frac{c_4}{n} \| \varphi \|_p \tag{4.14}$$

$$\| \mathcal{P}_n(C_{2,n} + C_{4,n} - C_2 - C_4)\varphi \|_{p-1} = \| \mathcal{P}_n(\widetilde{C}_{2,n} + \widetilde{C}_{4,n} - \widetilde{C}_2 - \widetilde{C}_4)\mathcal{D}\varphi \|_{p-1} \leq \frac{c_5}{n} \| \varphi \|_p \tag{4.15}$$

for all trigonometric polynomials $\varphi \in X_n^3$.

Hence, combining the estimates (4.14), (4.15), the proof is completed.

Remark 4.2. *If the number $n \gg 1$, the equation* (4.4) *is expected to have a unique solution $\widetilde{\varphi}^n$. For the fully discrete collocation method, it is expected that the following estimate holds under certain conditions:*

$$\begin{aligned}&\| \widetilde{\varphi}^n - \varphi \|_p \\ &\leq C\{ \| \mathcal{P}_n\mathcal{T}_n\varphi - \mathcal{T}\varphi \|_{p-1} + \| \mathcal{P}_n[(\mathcal{H}_n + \mathcal{B}_n + C_n) - (\mathcal{H} + \mathcal{B} + C)]\varphi \|_{p-1} + \| (\mathcal{P}_n - \mathcal{I})w \|_{p-1}\},\end{aligned}$$

for all $p > \frac{3}{2}$ and some positive constant C.

From the remarks 4.1 and 4.2, we expect that the collocation method is convergent in $H^p[0, 2\pi]^3$ for each $p > \frac{3}{2}$. The convergence analysis of the proposed numerical method depends on the invertibility of the boundary integral system as well as the discretized system, especially for the analysis of the properties of the operator $\mathcal{T}$ and $\mathcal{T}_n$, which is currently under investigation.

5. Numerical experiment

For the smooth integrals, we use the trapezoidal rule:

$$\begin{aligned}&\int_0^{2\pi} f(\eta)d\eta \approx \frac{\pi}{n}\sum_{j=0}^{2n-1} f(\eta_j^{(n)}),\\&\int_0^{2\pi} Q(t,\eta)f(\eta)d\eta \approx \frac{\pi}{n}\sum_{j=0}^{2n-1} Q(t,\eta_j^{(n)})f(\eta_j^{(n)}).\end{aligned} \tag{5.1}$$

For the singular integrals, following the quadrature rules in [46, equation (3.93)], [42, equation (4.6)] and [38, equation (13.39)], we employ the following quadrature rules:

$$\begin{aligned}&\int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)f(\eta)d\eta \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t)f(\eta_j^{(n)})\\&\int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)Q(t,\eta)f(\eta)d\eta \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t)Q(t,\eta_j^{(n)})f(\eta_j^{(n)})\\&\int_0^{2\pi} \ln\Big(4\sin^2\frac{t-\eta}{2}\Big)Q(t,\eta)f'(\eta)d\eta \approx \sum_{j=0}^{2n-1}\sum_{m=0}^{2n-1} d_{m-j}^{(n)}R_m^{(n)}(t)Q(t,\eta_m^{(n)})f(\eta_j^{(n)})\\&\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\eta-t}{2}f(\eta)d\eta \approx \sum_{j=0}^{2n-1} U_j^{(n)}(t)f(\eta_j^{(n)})\\&\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\eta-t}{2}Q(t,\eta)f(\eta)d\eta \approx \sum_{j=0}^{2n-1} U_j^{(n)}(t)Q(t,\eta_j^{(n)})f(\eta_j^{(n)})\\&\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\eta-t}{2}Q(t,\eta)f'(\eta)d\eta \approx \sum_{j=0}^{2n-1}\sum_{m=0}^{2n-1} d_{m-j}^{(n)}U_m^{(n)}(t)Q(t,\eta_m^{(n)})f(\eta_j^{(n)})\end{aligned} \tag{5.2}$$

where Q is a continuous function, and the weight function is given by

$$\begin{aligned}&R_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\Big[m(t-\eta_j^{(n)})\Big] - \frac{\pi}{n^2}\cos\Big[n(t-\eta_j^{(n)})\Big],\\&U_j^{(n)}(t) = \frac{1}{2n}\Big[1-\cos n(\eta_j^{(n)}-t)\Big]\cot\frac{\eta_j^{(n)}-t}{2},\\&d_j^{(n)} = \begin{cases}\frac{(-1)^j}{2}\cot\frac{j\pi}{2n}, & j=\pm1,\pm2,\cdots\pm 2n-1,\\ 0, & j=0.\end{cases}\end{aligned}$$

Thus, the equation becomes

$$
\begin{aligned}
w_{1,i}^{(n)} &= \sum_{j=0}^{2n-1} X_{ij,p}^{n}\varphi_{1,j}^{(n)} + \sum_{j=0}^{2n-1} X_{ij,s}^{n}\varphi_{2,j}^{(n)} + \sum_{j=0}^{2n-1} X_{ij,a}^{n}\varphi_{3,j}^{(n)} \\
&\quad +\mu m_4(\eta_i^{(n)},\eta_i^{(n)})\varphi_{1,i}^{(n)} + \mu m_2(\eta_i^{(n)},\eta_i^{(n)})\varphi_{2,i}^{(n)} + \sum_{m=0}^{2n-1} d_{m-i}^{(n)}\varphi_{2,m}^{(n)}, \\
w_{2,i}^{(n)} &= \sum_{j=0}^{2n-1} Y_{ij,p}^{n}\varphi_{1,j}^{(n)} + \sum_{j=0}^{2n-1} Y_{ij,s}^{n}\varphi_{2,j}^{(n)} \\
&\quad + m_2(\eta_i^{(n)},\eta_i^{(n)})\varphi_{1,i}^{(n)} + \sum_{m=0}^{2n-1} d_{m-i}^{(n)}\varphi_{1,m}^{(n)} - m_4(\eta_i^{(n)},\eta_i^{(n)})\varphi_{2,i}^{(n)}, \\
w_{3,i}^{(n)} &= \sum_{j=0}^{2n-1} Z_{ij,p}^{n}\varphi_{1,j}^{(n)} + \sum_{j=0}^{2n-1} Z_{ij,s}^{n}\varphi_{2,j}^{(n)} - \frac{1}{\omega^2\rho_a}\sum_{j=0}^{2n-1} Z_{ij,a}^{n}\varphi_{3,j}^{(n)} \\
&\quad +|z'(\eta_i^{(n)})|\varphi_{1,i}^{(n)} + \frac{1}{\omega^2\rho_f}|z'(\eta_i^{(n)})|\varphi_{3,i}^{(n)}
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
X_{ij,p}^{(n)} &= -\kappa_p^2|z'(\eta_i^{(n)})|R_j^{(n)}(\eta_i^{(n)})s_p^1(\eta_i^{(n)},\eta_j^{(n)})|z'(\eta_j^{(n)})|\left(\mu m_1^2(\eta_i^{(n)},\eta_j^{(n)}) + (\mu+\lambda)\right) \\
&\quad + \mu\sum_{m=0}^{2n-1} d_{m-j}^{(n)}R_m^{(n)}(\eta_i^{(n)})k_p^1(\eta_i^{(n)},\eta_m^{(n)})m_3(\eta_i^{(n)},\eta_m^{(n)}) + \mu R_j^{(n)}(\eta_i^{(n)})k_p^1(\eta_i^{(n)},\eta_j^{(n)})m_4(\eta_i^{(n)},\eta_j^{(n)}) \\
&\quad - \mu\sum_{m=0}^{2n-1} d_{m-j}^{(n)}R_m^{(n)}(\eta_i^{(n)})h_p^2(\eta_i^{(n)},\eta_m^{(n)})m_1(\eta_i^{(n)},\eta_m^{(n)}) - \mu R_j^{(n)}(\eta_i^{(n)})h_p^2(\eta_i^{(n)},\eta_j^{(n)})m_2(\eta_i^{(n)},\eta_j^{(n)}) \\
&\quad - \frac{\pi}{n}\kappa_p^2 s_p^2(\eta_i^{(n)},\eta_j^{(n)})|z'(\eta_j^{(n)})|\left(\mu m_1^2(\eta_i^{(n)},\eta_j^{(n)}) + (\mu+\lambda)\right) \\
&\quad + \frac{\pi}{n}\mu\sum_{m=0}^{2n-1} d_{m-j}^{(n)}k_p^2(\eta_i^{(n)},\eta_m^{(n)})m_3(\eta_i^{(n)},\eta_m^{(n)}) + \frac{\pi}{n}\mu k_p^2(\eta_i^{(n)},\eta_j^{(n)})m_4(\eta_i^{(n)},\eta_j^{(n)}) \\
&\quad - \frac{\pi}{n}\mu\sum_{m=0}^{2n-1} d_{m-j}^{(n)}\left(\widetilde{h}^1(\eta_i^{(n)},\eta_m^{(n)}) + h_p^3(\eta_i^{(n)},\eta_m^{(n)})\right)m_1(\eta_i^{(n)},\eta_m^{(n)}) \\
&\quad - \frac{\pi}{n}\mu\left(\widetilde{h}^1(\eta_i^{(n)},\eta_j^{(n)}) + h_p^3(\eta_i^{(n)},\eta_j^{(n)})\right)m_2(\eta_i^{(n)},\eta_j^{(n)}) \\
&\quad - \mu\sum_{m=0}^{2n-1} d_{m-j}^{(n)}U_m^{(n)}(\eta_i^{(n)})m_1(\eta_i^{(n)},\eta_m^{(n)}) - \mu U_j^{(n)}(\eta_i^{(n)})m_2(\eta_i^{(n)},\eta_j^{(n)})
\end{aligned}
$$

$$
\begin{aligned}
X_{ij,s}^{(n)} =& \mu\kappa_s^2 |z'(\eta_i^{(n)})| R_j^{(n)}(\eta_i^{(n)}) s_s^1(\eta_i^{(n)},\eta_j^{(n)}) m_3(\eta_i^{(n)},\eta_j^{(n)}) m_1(\eta_i^{(n)},\eta_j^{(n)}) |z'(\eta_j^{(n)})| \\
&+ \mu \sum_{m=0}^{2n-1} d_{m-j}^{n} R_m^{(n)}(\eta_i^{(n)}) k_s^1(\eta_i^{(n)},\eta_m^{(n)}) m_1(\eta_i^{(n)},\eta_m^{(n)}) + \mu R_j^{(n)}(\eta_i^{(n)}) k_s^1(\eta_i^{(n)},\eta_j^{(n)}) m_2(\eta_i^{(n)},\eta_j^{(n)}) \\
&+ \mu \sum_{m=0}^{2n-1} d_{m-j}^{(n)} R_m^{(n)}(\eta_i^{(n)}) h_s^2(\eta_i^{(n)},\eta_m^{(n)}) m_3(\eta_i^{(n)},\eta_m^{(n)}) + \mu R_j^{(n)}(\eta_i^{(n)}) h_s^2(\eta_i^{(n)},\eta_j^{(n)}) m_4(\eta_i^{(n)},\eta_j^{(n)})] \\
&+ \frac{\pi}{n}\mu\kappa_s^2 |z'(\eta_i^{(n)})| s_s^2(\eta_i^{(n)},\eta_j^{(n)}) m_3(\eta_i^{(n)},\eta_j^{(n)}) m_1(\eta_i^{(n)},\eta_j^{(n)}) |z'(\eta_j^{(n)})| \\
&+ \frac{\pi}{n}\mu \sum_{m=0}^{2n-1} d_{m-j}^{(n)} k_s^2(\eta_i^{(n)},\eta_m^{(n)}) m_1(\eta_i^{(n)},\eta_m^{(n)}) + \frac{\pi}{n}\mu k_s^2(\eta_i^{(n)},\eta_j^{(n)}) m_2(\eta_i^{(n)},\eta_j^{(n)}) \\
&+ \frac{\pi}{n}\mu \sum_{m=0}^{2n-1} d_{m-j}^{(n)} \Big(\widetilde{h}^1(\eta_i^{(n)},\eta_m^{(n)}) + h_s^3(\eta_i^{(n)},\eta_m^{(n)})\Big) m_3(\eta_i^{(n)},\eta_m^{(n)}) \\
&+ \frac{\pi}{n}\mu \Big(\widetilde{h}^1(\eta_i^{(n)},\eta_j^{(n)}) + h_s^3(\eta_i^{(n)},\eta_j^{(n)})\Big) m_4(\eta_i^{(n)},\eta_j^{(n)}) \\
&+ \mu \sum_{m=0}^{2n-1} d_{m-j}^{(n)} U_m^{(n)}(\eta_i^{(n)}) m_3(\eta_i^{(n)},\eta_m^{(n)}) + \mu U_j^{(n)}(\eta_i^{(n)}) m_4(\eta_i^{(n)},\eta_j^{(n)})
\end{aligned}
$$

$$
X_{ij,a}^{(n)} = R_j^{(n)}(\eta_i^{(n)}) |z'(\eta_i^{(n)})| s_a^1(\eta_i^{(n)},\eta_j^{(n)}) |z'(\eta_j^{(n)})| + \frac{\pi}{n} |z'(\eta_i^{(n)})| s_a^2(\eta_i^{(n)},\eta_j^{(n)}) |z'(\eta_j^{(n)})|
$$

$$
\begin{aligned}
Y_{ij,p}^{(n)} =& \kappa_p^2 |z'(\eta_i^{(n)})| R_j^{(n)}(\eta_i^{(n)}) s_p^1(\eta_i^{(n)},\eta_j^{(n)}) m_3(\eta_i^{(n)},\eta_j^{(n)}) m_1(\eta_i^{(n)},\eta_j^{(n)}) |z'(\eta_j^{(n)})| \\
&+ \sum_{m=0}^{2n-1} d_{m-j}^{n} R_m^{(n)}(\eta_i^{(n)}) k_p^1(\eta_i^{(n)},\eta_m^{(n)}) m_1(\eta_i^{(n)},\eta_m^{(n)}) + R_j^{(n)}(\eta_i^{(n)}) k_p^1(\eta_i^{(n)},\eta_j^{(n)}) m_2(\eta_i^{(n)},\eta_j^{(n)}) \\
&+ \sum_{m=0}^{2n-1} d_{m-j}^{(n)} R_m^{(n)}(\eta_i^{(n)}) h_p^2(\eta_i^{(n)},\eta_m^{(n)}) m_3(\eta_i^{(n)},\eta_m^{(n)}) + R_j^{(n)}(\eta_i^{(n)}) h_p^2(\eta_i^{(n)},\eta_j^{(n)}) m_4(\eta_i^{(n)},\eta_j^{(n)}) \\
&+ \frac{\pi}{n}\kappa_p^2 |z'(\eta_i^{(n)})| s_p^2(\eta_i^{(n)},\eta_j^{(n)}) m_3(\eta_i^{(n)},\eta_j^{(n)}) m_1(\eta_i^{(n)},\eta_j^{(n)}) |z'(\eta_j^{(n)})| \\
&+ \frac{\pi}{n} k_p^2(\eta_i^{(n)},\eta_j^{(n)}) \sum_{m=0}^{2n-1} d_{m-j}^{(n)} k_p^2(\eta_i^{(n)},\eta_m^{(n)}) m_1(\eta_i^{(n)},\eta_m^{(n)}) + \frac{\pi}{n} k_p^2(\eta_i^{(n)},\eta_j^{(n)}) m_2(\eta_i^{(n)},\eta_j^{(n)})] \\
&+ \frac{\pi}{n} \sum_{m=0}^{2n-1} d_{m-j}^{(n)} \Big(\widetilde{h}^1(\eta_i^{(n)},\eta_m^{(n)}) + h_p^3(\eta_i^{(n)},\eta_m^{(n)})\Big) m_3(\eta_i^{(n)},\eta_m^{(n)}) \\
&+ \frac{\pi}{n} \Big(\widetilde{h}^1(\eta_i^{(n)},\eta_j^{(n)}) + h_p^3(\eta_i^{(n)},\eta_j^{(n)})\Big) m_4(\eta_i^{(n)},\eta_j^{(n)}) \\
&+ \sum_{m=0}^{2n-1} d_{m-j}^{(n)} U_m^{(n)}(\eta_i^{(n)}) m_3(\eta_i^{(n)},\eta_m^{(n)}) + U_j^{(n)}(\eta_i^{(n)}) m_4(\eta_i^{(n)},\eta_j^{(n)})
\end{aligned}
$$

$$
\begin{aligned}
Y_{ij,s}^{(n)} =& \kappa_s^2 |z'(\eta_i^{(n)})| R_j^{(n)}(\eta_i^{(n)}) s_s^1(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^{(n)})| m_1^2(\eta_i^{(n)}, \eta_j^{(n)}) \\
& - \sum_{m=0}^{2n-1} d_{m-j}^{(n)} R_m^{(n)}(\eta_i^{(n)}) k_s^1(\eta_i^{(n)}, \eta_m^{(n)}) m_3(\eta_i^{(n)}, \eta_m^{(n)}) - R_j^{(n)}(\eta_i^{(n)}) k_s^1(\eta_i^{(n)}, \eta_j^{(n)}) m_4(\eta_i^{(n)}, \eta_j^{(n)}) \\
& + \sum_{m=0}^{2n-1} d_{m-j}^{(n)} R_j^{(n)}(\eta_i^{(n)}) h_s^2(\eta_i^{(n)}, \eta_m^{(n)}) m_1(\eta_i^{(n)}, \eta_m^{(n)}) + R_j^{(n)}(\eta_i^{(n)}) h_s^2(\eta_i^{(n)}, \eta_j^{(n)}) m_2(\eta_i^{(n)}, \eta_j^{(n)}) \\
& + \frac{\pi}{n} \kappa_s^2 |z'(\eta_i^{(n)})| s_s^2(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^{(n)})| m_1^2(\eta_i^{(n)}, \eta_j^{(n)}) \\
& - \frac{\pi}{n} \sum_{m=0}^{2n-1} d_{m-j}^{(n)} k_s^2(\eta_i^{(n)}, \eta_m^{(n)}) m_3(\eta_i^{(n)}, \eta_m^{(n)}) - \frac{\pi}{n} k_s^2(\eta_i^{(n)}, \eta_j^{(n)}) m_4(\eta_i^{(n)}, \eta_j^{(n)}) \\
& + \frac{\pi}{n} \sum_{m=0}^{2n-1} d_{m-j}^{(n)} \left(\widetilde{h}^1(\eta_i^{(n)}, \eta_m^{(n)}) + h_s^3(\eta_i^{(n)}, \eta_m^{(n)})\right) m_1(\eta_i^{(n)}, \eta_m^{(n)}) \\
& + \frac{\pi}{n} \left(\widetilde{h}^1(\eta_i^{(n)}, \eta_j^{(n)}) + h_s^3(\eta_i^{(n)}, \eta_j^{(n)})\right) m_2(\eta_i^{(n)}, \eta_j^{(n)}) \\
& + \sum_{m=0}^{2n-1} d_{m-j}^{(n)} U_m^{(n)}(\eta_i^{(n)}) m_1(\eta_i^{(n)}, \eta_m^{(n)}) + U_j^{(n)}(\eta_i^{(n)}) m_2(\eta_i^{(n)}, \eta_j^{(n)}) \\
Z_{ij,p}^{(n)} =& R_j^{(n)}(\eta_i^{(n)}) k_p^1(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^n)| + \frac{\pi}{n} k_p^2(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^n)| \\
Z_{ij,s}^{(n)} =& R_j^{(n)}(\eta_i^{(n)}) h_s^2(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^n)| + U_j^{(n)}(\eta_i^{(n)}) |z'(\eta_j^n)| + \frac{\pi}{n} [\widetilde{h}^1(\eta_i^{(n)}, \eta_j^{(n)}) + h_s^3(\eta_i^{(n)}, \eta_j^{(n)})] \\
Z_{ij,a}^{(n)} =& R_j^{(n)}(\eta_i^{(n)}) k_a^1(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^n)| + \frac{\pi}{n} k_a^2(\eta_i^{(n)}, \eta_j^{(n)}) |z'(\eta_j^n)|
\end{aligned}
$$

An exact solution is available for the evaluation of the accuracy. We use the boundary integral equation method to get the numerical solution of this problem. The domain Ω is a disk, whose radius is R_0. Ω is a linear and isotropic elastic solid body, and the incident plane wave is given by $u^{inc} = e^{ik_a x\cdot \boldsymbol{d}}$ with the unit direction $\boldsymbol{d} = (1, 0)$. The exact solution for this model can be written as

$$
u^s(R,\theta) = \sum_{n=0}^{\infty} a_n H_n(k_a R_0) \cos(n\theta),
$$

$$
\boldsymbol{u} = \nabla u_p + \nabla \times u_s,
$$

where

$$
u_p(R,\theta) = \sum_{n=0}^{\infty} b_n J_n(k_p R_0) \cos(n\theta),
$$

$$
u_s(R,\theta) = \sum_{n=1}^{\infty} c_n J_n(k_s R_0) \sin(n\theta).
$$

The coefficients a_n, b_n, c_n can be determined from the transmission conditions on $\partial\Omega$ by the collocation method. We can get a linear system as follows:

$$
E_n X_n = e_n,
$$

where $X_n = (a_n, b_n, c_n)^\top$, $E_n = [E_n^{ij}]$, $e_n = [e_n^j]$, $i, j = 1, 2, 3$. The elements are given by the following formulations:

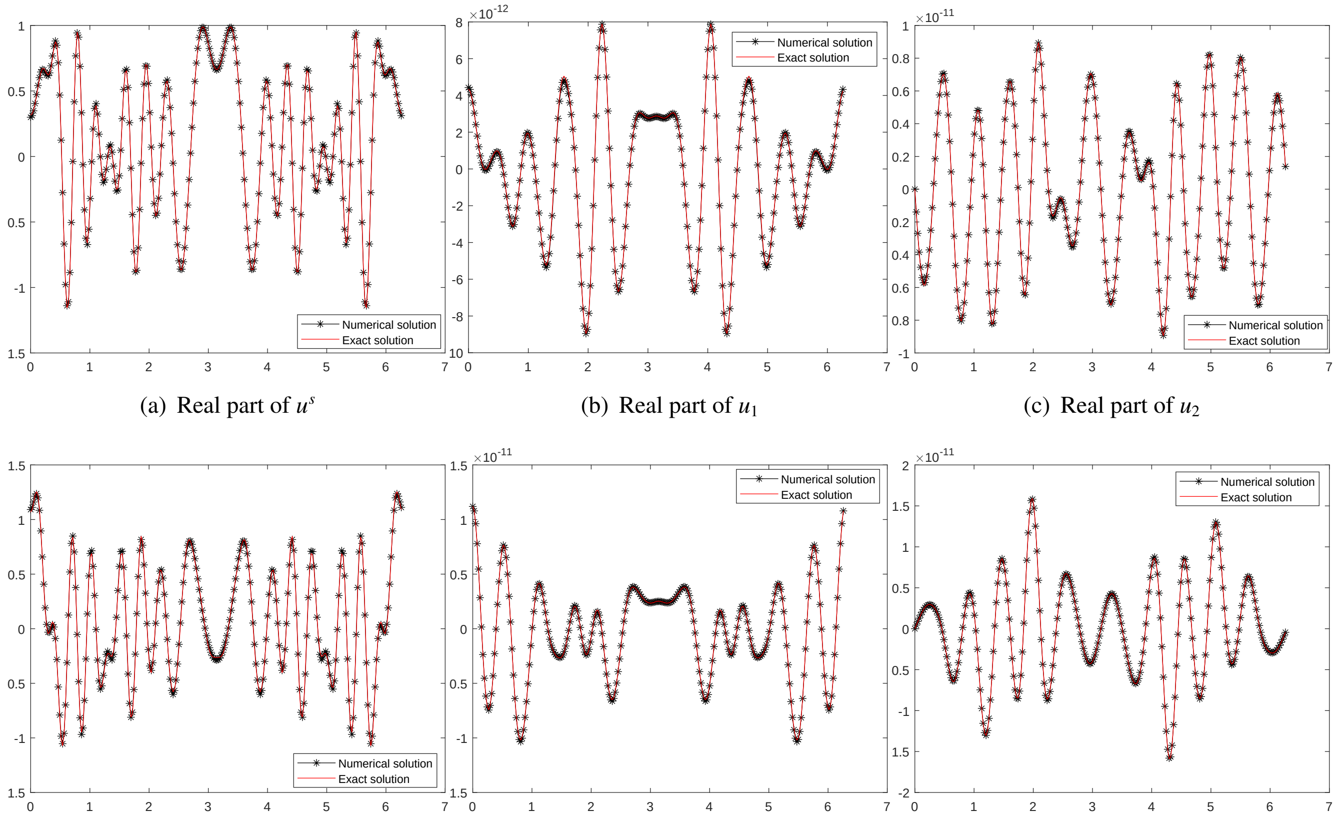
$$
E_n^{11} = H_n^{(1)}(k_a R_0),
$$

$$
\begin{aligned}
E_n^{12} &= \left[\frac{\mu(n^2+n)}{R_0^2} - (\lambda+2\mu)k_p^2\right] J_n(k_pR_0) - \frac{\mu k_p}{R_0} J_{n-1}(k_pR_0), \\
E_n^{13} &= \frac{\mu n k_s}{R_0} J_{n-1}(k_sR_0) - \frac{\mu(n^2+n)}{R_0^2} J_n(k_sR_0), \quad E_n^{21} = 0, \\
E_n^{22} &= \frac{n k_p}{R_0} J_{n-1}(k_pR_0) - \frac{n^2+n}{R_0^2} J_n(k_pR_0), \\
E_n^{23} &= \left(\frac{n^2+n}{R_0^2} - k_s^2\right) J_n(k_sR_0) - \frac{k_s}{R_0} J_{n-1}(k_sR_0), \\
E_n^{31} &= -H_{n-1}(k_aR_0) + \frac{n}{kR_0} H_n(k_aR_0), \\
E_n^{32} &= \rho_f \omega^2 \frac{k_p}{k}\left[J_{n-1}(k_pR_0) - \frac{n}{k_pR_0} J_n(k_pR_0)\right], \\
E_n^{33} &= \rho_f \omega^2 \frac{n}{kR_0} J_n(k_sR_0), \\
e_n^1 &= -\epsilon_n i^n J_n(k_aR_0), \\
e_n^2 &= 0, \\
e_n^3 &= -\epsilon_n i^n J_{n-1}(k_aR_0) - \frac{n}{kR_0} J_n(k_aR_0),
\end{aligned}
$$

where $\epsilon_n = 1$, if $n = 0$ else $\epsilon_n = 2$.

In this example, the numbers of the coefficients for a_n, b_n, c_n are all 25, and the radius is $R_0 = 2$ m. The sound speed in the water is $c = 1480$ m/s, and the density of the water is $\rho_f = 1000$ kg/m^3. The density of copper alloy is $\rho = 8100$ kg/m^3, and the frequency is $\omega = 5k\pi Hz$. The wave speeds of the pressure wave and the shear wave in the copper alloy are $c_p = 4840$ m/s and $c_s = 2270$ m/s. The number of the collocation points is $M = 128$, and these points are uniformly distributed on the circle. We observe the acoustic scattered field on the circle with radius $1.1R_0$ and observe the elastic displacement field on the circle with radius $0.9R_0$.

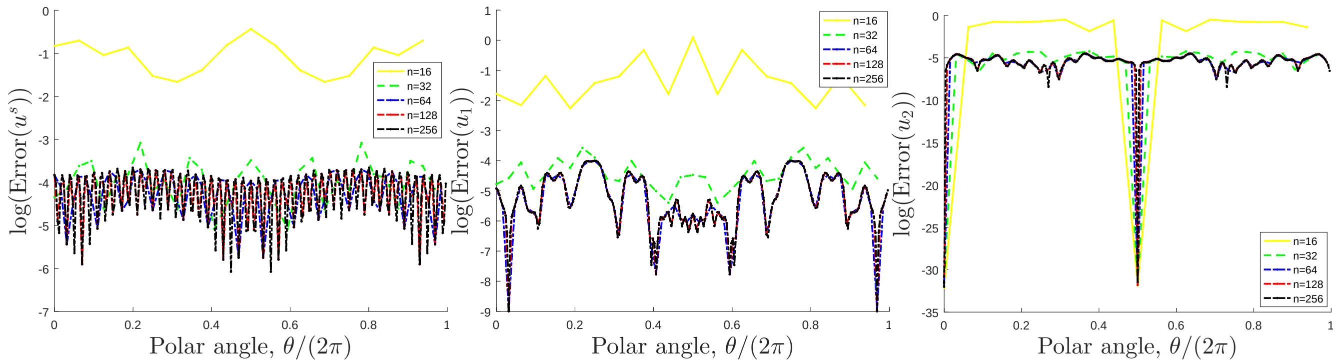
Figure 1 shows the numerical solutions and the analytic solution. We can observe that the numerical solution can stably approximate the analytic solution even if the displacement and the acoustic scattering field are multi-level valued. Figure 2 presents the relative errors between the exact solution and the numerical solution with different numbers of the collocation points. We can see that the presented method will give accurate results. It also can be seen that the errors do not decrease for $n \in \{64, 128, 256\}$.



(a) Real part of u^s (b) Real part of u_1 (c) Real part of u_2

(d) Imaginary part of u^s (e) Imaginary part of u_1 (f) Imaginary part of u_2

Figure 1. The numerical solutions and the exact solution for $R_0 = 2$ in Example 1.



(a) Error: u^s (b) Error: u_1 (c) Error: u_2

Figure 2. The accuracy by changing the numbers of the boundary collocation points.

6. Concluding remarks

In this paper, we have studied in two dimensions the fluid-solid interaction scattering problem by a boundary integral equation method. The effectiveness of the method has been shown by solving some examples. In the numerical example, we construct the exact solution to check the feasibility and accuracy of the presented method. From the numerical results, we can see that the proposed method is effective. We give some theoretical results for the discretized singular operators. Other potential methods for the corresponding problem are considered, such as the singular boundary method [49–51], etc.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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