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Research article

Solutions for a class of problems driven by an anisotropic (p,q)-Laplacian type operator

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Abstract: In this manuscript, existence and multiplicity results are obtained for a problem involving an anisotropic (p,q)-Laplacian-type operator by means of sub-supersolutions and variational techniques. This problem arises in various applications such as in the study of the enhancement of images, the spread of epidemic disease and in the dynamic of fluids. Under a general condition, the existence of a solution is proved, and the multiplicity of solutions is obtained by considering an additional natural hypothesis.

Keywords: (p,q)-Laplacian type operator; anisotropic operator, sub-supersolutions; existence; multiplicity
Mathematics Subject Classification: 35A15, 35J60

1. Introduction

In this paper it is considered the existence and multiplicity of nonnegative solutions for the nonhomogeneous anisotropic (p,q)-Laplacian type problem given by

$$\begin{cases} -\Delta_{\vec{p}}u - \Delta_{\vec{q}}u = k(x)u^{\alpha-1} + f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{P}$$

with $\Omega \subset \mathbb{R}^N (N \geq 3)$ a bounded domain with smooth boundary, $1 < \alpha$, where

$$\Delta_{\vec{p}}u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right),$$

$$\Delta_{\vec{q}}u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right),$$

and

$$1 < p_1 \leq p_2 \leq ... \leq p_N < p^*, \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$

$$1 < q_1 \leq q_2 \leq ... \leq q_N < q^*, \sum_{i=1}^{N} \frac{1}{q_i} > 1,$$

where $p^* := \dfrac{N\overline{p}}{N-\overline{p}}$, $q^* := \dfrac{N\overline{q}}{N-\overline{q}}$, $\overline{p} := \dfrac{N}{\sum_{i=1}^{N} \frac{1}{p_i}}$, $\overline{q} := \dfrac{N}{\sum_{i=1}^{N} \frac{1}{q_i}}$, $f : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ is a continuous function and

(H) $k \in L^\infty(\Omega)$ and $k(x) > 0$ a.e. in Ω;
(f_1) There exists $\delta > 0$ such that $f(x,t) \geq (1 - t^{\alpha-1})k(x)$, for all $0 \leq t \leq \delta$, a.e. in Ω;
(f_2) There is $r > 1$ with $|f(x,t)| \leq k(x)(1 + |t|^{r-1})$, for all $t \geq 0$, a.e. in Ω.

Consider $X := W_0^{1,\vec{p}}(\Omega) \cap W_0^{1,\vec{q}}(\Omega)$ with the norm $\|\cdot\|$ defined in Section 3. We say that $u \in X$ is a weak solution for (P) if

$$\int_\Omega \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \int_\Omega \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega k(x)u^{\alpha-1}\phi + f(x,u)\phi,$$

for all $\phi \in X$.

Let $\|\cdot\|_\infty$ denote the norm in the space $L^\infty(\Omega)$. By applying sub-supersolutions and a minimization argument, it is possible to obtain the existence result provided below.

Theorem 1.1. *Consider the hypotheses (H) and (f_1) − (f_2). If $\|k\|_\infty$ is small enough, then (P) has a nonnegative solution.*

Under subcriticial growth and a Ambrosetti-Rabinowitz type condition

(f_3) It holds that $r < \min\{p^*, q^*\}$ and $\alpha \leq \min\{p_1, q_1\}$ or it holds simultaneously that $\max\{p_N, q_N\} < \alpha$ and there is $t_0 > 0$ with

$$0 < \theta F(x,t) \leq t f(x,t), \quad \text{a.e. in } \Omega \text{ for all } t \geq t_0,$$

where $\theta > \max\{p_N, q_N\}$ and F is the primitive of f, that is, $F(x,t) = \displaystyle\int_0^t f(x,\tau)\,d\tau$.

it is possible to obtain a multiplicity result.

Theorem 1.2. *Consider the hypotheses (H) and (f_1) − (f_3). If $\|k\|_\infty$ is small enough, then (P) admits at least two nonnegative weak solutions.*

Partial Differential equations involving anisotropic operators have significant relevance in several domains of Science and Technology. For example in the reference [1] it was considered a mathematical model which was applied for both image enhancement and denoising in terms of anisotropic equations as well as allowing the preservation of significant characteristics of the image. We also quote that anisotropic differential equations are considered in models that describe the spread of epidemic disease in heterogeneous environments. In Physics such operators can be applied to describe the dynamics of fluids with different conductivities in different directions. For more details concerning such applications see [1–4].

On the other hand differential equations with $(p, q)-$Laplacian type operators in (P) arose due to its applicability in several relevant models in biophysics, plasma physics and chemical reaction design which are driven by the parabolic reaction-diffusion system

$$u_t - \operatorname{div}[(|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u] = c(x, u). \tag{1.1}$$

In such applications, the function u plays an important role as it provides the concentration of the substance being studied. The divergent term, on the other hand, provides important informations regarding the diffusion process incorporated into the model. Additionally, the term c introduces the reaction component and encompasses relevant informations such as the source and the loss occurring in the process. It is worth noting that in Biology and Chemistry applications, the reaction term $c(x, u)$ exhibits a polynomial behavior with respect to the variable u and has variable coefficients. For more technical details concerning these applications, we point out the following references [5, 6].

Recently, anisotropic equations with a (p, q)-Laplacian type operator have been under consideration in the literature. For example in the reference [7] the authors obtained, under certain conditions and through a sub-supersolution approach and minimization arguments in convex sets, the existence of a positive solution for the problem

$$\begin{cases} -\Delta_{\vec{p}} u - \Delta_{\vec{q}} u &= \lambda u^{\gamma-1} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $1 < q_i \leq p_i, i = 1, ..., N$, λ is a parameter and $\gamma > 1$. In the case $1 < \gamma < q_1$ it was obtained that the above problem has a solution if and only if $\lambda > 0$. In the case $q_1 \leq \gamma < p_N$ it was proved that there exists $\sigma > 0$, such that the problem does not have positive solutions for $\lambda < \sigma$ and at least one positive solution for $\lambda > \sigma$.

In [8], by means of the condition $1 < q_i \leq p_i, i = 1, .., N$, results for weighted anisotropic Sobolev spaces and the Galerkin technique, it was considered the weighted problem

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} \pm b(x)\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right) &= f(x, u, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

and existence results were obtained.

We also mention that in [9], the application of genus theory and Clark's theorem led to the multiplicity and the existence of infinitely many solutions for anisotropic problems with a (p, q)-Laplacian type operator with a subcritical growth. Such results depend on the behavior of the considered nonlinearity. Moreover, the same method was applied to obtain infinitely many solutions for the critical case.

Regarding the isotropic case, we refer to the recent references [10, 11], whose authors obtained existence and multiplicity results for a class of problems driven by an operator with non-standard growth, that includes the $(p, q)-$Laplacian. For a survey on isotropic problems involving the $(p, q)-$Laplacian operator, see [12].

An important fact is that the $(p, q)-$Laplacian operator is a particular case of the double-phase operator

$$\mathcal{L}^a_{p,q}(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u),$$

where $p, q > 1$ and a is a nonnegative essentially bounded function. In the last years there is an increasing interest in problems involving the operator mentioned above. For example in [13], multiplicity of solutions for double phase problem involving the above operator, a Kirchhoff term, a singular function and a subcritical nonlinearity is obtained via the fibering method in form of the Nehari manifold. In the reference [14], the existence of a pair of nontrivial nonnegative and nonpositive solutions for a double-phase type problem in $\mathbb{R}^N$ involving variable exponents is obtained using variational methods. We also quote the related reference [15], where the authors obtained existence and uniqueness of solutions for a parabolic problem involving a double-phase operator with variable exponents.

Regarding the motivations to study (P) we quote the recent references [7–9] and [16], where in this last one existence and multiplicty results were proved for an anisotropic problem related to (P) with $\alpha = 2$. We also point out that this work was motivated by [17], where a problem related to (P) was considered in a variable exponents setting. This paper was also motivated by [18], where the authors studied an isotropic version of the problem (P), considering a differential operator that allows to consider several cases such as the usual $p\&q-$Laplacian operator. The main difference of the present manuscript with respect to [18] is that different estimates and new results were needed, see for instance Lemmas 3.1, 3.3, 3.5, 3.6, and Corollary 3.2. Regarding the results obtained in this manuscript, we point out that differently from [7–9], the problem (P) allows not to assume that $q_i \leq p_i, i = 1, ..., N$, thanks to Lemma 3.1 and Corollary 3.2. By using a global minimization argument, Theorem 1.1 allows to handle a wide class of nonlinearities such as supercritical and critical nonlinearities, that are challenging cases in the study of elliptic problems. By additionally considering an Ambrosetti-Rabinowitz type condition and a subcritical growth, Theorem 1.2 provides the existence of at least two solutions for (P). A common aspect in such results is the appropriate construction of sub-supersolutions, which is possible due to Lemma 3.6. Moreover, we quote that such approach cannot be directly applied and it is rare in the literature in anisotropic problems due to several technicalities, such as the lack of homogeneity of anisotropic operators.

The results of this manuscript permits to consider in the problem (P) the function

$$f(x,t) = \begin{cases} k(x)(1 - t^{\alpha-1}), & 0 \leq t \leq s_0, \\ k(x)((1 - s_0^{\alpha-1}) + (t - s_0)^{r-1}), & t > s_0, \end{cases}$$

where $0 < s_0 < 1$ is fixed and $1 < \min\{\alpha, r\} \leq \max\{\alpha, r\} < \min\{p^\star, q^\star\}$.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. Consider $1 < p_1 \leq p_2 \leq ... \leq p_N$ real numbers and $\overrightarrow{p} := (p_1, ..., p_N) \in \mathbb{R}^N$. We denote by $W^{1,\overrightarrow{p}}(\Omega)$ the space given by

$$W^{1,\overrightarrow{p}}(\Omega) := \left\{ u \in L^{p_N}(\Omega); \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, ..., N \right\},$$

which is a Banach space when endowed with the norm

$$|u|_{1,\overrightarrow{p}} := \|u\|_{L^{p_N}} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}, \tag{2.1}$$

where $\|\cdot\|_{L^{p_i}}$ denotes the usual norm of $L^{p_i}(\Omega)$, $i = 1, ..., N$. It will be denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$ the Banach space defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(\Omega)$ with respect to the norm $\|\cdot\|_{1,\overrightarrow{p}}$.

Consider $\overline{p}$ the harmonic mean of p_i, $i = 1, ..., N$, that is

$$\overline{p} := N / \sum_{i=1}^{N} \frac{1}{p_i}.$$

Suppose that $\overline{p} < N$ and consider $p^* := \dfrac{N\overline{p}}{N - \overline{p}}$. If $p_N < p^\star$, then there exists an embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$, which is continuous for $q \in [1, p^*]$ and compact in the case $q \in [1, p^*)$, see [19] . Thus, it follows that the norm,

$$|u| := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}, u \in W_0^{1,\overrightarrow{p}}(\Omega),$$

is equivalent to the one given in (2.1).

3. Auxiliary results

In this section it will be presented several results that will play an important role in the study of (P). Applying the same reasoning of the proof of [20, Theorem 2.2], we obtain the result below.

Lemma 3.1. *The space $W_0^{1,\overrightarrow{p}}(\Omega)$ equipped with the norm $\|u\|_{1,\overrightarrow{p}} := \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}^2 \right)^{1/2}$, $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a uniformly convex Banach space.*

Proof. It is clear that $W_0^{1,\overrightarrow{p}}(\Omega)$ is a Banach space with the given norm. From [21, Theorem 5.2.25] (see also [22, Remark 1]) it holds that $W := L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$ with the norm $|||(w_1, ..., w_N)||| := \left(\sum_{i=1}^{N} \|w_i\|_{L^{p_i}}^2 \right)^{1/2}$ is a uniformly convex Banach space. Note that the inclusion

$$\begin{aligned} I : (W_0^{1,\overrightarrow{p}}(\Omega), |\cdot|) &\longrightarrow (W, |||\cdot|||) \\ u &\longrightarrow u \end{aligned}$$

is an isometric embedding. Since every linear subspace of a uniformly convex linear normed space is also uniformly convex, then $(W_0^{1,\overrightarrow{p}}(\Omega), |\cdot|)$ is uniformly convex.

The previous result implies the next one.

Corollary 3.2. *The linear space X equipped with the norm $\|u\| := \max\{\|u\|_{1,\overrightarrow{p}}, \|u\|_{1,\overrightarrow{q}}\}$, $u \in X$ is a uniformly convex Banach space, and consequently it is reflexive.*

Below, we present an existence result. Its proof is based on adapting the proofs of [16, Lemma 2.1] and [23, Lemma 3.1].

Lemma 3.3. *Let $a \in X'$.Then the problem*

$$\begin{cases} -\Delta_{\overrightarrow{p}} u - \Delta_{\overrightarrow{q}} u &= a \ \ in \ \ \Omega, \\ u &= 0 \ \ on \ \ \partial\Omega, \end{cases}$$

has a unique solution in X.

Proof. Let $T : X \to X'$ be the continuous map given by

$$\left\langle Tu, \phi \right\rangle = \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}$$

Since $q_i, p_i > 1, i = 1, ..., N$, we have from Simon's inequality [24, page 210]

$$\langle |x|^{l-2}x - |y|^{l-2}y, x - y\rangle \geq \begin{cases} C\dfrac{|x-y|^2}{(1+|x|+|y|)^{2-l}} & \text{if } \ 1 \leq l \leq 2, \\ C|x-y|^l & \text{if } \ l \geq 2, \end{cases} \tag{3.1}$$

for some constant $C > 0$ and all $x, y \in \mathbb{R}^N$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$, that

$$\left\langle Tu - Tv, u - v \right\rangle > 0 \ \text{ for all } \ u, v \in X \ \text{ with } \ u \neq v.$$

Let $(u_n) \subset X$ be a sequence with $\|u_n\| \to +\infty$. As in the proof of [25, Theorem 36], for each $i \in \{1, ..., N\}$ and $n \in \mathbb{N}$ consider

$$\alpha_{i,n} := \begin{cases} p_N, & \text{if } \ \left\|\dfrac{\partial u_n}{\partial x_i}\right\|_{p_i} \leq 1, \\ p_1, & \text{if } \ \left\|\dfrac{\partial u_n}{\partial x_i}\right\|_{p_i} > 1. \end{cases}$$

Since $(a_1 + ... + a_N)^\beta \leq C(a_1^\beta + ... + a_N^\beta)$ for $\beta \geq 1$ and $a_i \geq 0, i = 1, ..., N$ for some constant $C > 0$, we obtain that

$$\begin{aligned} \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} &\geq \sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}^{\alpha_{i,n}} \\ &= \sum_{\{i;\alpha_{i,n}=p_N\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}^{\alpha_{i,n}} + \sum_{\{i;\alpha_{i,n}=p_1\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}^{\alpha_{i,n}} \\ &\geq C_1\left(\sum_{\{i;\alpha_{i,n}=p_N\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}\right)^{p_N} + C_2\left(\sum_{\{i;\alpha_{i,n}=p_1\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}\right)^{p_1} \\ &= C_1\left(\sum_{\{i;\alpha_{i,n}=p_N\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}\right)^{p_N} + C_2\left(\sum_{\{i;\alpha_{i,n}=p_1\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}\right)^{p_1} \\ &+ C_2\left(\sum_{\{i;\alpha_{i,n}=p_N\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}\right)^{p_1} - C_2\left(\sum_{\{i;\alpha_{i,n}=p_N\}} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}\right)^{p_1} \\ &\geq C_3\|u\|_{1,\overrightarrow{p}}^{p_1} - C_4, \end{aligned} \tag{3.2}$$

where $C_i > 0, i = 1, ..., 4$ are constants, and

$$\int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{q_i} \geq A_1 \|u\|_{1,\overrightarrow{q}}^{q_1} - A_2,$$

where $A_1, A_2 > 0$ are constants. Thus, we conclude that

$$\int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i} + \int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{q_i} \geq A_3 \min\{\|u\|^{p_1}, \|u\|^{q_1}\} - A_4,$$

where $A_3, A_4 > 0$ are constants.

Therefore

$$\lim_{n \to +\infty} \frac{\left\langle T u_n, u_n \right\rangle}{\|u_n\|} = +\infty.$$

Therefore, by using Minty-Browder's Theorem [26, Theorem 5.16], we conclude that there exists a unique function $u \in X$ such that $Tu = a$.

Definition 3.4. *Consider $u, v \in X$. It will be denoted by $-\Delta_{\overrightarrow{p}} u - \Delta_{\overrightarrow{q}} u \leq -\Delta_{\overrightarrow{p}} v - \Delta_{\overrightarrow{q}} v$ in Ω if*

$$\int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} + \int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} \leq \int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial v}{\partial x_i} \right|^{p_i - 2} \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_i} + \int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial v}{\partial x_i} \right|^{q_i - 2} \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_i},$$

holds for all $\phi \in X$ with $\phi(x) \geq 0$ a.e. in Ω.

By adapting the idea of the proof of [16, Lemma 2.2], we can obtain the next result, which consists in a weak comparison principle.

Lemma 3.5. *Let Ω be a bounded domain and consider $u, v \in X$ satisfying*

$$\begin{cases} -\Delta_{\overrightarrow{p}} u - \Delta_{\overrightarrow{q}} u & \leq & -\Delta_{\overrightarrow{p}} v - \Delta_{\overrightarrow{q}} v \ \text{ in } \ \Omega, \\ u & \leq & v \text{ on } \ \partial\Omega, \end{cases}$$

where $u \leq v$ on $\partial\Omega$ means that $(u - v)^+ := \max\{0, u - v\} \in X$. Then $u(x) \leq v(x)$ a.e. in Ω.

Proof. Using the test function $\phi = (u - v)^+ := max\{u - v, 0\} \in X$ it follows that

$$\int_{\Omega \bigcap [u \geq v]} \sum_{i=1}^{N} \left[\left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p_i - 2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) + \left(\left| \frac{\partial u}{\partial x_i} \right|^{q_i - 2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{q_i - 2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \right] \leq 0.$$

Thus, from inequality (3.1) we obtain that

$$\int_\Omega \left|\frac{\partial}{\partial x_i}(u-v)^+\right|^{p_i} = 0, \text{ if } 2 \le p_i,$$
$$\int_\Omega \left|\frac{\partial}{\partial x_i}(u-v)^+\right|^{q_i} = 0, \text{ if } 2 \le q_i,$$
$$\int_\Omega \frac{\left|\frac{\partial}{\partial x_i}(u-v)^+\right|^{2}}{\left(1+\left|\frac{\partial u}{\partial x_i}\right|+\left|\frac{\partial v}{\partial x_i}\right|\right)^{2-p_i}} = 0, \text{ if } 1 < p_i \le 2,$$
$$\int_\Omega \frac{\left|\frac{\partial}{\partial x_i}(u-v)^+\right|^{2}}{\left(1+\left|\frac{\partial u}{\partial x_i}\right|+\left|\frac{\partial v}{\partial x_i}\right|\right)^{2-q_i}} = 0, \text{ if } 1 < q_i \le 2.$$

Therefore $\|(u-v)^+\| = 0$, which finishes the proof of the result.

Reasoning as in [27, Lemma 4.1] we obtain the result below.

Lemma 3.6. *Let Ω be a bounded domain in $\mathbb{R}^N (N \ge 3)$ such that the embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$ holds. Let $\lambda > 0$ and consider $u_\lambda \in X$ the unique solution of the problem*

$$\begin{cases} -\Delta_{\overrightarrow{p}} u - \Delta_{\overrightarrow{q}} u &= \lambda \ \text{in} \ \Omega, \\ u &= 0 \ \text{in} \ \Omega. \end{cases} \tag{P_λ}$$

Consider C_0 the best constant of the embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$. Define $h := \dfrac{p_1}{2|\Omega|^{\frac{1}{N}} C_0}$, and $t := \dfrac{q_1}{2|\Omega|^{\frac{1}{N}} C_0}$. If $\lambda \ge h$, then $u \in L^\infty(\Omega)$ with $\|u\|_\infty \le C^{\overrightarrow{p}} \lambda^{\frac{1}{p_1-1}}$, and $\|u\|_\infty \le C_{\overrightarrow{p}} \lambda^{\frac{1}{p_N-1}}$ when $\lambda < h$. If $\lambda \ge t$, then $u \in L^\infty(\Omega)$ with $\|u\|_\infty \le C^{\overrightarrow{q}} \lambda^{\frac{1}{q_1-1}}$, and $\|u\|_\infty \le C_{\overrightarrow{q}} \lambda^{\frac{1}{q_N-1}}$ when $\lambda < t$. Here $C^{\overrightarrow{p}}, C_{\overrightarrow{p}}$ are constants depending only on $p_1, p_N, |\Omega|$ and C_0 and $C^{\overrightarrow{q}}, C_{\overrightarrow{q}}$ are constants that depends only on $q_1, q_N, |\Omega|$ and C_0.

Proof. Consider the solution u_λ of (P_λ). It is clear that u_λ is a nontrivial function satisfying $u(x) \ge 0$ a. e. in Ω. For $k \ge 0$, l define the set $A_k := \{x \in \Omega; u(x) > k\}$. Now, consider $\epsilon > 0$. By using the test function $(u-k)^+ \in X$ in (P_λ), we can apply Young's inequality, yielding to

$$\begin{aligned}\int_{A_k}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx \leq &\int_{A_k}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx + \int_{A_k}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\\ =&\lambda\int_{A_k}(u-k)dx\\ \leq&\lambda|A_k|^{\frac{1}{N}}\|(u-k)^+\|_{L^{\frac{N}{N-1}}(\Omega)}\\ \leq&\lambda|A_k|^{\frac{1}{N}}C_0\int_{A_k}|\nabla u|dx\\ \leq&\lambda|A_k|^{\frac{1}{N}}C_0\int_{A_k}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|dx\\ \leq&\lambda|A_k|^{\frac{1}{N}}C_0\sum_{i=1}^{k}\int_{A_k}\frac{\left(\left|\epsilon\frac{\partial u}{\partial x_i}\right|\right)^{q_i}}{q_i}dx + \lambda|A_k|^{\frac{1}{N}}C_0\sum_{i=1}^{k}\int_{A_k}\frac{(\epsilon^{-1})^{(q_i)'}}{q_i{}'}dx.\end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned}\lambda|A_k|^{\frac{1}{N}}C_0\sum_{i=1}^{k}\int_{A_k}\frac{\left(\epsilon\left|\frac{\partial u}{\partial x_i}\right|\right)^{q_i}}{q_i} \leq&\frac{M|A_k|^{\frac{1}{N}}C_0}{q_1}\int_{A_k}\epsilon^{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\\ \leq&\frac{M|\Omega|^{\frac{1}{N}}C_0}{p_1}\sum_{i=1}^{k}\int_{A_k}\epsilon^{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx.\end{aligned}$$

Let $t := \dfrac{q_1}{2|\Omega|^{\frac{1}{N}}C_0}$. Suppose that $M \geq t$. Define

$$\epsilon := \left(\frac{q_1}{2M|\Omega|^{\frac{1}{N}}C_0}\right)^{\frac{1}{q_1}}.$$

Thus, $\epsilon \leq 1$ and

$$\begin{aligned}\frac{\lambda|\Omega|^{\frac{1}{N}}C_0}{q_1}\sum_{i=1}^{k}\int_{A_k}\epsilon^{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx \leq& \frac{\lambda|\Omega|^{\frac{1}{N}}C_0\epsilon^{q_1}}{q_1}\sum_{i=1}^{k}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\\ =&\frac{1}{2}\sum_{i=1}^{k}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we have that

$$\begin{aligned}\sum_{i=1}^{k}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx \leq&\frac{2\lambda|A_k|^{\frac{1}{N}}C_0}{(q_N)'}\sum_{i=1}^{k}\int_{A_k}\epsilon^{-(q_i)'}dx\\ \leq&\gamma|A_k|^{1+\frac{1}{N}},\end{aligned}$$

where

$$\gamma := \frac{2N\epsilon^{-(q_1)'}C_0}{(q_N)'},$$

which implies that

$$\int_{A_k}(u-k)dx=\frac{1}{\lambda}\sum_{i=1}^{N}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\leq\gamma|A_k|^{1+\frac{1}{N}}.$$

Therefore by the L^∞ estimates in [28, Lemma 5.1-Chapter 2] we get

$$\begin{aligned}\|u\|_\infty\leq&\gamma(N+1)|\Omega|^{\frac{1}{N}}\\=&C^{\vec{q}}\lambda^{\frac{1}{q_1-1}},\end{aligned}$$

where $C^{\vec{q}}$ is a constant that has the dependence described in the statement of the result. The case $\lambda<t$ follows by repeating the previous arguments with

$$\epsilon:=\left(\frac{q_1}{2\lambda|\Omega|^{\frac{1}{N}}C_0}\right)^{\frac{1}{q_N}}.$$

The other parts of the result follows by reasoning as in the previous cases.

4. Proof of Theorem 1.1

We will now present the notion of sub-supersolution that we will consider, and a lemma involving such a concept.

A pair $(\underline{u},\overline{u})\in(X\cap L^\infty(\Omega))\times(X\cap L^\infty(\Omega))$ is said to be a sub-supersolution for (P) if $\underline{u}(x)\leq\overline{u}(x)$ a.e. in Ω and the inequalities

$$\begin{aligned}&\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial\underline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial\underline{u}}{\partial x_i}\frac{\partial\phi}{\partial x_i}+\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial\underline{u}}{\partial x_i}\right|^{q_i-2}\frac{\partial\underline{u}}{\partial x_i}\frac{\partial\phi}{\partial x_i}\leq\int_\Omega k(x)\underline{u}^{\alpha-1}\phi+\int_\Omega f(x,\underline{u})\phi,\\&\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial\overline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial\overline{u}}{\partial x_i}\frac{\partial\phi}{\partial x_i}+\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial\overline{u}}{\partial x_i}\right|^{q_i-2}\frac{\partial\overline{u}}{\partial x_i}\frac{\partial\phi}{\partial x_i}\geq\int_\Omega k(x)\overline{u}^{\alpha-1}\phi+\int_\Omega f(x,\overline{u})\phi,\end{aligned}\tag{4.1}$$

hold for all $\phi\in X$, with $\phi(x)\geq0$ a.e. in Ω.

For $\|k\|_\infty$ small enough, the following result, where the Lemmas 4.1, 3.6 and 3.5 plays an importat role in its proof, provides the existence of a sub-supersolution for (P).

Lemma 4.1. *Consider the hypotheses* (H) *and* $(f_1)-(f_2)$*. Then, there exists* $\eta>0$ *such that* (P) *has a sub-supersolution* $(\underline{u},\overline{u})\in(X\cap L^\infty(\Omega))\times(X\cap L^\infty(\Omega))$*, with* $\|\underline{u}\|_\infty\leq\delta$*, where* δ *was provided in* (f_1)*, whenever* $\|k\|_\infty<\eta$*.*

Proof. From Lemmas 3.3 and 3.6 there are unique nonnegative functions $\underline{u},\overline{u}\in X\cap L^\infty(\Omega)$ satisfying

$$\begin{cases}-\Delta_{\vec{p}}\underline{u}-\Delta_{\vec{q}}\underline{u}&=&k(x)\ \text{in}\ \Omega,\\\underline{u}&=&0\ \text{on}\ \partial\Omega\end{cases}\quad\text{and}\quad\begin{cases}-\Delta_{\vec{p}}\overline{u}-\Delta_{\vec{q}}\overline{u}&=&1+k(x)\ \text{in}\ \Omega,\\\overline{u}&=&0\ \text{on}\ \partial\Omega,\end{cases}\tag{4.2}$$

respectively with

$$\|\underline{u}\|_\infty\leq\max\{C^{\vec{p}}\|k\|_\infty^{\frac{1}{p_1-1}},C_{\vec{p}}\|k\|_\infty^{\frac{1}{p_N-1}},C^{\vec{q}}\|k\|_\infty^{\frac{1}{q_1-1}},C_{\vec{q}}\|k\|_\infty^{\frac{1}{q_N-1}}\}$$

and

$$\|\overline{u}\|_\infty \le \max\{C^{\overrightarrow{p}}\|1+k\|_\infty^{\frac{1}{p_1-1}}, C_{\overrightarrow{p}}\|1+k\|_\infty^{\frac{1}{p_N-1}}, C^{\overrightarrow{q}}\|1+k\|_\infty^{\frac{1}{q_1-1}}, C_{\overrightarrow{q}}\|1+k\|_\infty^{\frac{1}{q_N-1}}\},$$

where the constants $C^{\overrightarrow{p}}, C^{\overrightarrow{q}}, C_{\overrightarrow{p}}$, and $C_{\overrightarrow{q}}$ are provided by Lemma 3.6. Moreover, by applying Lemma 3.6, it is possible to choose $\eta > 0$, depending only on $C^{\overrightarrow{p}}, C^{\overrightarrow{q}}, C_{\overrightarrow{p}}$, and $C_{\overrightarrow{q}}$ such that $\|\underline{u}\|_\infty \le \delta/2$ and $\|k\|_\infty \max\{\|\overline{u}\|_\infty^{\alpha-1}, \|\overline{u}\|_\infty^{r-1}\} \le 1$ for $\|k\|_\infty < \eta$. From Lemma 3.5 and (4.2) we get $0 < \underline{u}(x) \le \overline{u}(x)$ a.e. in Ω.

Consider $\phi \in X$ with $\phi(x) \ge 0$ a.e. in Ω. From (f_1) and (4.2) we get

$$\begin{aligned}\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{q_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} - \int_\Omega k(x)\underline{u}^{\alpha-1}\phi - \int_\Omega f(x,\underline{u})\phi &\le \int_\Omega k(x)\phi - \int_\Omega k(x)\underline{u}^{\alpha-1}\phi \\ &\quad - \int_\Omega (1-\underline{u}^{\alpha-1})k(x)\phi \\ &= 0.\end{aligned}$$

Combining (f_2), (4.2), and considering the choice of η that was made before, we have

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{q_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} - \int_\Omega k(x)\overline{u}^{\alpha-1}\phi - \int_\Omega f(x,\overline{u})\phi \ge \int_\Omega (1-\|k\|_\infty \max\{\|\overline{u}\|_\infty^{\alpha-1}, \|\overline{u}\|_\infty^{r-1}\})\phi \ge 0,$$

for all $\phi \in X$ with $\phi(x) \ge 0$ a.e in Ω, which finishes the proof.

Proof of Theorem 1.1. Let $\underline{u}, \overline{u} \in X$ be the functions provided in Lemma 4.2. Consider

$$z(x,t) := \begin{cases} k(x)\overline{u}^{\alpha-1} + f(x,\overline{u}(x)), & t > \overline{u}(x), \\ k(x)t^{\alpha-1} + f(x,t), & \underline{u}(x) \le t \le \overline{u}(x), \\ k(x)\underline{u}^{\alpha-1} + f(x,\underline{u}(x)), & t < \underline{u}(x), \end{cases} \tag{4.3}$$

for $(x,t) \in \Omega \times \mathbb{R}$, and the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u - \Delta_{\overrightarrow{q}} u = z(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\widetilde{P}$}$$

whose solutions are the critical points of the $C^1((X,\|\cdot\|),\mathbb{R})$ functional given by

$$J(u) := \int_\Omega \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} + \int_\Omega \sum_{i=1}^N \frac{1}{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i} - \int_\Omega Z(x,u), u \in X,$$

where $Z(x,t) := \int_0^t z(x,\tau)d\tau$. We affirm that J is a sequentially weakly lower semicontinuous functional. In fact, let $(u_n) \subset X$ be a sequence with $u_n \rightharpoonup u$ in X. From the weakly lower semicontinuity of the norm, we obtain $\|u\| \le \liminf_{n\to+\infty} \|u_n\|$. The continuous embedding $X \hookrightarrow W_0^{1,\overrightarrow{p}}(\Omega)$ and the compact embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [1, \overline{p}^\star)$, imply, up to a subsequence, that $u_n(x) \to u(x)$ a.e. in Ω. Thus, by applying the fact that $z \in L^\infty(\Omega)$ and using the Lebesgue Dominated Convergence theorem, we obtain

$$\int_\Omega Z(x,u_n) \to \int_\Omega Z(x,u).$$

Therefore

$$\begin{aligned}
J(u) &= \int_\Omega \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} + \int_\Omega \sum_{i=1}^N \frac{1}{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i} - \int_\Omega Z(x,u)\\
&\leq \liminf_{n\to+\infty} \int_\Omega \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} + \int_\Omega \sum_{i=1}^N \frac{1}{q_i}\left|\frac{\partial u_n}{\partial x_i}\right|^{q_i} - \int_\Omega Z(x,u_n)\\
&= \liminf_{n\to+\infty} J(u_n),
\end{aligned}$$

which proves the claim.

Note that J is coercive, which implies that J is bounded below. Thus, there exists $\rho := \inf_{u\in X} J(u)$ and a sequence $(u_n) \subset X$ such that $J(u_n) \to \rho$. Since J is coercive, it follows that (u_n) is bounded in X. Then, up to a subsequence, it holds that $u_n \rightharpoonup u$ for some $u \in X$. From the sequential weakly lower semicontinuity, we have

$$\alpha \leq J(u) \leq \liminf_{n\to+\infty}(u_n) = \rho.$$

Thus, u is a critical point of J, which implies that it is a solution of $(\widetilde{P})$. We affirm that $\underline{u}(x) \leq u(x) \leq \underline{u}(x)$ a.e. in Ω. Since u solves $(\widetilde{P})$, we obtain for the test function $\phi := (\underline{u}-u)^+ \in X$ that

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial(\underline{u}-u)^+}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i}\frac{\partial(\underline{u}-u)^+}{\partial x_i} = \int_\Omega z(x,u)(\underline{u}-u)^+.$$

Using the previous equality, and the first inequality of (4.1) we obtain that

$$\begin{aligned}
&\int_{\Omega\bigcap[\underline{u}\geq u]} \sum_{i=1}^N \left[\left(\left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial \underline{u}}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) + \left(\left|\frac{\partial \underline{u}}{\partial x_i}\right|^{q_i-2}\frac{\partial \underline{u}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial \underline{u}}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\right]\\
&\leq \int_{\Omega\bigcap[\underline{u}\geq u]} (k(x)\underline{u}^{\alpha-1} + f(x,\underline{u}) - z(x,u))(\underline{u}-u)^+\\
&= 0.
\end{aligned}$$

From the inequality (3.1) we obtain that

$$\int_{\Omega}\left|\frac{\partial}{\partial x_i}(\underline{u}-u)^+\right|^{p_i}=0,\ \text{if}\ 2\le p_i,$$
$$\int_{\Omega}\left|\frac{\partial}{\partial x_i}(\underline{u}-u)^+\right|^{q_i}=0,\ \text{if}\ 2\le q_i,$$
$$\int_{\Omega}\frac{\left|\frac{\partial}{\partial x_i}(\underline{u}-u)^+\right|^2}{\left(1+\left|\frac{\partial\underline{u}}{\partial x_i}\right|+\left|\frac{\partial u}{\partial x_i}\right|\right)^{2-p_i}}=0,\ \text{if}\ 1<p_i\le 2,$$
$$\int_{\Omega}\frac{\left|\frac{\partial}{\partial x_i}(\underline{u}-u)^+\right|^2}{\left(1+\left|\frac{\partial\underline{u}}{\partial x_i}\right|+\left|\frac{\partial u}{\partial x_i}\right|\right)^{2-q_i}}=0,\ \text{if}\ 1<q_i\le 2.$$

Therefore $\|(\underline{u}-u)^+\|=0$, which provides that $\underline{u}(x)\le u(x)$ a.e. in Ω. On the other hand we have

$$\int_{\Omega\bigcap[u\ge\overline{u}]}\sum_{i=1}^{N}\left[\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}-\left|\frac{\partial\overline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial\overline{u}}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i}-\frac{\partial\overline{u}}{\partial x_i}\right)+\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}-\left|\frac{\partial\overline{u}}{\partial x_i}\right|^{q_i-2}\frac{\partial\overline{u}}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i}-\frac{\partial\overline{u}}{\partial x_i}\right)\right]$$
$$\le\int_{\Omega\bigcap[u\ge\overline{u}]}(z(x,u)-k(x)\overline{u}^{\alpha-1}-f(x,\overline{u}))(u-\overline{u})^+$$
$$=0,$$

which provides that $\|(u-\overline{u})^+\|=0$. Therefore $u(x)\le\overline{u}(x)$ a.e. in Ω. Thus, u solves (P).

5. Proof of Theorem 1.2

Consider $\underline{u}\in X$ obtained in Lemma 4.2, the function

$$w(x,t)=\begin{cases}k(x)t^{\alpha-1}+f(x,t), & t\ge\underline{u}(x),\\ k(x)\underline{u}(x)^{\alpha-1}+f(x,\underline{u}(x)) & t<\underline{u}(x),\end{cases}$$

and the problem

$$\begin{cases}-\Delta_{\overrightarrow{p}}u-\Delta_{\overrightarrow{q}}u &=& w(x,u)\ \text{in}\ \Omega,\\ u &=& 0\ \text{on}\ \partial\Omega,\end{cases}\tag{5.1}$$

whose solutions coincide with the critical points of the $C^1((X,\|\cdot\|),\mathbb{R})$ functional

$$L(u):=\int_{\Omega}\sum_{i=1}^{N}\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}+\int_{\Omega}\sum_{i=1}^{N}\frac{1}{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}-\int_{\Omega}W(x,u),\ u\in X,\tag{5.2}$$

where $W(x,t):=\int_0^t w(x,\tau)d\tau$.

Lemma 5.1. *The functional L satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset X$ be a sequence such that $L'(u_n) \to 0$ and $L(u_n) \to c$ for some $c \in \mathbb{R}$.

The proof will start by considering the case where $\max\{p_N, q_N\} < \alpha$. We have that (f_3) holds for $\theta' > 0$ satisfying $\max\{p_N, q_N\} < \theta' < \min \alpha, \theta$. By combining $(f_2) - (f_3)$, the boundedness of $\underline{u}$ and the continuous embedding $X \hookrightarrow L^1(\Omega)$, we can find positive constants $C_i > 0, i = 1, 2, 3$, such that

$$\begin{aligned} C_1 + \|u_n\| &\geq L(u_n) - \frac{1}{\theta'} L'(u_n)u_n \\ &\geq C_2(\|u_n\|^{p_1}_{1,\vec{p}} + \|u_n\|^{q_1}_{1,\vec{q}}) + \int_{\{u_n \geq \underline{u}\}} \left(\frac{1}{\theta'} - \frac{1}{\alpha}\right) k(x) u_n^\alpha - C_3\|u_n\| \\ &\geq C_2(\|u_n\|^{p_1}_{1,\vec{p}} + \|u_n\|^{q_1}_{1,\vec{q}}) - C_3\|u_n\|, \end{aligned}$$

which imply the boundedness of (u_n) in X.

Regarding the case $\alpha < \min\{p_1, q_1\}$ note that (f_2), (f_3), the boundedness of $\underline{u}$ and the continuous embeddings $X \hookrightarrow L^\alpha(\Omega)$ implies

$$\begin{aligned} K_1 + \|u_n\| &\geq L(u_n) - \frac{1}{\theta} L'(u_n)u_n \\ &\geq K_2(\|u_n\|^{p_1}_{1,\vec{p}} + \|u_n\|^{q_1}_{1,\vec{q}}) - K_3 \int_\Omega k(x)|u_n|^\alpha - K_3\|u_n\| \\ &\geq K_2(\|u_n\|^{p_1}_{1,\vec{p}} + \|u_n\|^{q_1}_{1,\vec{q}}) - K_4\|k\|_\infty \|u_n\|^\alpha - K_3\|u_n\|, \end{aligned} \tag{5.3}$$

for some constants $K_i > 0, i = 1, ..., 4$, which do not depend on $n \in \mathbb{N}$. Since $\alpha < \min\{p_1, q_1\}$ we have that (u_n) is bounded in X.

Suppose that $\alpha = p_1$. Let $\|k\|_\infty$ be small enough such that $K_2 - K_4\|k\|_\infty > 0$. By reasoning as in (5.3) and applying the continuous embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^\alpha(\Omega)$, we can obtain that

$$K_1 + \|u_n\| \geq (K_2 - K_4\|k\|_\infty)\|u_n\|^{p_1}_{1,\vec{p}} + K_2\|u_n\|^{q_1}_{1,\vec{q}} - K_3\|u_n\|,$$

which provides that (u_n) is bounded in X . To obtain the boundedness of the sequence (u_n) in X when $\alpha = q_1$, we can apply a similar argument as before.

From Corollary 3.2 we obtain, up to a subsequence, that

$$\begin{cases} u_n & \rightharpoonup u \text{ in } X, \\ u_n(x) & \to u(x) \text{ a.e. in } \Omega, \\ u_n & \to u \text{ in } L^q(\Omega), \text{ for all } 1 \leq q < \min\{p^*, q^*\}, \end{cases}$$

for some $u \in X$. Thus, from the Lebesgue's Dominated Convergence Theorem we get

$$\int_\Omega \sum_{i=1}^N \left[\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) + \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\right]$$
$$= o_n(1).$$

The inequality (3.1) and the previous equality we obtain, up to a subsequence, that

$$\int_\Omega \left|\frac{\partial}{\partial x_i}(u_n - u)\right|^{p_i} = \int_\Omega \left|\frac{\partial}{\partial x_i}(u_n - u)\right|^{q_i} \to 0, i = 1, ..., N,$$

which implies that $\|u_n - u\| \to 0$.

The Mountain Pass Geometry for the functional L defined in (5.2) is obtained in the next result.

Lemma 5.2. *For $\|k\|_\infty$ small enough the assertions below hold.*

(i) *There are constants R and σ with $R > \|\underline{u}\|$, satisfying*

$$L(\underline{u}) < 0 < \sigma \leq \inf_{u \in \partial B_R(0)} L(u).$$

(ii) *There is $e \in X \setminus \overline{B_{2R}(0)}$ satisfying $L(e) < \sigma$.*

Proof. Since $p_1, q_1 > 1$ we have $L(\underline{u}) < 0$. Let $u \in X$ be a function. From the continuous embeddings $X \hookrightarrow L^\xi(\Omega), \xi = \alpha, r$, and arguing as in 3.2 we have

$$L(u) \geq K_1 \min\{\|u\|^{p_1}, \|u\|^{q_1}\} - K_2\|u\| - K_3\|k\|_\infty(\|u\|^\alpha + \|u\|^r) - K_4,$$

for constants $K_i > 0, i = 1,2,3,4$. If necessary, decrease $\|k\|_\infty$ in such a way that $\|u\| = \max\{\|\underline{u}\|_{1,p}, \|\underline{u}\|_{1,q}\} < 1$, which is possible by choosing the test function $\phi = \underline{u}$ in the first inequality of (4.1) and applying Lemma 3.6.

Fix $\sigma > 0$ and consider $R \geq 1 > \|\underline{u}\|$ large enough satisfying $K_1 \min\{R^{p_1}, R^{q_1}\} - K_2 R - K_4 \geq 2\sigma$. Decreasing $\|k\|_\infty$ such that $K_3\|k\|_\infty(R^\alpha + R^r) \leq \sigma$, we get $L(u) \geq \sigma$ for all $u \in X$ such that $\|u\| = R$, which proves (i).

Regarding the proof of (ii) note that (f_3) and the inequality $\max\{p_n, q_N\} < \theta$ provides that $L(t\underline{u}) \leq K_1(t^{p_N} + t^{q_N}) - K_2 t^\alpha - K_3 t^\theta + K_4 < 0$ for some constants $C_i > 0, i = 1, ..., 4$ and $t > 0$ large enough.

Proof of Theorem 1.2. Let $\underline{u}, \overline{u} \in X$ be the functions given in Lemma 4.2 and denote by $u_1 \in X$ the solution of (P) obtained in Theorem 1.1, which minimizes J that was defined in (4.4). Applying Lemma 5.1 and 5.2 we obtain that the conditions of the Mountain Pass Theorem [29, Theorem 2.1] are satisfied by the functional L. Therefore,

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} L(\gamma(t)), \text{ where } \Gamma := \{\gamma \in C([0,1], X); \gamma(0) = \underline{u}, \gamma(1) = e\},$$

is a critical value of the functional L, that is, $L'(u_2) = 0$ and $L(u_2) = c$, for some $u_2 \in X$. The definition of z provided in (4.3) provides that $J(u) = L(u)$ for $u \in \{h \in X; 0 \leq h(x) \leq \overline{u}(x) \text{ a.e in } \Omega\}$. Hence $J(\underline{u}) = L(\underline{u})$ and $L(u_1) = J(u_1) = \inf_{u \in X} J(u)$. Recall from the proof of Lemma 5.2 that $L(\underline{u}) < 0$. Thus, if $u_2(x) \geq \underline{u}(x)$ a.e. in Ω, then it will follow that (P) has two solutions $u_1, u_2 \in X$ with $L(u_1) \leq L(\underline{u}) < 0 < \sigma \leq c = L(u_2)$, where $\sigma > 0$ is provided in Lemma 5.2.

We affirm that the inequality $u_2(x) \geq \underline{u}(x)$ a.e. in Ω holds. In fact, by applying in (5.1) the test function $(\underline{u} - u_2)^+ \in X$ we get

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_2}{\partial x_i}\right|^{p_i-2} \frac{\partial u_2}{\partial x_i} \frac{(\underline{u} - u_2)^+}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_2}{\partial x_i}\right|^{q_i-2} \frac{\partial u_2}{\partial x_i} \frac{(\underline{u} - u_2)^+}{\partial x_i} = \int_\Omega w(x, u_2)(\underline{u} - u_2)^+ \\
&= \int_\Omega (k(x)\underline{u}(x)^{\alpha-1} + f(x, \underline{u}(x)))(\underline{u} - u_2)^+ \\
&\geq \int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial(\underline{u} - u_2)^+}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{q_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial(\underline{u} - u_2)^+}{\partial x_i}.
\end{aligned}$$

Thus, it follows from (3.1) that

$$\begin{aligned}
\int_\Omega \left|\frac{\partial}{\partial x_i}(\underline{u} - u_2)^+\right|^{p_i} &= 0,\ \text{if}\ \ 2 \le p_i,\\
\int_\Omega \left|\frac{\partial}{\partial x_i}(\underline{u} - u_2)^+\right|^{q_i} &= 0,\ \text{if}\ \ 2 \le q_i,\\
\int_\Omega \frac{\left|\frac{\partial}{\partial x_i}(\underline{u} - u_2)^+\right|^2}{\left(1 + \left|\frac{\partial \underline{u}}{\partial x_i}\right| + \left|\frac{\partial u_2}{\partial x_i}\right|\right)^{2-p_i}} &= 0,\ \text{if}\ \ 1 < p_i \le 2,\\
\int_\Omega \frac{\left|\frac{\partial}{\partial x_i}(\underline{u} - u_2)^+\right|^2}{\left(1 + \left|\frac{\partial \underline{u}}{\partial x_i}\right| + \left|\frac{\partial u_2}{\partial x_i}\right|\right)^{2-q_i}} &= 0,\ \text{if}\ \ 1 < q_i \le 2,
\end{aligned}$$

which provides that $\|(\underline{u} - u_2)^+\| = 0$. Therefore $u_2(x) \ge \underline{u}(x)$ a.e. in Ω.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare there is no conflict of interest.

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