Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 15(3): 515–532.
DOI: 10.3934/cam.2023025
Received: 20 April 2023
Revised: 07 August 2023
Accepted: 16 August 2023
Published: 23 August 2023

Research article

Harmonic maps into sub-Riemannian Lie groups

Erlend Grong*, Irina Markina

Department of Mathematics, University of Bergen, P.O. Box 7803, 5020 Bergen, Norway

* **Correspondence:** Email: erlend.grong@uib.no.

Abstract: We define harmonic maps between sub-Riemannian manifolds by generalizing known definitions for Riemannian manifolds. We establish conditions for when a horizontal map into a Lie group with a left-invariant metric structure is a harmonic map. We show that sub-Riemannian harmonic maps can be abnormal or normal, just as sub-Riemannian geodesics. We illustrate our study by presenting the equations for harmonic maps into the Heisenberg group.

Keywords: Sub-Riemannian manifolds; horizontal maps; harmonic maps; Darboux derivative
Mathematics Subject Classification: 58E20, 53C17

1. Introduction

A harmonic map between Riemannian manifolds (M, g), $\dim(M) = m$, and (N, h), $\dim(N) = n$, are smooth maps giving the minimum to the energy functional

$$\mathscr{E}(f) = \int_M e(f) d\mu, \quad e(f)(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} h(w_j, df(v_i))^2, \quad f: M \to N, \tag{1.1}$$

where $d\mu$ is the Riemannian volume density on M, $\{v_i\}_{i=1}^m$ is an orthonormal basis in T_xM, and $\{w_j\}_{j=1}^n$ is an orthonormal basis in $T_{f(x)}N$. Particular examples are maps $f: [a,b] \to N$, describing the Riemannian geodesics in N and harmonic functions $f: M \to \mathbb{R}$. Other examples are minimal surfaces. For instance, a minimal surface in $\mathbb{R}^3$ can be seen as a harmonic map $f: [a,b] \times [a,b] \to \mathbb{R}^3$; see, e.g., [1–3] or more recent survey for minimal submanifolds [4]. The Euler-Lagrange equations of (1.1) correspond to the solution of $\tau(f) = 0$, where $\tau(f) = \operatorname{tr}_g \boldsymbol{\nabla}_\times df(\times)$ denotes the tension field of f, defined by using an induced connection on $T^*M \otimes f^*TN$ from the Levi-Civita connections on respectively M and N. The celebrated result of [5] states that any smooth map $f \in C^\infty(M, N)$ from a compact Riemannian manifold M to a manifold N of non-positive scalar curvature can be deformed to a harmonic map.

A generalization of this terminology has been suggested for sub-Riemannian manifolds. A sub-Riemannian manifold is a triplet (M, D, g) consisting of a smooth, connected manifold M, a subbundle D of the tangent bundle TM, and a sub-Riemannian metric g defined only on vectors in D. We assume that D is bracket-generating, meaning that sections of D and a sufficient number of their Lie brackets span T_xM at each point $x \in M$. Studies of harmonic maps $f\colon M \to N$ from a sub-Riemannian (M, D, g) into a Riemannian manifold (N, h) of non-positive curvature was made, for instance, in [6–9]. Here, the energy functional (1.1) is modified by letting $v_1, \dots, v_m$ be an orthonormal basis of D_x, and the corresponding equation $\tau(f) = 0$ turns to be of a hypoelliptic type. The existence and regularity of the solution to $\tau(f) = 0$ was established in [7] under some convexity condition on N, and uniqueness has been shown in [10].

In the present paper we consider harmonic maps allowing the target space to be a sub-Riemannian manifold. Already the study of curves in sub-Riemannian manifolds shows that it is not sufficient to deal exclusively with the Euler-Lagrange equation when it comes to minimizers of (1.1). More precisely, there are examples of curves that are energy minimizers, and hence the length minimizers, which are not solutions to the Euler-Lagrange equation. Such curves are necessarily singular points in the space of curves of finite sub-Riemannian length, also called horizontal curves, fixing two given points. Minimizers that are solutions of the Euler-Lagrange equation are called normal, and they are smooth [11, 12]. There are several open questions related to the regularity of minimizers which are singular curves [13]; see also [14]. To simplify the exposition we choose the target sub-Riemannian manifold (N, E, h) to be a Lie group with a left-invariant sub-Riemannian structure (E, h); see also [15], where the target space is a Carnot group. The restriction of the target space to a Lie group allows one to avoid some of the complications of L^2 and Sobolev maps between general manifolds; see, e.g., [16, Section 4]. Furthermore, applying the Maurer-Cartan form on a Lie group simplifies calculations and prevents the need of to choose an explicit connection for the target manifold as well. The map $f\colon M \to N$ is required to be horizontal, that is $df(D) \subset E$. We consider the harmonic maps to be analogous of "normal" and singular geodesics, based on the study of the maps that are regular or singular points of an analogue of the end-point-map. We finally produce equations for both types of horizontal maps: the singular (or abnormal) maps and the normal, latter being solutions of the Euler-Lagrange equation. We will not address conditions for existence or non-existence of such harmonic maps, rather leaving such questions for future research.

We emphasize that we consider a sub-Riemannian analogue of (1.1) which is only defined for horizontal maps and a map f is harmonic if it is a critical value under *horizontal variations*. See (4.1) for the definition of the sub-Riemannian energy functional. Such an approach can be considered as the limiting case when the length of vectors outside of E in TN approach infinity. This is in contrast to work in [17, Proposition 5.1] on CR manifolds, which uses an orthogonal projection to define an energy functional for all maps, and where maps are considered harmonic if it is a critical value relative to *all variations*. The latter can be considered as a limiting case where the length of vectors orthogonal to E in TN approach zero. However, we note that if a map f is horizontal *and* harmonic in the sense of the definition in [17], then f will also be harmonic according to our definition, as being critical under all variations implies that f is also critical with respect to horizontal variations.

The structure of the paper is as follows. In Section 2 we introduce sub-Riemannian manifolds, sub-Riemannian measure spaces, and connections compatible with such structures. In Section 3, we define horizontal maps from a compact sub-Riemannian measure space into a Lie group with a left-invariant sub-Riemannian structure, and we show the Hilbert manifold structure of the space of these maps. For the rest of the paper, we use the the convention that M is compact, which ensures that the functional in (1.1) is finite. Similar to what is done for Riemannian harmonic maps (see, e.g., [18, Section 2]) the case of M non-compact can be considered by calling f harmonic if it is a critical value of the energy functional when restricted to any (relatively) compact subdomain. For simplicity, we will also assume that M is simply connected. See Remark 4.6 where we suggest modifications for a non-simply connected manifold. We introduce the idea of regular and singular maps, as well as some conditions for these maps. Finally, in Section 4, we find equations for both the normal and abnormal harmonic maps. We show that these equations are a natural generalization of above-mentioned cases of maps into Riemannian manifolds, as well as abnormal and normal sub-Riemannian geodesics. We also give an explicit differential equation for harmonic maps into the Heisenberg group.

2. Sub-Riemannian geometry

2.1. Sub-Riemannnian measure space

A sub-Riemannian manifold is a triple (M, D, g) where M is a connected manifold, D is a subbundle of TM and $g = \langle\cdot\,,\cdot\rangle_g$ is a metric tensor defined on sections of D. Throughout the paper, unless otherwise stated, the subbundle D is assumed to be *bracket-generating*, meaning that the sections of D and their iterated brackets span the tangent space at each point of M. This condition is sufficient to ensure that any pair of points x_0 and x_1 in M can be connected by *a horizontal curve* γ, i.e., an absolutely continuous curve such that $\dot\gamma(t) \in D_{\gamma(t)}$ for almost every t; see [19,20]. Thus, the distance

$$d_g(x_0, x_1) = \inf\left\{ \int_0^1 |\dot\gamma(t)|_g \, dt \ : \ \begin{array}{c} \gamma \text{ is horizontal} \\ \gamma(0) = x_0, \quad \gamma(1) = x_1 \end{array} \right\} \tag{2.1}$$

is well defined. Furthermore, the metric topology with respect to d_g coincides with the manifold topology on M. We do not exclude the possibility $D = TM$.

Associated with the sub-Riemannian metric g, there is a vector bundle morphism

$$\sharp^g : T^*M \to D, \tag{2.2}$$

defined by

$$\alpha(v) = \langle \sharp^g \alpha, v\rangle_g$$

for any $x \in M$, $\alpha \in T^*_x M$, and $v \in D_x$. Define a cometric $g^* = \langle\cdot\,,\cdot\rangle_{g^*}$ on T^*M by

$$\langle \alpha,\beta\rangle_{g^*} = \alpha(\sharp^g\beta) = \langle \sharp^g\alpha, \sharp^g\beta\rangle_g, \qquad \alpha,\beta \in \Gamma(T^*M).$$

This cometric is exactly degenerated along the subbundle $\mathrm{Ann}(D) \subseteq T^*M$ of covectors vanishing on D. Conversely, given a cometric g^* on T^*M that is degenerated along a subbundle of T^*M, we can define the subbundle D of TM as the image of the map $\sharp^g : \alpha \mapsto \langle \alpha, \cdot\rangle_{g^*}$ in (2.2), and a metric g on D by the relation

$$\langle \sharp^g\alpha, \sharp^g\beta\rangle_g = \langle \alpha,\beta\rangle_{g^*}.$$

Hence, a sub-Riemannian manifold can equivalently be defined as a connected manifold with a symmetric positive semi-definite tensor g^* on $\Gamma(TM^{\otimes 2})$ degenerating on a subbundle of T^*M. In what follows, we will speak about a sub-Riemannian structure interchangeably as (D, g) or g^*, assuming that the subbundle D is bracket-generating. For more on sub-Riemannian manifolds, see, e.g., [14,21].

Definition 2.1. *A sub-Riemannian measure space* $(M, D, g, d\mu)$ *is a sub-Riemannian manifold* (M, D, g) *with a choice of smooth volume density* $d\mu$ *on* M*. If* $D = TM$*, then* $d\mu$ *is the volume density of the Riemannian metric* g*.*

On a sub-Riemannian measure space $(M, D, g, d\mu)$ there is a unique choice of second order operator

$$\Delta_{g,d\mu}\phi = \operatorname{div}_{d\mu} \sharp^g d\phi, \qquad \phi \in C^\infty(M). \tag{2.3}$$

We call the operator in (2.3) *the sub-Laplacian* of the sub-Riemannian measure space. Since D is bracket-generating, the classical result of Hörmander [22] states that $\Delta_{g,d\mu}$ is a hypoelliptic operator. If the measure $d\mu$ is clear from the context, we simply write Δ_g. We also denote the sub-Riemannian measure space as $(M, g^*, d\mu)$.

We say that a Riemannian metric $\bar{g} = \langle \cdot\,, \cdot \rangle_{\bar{g}}$ is *a taming metric* of $(M, D, g, d\mu)$ if g is the restriction $\bar{g}|D$ of $\bar{g}$ to D and the volume density of $\bar{g}$ equals $d\mu$.

Lemma 2.2. *Any sub-Riemannian measure space has a taming Riemannian metric.*

Proof. Let $(M, D, g, d\mu)$ be any sub-Riemannian measure space with $\dim M = m$ and $\operatorname{rank} D = k$. If $k = m$, then by convention $\bar{g} = g$ is a taming Riemannian metric. For $k < m$, we take an arbitrary Riemannian metric $\bar{g}_0$ on M and let $D^\perp$ denote the orthogonal complement of D with respect to $\bar{g}_0$. The rank of $D^\perp$ equals $m - k$. Define a Riemannian metric $\bar{g}_1$ such that D and $D^\perp$ are still orthogonal with respect to $\bar{g}_1$, and $\bar{g}_1|D = g$, $\bar{g}_1|D^\perp = \bar{g}_0|D^\perp$. Let $d\bar{\mu}$ be the Riemannian volume density with respect to $\bar{g}_1$, and write $d\bar{\mu} = \rho d\mu$.

Finally, we define the metric $\bar{g}$ to be such that D and $D^\perp$ are orthogonal with respect to $\bar{g}$ and

$$\bar{g}|D = g, \qquad \bar{g}|D^\perp = \rho^{-1/(m-k)}\bar{g}_1|D^\perp,$$

which gives us the desired Riemannian metric.

Remark 2.3 (Hausdorff and Popp's measure). A manifold M carries a measure dx which is the push-forward of the Lebesgue measure by the chart map. The distance d_g in (2.1) generated by the sub-Riemannian metric tensor g produces the Hausdorff measure dH. Relative to any coordinate system defined sufficiently close to a regular point, $dH = q(x)dx$ is absolutely continuous with respect to dx. It is not clear whether q is a smooth function. Another construction of a measure near regular point has been provided by O. Popp (see [21, Chapter 10]) which gives a measure $d\mu$ with a smooth Radon-Nikodym derivative with respect to dx. The latter allows one to define the sub-Laplacian by making use of the integration by parts with respect to the smooth measure $d\mu$, which leads to the sub-Laplacian introduced in [23]. For the case of the Carnot groups, both the Hausdorff and the Haar measures are equal up to a constant, and are hence all smooth.

2.2. *Compatible connections on a sub-Riemannian measure space*

Consider a sub-Riemannian structure g^* on M. For a two-tensor field $\xi \in \Gamma(T^*M^{\otimes 2})$ we write

$$\operatorname{tr}_g \xi(\times, \times) = \xi(g^*), \quad \text{i.e.,} \quad \operatorname{tr}_g \xi(\times, \times)(x) = \sum_{i=1}^k \xi(v_i, v_i)$$

for an arbitrary orthonormal basis $v_1, \ldots, v_k$ of D_x with $k = \operatorname{rank} D$. We want to consider connections on sub-Riemannian manifolds and sub-Riemannian measure spaces. We begin with the following definition of a connection on tensor fields; see, for instance, [24, Chapter 4].

Definition 2.4. *Let ∇ be an affine connection on TM.*

(a) *We say that ∇ is compatible with (D, g) (equiv. g^*) if it satisfies the following equivalent conditions:*

  (i) $\nabla g^* = 0$,

  (ii) $\nabla \sharp^g = \sharp^g \nabla$,

  (iii) *For any $X_1, X_2 \in \Gamma(D)$, $Z \in \Gamma(TM)$, we have that $\nabla_Z X_1 \in \Gamma(D)$ and*

$$Z\langle X_1, X_2\rangle_g = \langle \nabla_Z X_1, X_2\rangle_g + \langle X_1, \nabla_Z X_2\rangle_g .$$

(b) *We say that ∇ is compatible with $(D, g, d\mu)$ (equiv. $(g^*, d\mu)$) if ∇ is compatible with (D, g) (equiv. g^*) and for any $\phi \in C^\infty(M)$*

$$\operatorname{tr}_g \nabla_\times d\phi(\times) = \sum_{i=1}^{k} \nabla_{v_i} d\phi(v_i) = \Delta_{g,d\mu}\phi$$

for an orthonormal basis $v_1, \dots, v_k$ of D_x.

The following is known on sub-Riemannian manifolds.

Proposition 2.5. *[25] Let g^* be a sub-Riemannian structure and $d\mu$ a volume density on M. Then $(g^*, d\mu)$ has a compatible connection.*

We also prove the following result.

Lemma 2.6. (a) *A connection ∇ is compatible with g^* if and only if for every point $x \in M$ there exists a local orthonormal frame $X_1, \dots, X_k$ of D around x such that $\nabla X_j(x) = 0$.*

(b) *A connection ∇ is compatible with $(g^*, d\mu)$ if and only if for every point $x \in M$ there exists a local orthonormal frame $X_1, \dots, X_k$ of D around x such that $\nabla X_j(x) = 0$ and $\operatorname{div}_{d\mu} X_i(x) = 0$.*

Proof. If ∇ preserves D, then $\nabla|D$ is a connection on D preserving the inner product g. Hence there is a local orthonormal frame of D that is parallel with respect to ∇ at a given point x; see, e.g., [26, Theorem 2.1 and Remark 2.2] for details. Conversely, let α be an arbitrary one-form and $x \in M$ an arbitrary point. Assume that there exists an orthonormal frame $X_1, \dots, X_k$ of D around x such that it is ∇-parallel at x. Completing calculations at x, we obtain

$$X|\alpha|^2_{g^*}(x) = \sum_{i=1}^{k} X(\alpha(X_i))^2(x) = 2\sum_{i=1}^{k} \alpha(X_i)(x)(\nabla_X\alpha)(X_i)(x) = 2\langle \alpha, \nabla_X \alpha\rangle_{g^*}(x).$$

If we can find such a basis for every point in M, it follows that ∇ is compatible with g^*. This proves (a).

The result in (b) follows from the identity

$$\Delta_{g,d\mu} f = \sum_{i=1}^{k} X_i^2 f + \sum_{i=1}^{k} (X_i f) \operatorname{div}_{d\mu} X_i,$$

that holds for any local orthonormal basis of M.

Corollary 2.7. *Let ∇ be a connection compatible with (g^*, μ) and let X be a horizontal vector field. Then*

$$\operatorname{div}_{d\mu} X = \operatorname{tr}_g \langle \nabla_\times X, \times \rangle_g.$$

Proof. For a given point $x \in M$, choose an orthonormal frame $X_1, \dots, X_k$ of D around x with $\nabla X_i(x) = 0$ and $\operatorname{div}_{d\mu} X_i(x) = 0$. Write $X = \sum_{i=1}^k f_i X_i$. Then

$$\begin{aligned} \operatorname{div}_{d\mu} X(x) &= \sum_{i=1}^k (X_i f_i(x) + f_i(x) \operatorname{div}_{d\mu} X_i(x)) \\ &= \sum_{i=1}^k X_i f_i(x) = \sum_{i=1}^k \langle \nabla_{X_i} X, X_i \rangle_g(x). \end{aligned}$$

Since $x \in M$ was arbitrary, the result follows.

2.3. Left-invariant sub-Riemannian structures

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let (E, h) be a sub-Riemannian structure on G. We say that the sub-Riemannian structure is left-invariant if E is a left-invariant distribution and if

$$\langle v, w \rangle_h = \langle a \cdot v, a \cdot w \rangle_h, \qquad \text{for any } a \in G, \ v, w \in E_1 = \mathfrak{e} \subseteq \mathfrak{g},$$

where we denote by $a \cdot v$ the action on $v \in \mathfrak{g}$ by the differential of the left translation by $a \in G$, and $1 \in G$ is the identity element. Equivalently, let ω be *the left Maurer-Cartan form*, given by $\omega(v) = a^{-1} \cdot v \in \mathfrak{g}$ for any $v \in T_a G$. Then $v \in E$ if and only if $\omega(v) \in \mathfrak{e} = E_1 \subseteq \mathfrak{g}$. We then say that (E, h) is obtained by left translation of $(\mathfrak{e}, \langle \,\cdot\,, \cdot \rangle)$.

Example 2.8 (The Heisenberg group). We consider the space $H^n = \mathbb{R}^{2n+1}$ with coordinates $(a, b, c) = (a_1, \dots, a_n, b_1, \dots, b_n, c)$. We give this space a global frame

$$A_j = \partial_{a_j} - \frac{1}{2} b_j \partial_c, \qquad B_j = \partial_{b_j} + \frac{1}{2} a_j \partial_c, \qquad C = \partial_c. \tag{2.4}$$

The corresponding coframe is given by da_j, db_j and $\theta = dc + \frac{1}{2} \sum_{j=1}^n (b_j da_j - a_j db_j)$. Note the bracket relations

$$[C, A_i] = [C, B_j] = [A_i, A_j] = [B_i, B_j] = 0, \qquad [A_i, B_j] = \delta_{ij} C. \tag{2.5}$$

Hence, these vector fields form a Lie algebra which we will write as $\mathfrak{h}_n$. We can give $\mathbb{R}^{2n+1}$ a group structure such that the vector fields in (2.4) become left-invariant. The group multiplication is given by

$$(a, b, c) \cdot (\tilde{a}, \tilde{b}, \tilde{c}) = \left(a + \tilde{a}, b + \tilde{b}, c + \tilde{c} + \frac{1}{2} (\langle a, \tilde{b} \rangle_{\mathbb{R}^n} - \langle \tilde{a}, b \rangle_{\mathbb{R}^n}) \right).$$

We will define a sub-Riemannian structure (E, h) on H^n by letting $A_1, \dots, A_n$, $B_1, \dots, B_n$ be an orthonormal basis.

3. Horizontal maps into Lie groups

3.1. Maps into Lie groups and the Darboux derivative

In what follows, we will let $\Omega^p(M,\mathfrak{g})$ be the space of $\mathfrak{g}$-valued differential p-forms on a manifold M. We recall the definition and properties of the Darboux derivative; referring to [27] for more details. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\omega \in \Omega^1(G,\mathfrak{g})$ be the left Maurer-Cartan form as defined in Section 2.3. This form satisfies the left Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega,\omega] = 0, \tag{3.1}$$

with $[\omega,\omega]$ being the two-form $(v,w) \mapsto 2[\omega(v),\omega(w)]$. See Appendix A for more details. If M is a given manifold and $f\colon M \to G$ is a smooth map, we say that $\alpha_f := f^*\omega$ is the *left Darboux derivative* of f. It follow from definition that α_f satisfies (3.1). Conversely, if $\beta \in \Omega^1(M,\mathfrak{g})$ satisfies $d\beta + \frac{1}{2}[\beta,\beta] = 0$, then locally β is the Darboux derivative of some function. If the monodromy representation of β is trivial (see [27, Chapter 3, Theorem 7.14]) then the structural equation implies that $\beta = \alpha_f$ for some map $f\colon M \to G$. Particularly, for a connected, simply connected manifold M the monodromy representation of any $\mathfrak{g}$-valued one-form is trivial, meaning that any form satisfying the left Maurer-Cartan equation can be represented as a Darboux derivative. Through the rest of the paper, we assume that M is connected and simply connected.

Denote by $\mathcal{A} \subseteq \Omega^1(M,\mathfrak{g})$ the collection of forms α satisfying $d\alpha + \frac{1}{2}[\alpha,\alpha] = 0$, and define

$$T_\alpha\mathcal{A} = \left\{ \dot\beta(0) \; : \; \begin{array}{c} \beta : (-\varepsilon,\varepsilon) \to \Omega^1(M,\mathfrak{g}) \text{ is smooth,} \\ \beta(0) = \alpha, \beta(t) \in \mathcal{A} \text{ for any } t \in (-\varepsilon,\varepsilon). \end{array} \right\}.$$

Lemma 3.1. *We have*

$$T_\alpha\mathcal{A} = \{dF + [\alpha,F] \; : \; F \in C^\infty(M,\mathfrak{g})\}.$$

Proof. Let $\beta(t)$ be a differentiable curve in $\Omega^1(M,\mathfrak{g})$ and assume that $d\beta(t) + \frac{1}{2}[\beta(t),\beta(t)] = 0$. If we differentiate this relation and assume that $\beta(0) = \alpha$ and $\dot\beta(0) = \eta$, then

$$d\eta + [\alpha,\eta] = 0.$$

If $\alpha = f^*\omega$ and we write $\eta = \mathrm{Ad}(f^{-1})\tilde\eta$, then

$$\mathrm{Ad}(f^{-1})d\tilde\eta = 0. \tag{3.2}$$

We remark that here we are abusing notation to write $\eta|_x = \mathrm{Ad}(f(x)^{-1})\tilde\eta|_x$, where $f(x)^{-1}$ is the inverse of $f(x)$ with respect to the group operation in G. To see that (3.2) holds, recall first that for any curve $A(t)$ in a Lie group G, we have that $\frac{d}{dt}\mathrm{Ad}(A(t)^{-1}) = -\mathrm{ad}(\omega(\dot A(t)))\,\mathrm{Ad}(A(t)^{-1})$, where ω is the Maurer-Cartan form for the group G. Considering the special case where $A(t) = f(\gamma(t))$ for an arbitrary smooth curve $\gamma(t)$ in M with $f^*\omega = \alpha$, we get the formula for the differential

$$d\,\mathrm{Ad}(f^{-1}) = -\mathrm{ad}(f^*\omega)\,\mathrm{Ad}(f^{-1}) = -\mathrm{ad}(\alpha)\,\mathrm{Ad}(f^{-1}).$$

Using the definition of $\eta = \mathrm{Ad}(f^{-1})\tilde\eta$, this leads to

$$0 = d\eta + [\alpha,\eta] = (d\,\mathrm{Ad}(f^{-1})) \wedge \tilde\eta + \mathrm{Ad}(f^{-1})d\tilde\eta + [\alpha,\mathrm{Ad}(f^{-1})\tilde\eta] = \mathrm{Ad}(f^{-1})d\tilde\eta.$$

In summary, the form $\tilde{\eta} \in \Omega^1(M, \mathfrak{g})$ is closed and we can find a function $\tilde{F}\colon M \to \mathfrak{g}$ such that $\tilde{\eta} = d\tilde{F}$ due to the vanishing de Rham cohomology; see [28, Theorem 11.14]. Furthermore, if we define $F = \mathrm{Ad}(f^{-1})\tilde{F}$, then

$$\beta = \mathrm{Ad}(f^{-1})d\tilde{F} = dF + [\alpha, F].$$

Conversely, for any $F \in C^\infty(M, \mathfrak{g})$, we can define a curve $g(t) = f \cdot \exp(tF)$ in the space of smooth maps $M \to G$ and $\beta(t) = g(t)^*\omega$. Here $\exp\colon \mathfrak{g} \to G$ is the group exponential. Let $v \in T_xM$, $x \in M$, be arbitrary and define $\gamma(s)\colon (-\epsilon, \epsilon) \to M$, $\epsilon > 0$, as a curve with $\gamma(0) = x$ and $\partial_s\gamma(0) = v$. If we set $\Gamma(s, t) = g(t)(\gamma(s))$, then we compute

$$\begin{aligned}
\dot{\beta}(0)(v) &= \partial_t\omega(\partial_s\Gamma(s, t))|_{(s,t)=(0,0)} = \partial_t\Gamma^*\omega(\partial_s)|_{(s,t)=(0,0)} \\
&= \left(\partial_s\Gamma^*\omega(\partial_t) - d(\Gamma^*\omega)(\partial_s, \partial_t)\right)|_{(s,t)=(0,0)} \\
&= \left(\partial_s(\omega(\partial_t\Gamma)) - d(\Gamma^*\omega)(\partial_s, \partial_t)\right)|_{(s,t)=(0,0)} \\
&= \partial_sF(\gamma(s))|_{s=0} + [\omega(\partial_s\Gamma), \omega(\partial_t\Gamma)]|_{(s,t)=(0,0)} \\
&= dF(v) + [\alpha(v), F(x)].
\end{aligned}$$

Recall that $\alpha = f^*\omega$. The result follows.

We want to close our space of Darboux derivatives into a Hilbert space. Let $(M, D, g, d\mu)$ be a sub-Riemannian measure space and let $\bar{g}$ be a taming Riemannian metric. Extend the inner product on $\mathfrak{e}$ to a full inner product on $\mathfrak{g}$. These choices give us an induced inner product on $\wedge^kT^*M \otimes \mathfrak{g}$, which allows us to define an $L2$-inner product $\langle \beta, \beta \rangle = \int_M \langle \beta(x), \beta(x) \rangle \, d\mu(x)$ for any $\beta \in \Omega^k(M, \mathfrak{g})$. With this definition, we consider $L^2\Omega(M, \mathfrak{g}) = \oplus_{k=0}^{\dim M} L^2\Omega^k(M, \mathfrak{g})$ as the space of L^2-forms with values in $\mathfrak{g}$. We remark that since M is compact and $\mathfrak{g}$ is finite dimensional, any other choice of taming Riemannian metric $\bar{g}$ and inner product of $\mathfrak{g}$ will give us an equivalent L^2-inner product, meaning in particular that L^2-forms are independent of these choices. More about the theory of L^p forms can be found, for instance in [29].

Corollary 3.2. *Assume that M is simply connected and compact. Then the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $L^2\Omega^1(M, \mathfrak{g})$ is a Hilbert submanifold of $L^2\Omega^1(M, \mathfrak{g})$ with tangent space $\overline{T_\alpha\mathcal{A}} \subseteq L^2\Omega^1(M, \mathfrak{g})$.*

Proof. Let $\alpha \in \mathcal{A}$ be the Darboux derivative $\alpha = \alpha_f = f^*\omega$ of a map f. We consider an arbitrary curve $\beta \in \mathcal{A}$ that can be written as $\beta = dF + [\alpha, F]$ for some $F \in C^\infty(M, \mathfrak{g})$. From the proof of Lemma 3.1, we note that if $F = \mathrm{Ad}(f^{-1})\tilde{F}$, then $\mathrm{Ad}(f^{-1})\beta = d\tilde{F}$. We denote by $\tilde{F}_\beta$ a unique solution to this equation satisfying $\int_M \tilde{F}_\beta d\mu = 0$, and we define $F_\beta = \mathrm{Ad}(f^{-1})\tilde{F}_\beta$. Then,

$$\begin{aligned}
\|F_\beta\|_{L^2} &\leq \|\,\mathrm{Ad}(f^{-1})\|_{L^\infty}\|\tilde{F}_\beta\|_{L^2} \\
&\overset{\text{Poincaré}}{\leq} C\|\,\mathrm{Ad}(f^{-1})\|_{L^\infty}\|d\tilde{F}_\beta\|_{L^2} \leq C\|\,\mathrm{Ad}(f^{-1})\|_{L^\infty}\|\,\mathrm{Ad}(f)\|_{L^\infty}\|\beta\|_{L^2}
\end{aligned}$$

for some constant $C > 0$. Note that the linear map $\beta \mapsto \tilde{F}_\beta$ is bounded and invertible with respect to the L^2 metric, which, in particular, is smooth. Here we have used the Poincaré inequality for compact Riemannian manifolds found in, e.g., [30, Theorem 2.10]. It follows that this map can be extended by limits to be well defined as a map from $\overline{T_\alpha\mathcal{A}}$ to $\{F \in L^2(M, \mathfrak{g}) \,:\, \int_M \mathrm{Ad}(f)F \, d\mu = 0\}$.

Continuing, we introduce a map $\Phi\colon T_\alpha\mathcal{A} \mapsto \mathcal{A}$ as

$$\Phi(\beta) = (f \cdot e^{F_\beta})^*\omega, \quad \text{for} \quad \beta \in T_\alpha\mathcal{A}.$$

We observe then that for any $v \in T_xM$, we can apply the formula for the differential of the Lie group exponential to obtain

$$\begin{aligned}\Phi(\beta)(v) &= e^{-F_\beta(x)} f(x)^{-1} df(v) e^{F_\beta(x)} + \frac{1 - e^{-\operatorname{ad}(F_\beta(x))}}{\operatorname{ad}(F_\beta(x))} dF_\beta(v) \\ &= e^{-\operatorname{ad}(F_\beta(x))}\alpha + \frac{1 - e^{-\operatorname{ad}(F_\beta(x))}}{\operatorname{ad}(F_\beta(x))}(\beta(v) + \operatorname{ad}(F_\beta(x))\alpha(v)) \\ &= \alpha(v) + \frac{1 - e^{-\operatorname{ad}(F_\beta(x))}}{\operatorname{ad}(F_\beta(x))}\beta(v),\end{aligned}$$

meaning that

$$\Phi(\beta) = \alpha + \frac{1 - e^{-\operatorname{ad}(F_\beta)}}{\operatorname{ad}(F_\beta)}\beta = \alpha + \sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{ad}(F_\beta)^n}{(n+1)!}\beta.$$

This map is well defined for any $\beta \in \overline{T_\alpha \mathcal{A}}$, giving a smooth map $\Phi : \overline{T_\alpha \mathcal{A}} \to \overline{\mathcal{A}} \subseteq L^2\Omega^1(M, \mathfrak{g})$. Furthermore, we see that its Fréchet differential at $\beta = 0$ is given by

$$D\Phi|_0(\beta) = \beta.$$

Thus, Φ is locally injective, so it can be used as a chart close to $0 \in \overline{T_\alpha \mathcal{A}}$. Since $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ the result follows.

3.2. Horizontal maps

For the rest of this section, $(M, D, g, d\mu)$ will be a simply connected, compact sub-Riemannian measure space while G will be a Lie group with Lie algebra $\mathfrak{g}$ and left Maurer-Cartan form ω. The structure (E, h) on G will be defined by left translation of $(\mathfrak{e}, \langle \cdot\,, \cdot \rangle)$. We introduce the following concept.

Definition 3.3. *Let (M, D, g) and (N, E, h) be two sub-Riemannian manifolds. We say that a smooth map $f : M \to N$ is* horizontal *if $df(D) \subseteq E$.*

To simplify the discussion in this paper, we only consider the case when $N = G$ is a Lie group G with a left-invariant sub-Riemannian structure (E, g) that is the left translation of a vector space $\mathfrak{e} \subset \mathfrak{g}$ and a scalar product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{e}$. Then, $f : M \to G$ is horizontal if and only if $\alpha_f = f^*\omega$ sends D into $\mathfrak{e}$. We write $\mathcal{A}_{D,E}$ for the collection of such forms α_f.

Consider $\Omega^1(D, V) = \Gamma(D^* \otimes V)$ as partial one-forms only defined on D with values in a vector space V. Write $L^2\Omega^1(D, V)$ for its L^2-closure. Consider $\overline{\mathcal{A}_{D,E}} \subseteq \overline{\mathcal{A}}$. Define a linear map

$$P : L^2\Omega^1(M, \mathfrak{g}) \to L^2\Omega^1(D, \mathfrak{g}/\mathfrak{e}) \quad \text{by} \quad P(\alpha) = \alpha|D \mod \mathfrak{e}. \tag{3.3}$$

Then $\overline{\mathcal{A}_{D,E}} = \ker P \cap \overline{\mathcal{A}}$.

Definition 3.4. *We say that $\alpha \in \overline{\mathcal{A}_{D,E}}$ is regular (respectively, singular) if α is a regular (respectively, singular) point of $P|_{\mathcal{A}}$; that is the differential of the map P is surjective (not surjective) at $\alpha \in \mathcal{A}$. We say that a sub-Riemannian horizontal map $f : M \to G$ is regular (respectively, singular), if its Darboux derivative $\alpha_f \in \mathcal{A}_D$ is regular (respectively, singular).*

Since $\overline{\mathcal{A}_{D,E}} = \ker P \cap \overline{\mathcal{A}}$, the implicit function theorem implies that $\overline{\mathcal{A}_{D,E}}$ has the local structure of a manifold around any regular α.

3.3. Strong bracket generating condition

We list the conditions for distributions on M and G, which guarantee the absence of singular morphisms.

Definition 3.5. *We say that $\mathfrak{e} \subset \mathfrak{g}$ is a strongly q-bracket generating subspace of $\mathfrak{g}$ if for any $1 \leq l \leq q$ and any set of linearly independent vectors $A_1, \dots, A_l \in \mathfrak{e}$ and any $Z_1, \dots, Z_l \in \mathfrak{g}$, there exists an element $B \in \mathfrak{e}$ such that*

$$Z_j - [A_j, B] \in \mathfrak{e}, \qquad j = 1, \dots, l.$$

Example 3.6. Let θ be a left-invariant one-form on a $(2n + 1)$-dimensional Lie group G and define $\ker \theta|_1 = \mathfrak{e} \subseteq \mathfrak{g}$. Assume that $d\theta|(\wedge^2 \mathfrak{e})$ is non-degenerate, i.e., θ is a contact form on G. We can find a basis $A_1, \dots, A_n, B_1, \dots, B_n$ of $\mathfrak{e}$ such that $d\theta(A_i, A_j) = d\theta(B_i, B_j) = 0$ and $d\theta(B_i, A_j) = \delta_{ij}$. Let $Z \in \mathfrak{g}$ be the unique element satisfying $\theta(Z) = 1$ and $d\theta(Z, \cdot\,) = 0$. To find $B \in \mathfrak{e}$, we need to solve the equations

$$[p_j A_i, B] = \tilde{p}_j Z, \qquad [q_j B_i, B] = \tilde{q}_j Z, \qquad p_j \neq 0, \quad q_j \neq 0.$$

One can easily check that

$$B = \sum_{i=1}^{n} \left(\frac{\tilde{p}_i}{p_i} B_i - \frac{\tilde{q}_i}{q_i} A_i \right).$$

is a solution. This shows that such structures are strongly $2n$-bracket generating. In particular, we note that the Heisenberg group H^n is has a strong $2n$-bracket generating distribution.

Proposition 3.7. *Let (M, D, g) be a sub-Riemannian manifold, where M is simply connected and D has rank $k \geq 2$. Let $\mathfrak{g}$ be a Lie algebra with a generating subspace $\mathfrak{e} \subseteq \mathfrak{g}$ of positive codimension. Let $\alpha \in \Omega^1(M, \mathfrak{g})$ be a one-form satisfying $\alpha(D) \subseteq \mathfrak{e}$.*

(a) *Assume that there exists a non-intersecting horizontal loop $\gamma : [0, 1] \to M$ such that $\alpha(\dot{\gamma}(t)) = 0$ for almost every $t \in [0, 1]$. Then α is singular.*

(b) *Assume that $\mathfrak{e} \subseteq \mathfrak{g}$ is strongly k-bracket generating. If $\alpha|D$ is injective at every point, then α is regular.*

Proof. Choose a complement $\mathfrak{f}$ to $\mathfrak{e}$ in $\mathfrak{g}$. Let $F \in C^\infty(M, \mathfrak{g})$ be a function and write $F = F_{\mathfrak{e}} + F_{\mathfrak{f}}$ according to the decomposition $\mathfrak{g} = \mathfrak{e} \oplus \mathfrak{f}$. Recall that the regularity of α is equivalent to the assumption that for any one-form $\psi \in \Omega^1(M, \mathfrak{g})$, one can choose $F_{\mathfrak{e}}$ and $F_{\mathfrak{f}}$ such that

$$dF_{\mathfrak{f}}|D + [\alpha|D, F_{\mathfrak{e}}] + [\alpha|D, F_{\mathfrak{f}}] = \psi|D \mod \mathfrak{e}. \tag{3.4}$$

(a) If $\gamma : [0, 1] \to M$ is a non-intersecting horizontal loop, then $x_1 = \gamma(1/2) \neq x_0 = \gamma(0) = \gamma(1)$ by assumption. Define $\gamma_1, \gamma_2 : [0, 1] \to M$ by $\gamma_1(t) = \gamma(t/2)$ and $\gamma_2(t) = \gamma(1 - t/2)$, which are non-intersecting horizontal curves from x_0 to x_1. Let U be an open set that does not intersect γ_2, but intersects with a subset of γ_1 of positive length. Let $\psi = \psi_0 \otimes Z$, where $Z \in \mathfrak{f}$, $Z \neq 0$, and ψ_0 denotes a real valued one-form with support in U such that $C = \int_0^1 \psi_0(\dot{\gamma}(t))\, dt > 0$. If we find a function $F = F_{\mathfrak{e}} + F_{\mathfrak{f}}$ solving (3.4) then

$$F_{\mathfrak{f}}(x_1) - F_{\mathfrak{f}}(x_0) = \int_{\gamma_1} \psi \; = CZ \neq 0.$$

However, in order for (3.4) to hold, we would also need to $F_{\mathfrak{f}}(x_1) - F_{\mathfrak{f}}(x_0)$ tp equal $\int_{\gamma_2} \psi$ which is clearly 0 by the definition of ψ, giving us a contradiction.

(b) If $\alpha|D$ is injective, then we can choose $F_{\mathfrak{f}} = 0$. To show the regularity of α we need to solve the equation $[\alpha|D, F_{\mathfrak{e}}] = \psi|D \mod \mathfrak{e}$. The assumption of D being strongly k-bracket generating implies that the equation $[\alpha|D, F_{\mathfrak{e}}] = \psi|D \mod \mathfrak{e}$ has a solution for any one-form $\psi \in \Omega^1(M, \mathfrak{g})$. To be more precise, let $X_1, \ldots, X_k$ be a local basis of D and $\psi \in \Omega^1(M, \mathfrak{g})$. We respectively define $A_j \in C^\infty(M, \mathfrak{e})$ and $Z_j \in C^\infty(M, \mathfrak{g})$ by $A_j = \alpha(X_j)$ and $Z_j = \psi(X_j)$, $j = 1, \ldots, k$. Then we can then define $F_{\mathfrak{e}}$ such that $[A_j, F_{\mathfrak{e}}] = Z_j$ by the strongly k-bracket generating condition on D.

4. Harmonic maps

4.1. Normal and abnormal harmonic maps

Let $(M, D, g, d\mu)$ be a given sub-Riemannian measure space and let (G, E, h) be a Lie group with a left-invariant sub-Riemannian structure. For a horizontal map $f\colon M \to G$ with Darboux derivative α_f, we define its energy as

$$\begin{aligned} \mathscr{E}(f) &= \frac{1}{2}\int_M |df|^2_{g^*\otimes f^*h} d\mu = \frac{1}{2}\int_M \operatorname{tr}_g(f^*h)(\times,\times)\, d\mu \\ &= \frac{1}{2}\int_M |\alpha_f|^2_{g^*} d\mu =: \hat{\mathscr{E}}(\alpha_f). \end{aligned} \tag{4.1}$$

We note that if $v_1, \ldots, v_k \in D_x$ and $w_1, \ldots, w_n \in E_{f(x)}$ are respective orthonormal bases then

$$|df|^2_{g^*\otimes f^*h}(x) = \sum_{i=1}^k |df(v_i)|^2_h = \sum_{j=1}^n \sum_{i=1}^k \langle w_j, df(v_i)\rangle^2_h.$$

We would generalize the definition of harmonic maps from [5] to the sub-Riemannian setting, saying that f is harmonic if it is a critical value of $\mathscr{E}$. Instead, we use the Darboux derivative to make this definition precise. For $\alpha \in \overline{\mathcal{A}_{D,E}}$, we define a variation α_s of α as a differentiable curve $(-\varepsilon, \varepsilon) \to \overline{\mathcal{A}_{D,E}}$, $s \mapsto \alpha_s$, such that $\alpha_0 = \alpha$.

Definition 4.1. *We say that $\alpha \in \overline{\mathcal{A}_{D,E}}$ is harmonic if it is a critical point of $\hat{\mathscr{E}}$, i.e., for every variation α_s of α, we have $\frac{d}{ds}\hat{\mathscr{E}}(\alpha_s)|_{s=0} = 0$. We say that f is harmonic if α_f is harmonic.*

We have the following result.

Theorem 4.2. *Let M be a simply connected, compact manifold, and ∇ a connection compatible with the sub-Riemannian measure space $(M, g^*, d\mu)$. Let the map $\sharp = \sharp^h_1 : \mathfrak{g}^* \to \mathfrak{e}$ correspond to the sub-Riemannian metric h at the identity. Assume that $\alpha \in \mathcal{A}_{D,E}$ is harmonic. Then at least one of the following statements holds.*

(a) (Abnormal case) *There exists form a $\eta \in \Omega^1(M, \mathfrak{g}^*)$ with $\eta|D$ non-zero, satisfying $\sharp\eta|D = 0$ and*

$$\delta_D\eta - \operatorname{tr}_g \operatorname{ad}^*(\alpha(\times))\eta(\times) = 0.$$

with $\delta_D\eta = -\operatorname{tr}_g \nabla_\times\eta(\times)$.

(b) (Normal case) *There exists a form* $\lambda \in \Omega^1(M, \mathfrak{g}^*)$ *satisfying* $\sharp\lambda|D = \alpha|D$ *and*

$$\delta_D \lambda - \operatorname{tr}_g \operatorname{ad}^*(\alpha(\times))\lambda(\times) = 0.$$

with $\delta_D \lambda = -\operatorname{tr}_g \nabla_\times \lambda(\times)$.

Recall that ad^* denotes the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ given by $(\operatorname{ad}^*(A)\beta)(B) = -\beta([A, B])$ for any $A, B \in \mathfrak{g}, \beta \in \mathfrak{g}^*$.

Remark 4.3. We remark the following about the result of Theorem 4.2.

- Case (a), which we call *abnormal*, occurs when $\alpha = \alpha_f \in \mathcal{A}_{D,E}$ is singular. It is a property that holds for all singular elements, and it is not related to optimality. The proof of the result in (a) does not use the property that α is a harmonic form.
- Case (b), which is called *normal*, occurs when $\alpha = \alpha_f \in \mathcal{A}_{D,E}$ is both regular and is a critical value of $\hat{\mathscr{E}}$. However, there are also singular forms $\alpha \in \mathcal{A}_{D,E}$ that are critical values of $\hat{\mathscr{E}}$ and also have a corresponding $\lambda \in \Omega^1(M, \mathfrak{g}^*)$ satisfying the equations as in (b), but such an extremal form α is not called normal. Thus, Cases (a) and (b) are not completely disjoint.
- We remark also that the results of Theorem 4.2 only depend on restrictions $\alpha|D$ and $\lambda|D$ of forms to D and do not depend on their extension to the entire tangent bundle. Hence, we could have stated Theorem 4.2 by using only the restrictions $\alpha|D$ and $\lambda|D$.

Before proceeding to the proof, we observe how the result of Theorem 4.2 satisfies known examples in literature.

Example 4.4. We note that if $E = TG$, so that G is a Riemannian Lie group, then there cannot exist any abnormal harmonic maps. Indeed, since $\sharp$ is now is a bijective map, we cannot have that $\eta|D \neq 0$, while still having that $\sharp\eta|D = 0$.

For the normal case, we have $\sharp\lambda = \alpha$ and if $\alpha = \alpha_f$, then

$$\delta_D \alpha + \operatorname{tr}_g \operatorname{ad}(\alpha(\times))^\dagger \alpha(\times) = 0, \tag{4.2}$$

where $\operatorname{ad}(\alpha(\times))^\dagger$ is the transpose map with respect to left-invariant metric h on G. This is just the classical tension field equation for harmonic maps. In order to explain this, we write (4.2) as

$$\tau(f) = \operatorname{tr}_g \boldsymbol{\nabla}_\times df(\times) = 0, \qquad \boldsymbol{\nabla} = \nabla \otimes f^*\nabla^h. \tag{4.3}$$

Here, ∇^h is the Levi-Civita connection on G, which for left-invariant vector fields, can be written as,

$$2\nabla^h_A B = \operatorname{ad}(A)B - \operatorname{ad}(A)^\dagger B - \operatorname{ad}(B)^\dagger A.$$

Furthermore, $\boldsymbol{\nabla}$ is the induced connection on $T^*M \otimes f^*TG$, which can be described by

$$(\boldsymbol{\nabla}_X\, df)(Y) = \nabla^h_{df(X)} df(Y) - df(\nabla_X Y).$$

Equation (4.3) coincides with the tension field $\tau(f)$ for maps between the Riemannian manifolds in [5] or from sub-Riemannian manifolds to Riemannian manifolds in [7, 9]. For the special case $G = \mathbb{R}$, we have that $\tau(f) = \Delta_{g,d\mu} f$.

Example 4.5. Consider $M = [0, 1]$. Although this is not within the scope of the theorem, as M is a Riemannian manifold with boundary, the theorem is still valid under the assumption that any variation is constant on $\partial M = \{0, 1\}$. Define ∇^ℓ to be the left-invariant connection on G, i.e., the connection such that $\nabla^\ell A = 0$ for any left-invariant vector field A. This connection is compatible with the sub-Riemannian structure (E, h). Let T^ℓ be the torsion of ∇^ℓ, given for left-invariant vector fields by

$$T^\ell(A, B) = -\operatorname{ad}(A)B.$$

We say that the *adjoint connection* to ∇^ℓ is given by $\hat{\nabla}^\ell_A B = \nabla^\ell_A B - T^\ell(A, B)$. For the special case of ∇^ℓ, its adjoint will be the right invariant connection. If $f\colon [0, 1] \to G$, then the equation in Theorem 4.2 (a) is written as

$$\hat{\nabla}_{\dot{f}}\eta = 0, \qquad \sharp^h\eta = 0,$$

where η is a one-form along $f(t)$. The equation in Theorem 4.2 (b) becomes

$$\hat{\nabla}_{\dot{f}}\lambda = 0, \qquad \sharp^h\lambda = \dot{f}.$$

These are the respective equations for abnormal curves and normal geodesics see [31, 32] for details.

Proof of Theorem 4.2. Recall that we have defined L^2-forms with respect to a taming metric $\bar{g}$ and an inner product on $\mathfrak{g}$. Assume that $\alpha \in \mathcal{A}_{D,E}$ is harmonic. Write

$$\begin{aligned} Q &= \{\eta \in \Omega^1(M, \mathfrak{g}^*)\ : \eta(D) \subseteq \operatorname{Ann}(\mathfrak{e})\}, \\ \Lambda_\alpha &= \{\lambda \in \Omega^1(M, \mathfrak{g}^*)\ :\ \sharp\lambda|D = \alpha|D\}. \end{aligned}$$

Note that Q is a vector space, while Λ_α is an affine space with $\lambda_1 - \lambda_2 \in Q$ for $\lambda_1, \lambda_2 \in \Lambda_\alpha$. We first observe the following. Consider the operator $L_\alpha F := dF + [\alpha, F]$. Then for any $\eta \in L^2\Omega^1(M, \mathfrak{g}^*)$, we note that

$$\int_M \operatorname{tr}_g \eta(\times) dF(\times) d\mu = \int_M \operatorname{tr}_g \eta(\times)\nabla_\times F d\mu = \int_M (\delta_D\eta) F d\mu;$$

hence

$$\begin{aligned} \int_M \operatorname{tr}_g \eta(\times)(L_\alpha F)(\times)\, d\mu &= \int_M \operatorname{tr}_g \eta(\times)(dF(\times) + [\alpha(\times), F]) \\ &= \int_M (\delta_D\eta - \operatorname{tr}_g \operatorname{ad}^*(\alpha(\times))\eta(\times)) F d\mu =: \int_M (L^*_\alpha\eta) F d\mu. \end{aligned}$$

If $\alpha \in \overline{\mathcal{A}_{D,E}}$ is singular, then there is a non-zero form $\check{\eta} \in L^2\Omega^1(D, \mathfrak{g}/\mathfrak{e})$ orthogonal to the image of $D_\alpha P(T_\alpha\mathcal{A})$ where P is given in (3.3). Define $\eta = \langle\check{\eta}, D_\alpha P\cdot\rangle_{L^2} \in \bar{Q}$. Then for any element $L_\alpha F$ in $T_\alpha\mathcal{A}$, $F \in C^\infty(M, \mathfrak{g})$, we have

$$0 = \langle\eta, D_\alpha P L_\alpha F\rangle = \int_M \eta(L_\alpha F)\, d\mu = \int_M (L^*_\alpha\eta) F\, d\mu.$$

As this holds for any $F \in C^\infty(M, \mathfrak{g})$, the result in (a) follows.

If α is regular, then $\overline{\mathcal{A}_{D,E}}$ is locally a manifold with $T_\alpha\overline{\mathcal{A}_{D,E}}$ being the closure of elements $L_\alpha F$ such that $L_\alpha F(D) \subseteq \mathfrak{e}$. In other words, elements in $T_\alpha\overline{\mathcal{A}_{D,E}}$ are in the closure of elements $L_\alpha F$, $F \in C^\infty(M, \mathfrak{g})$, that are orthogonal to Q, which can be written as

$$T_\alpha\overline{\mathcal{A}_{D,E}} = \overline{\{L_\alpha F \,:\, \langle F, \phi\rangle_{L^2} = 0 \text{ for any } \phi \in L_\alpha^* Q\}}.$$

Let F be an arbitrary such element in $C^\infty(M, \mathfrak{g})$ that is orthogonal to $L_\alpha^* Q$. For such a tangent vector in $T_\alpha\overline{\mathcal{A}_{D,E}}$, let α_s be the corresponding variation with $\alpha_0 = \alpha$ and $\frac{d}{ds}\alpha_s|_{s=0} = L_\alpha F$. We observe that for any smooth $\tilde{\lambda} \in \Lambda_\alpha$,

$$\frac{d}{ds}\hat{\mathscr{E}}(\alpha_s)|_{s=0} = \int_M \mathrm{tr}_g\langle \alpha(\times), dF(\times) + [\alpha(\times), F]\rangle d\mu = \int_M (L_\alpha^*\tilde{\lambda})F d\mu.$$

If this vanishes for all such variations, then $L_\alpha^*\tilde{\lambda} \in (C^\infty(M, \mathfrak{g}) \cap (L_\alpha^* Q)^\perp)^\perp$. We remark that since the elements of $L^2(M, \mathfrak{g})$ can be considered as equivalence classes of sequences $(\phi_n)_{n=1}^\infty$ of smooth functions convergent in L^2, we have

$$(C^\infty(M, \mathfrak{g}) \cap (L_\alpha^* Q)^\perp)^\perp = \overline{L_\alpha^* Q}, \qquad \text{and} \qquad \overline{L_\alpha^* Q} \cap C^\infty(M, \mathfrak{g}) = L_\alpha^* Q.$$

Furthermore, since $\tilde{\lambda}$ is smooth, then so is $L_\alpha^*\tilde{\lambda}$; hence we can write $L_\alpha^*\tilde{\lambda} = L_\alpha^*\eta \in L_\alpha^* Q$. By defining $\lambda = \tilde{\lambda} - \eta \in \Lambda_\alpha$, we find that λ satisfies $L_\alpha^*\lambda = 0$.

Remark 4.6. The results in Theorem 4.2 can be generalized to a non simply connected manifold. If M is not simply connected, we consider its universal cover $\Pi\colon \tilde{M} \to M$. We note that $\tilde{M}$ might not be compact, but, as mentioned in our introduction, we can consider compact subdomains. We can then lift functions from M to $\tilde{M}$ as $f \mapsto f \circ \Pi$. By using a partition of unity, we decompose an integral over $\tilde{M}$ or one of its compact subdomains as integrals over open sets that are mapped bijectively to an open set in M. It leads to the conclusion that if f is a harmonic map, then so is $f \circ \Pi$. Looking at the equations in Theorem 4.2, we see that they are all local and can hence be projected to M.

4.2. Harmonic maps into the Heisenberg group

We consider the case of harmonic maps $f\colon M \to H^n$. Let (a, b, c) be the coordinates on H^n as described in Example 2.8. We then have the following corollary.

Proposition 4.7. *Let $f\colon M \to H^n$ be a horizontal map from $(M, D, g, d\mu)$ into the Heisenberg group (H^n, E, h) with its standard sub-Riemannian structure. Write*

$$(u, v, w) = (a, b, c) \circ f, \qquad \zeta = u + iv.$$

Then f is a normal harmonic if and only if for some horizontal vector field $Y \in \Gamma(M)$ satisfying $\mathrm{div}_{d\mu} Y = 0$ *we have*

$$(\Delta_{g,d\mu} - iY)\zeta = 0.$$

We note that the operator $\Delta_{g,d\mu} - iY$ is hypoelliptic by [33]. However, recall that we are *also* assuming that f is horizontal, meaning that

$$\left(dw + \tfrac{1}{2}\textstyle\sum_{j=1}^n (v_j du_j - u_j dv_j)\right)|D = 0.$$

Note that if rank $D \leq 2n$, then all harmonic maps are normal by Example 3.6. and Proposition 3.7.

Proof. Since f is horizontal, then

$$\alpha|D = \alpha_f|D = \sum_{j=1}^{n} du_j|D \otimes A_j + \sum_{j=1}^{n} dv_j|D \otimes B_j.$$

From the requirement that $\sharp\lambda|_D = \alpha|D$, we have that

$$\lambda|D = \sum_{j=1}^{n} du_j|D \otimes da_j + \sum_{j=1}^{n} dv_j|D \otimes db_j + \lambda_0|D \otimes \theta,$$

where λ_0 is a one-form on M. Write $Y = \sharp^g \lambda_0$ as a vector field. The harmonic equation is given by

$$\begin{aligned}
0 &= \delta_D \lambda - \operatorname{tr}_g \operatorname{ad}^*(\alpha(\times))\lambda(\times) \\
&= -\sum_{j=1}^{n} \operatorname{tr}_g(\nabla_\times du_j)(\times) \otimes da_j - \sum_{j=1}^{n} \operatorname{tr}_g(\nabla_\times dv_j)(\times) \otimes db_j + (\nabla_\times \lambda_0)(\times) \otimes \theta \\
&\quad + \operatorname{tr}_g \lambda_0(\times)\left(du_j(\times) \otimes db_j\right) - \operatorname{tr}_g \lambda_0(\times)\left(dv_j(\times) \otimes da_j\right) \\
&= -\sum_{j=1}^{n} (\Delta_D u_j + Y v_j) \otimes da_j - \sum_{j=1}^{n} (\Delta_D v_j - Y u_j) \otimes db_j + (\operatorname{div}_\mu Y) \otimes \theta.
\end{aligned}$$

It follows that $\operatorname{div}_\mu Y = 0$ and

$$\Delta_{g,\mu}\zeta = (\Delta_{g,\mu} u_j + i\Delta_{g,\mu} v_j)_j = (-Y v_j + i Y u_j)_j = iY\zeta,$$

completing the proof.

Example 4.8. If we choose $M = [0, 1]$ in Proposition 4.7, allowing a manifold with boundary, we obtain the normal sub-Riemannian geodesics on the Heisenberg group. More precisely, the horizontality requirement for f can be written as

$$\dot{w} = -\frac{1}{2} \sum_{j=1}^{n} (v_j \dot{u}_j - u_j \dot{v}_j),$$

while $\zeta = u + iv$ now has to satisfy

$$\ddot{\zeta} + iy\dot{\zeta} = 0, \qquad \text{for some constant } y.$$

In other words $\dot{\zeta} = e^{iyt}\dot{\zeta}(0)$, which is exactly the equation for normal geodesics on the Heisenberg group with solutions being the horizontal lifts of circular arcs on the (u, v)-plain.

A. Forms with values in the Lie algebra

We recall here the definition of brackets of forms with values in a Lie algebra $\mathfrak{g}$. Let $\alpha \in \Omega^k(M, \mathfrak{g})$ be a $\mathfrak{g}$ valued k-form, that is a section of the vector bundle $\wedge^k T^*M \otimes \mathfrak{g}$. Note that all such elements can be written as a finite sum of elements $\check{\alpha} \otimes A$ where $\check{\alpha} \in \Omega^k(M)$ is a real valued k-form, and $A \in \mathfrak{g}$. For $\check{\alpha} \in \Omega^k(M), \check{\beta} \in \Omega^l(M)$ and $A, B \in \mathfrak{g}$, we define

$$\left[\check{\alpha} \otimes A, \check{\beta} \otimes B\right] = (\check{\alpha} \wedge \check{\beta}) \otimes [A, B].$$

We can extend this definition by linearity to arbitrary forms $\alpha \in \Omega^k(M, \mathfrak{g})$ and $\beta \in \Omega^l(M, \mathfrak{g})$, to obtain a form $[\alpha, \beta] \in \Omega^{k+l}(M, \mathfrak{g})$. We note that $[\alpha, \beta] = (-1)^{kl+1}[\beta, \alpha]$. We look at the particular case when $k = l = 1$. For $\alpha, \beta \in \Omega^1(M, \mathfrak{g})$ and a basis $A_1, \dots, A_n$ of $\mathfrak{g}$, we write $\alpha = \sum_{j=1}^n \check{\alpha}_j \otimes A_j$ and $\beta = \sum_{j=1}^n \check{\beta}_j \otimes A_j$. We then observe that for any $v, w \in TM$,

$$\begin{aligned}[\alpha, \beta](v, w) = [\beta, \alpha](v, w) &= \sum_{i,j=1}^{n} (\check{\alpha}_i(v)\check{\beta}_j(w) - \check{\beta}_j(v)\check{\alpha}_i(w))[A_i, A_j] \\ &= [\alpha(v), \beta(w)] - [\alpha(w), \beta(v)] = [\alpha(v), \beta(w)] + [\beta(v), \alpha(w)].\end{aligned}$$

In particular, $[\alpha, \alpha](v, w) = 2[\alpha(v), \alpha(w)]$. If α is a one-form and F is a zero-form, i.e., a function, then $[\alpha, F](v) = [\alpha(v), F(x)]$ for every $v \in T_x M$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author was by Trond Mohn Foundation-Grant TMS2021STG02 (GeoProCo). The work of both authors was partially supported by the project Pure Mathematics in Norway, funded by the Trond Mohn Foundation and Tromsø Research Foundation. We thank Pierre Pansu and Mauricio Godoy Molina for helpful discussions.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. Bochner, Harmonic surfaces in Riemann metric, *Trans. Amer. Math. Soc.*, **47** (1940), 146–154. https://doi.org/10.1090/S0002-9947-1940-0001632-1
2. R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, Interscience Publishers, Inc., New York, N.Y., 1950.
3. R. Osserman, *A survey of minimal surfaces*. Dover Publications, Inc., New York, 1986.
4. H. B Lawson. Jr, *Lectures on minimal submanifolds, Vol. I*, volume 9 of *Mathematics Lecture Series*. Publish or Perish, Inc., Wilmington, Del., second edition, 1980.
5. J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86** (1964),109–160. https://doi.org/10.2307/2373037

6. E. Barletta, S. Dragomir, H. Urakawa, Pseudoharmonic maps from nondegenerate CR manifolds to Riemannian manifolds, *Indiana Univ. Math. J.*, **50** (2001), 719–746. https://doi.org/10.1512/iumj.2001.50.1931

7. Y. Dong, Eells-Sampson type theorems for subelliptic harmonic maps from sub-Riemannian manifolds, *J. Geom. Anal.*, **31** (2021),3608–3655. https://doi.org/10.1007/s12220-020-00408-z

8. S. Dragomir, D. Perrone, Levi harmonic maps of contact Riemannian manifolds, *J. Geom. Anal.*, **24** (2014), 1233–1275. https://doi.org/10.1007/s12220-012-9371-8

9. J. Jost, C. J. Xu, Subelliptic harmonic maps, *Trans. Amer. Math. Soc.*, **350** (1998), 4633–4649. https://doi.org/10.1090/S0002-9947-98-01992-8

10. Z. R. Zhou, Uniqueness of subelliptic harmonic maps, *Ann. Global Anal. Geom.*, **17** (1999), 581–594.

11. U. Hamenstädt, Some regularity theorems for Carnot-Carathéodory metrics, *J. Differential Geom.*, **32** (1990), 819–850. https://doi.org/10.4310/jdg/1214445536

12. R. S. Strichartz, Sub-Riemannian geometry, *J. Differential Geom.*, **24** (1986), 221–263. https://doi.org/10.4310/jdg/1214440436

13. R. Monti, The regularity problem for geodesics of the control distance, In *Bruno Pini Mathematical Analysis Seminar 2018*, volume 9 of *Bruno Pini Math. Anal. Semin.*, Univ. Bologna, Alma Mater Stud., Bologna, 2018.

14. A. Agrachev, D. Barilari, U. Boscain, *A comprehensive introduction to sub-Riemannian geometry*, volume 181 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020. https://doi.org/10.1017/9781108677325

15. G. He, J. Li, P. Zhao, Liouville-type theorems for CC-F-harmonic maps into a Carnot group, *J. Geom. Anal.*, **31** (2021),4024–4050. https://doi.org/10.1007/s12220-020-00424-z

16. R. Schoen, K. Uhlenbeck, Boundary regularity and the Dirichlet problem for harmonic maps, *J. Differential Geom.*, **18** (1983),253–268. https://doi.org/10.4310/jdg/1214437663

17. R. Petit, Mok-Siu-Yeung type formulas on contact locally sub-symmetric spaces, *Ann. Global Anal. Geom.*, **35** (2009), 1–37. https://doi.org/10.1007/s10455-008-9120-1

18. J. Eells and L. Lemaire. Another report on harmonic maps, *Bull. London Math. Soc.*, **20** (1988), 385–524. https://doi.org/10.1112/blms/20.5.385

19. W. L. Chow, Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, *Math. Ann.*, **117** (1939), 98–105. https://doi.org/10.1007/BF01450011

20. P. K. Rashevskii, On joining any two points of a completely nonholonomic space by an admissible line, *Math. Ann.*, **3** (1938), 83–94.

21. R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

22. L. Hörmander, Hypoelliptic second order differential equations, *Acta Math.*, **119** (1967), 147–171. https://doi.org/10.1007/BF02392081

23. A. Agrachev, U. Boscain, J. P. Gauthier, F. Rossi, The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups, *J. Funct. Anal.*, **256** (2009),2621–2655. https://doi.org/10.1016/j.jfa.2009.01.006
24. J. M. Lee, *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. An introduction to curvature.
25. E. Grong, A. Thalmaier, Stochastic completeness and gradient representations for sub-Riemannian manifolds, *Potential Anal.*, **51** (2019),219–254. https://doi.org/10.1007/s11118-018-9710-x
26. E. Grong, Curvature and the equivalence problem in sub-Riemannian geometry, *Arch. Math. (Brno)*, **58** (2022),295–327. https://doi.org/10.5817/AM2022-5-295
27. R. W. Sharpe, *Differential geometry*, volume 166 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Cartan's generalization of Klein's Erlangen program, With a foreword by S. S. Chern.
28. J. M. Lee, *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.
29. C. Scott, L^p theory of differential forms on manifolds, *Trans. Amer. Math. Soc.*, **347**(1995), 2075–2096. https://doi.org/10.1090/S0002-9947-1995-1297538-7
30. E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. https://doi.org/10.1090/cln/005
31. M. Godoy Molina, E. Grong, Riemannian and sub-Riemannian geodesic flows, *J. Geom. Anal.*, **27** (2017),1260–1273. https://doi.org/10.1007/s12220-016-9717-8
32. E. Grong, Affine connections and curvature in sub-riemannian geometry, arXiv preprint arXiv:2001.03817, 2020.
33. B. Helffer, J. Nourrigat, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, volume 58 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.