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Research article

Comparative analysis of parameter estimation methods for stable distributions applied to commodity markets

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Abstract: This work presents a comparative analysis of the three main methods for estimating the parameters of stable distributions: McCulloch, Koutrouvelis and the maximum likelihood estimator applied to the returns of gold, silver, and uranium prices. The analysis focuses on four complementary dimensions: the estimation bias, the average quadratic error, the average absolute error, and the execution time. The results highlight the trade-offs between statistical precision and computational complexity for each method. This study provides practical recommendations for selecting an estimation method tailored to the unique characteristics of raw material markets.

Keywords: stable distribution; α-stable; parameter estimation; raw materials; gold; uranium; comparison methods; performance evaluation

JEL Codes: C10, C13, C15, C16, G10

1. Introduction

Estimating the parameters of stable distributions, or α-stables, is a fundamental challenge in the statistical modeling of financial securities, especially for mining activities whose returns are systematically determined by properties of asymmetric and heavy-tailed properties, as discussed in Mandelbrot (1963). Rachev et al. (2000) argued that stable distributions provide a rigorous theoretical framework for capturing the leptokurticity and extreme deviations observed in primary material markets, in contrast to the Gaussian hypothesis traditionally used in the modern theory of the portefeuille. This class of distributions, defined by four parameters (α, β, σ, μ), generalizes the normal law and includes special cases such as the Cauchy and Lévy distributions. It has the major theoretical advantage of being invariant by addition according to central limit theory, as discussed in Nolan (2020).

In the specific context of precious metals and nuclear raw materials, uranium has unique market dynamics linked to geopolitical and regulatory factors, while gold and silver retain their traditional status as safe havens during financial crises (see, for example; Baur et al. (2010)). Accurate modeling of their return distributions is essential for risk management, asset valuation, and the development of robust investment strategies. As Samorodnitsky et al. (1994) pointed out, "the inappropriate use of Gaussian models for data with heavy tails can lead to erroneous estimates of value at risk and catastrophic failures in risk management."

With the exception of a few special cases, the problem of estimating parameters stable distribution is particularly complicated due to the lack of a closed form for the probability density function. Three main approaches are used, each requiring a different trade-off between precision, robustness, and computational complexity. The quantile-based method proposed by McCulloch (1986) provides a quick and reliable estimator that is especially well-suited to large-scale measurements. The method proposed by Koutrouvelis (1980) is based on the regression of the characteristic function and is hereafter called the Koutrouvelis method, which provides a good balance between calculation time and accuracy. Finally, although being computationally demanding, the maximum likelihood estimator (MLE) yields the best results in terms of statistical efficiency and ideal asymptotic properties, which are shown in Nolan (2001).

Our study proposes a systematic comparative evaluation of these three approaches applied to gold, silver, and uranium production records from 2007 to 2025. The analysis is based on four complementary dimensions: the bias of estimators, which measures their systematic deviation from true parametric values; the mean squared error (MSE) and its square root (RMSE), which evaluate global precision; the mean absolute error (MAE), which is less sensitive to extreme values; and the execution time, which is a practical criterion for real-time applications. This multidimensional approach is based on the request made by Weron (2004) for complete assessments of estimators' properties, including both their statistical precision and computational feasibility.

The article is structured as follows: The theoretical framework of α-stable distributions and their parameterizations is presented in Section 2. The study's estimating techniques and methodology; are described in Section 3. Section 4 presents the findings and empirical methodology and analyzes the comparative outcomes. The conclusion summarizes the key contributions and suggests future research directions.

2. Preliminary studies and theoretical frameworks

stable distributions , also known as distributions of Lévy stables or distributions stables parétiennes, are a fundamental family of probability laws that characterize the normal distribution. Their significance in finance and economics stems from the central limit theory of Gnedenko et al. (1954), which states that stable distributions are the only non-trivial limits of normalized sums of independent and identically distributed variables.

A random variable X has an α-stable distribution, denoted by $X \sim S(\alpha, \beta, \sigma, \mu)$, if its characteristic function is:

$$\phi(t) = \mathbb{E}[e^{itX}] = \begin{cases} \exp\left\{-\sigma^{\alpha}|t|^{\alpha}\left[1 - i\beta sign(t)\tan\left(\frac{\pi\alpha}{2}\right)\right] + i\mu t\right\}, & \alpha \neq 1; \\ \exp\left\{-\sigma|t|\left[1 + i\beta\frac{2}{\pi}sign(t)\ln|t|\right] + i\mu t\right\}, & \alpha = 1. \end{cases} \tag{2.1}$$

The α-stable law is thus characterized by four real parameters $\Psi = (\alpha, \beta, \mu, \sigma)$. The parameter α, called the characteristic exponent or stability index, is an indicator of the degree of thickness of the tails of the distribution: the smaller it is, the thicker the tails are; which correspond to very large fluctuations. It is the most important parameter, and is between 0 and 2 ($0 < \alpha \leq 2$). Its maximum value $\alpha = 2$, corresponds to a particular stable law: the Gaussian law or normal law. β is the parameter of dissymmetry, which varies between -1 and 1 ($-1 \leq \beta \leq 1$) and when it is null, the distribution is symmetrical with respect to μ. When α approaches 2, β loses its effect leading to a trend toward the normal distribution. The parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the usual characteristics of position and scale respectively; with the remark that; for the Gaussian distribution, the standard deviation is $\sigma\sqrt{2}$. A random variable X of stable distribution will be noted, according to Zolotarev (1986), by : $X \sim S_{\alpha}(\beta, \mu, \sigma)$. The three exceptions mentioned above are the very famous Gaussian law $S_2(0, \mu, \sigma)$; and the lesser-known Cauchy law $S_1(0, \mu, \sigma)$ and Levy law $S_{\frac{1}{2}}(1, \mu, \sigma)$. The stable law has an additivity property according to which the sum of two independent stable random variables of the same stability index α is still stable with the same characteristic exponent α. This very interesting property is used in finance to study portfolios where two assets with the same value for α can be considered together. One of the particularities of the stable distribution is that it has infinite variance as soon as α is strictly less than 2. In fact, the moments of order $p \in \mathbb{N}$ of $X \sim S_{\alpha}(\beta, \mu, \sigma)$ are such that for $\alpha = 2$, $E|X|^p < +\infty, \forall p$.

$$E|X|^p = \begin{cases} < \infty & \text{if } p < \alpha > 0; \\ = \infty & \text{if } p \geq \alpha. \end{cases}$$

More precisely, it is shown that (see; for example Zolotarev (1986))

$$\lim_{t\to\infty} t^{\alpha}\mathbb{P}(X > t) = C_{\alpha}\frac{1+\beta}{2}\sigma^{\alpha} \quad \text{and} \quad \lim_{t\to\infty} t^{\alpha}\mathbb{P}(X < -t) = C_{\alpha}\frac{1-\beta}{2}\sigma^{\alpha};$$

where C_{α} is a constant given by : $C_{\alpha} = \left(\int_0^{\infty} x^{-\alpha}\sin x dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} & \text{if } \alpha \neq 1; \\ \frac{2}{\pi} & \text{if } \alpha = 1. \end{cases}$

$\Gamma(\theta)$ is the Euler gamma function defined; for $\theta > 0$, by : $\Gamma(\theta) = \int_0^{+\infty} x^{\theta-1}e^{-x}dx.$

We can thus see that the stable law takes into account the distribution tails that are often carriers of essential information, whereas the Gaussian law neglects these tails thus leading to an error that can be fatal for the investor. The disadvantage of the characteristic function (2.1) is that it is not continuous if $\alpha = 1$; which makes it not adapted to numerical calculations and; for these reasons Zolotarev (1986) proposed another parameterization called S_{α}^{0} which is usable for numerical calculations. To simulate stable laws, there is an algorithm developed by Chambers et al. (1976). This one allows us to generate a law $S_{\alpha}(\beta, 0, 1)$; with $\alpha \in]0, 2]$ and $\beta \in$ [-1, 1]. The parameters α and σ for this generator are very well-estimated by the method of McCulloch (1986). The parameters μ and β are correctly estimated by the method of McCulloch (1986) for small values of β, which is often the case for stock exchange chronicles. A bibliography of methods for estimating the parameters of an α-stable law has been compiled by Coulibaly et al. (2024); Coulibaly et al. (2022); Koutrouvelis (1980) and Press (1972).

The Figure 1 illustrates the influence of each parameter of the α-stable distribution on its probability density function (PDF).

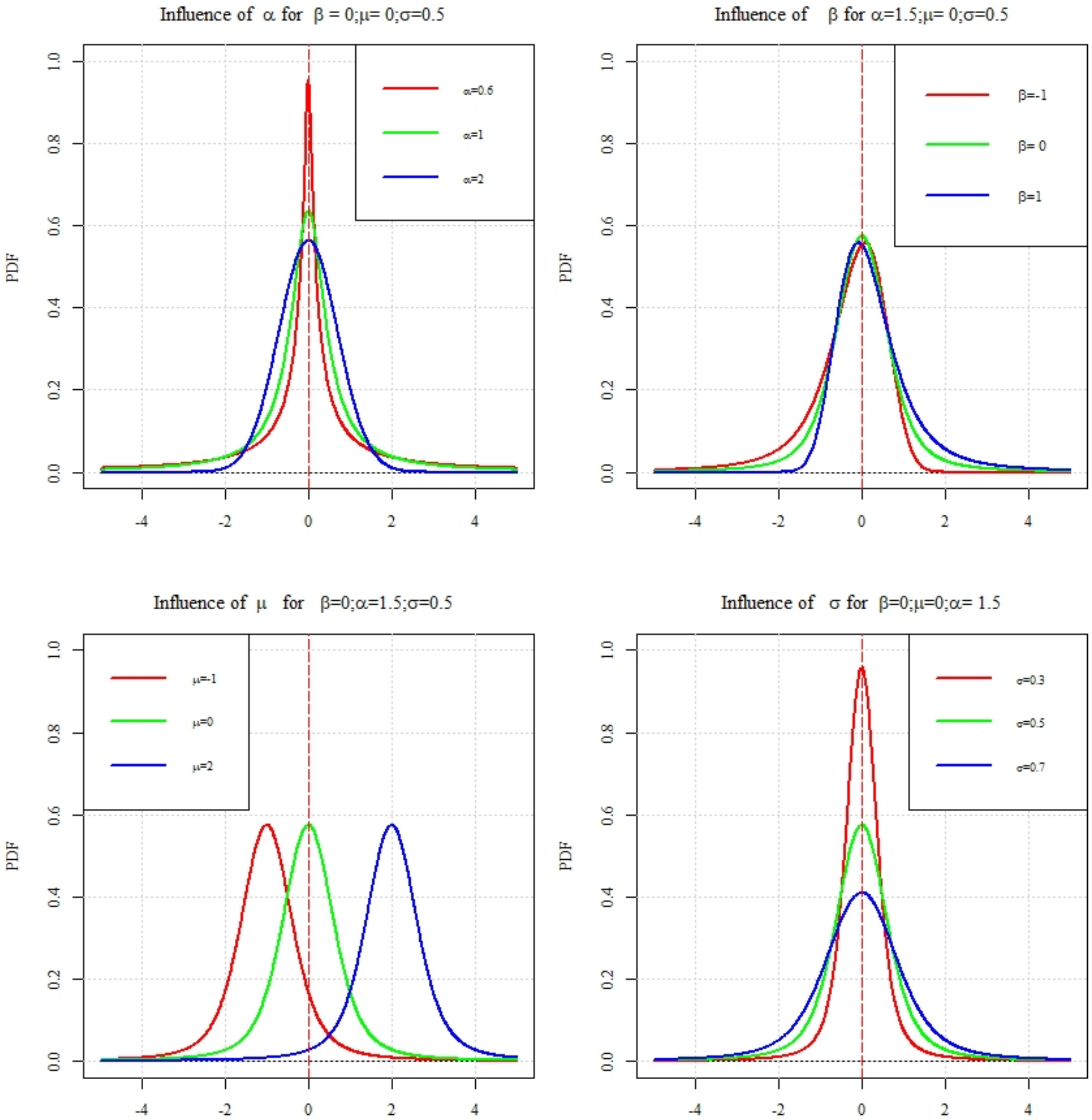

Figure 1. Influences of the parameters of the α-stable distribution on its PDF.

In the literature, several parameterizations coexist. We adopt Nolan's S_0 notation [Nolan (2020)], which guarantees the continuity of the parameters:

$$\phi(t) = \begin{cases} \exp\left\{-\sigma^{\alpha}|t|^{\alpha}\left[1 + i\beta sign(t)\tan\left(\frac{\pi\alpha}{2}\right)\left(|\sigma t|^{1-\alpha} - 1\right)\right] + i\mu_0 t\right\}, & \alpha \neq 1; \\ \exp\left\{-\sigma|t|\left[1 + i\beta\frac{2}{\pi} sign(t)\ln(\sigma|t|)\right] + i\mu_0 t\right\}, & \alpha = 1. \end{cases}$$

This parameterization is particularly well-suited to numerical calculations and parameter estimation.

Unlike the characteristic function, which has a closed form, the probability density function $f(x;\alpha,\beta,\sigma,\mu)$ and the distribution function $F(x;\alpha,\beta,\sigma,\mu)$ do not admit a simple analytical expression, except in special cases:

- **Normal distribution**: $\alpha = 2, \beta = 0$: $f(x; 2, 0, \sigma, \mu) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.

- **Cauchy distribution**: $\alpha = 1, \beta = 0$: $f(x; 1, 0, \sigma, \mu) = \frac{1}{\pi\sigma\left[1+\left(\frac{x-\mu}{\sigma}\right)^2\right]}$.

- **Lévy distribution**: $\alpha = \frac{1}{2}, \beta = 1$: $f(x; \frac{1}{2}, 1, \sigma, \mu) = \sqrt{\frac{\sigma}{2\pi}} \frac{1}{(x-\mu)^{3/2}} \exp\left(-\frac{\sigma}{2(x-\mu)}\right), x > \mu$.

For other parameter values, the density must be calculated numerically using the inverse Fourier transform: $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt$ or via Zolotarev's integral (Zolotarev (1986)):

$$f(x; \alpha, \beta) = \frac{\alpha}{\pi|\alpha - 1|}(x - \zeta) \int_{-\theta_0}^{\pi/2} V(\theta; \alpha, \beta) \exp\left(-(x - \zeta)^{\alpha/(\alpha-1)} V(\theta; \alpha, \beta)\right) d\theta;$$

where $\zeta = \beta \tan\left(\frac{\pi\alpha}{2}\right)$ and $V(\theta; \alpha, \beta)$ is an auxiliary function.

3. Methodology and main results

This section describes in detail the methodological framework adopted to conduct the comparative evaluation of the three methods for estimating the parameters of α-stable distributions. We first present the algorithms and theoretical principles underlying the methods of McCulloch (1986), Koutrouvelis (1980) and maximum Likelihood Estimation (MLE) (Nolan (2001)). There are other methods for estimating the parameters of an α-stable distribution, such as those proposed by Coulibaly et al. (2024) and Coulibaly et al. (2022). Next, we define the performance measures used to objectively compare these estimators: bias, mean squared error (MSE), mean absolute error (MAE), and execution time. Finally, we detail the specific characteristics of the data used—the gold, silver, and uranium markets—as well as the empirical approach followed, from data collection to comparative analysis.

3.1. Description of estimation methods

3.1.1. McCulloch's method

The McCulloch method, proposed by McCulloch in 1986, is a quantile-based estimator that offers an interesting compromise between speed and robustness. The algorithm proceeds in nine steps (McCulloch (1986)):

Algorithm 1 McCulloch estimator for α-stable distributions.

Input: Sample $\{x_1, x_2, \ldots, x_n\}$
Output: Estimated parameters $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\mu})$

1: Calculate empirical quantiles $\hat{x}_p$ for $p \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$
2: Calculate ratios $\hat{\Phi}_1 = \frac{\hat{x}_{0.95} - \hat{x}_{0.05}}{\hat{x}_{0.75} - \hat{x}_{0.25}}$
3: Calculate $\hat{\Phi}_2 = \frac{\hat{x}_{0.95} + \hat{x}_{0.05} - 2\hat{x}_{0.5}}{\hat{x}_{0.95} - \hat{x}_{0.05}}$
4: Use the tables of McCulloch (1986) to obtain $\hat{\alpha}$ and $\hat{\beta}$ based on $\hat{\Phi}_1$ and $\hat{\Phi}_2$
5: Calculate $\hat{\Phi}_3 = \frac{\hat{x}_{0.75} - \hat{x}_{0.25}}{\sigma}$ (Use the tables)
6: $\hat{\sigma} = \frac{\hat{x}_{0.75} - \hat{x}_{0.25}}{\hat{\Phi}_3}$
7: Calculate $\hat{\Phi}_4 = \frac{\mu - \hat{x}_{0.5}}{\sigma} + \beta \tan\left(\frac{\pi\alpha}{2}\right)$ (Use the tables)
8: $\hat{\mu} = \hat{x}_{0.5} + \hat{\sigma}\left(\hat{\Phi}_4 - \hat{\beta}\tan\left(\frac{\pi\hat{\alpha}}{2}\right)\right)$
9: **return** $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\mu})$

3.1.2. Koutrouvelis method (1980)

The Koutrouvelis method (Koutrouvelis (1980)) uses a regression approach based on the characteristic function. The estimator is obtained by minimizing the distance between the empirical and theoretical characteristic functions.

Algorithm 2 Koutrouvelis estimator for α-stable distributions.

Input: Sample $\{x_1, x_2, \ldots, x_n\}$, regularization parameter λ
Output: Estimated parameters $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\mu})$

1: Standardize data: $y_i = \frac{x_i - \mu_0}{\sigma_0}$ (with μ_0, σ_0 initial)
2: Calculate the empirical characteristic function $\phi_n(t) = \dfrac{1}{n}\sum_{j=1}^{n} e^{ity_j}$
3: For a set of points $t_1, t_2, \ldots, t_m$, perform regression:
4: $\log\left(-\text{łog}|\phi_n(t_j)|^2\right) = \log(2\sigma^{\alpha}) + \alpha \log|t_j| + \epsilon_j$
5: Obtain $\hat{\alpha}^{(1)}$ and $\hat{\sigma}^{(1)}$ by least squares
6: Perform the second regression:
7: $\arg(\phi_n(u_k)) = \mu u_k + \beta\hat{\sigma}^{(1)}|u_k|^{\hat{\alpha}^{(1)}} \operatorname{sign}(u_k)\omega(u_k, \hat{\alpha}^{(1)}) + \nu_k$
8: Obtain $\hat{\beta}^{(1)}$ and $\hat{\mu}^{(1)}$
9: Repeat steps 3, 7 until convergence
10: **return** Converged parameters $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\mu})$

3.1.3. Maximum likelihood estimator (MLE)

The maximum likelihood estimator maximizes the log-likelihood (Nolan (2001)):

$$\hat{\theta}_{\text{MLE}} = \arg\max_{\theta\in\Theta} \sum_{i=1}^{n} \log f(x_i; \theta);$$

where $f(x; \theta)$ is the density of the numerically calculated α-stable distribution.

To compare the three methods, we use the following measures calculated on N Monte Carlo replications:
• **Bias:** The bias of an estimator $\hat{\theta}$ of a parameter θ is defined by: $\text{Bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$.

An estimator is said to be unbiased if $\mathbb{E}[\hat{\theta}] = \theta$ for all values of θ. In the context of α-stable distributions, most estimators are biased for finite samples, but converge asymptotically.

•**Mean squared error (MSE) and root mean squared error (RMSE):** The mean squared error combines bias and variance: $\text{MSE}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2] = \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2$.
The root mean squared error (RMSE) is expressed in the same units as the parameter:
$\text{RMSE}(\hat{\theta}) = \sqrt{\text{MSE}(\hat{\theta})}$.
• **Mean absolute error (MAE)** is less sensitive to extreme values than the MSE: $\text{MAE}(\hat{\theta}) = \mathbb{E}[|\hat{\theta} - \theta|]$.
• **Execution time:** Computational time is an important practical consideration, especially for real-time applications. We measure the execution time in seconds for each method.
• **Quantile-quantile plots (QQ plots):** QQ plots allow for a visual assessment of the fit by comparing empirical quantiles to theoretical quantiles.
For an ordered sample $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and probabilities $p_i = \frac{i-0.5}{n}$, the QQ plot represents the points:

$$\left(F^{-1}(p_i; \hat{\theta}), x_{(i)}\right), \quad i = 1, \ldots, n;$$

where F^{-1} is the quantile function of the α-stable distribution with estimated parameters $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\mu})$. Table 1 provides a summary of all the statistical criteria used to measure and compare the performance of the various estimation methods examined.

Table 1. Performance measures for estimators.

Measure	Mathematical definition
Bias	$\text{Bias}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta_0)$
MSE	$\text{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta_0)^2$
RMSE	$\text{RMSE}(\hat{\theta}) = \sqrt{\text{MSE}(\hat{\theta})}$
MAE	$\text{MAE}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\lvert\hat{\theta}_i - \theta_0\rvert$
Execution time	$T(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N} t_i$ (in seconds)
Success rate	Proportion of samples where the estimate converged

4. Application results and discussion

This section presents and analyzes the results of the comparative evaluation of the three estimation methods (McCulloch, Koutrouvelis, MLE) applied to the returns on gold; silver and uranium over the period from January 2007 to December 2025, i.e; a sample size of 4777 observations for each of the financial assets. The analysis is based on descriptive statistics, performance measures from Monte Carlo simulations, and diagnostic visualizations. We performed our analysis in the context of our investigation using the statistical program R version 4.5.2 (R Foundation for Statistical Computing, Vienna, Austria). For cross-validation and result display, respectively, we also used the extra packages caret Kharrat et al. (2016) and ggplot2. In terms of the computer's specifications, we conducted our assessments on a machine with an Intel Core i7 processor and 16 GB of RAM. The size of the dataset and the complexity of the models affected the execution durations, but overall, the computer's processing capability allowed our analysis to be completed quickly. The actual dataset utilized in this work is freely accessible online. The author can provide the R codes for the simulation study upon reasonable request. Figures 2, 3, 4, 5, 6, 7, and 8 show the historical trends in gold, silver, and uranium prices over the period from January 2007 to December 2025. A visual analysis reveals distinct dynamics across commodities, reflecting heterogeneous market behavior. The price of gold shows a strong overall upward trend throughout the period. After an initial growth phase between 2007 and 2012, followed by a slight correction between 2012 and 2018, prices began rising again in 2019, reaching record levels in 2025. This trend confirms gold's role as a safe-haven asset during periods of economic and geopolitical uncertainty. Silver follows a trajectory similar to that of gold, but is characterized by greater volatility. The fluctuations observed between 2010 and 2013 reflect increased sensitivity to economic cycles and industrial demand. Starting in 2024, there is a significant acceleration in price growth, leading to a historic peak at the end of the period.

Figure 2. Evolution of raw material prices between 2007-01-01 and 2025-12-01.

In contrast, the uranium market was marked by a sharp decline in prices between 2011 and 2018, followed by a long period of stagnation. This trend can be linked to structural adjustments in the nuclear sector following the 2011 Fukushima accident. Starting in 2021, prices have been gradually recovering, supported by renewed global interest in nuclear energy amid the energy transition and concerns about energy security. In general, the three series exhibit non-stationary behavior characterized by long-term trends, structural breaks, and periods of varying volatility. These characteristics suggest the presence of complex dynamics requiring the use of econometric models capable of accounting for regime shifts, heteroscedasticity, and time dependencies in order to effectively model and forecast changes in commodity prices.

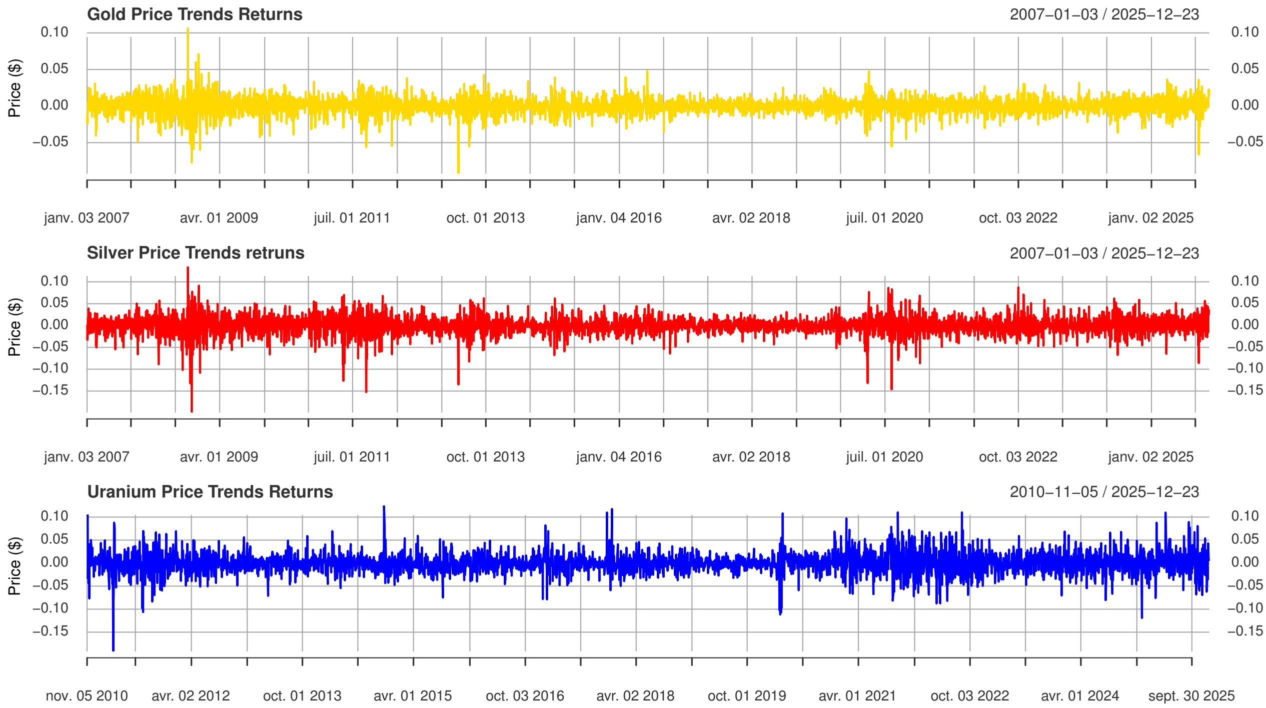

Figure 3. Commodity price returns between 2007-01-01 and 2025-12-01.

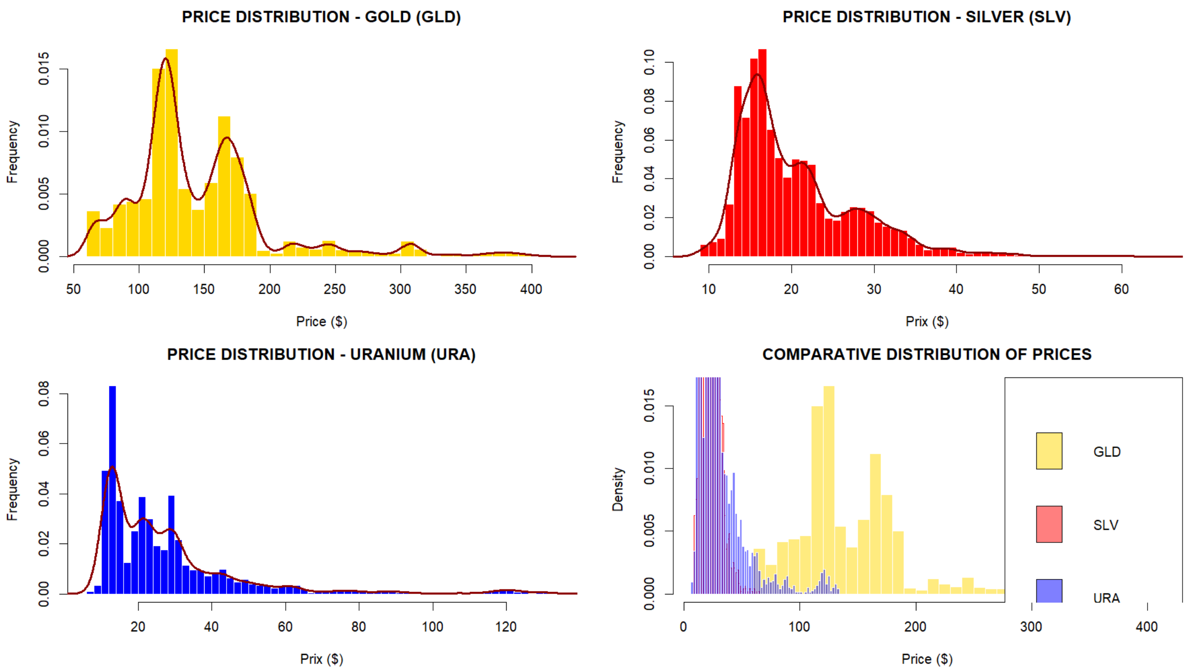

Figure 4. Histogram of relative performance of commodity prices between 2007-01-01 and 2025-12-01.

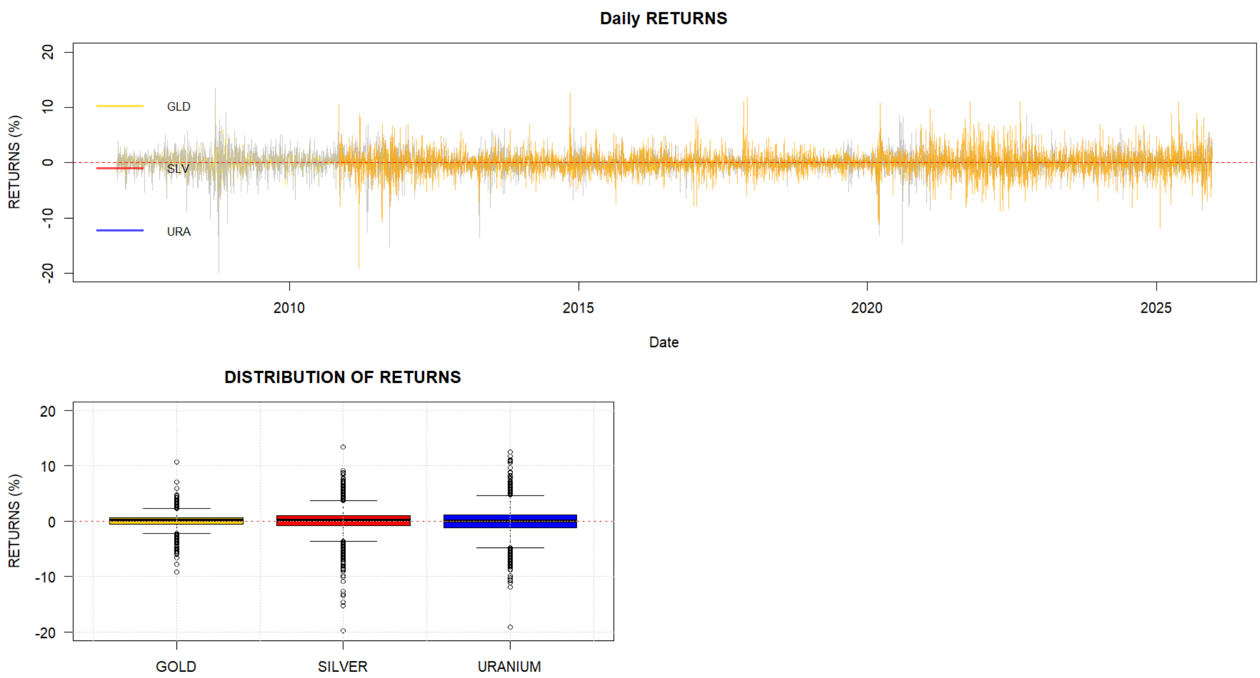

Figure 5. Daily returns and the distribution of returns of commodity prices between 2007-01-01 and 2025-12-01.

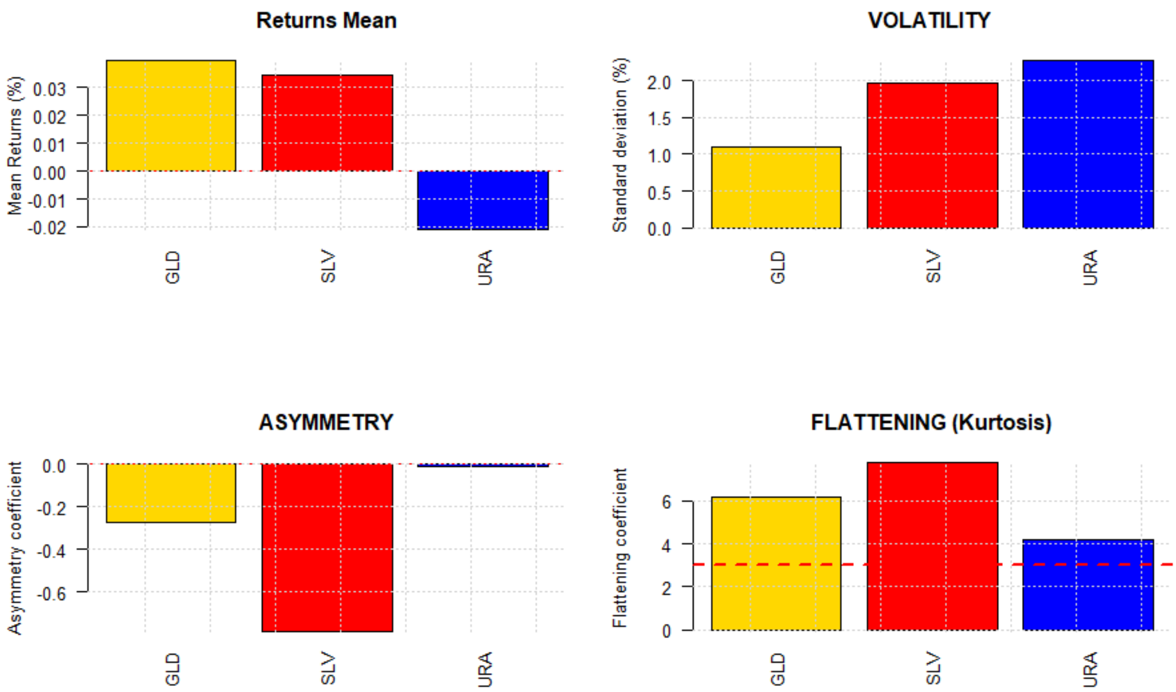

Figure 6. Descriptive statistics of returns.

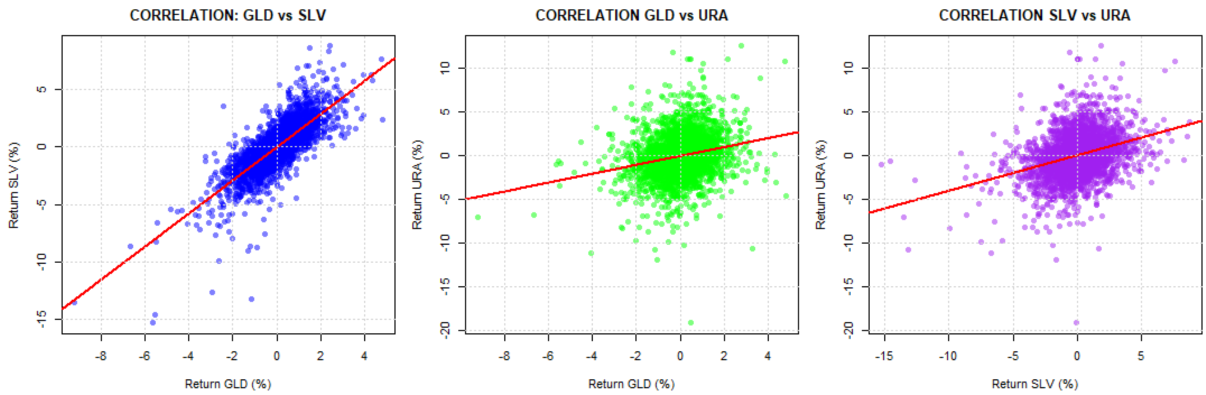

Figure 7. Correlation between assets.

Figure 8. Cumulative performance, total returns, volatility; and Sharpe ratio of assets.

4.1. Comparative performance of methods

Table 2 presents the parameter estimates of the α-stable distribution obtained using the McCulloch, Koutrouvelis, and maximum likelihood estimation (MLE) methods for gold, silver, and uranium price series. The results reveal noticeable differences across estimation procedures, both in terms of parameter values and computational efficiency. For the gold series, the McCulloch and MLE methods provide nearly identical estimates of the stability parameter ($\alpha = 1.4870$), the skewness parameter ($\beta = 0.0250$), and the scale and location parameters. In contrast, the Koutrouvelis method yields a substantially higher estimate of α (1.8322) and a negative skewness coefficient ($\beta = -0.1722$). Nevertheless, all estimated values of α remain below 2, indicating the presence of heavy tails and confirming the inadequacy of the Gaussian assumption for modeling gold price fluctuations. For the silver series, the estimated stability parameter ranges from 1.6840 to 1.8369 across methods. While McCulloch and MLE indicate a moderate positive skewness ($\beta = 0.2880$), the Koutrouvelis method suggests a negative asymmetry ($\beta = -0.3849$). These discrepancies highlight the sensitivity of the skewness parameter to the estimation technique employed. The uranium series exhibits the largest differences among methods. McCulloch and Koutrouvelis estimate a strong negative skewness ($\beta \approx -0.95$), whereas MLE provides a slightly less pronounced value ($\beta = -0.8507$). Moreover, the MLE estimate of the stability parameter ($\alpha = 1.9377$) is considerably closer to the Gaussian benchmark value of 2 than the estimates obtained by the other methods. However, since α remains below 2, the uranium price distribution still exhibits heavy-tailed behavior.

Table 2. Estimation of the parameters of the stable distribution.

		Gold					
Method	α	β	σ	μ	**AIC**	**BIC**	**Time (seconds)**
McCulloch	1.4870	0.0250	0.0068	0.0020	-	-	0.07
Koutrouvelis	1.8322	−0.1722	0.0075	0.0025	-	-	0.144
MLE	1.4870	0.0250	0.0071	0.0023	−1457.19	−1443.20	105.638
		Silver					
Method	α	β	σ	μ	**AIC**	**BIC**	**Time (seconds)**
McCulloch	1.684	0.288	0.0109	0.0028	-	-	0.01
Koutrouvelis	1.8369	−0.3849	0.0113	0.0018	-	-	0.02
MLE	1.6840	0.2880	0.0110	0.0032	−1457.19	−1443.20	336.79
		URANIUM					
Method	α	β	σ	μ	**AIC**	**BIC**	**Time (seconds)**
McCulloch	1.855	−0.95	0.0183	0.0053	-	-	0.01
Koutrouvelis	1.8313	−0.0045	0.0176	0.0032	-	-	0.01
MLE	1.9377	−0.8507	0.0192	0.0032	−1037.01	−1023.02	293.35

Regarding model assessment, only the MLE method provides likelihood-based criteria such as Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The reported values indicate an adequate fit of the α-stable distribution to the commodity price series. However, this increased statistical efficiency comes at a substantial computational cost. While the McCulloch and Koutrouvelis methods require only a few hundredths of a second, the MLE procedure takes more than 100 seconds for gold and nearly 300 seconds for silver and uranium. Overall, the results confirm that the distributions of gold, silver, and uranium prices deviate significantly from normality and exhibit heavy-tailed characteristics. The McCulloch and Koutrouvelis methods offer attractive computational efficiency, whereas the MLE approach remains the most comprehensive framework due to its likelihood-based inference and goodness-of-fit assessment capabilities.

Figures 9, 10, 11 and; the QQ plots allow for granular analysis and visual validation. Gold (GLD): Its relative stability allows all three methods to provide a visually acceptable fit (Figure 9). Koutrouvelis' method excels here, capturing the slight asymmetry well at a moderate computational cost.

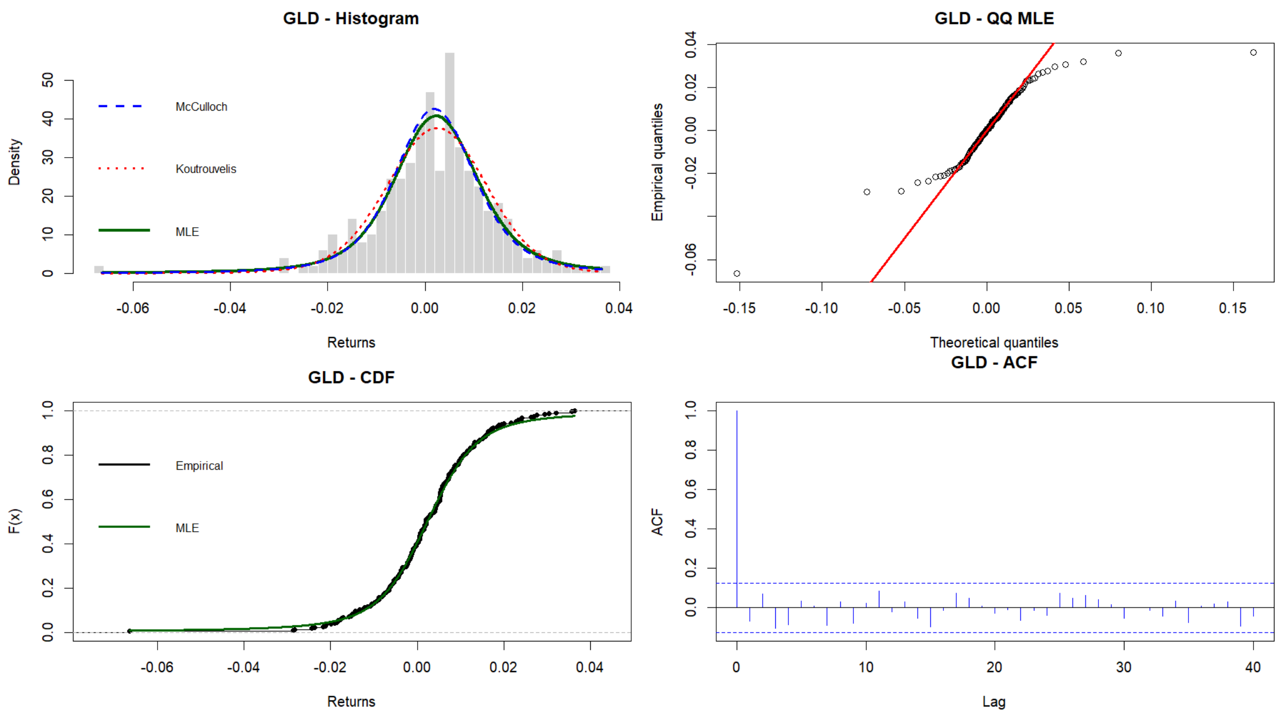

Figure 9. Gold: α-stable adjustment using estimation methods.

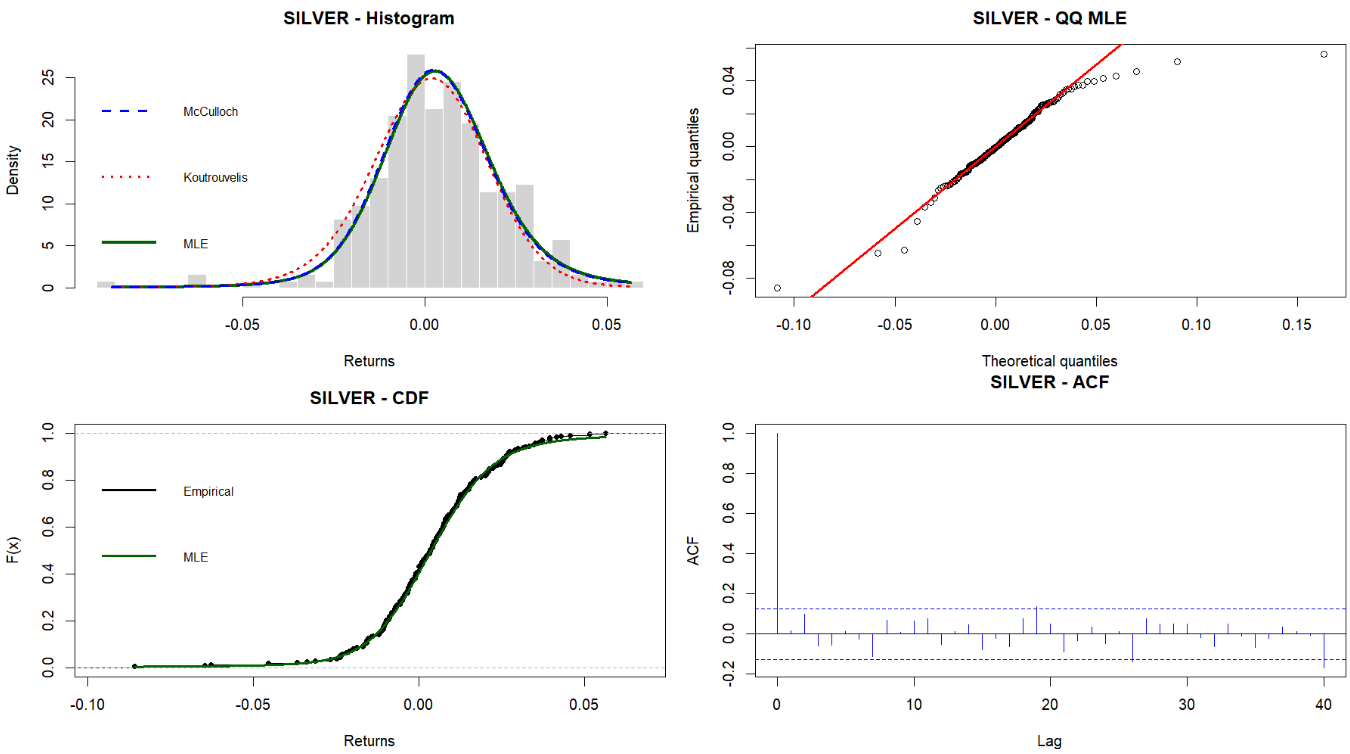

Figure 10. Silver: α-stable adjustment using estimation methods.

Silver (SLV): Its high volatility and negative asymmetry constitute a more demanding test (Figure 10). McCulloch's method shows its limitations here, with a poorer fit to the tails. The MLE and, to a lesser extent, Koutrouvelis, are able to model these extreme characteristics well. Uranium (URA): As the most volatile asset with complex dynamics, it is the most difficult to model, see Figure 11. The discrepancy between the fits of the different methods is more noticeable. Validation by QQ plots (Figures 9, 10; and 11): These graphs provide crucial validation. The alignment of points on the bisector is average for McCulloch, with notable deviations at the extremes (tails), good for Koutrouvelis, showing a solid fit across the entire distribution, and excellent for the MLE, where the points adhere closely to the line, attesting to the superior quality of the fit, particularly in the critical regions of the tails.

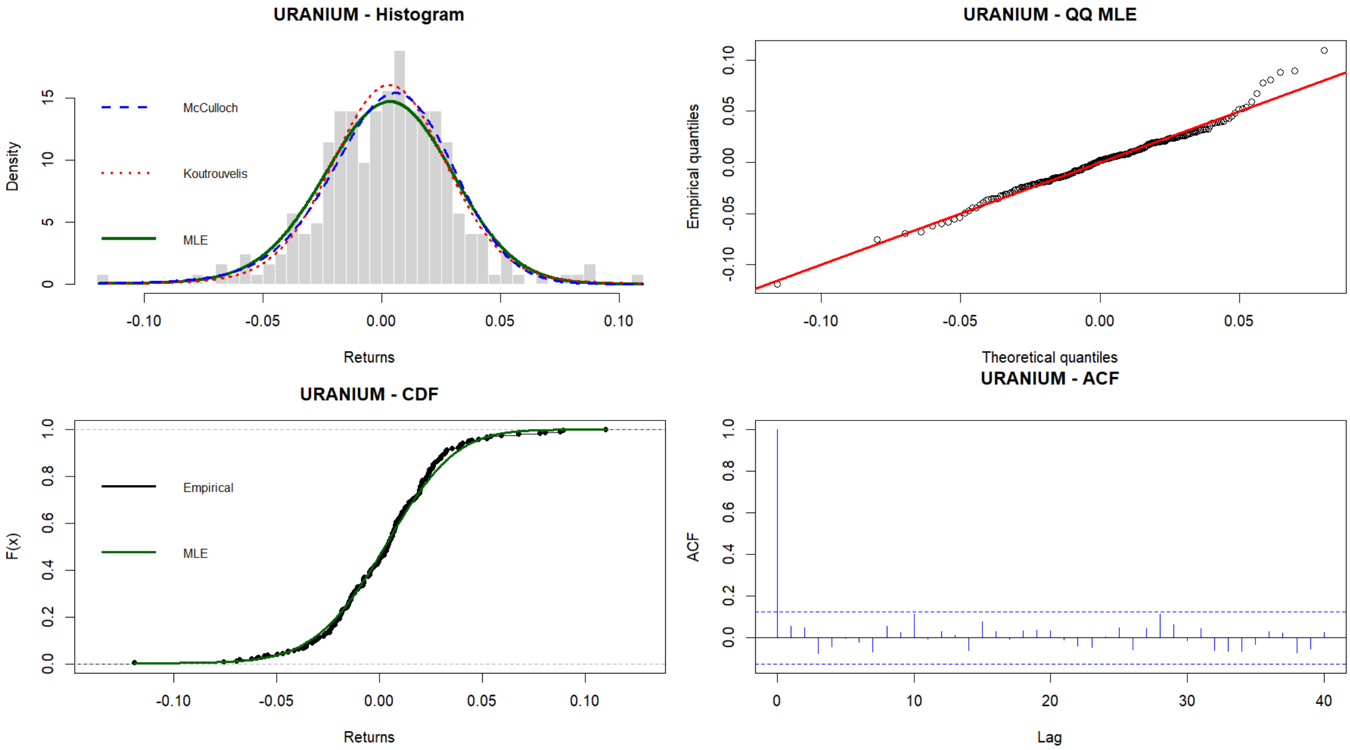

Figure 11. Uranium: α-stable adjustment using estimation methods.

Tables 3, 4, and 5 summarize the results of a Monte Carlo simulation (4,500 replications) designed to compare the effectiveness of three estimation methods (McCulloch, Koutrouvelis, and maximum likelihood estimation-MLE) for the four parameters $(\alpha, \beta, \sigma, \mu)$ of an α-stable distribution applied to gold, silver; and uranium price data. Analysis of the performance metrics (Bias, MAE, MSE, RMSE, Std.Dev) highlights the following points (Table 3). The Koutrouvelis and MLE methods significantly outperform McCulloch's method. Koutrouvelis offers the best accuracy for α with a very low bias (0.0131) and a minimum MSE (0.0083). For the asymmetry parameter β, the MLE method exhibits the lowest bias (0.0581), although its variance (Std. Dev. at 0.6224) is slightly higher than that of its competitors. Estimation of the scale (σ) and position (μ) parameters: The Koutrouvelis and MLE methods exhibit near-perfect and equivalent performance. Their bias, MSE, and RMSE are extremely close to zero (on the order of 10^{-4}), indicating optimal convergence and robustness for modeling the dispersion and central trend of gold returns.

Table 3. Monte Carlo simulation for performance-metric calculation with 4500 replications.

		Gold				
Parameter	**Bias**	**MAE**	**MSE**	**RMSE**	**Std.Dev**	**Method**
α	−0.3586	0.3835	0.3937	0.6275	0.5427	McCulloch
β	0.1309	0.3095	0.1591	0.3989	0.3971	McCulloch
σ	0.1981	0.1988	0.1969	0.4437	0.4185	McCulloch
μ	−0.0007	0.0010	0.0000	0.0013	0.0011	McCulloch
α	0.0131	0.0726	0.0083	0.0913	0.0952	Koutrouvelis
β	−0.1824	0.3233	0.1950	0.4416	0.4239	Koutrouvelis
σ	−0.0001	0.0003	0.0000	0.0004	0.0004	Koutrouvelis
μ	−0.0007	0.0011	0.0000	0.0013	0.0012	Koutrouvelis
α	0.0180	0.0579	0.0061	0.0780	0.0800	MLE
β	0.0581	0.5004	0.3521	0.5933	0.6224	MLE
σ	0.0000	0.0002	0.0000	0.0003	0.0003	MLE
μ	−0.0004	0.0005	0.0000	0.0006	0.0005	MLE

Table 4. Monte Carlo simulation for performance-metric calculation with 4500 replications.

		Silver				
Parameter	**Bias**	**MAE**	**MSE**	**RMSE**	**Std.Dev**	**Method**
α	−0.5536	0.5536	0.5887	0.7673	0.5600	McCulloch
β	−0.6400	0.6400	0.4633	0.6806	0.2443	McCulloch
σ	0.2959	0.2974	0.2931	0.5414	0.4779	McCulloch
μ	−0.0006	0.0016	0.0000	0.0021	0.0021	McCulloch
α	0.0250	0.0722	0.0072	0.0850	0.0857	Koutrouvelis
β	−0.6313	0.6941	0.8028	0.8960	0.6702	Koutrouvelis
σ	0.0045	0.0054	0.0001	0.0115	0.0112	Koutrouvelis
μ	0.0009	0.0026	0.0000	0.0037	0.0037	Koutrouvelis
α	0.0281	0.0706	0.0068	0.0825	0.0818	MLE
β	−0.5320	0.6817	0.7698	0.8774	0.7354	MLE
σ	−0.0001	0.0006	0.0000	0.0007	0.0007	MLE
μ	0.0004	0.0010	0.0000	0.0011	0.0011	MLE

Table 5. Monte Carlo simulation for performance-metric calculation with 4500 replications.

		Uranium				
Parameter	**Bias**	**MAE**	**MSE**	**RMSE**	**Std.Dev**	**Method**
α	−0.4243	0.4501	0.4851	0.6965	0.5822	McCulloch
β	0.1272	0.3600	0.2385	0.4883	0.4969	McCulloch
σ	0.2948	0.2955	0.2897	0.5382	0.4747	McCulloch
μ	−0.0011	0.0024	0.0000	0.0026	0.0025	McCulloch
α	0.0052	0.1517	0.0290	0.1704	0.1795	Koutrouvelis
β	−0.2341	0.4940	0.3632	0.6026	0.5853	Koutrouvelis
σ	0.0015	0.0023	0.0000	0.0036	0.0035	Koutrouvelis
μ	−0.0009	0.0023	0.0000	0.0026	0.0026	Koutrouvelis
α	−0.2274	0.2443	0.2547	0.5047	0.4749	MLE
β	−0.0720	0.3927	0.2328	0.4825	0.5029	MLE
σ	0.0001	0.0013	0.0000	0.0019	0.0020	MLE
μ	0.0029	0.0043	0.0001	0.0095	0.0096	MLE

5. Conclusions

Empirical results validate the existence of a fundamental trade-off between statistical efficiency and computational feasibility. No single method is universally "best". The choice must be guided by the application context: McCulloch for speed, Koutrouvelis for balance, and MLE for optimal accuracy. This performance hierarchy is consistent across the three commodity markets studied, although the relative advantage of MLE increases with the volatility and complexity of the series (silver, uranium). Based on these results, we recommend that for assets exhibiting high asymmetry and heavy tails-such as uranium-the MLE method is preferred, despite its computational complexity, as it better captures extremes. Conversely, the McCulloch method can be used as a quick approximation during the exploratory phase, but it must be validated by a more robust approach before being used in an operational setting. This study contributes to the literature on estimating α-stable distributions by providing a rigorous, multidimensional comparative evaluation of the three main methods, with a specific application to commodity markets. Our results highlight that no single method is universally superior, but that the optimal choice depends on a balance between statistical accuracy, computational constraints, and application objectives.

As commodity markets continue to evolve with the emergence of new technologies (green energy, digital) and new sources of volatility (climate change, geopolitical transitions), accurately modeling their return distributions will remain a crucial challenge. Methods for estimating α-stable distributions, with their unique ability to capture heavy tails and asymmetry, are essential tools in this methodological arsenal.

As Benoît Mandelbrot, a pioneer in the study of fat-tailed distributions in finance, pointed out, “financial asset prices follow laws that are more complex than simple normal distributio”. Our study confirms this intuition while providing practical recommendations for navigating this complexity in the specific context of the gold, silver; and uranium markets.

The path toward more robust and accurate models remains open, requiring both theoretical advances and computational innovations. This is the direction taken by the research prospects identified, with the ultimate goal of better understanding, measuring; and managing the risks inherent in commodity markets.

Several limitations of our study open up promising avenues for further research:

1. Sample size: Our study focuses on moderate to large samples. An extension to very small samples ($n < 100$) would be relevant for studies on emerging markets or limited historical periods.

2. Temporal dependence: The methods studied assume that observations are independent, an assumption that is often violated in financial series. The integration of Generalized autoRegressive conditional heteroskedasticity stable models (GARCH-stable models) or Lévy processes could enrich the analysis.

3. Incomplete comparison: Our study does not include certain recent methods such as continuous generalized method of moments (CGMM) estimation or Bayesian approaches.

Author contributions

Bakary D. Coulibaly, Marcel Sihintoé Badiane, and Aguemon Wiwegnon Uriel-Longin: Conceptualization, Methodology, Software, Validation, Data Curation, Writing-Original.

Chaibi Ghizlane, Siba Kalivogui, and Nouhan Traoré: Writing-Review and Editing, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors want to make it clear that no Artificial Intelligence (AI) technologies or methodologies were used in the conception, research, drafting, or any other stage of this article's creation. The writers have painstakingly created all of the content without using automation or AI support.

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Conflict of interest

The authors declare that they have no conflict of interest. No funding was obtained for this study.

Supplementary materials

The real dataset used in this work is available online and free of charge. The R codes for the simulation study are available from the corresponding author upon reasonable request.

References

Baur DG, McDermott TK (2010) Is gold a safe haven? International evidence. *J Bank Financ* 34: 1886–1898. https://doi.org/10.1016/j.jbankfin.2010.03.008

Chambers JM, Mallows CL, Stuck BW (1976) A method for simulating stable random variables. *J Am Stat Assoc* 71: 340–344. https://doi.org/10.1080/01621459.1976.10480344

Coulibaly BD, Chaibi G, El Khomssi M (2022) Parameters stable distribution estimate. *Adv Appl Stat* 80: 1–21. https://doi.org/10.17654/0972361722064

Coulibaly BD, Chaibi G, El Khomssi M (2024) Parameter estimation of the alpha-stable distribution and applications to financial data. *Chil J Stat* 15: 60-80. https://doi.org/10.32372/chjs.15-01-04

Gnedenko BV, Kolmogorov AN (1954) *Limit distributions for sums of independent random variables*. Addison-Wesley, Cambridge, MA.

Kharrat T, Boshnakov GN (2016) Package 'StableEstim'. *R Package Version* 2: 88.

Koutrouvelis IA (1980) Regression-type estimation of the parameters of stable laws. *J Am Stat Assoc* 75: 918–928. https://doi.org/10.1080/01621459.1980.10477573

Mandelbrot B (1963) The variation of certain speculative prices. *J Bus* 36: 394–419. https://doi.org/10.1086/294632

McCulloch JH (1986) Simple consistent estimators of stable distribution parameters. *Commun Stat Simul Comput* 15: 1109–1136. https://doi.org/10.1080/03610918608812563

Nolan JP (2001) Maximum likelihood estimation and diagnostics for stable distributions. In: Barndorff-Nielsen OE, Mikosch T, Resnick SI (eds) *Lévy processes: Theory and applications*. Birkhäuser, Boston, 379–400. https://doi.org/10.1007/978-1-4612-0197-7_17

Nolan JP (2020) *Stable distributions: Models for heavy-tailed data*. Birkhäuser, Boston.

Press SJ (1972) Multivariate stable distributions. *J Multivar Anal* 2: 444–462. https://doi.org/10.1016/0047-259X(72)90038-3

Rachev ST, Mittnik S (2000) *Stable Paretian models in finance*. Wiley, New York.

Samorodnitsky G, Taqqu MS (1994) *Stable non-Gaussian random processes: Stochastic models with infinite variance*. Chapman & Hall, New York.

Weron R (2004) Computationally intensive value at risk calculations. In: Gentle JE, Härdle W, Mori Y (eds) *Handbook of computational statistics*. Springer, Berlin, 911–950. https://hdl.handle.net/10419/22205

Zolotarev VM (1986) *One-dimensional stable distributions*. American Mathematical Society, Providence, RI.

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