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Research article

Explicit solutions to the Sharma-Tasso-Olver equation

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Abstract: We present new exact traveling wave solutions of generalized Sharma-Tasso-Olver (STO) with variable coefficients using three different methods, namely the extended F-expansion, the new sub-equations, and generalized Kudryashov expansion. We obtain new solutions with the form of solitons, triangular and rational functions. Computational results indicate that these methods are very useful and easily applicable for solving diverse types of differential equations in nonlinear science.

Keywords: extended F-expansion; new sub-equations; generalized Kudryashov expansion method; Sharma-Tasso-Olver equation
Mathematics Subject Classification: 35A09, 35E05

1. Introduction

Research on exact solutions of nonlinear differential equations with variable coefficients has been a significant area for recent decades, see e.g. [1−25]. We consider nonlinear STO equation with variable coefficients [26, 27].

$$u_t + f(t)\left(uu_x + \frac{1}{3}u^3\right)_x + g(t)\,u_{xxx} = 0, \tag{1.1}$$

In which $f(t) \neq 0$, $g(t) \neq 0$ are functions of t. In the scientific literature there are a various number of effective methods for the exact solutions of nolinear PDEs. Among these methods, similarity reduction [1], Adomian decomposition [13], Backlund transformation [2], Painleve expansion [3], homogeneous balance [15], Jacobi elliptic function [5, 6], tanh function [16], F-expansion [17−20], variational iteration [9−12], homotopy analysis [14] and Exp-function [21−23]. Riemann-Hilbert method [28–31], Lie symmetry [32], Hirota bilinear [33], Darboux method [34], variable-coefficient fractional Y-expansion method [35], Riccati equation method [36], fractional riccati method [37], fractional dual-function method [38]. Noether symmetries [39], Kudryashov method [40,41], Simplest equation method [42].

In order to study the traveling wave propagation solution of STO [43,44], let us introduce:

$$\zeta = x + \frac{\omega}{\alpha}\int_0^t g(t')\,dt',\ u(x,t) = u(\zeta), \tag{1.2}$$

in which α is a parameter and ω is wave speed. By Eq (1.2), Eq (1.1) is written

$$\frac{\omega}{\alpha}u_\zeta + u_{\zeta\zeta\zeta} + 3\left(uu_\zeta + \frac{1}{3}u^3\right)_\zeta = 0, \tag{1.3}$$

in which $f(t)$and $g(t)$ satisfy $f(t) = 3g(t)$. Integrating Eq (1.3), we get

$$\frac{\omega}{\alpha}u + u_{\zeta\zeta} + 3\left(uu_\zeta + \frac{1}{3}u^3\right) = 0. \tag{1.4}$$

In this study we get solitary wave and the periodic wave solutions by using algebraic direct method, Sub-equations method and F-expansion method. In the next two sections, the new proposed methods are presented and different types of exact solutions of STO are written down. Section 4 is devoted to the conclusion.

2. Methodology

Let Z be a polynomial function of x, and t. Consider the nonlinear PDE

$$Z(u, u_x, u_t, u_{xx,...}) = 0. \tag{2.1}$$

Let

$$u(x,t) = u(\zeta), \zeta = k(x + \lambda t), \tag{2.2}$$

where k, λ are constants. Inserting Eq (2.2) into Eq (2.1), we get the ODE in terms of $u(\xi)$

$$\chi\left(u, ku', \lambda ku', k^2u'', ...\right) = 0. \tag{2.3}$$

2.1. Extended F-Expansion method

Let the solution be written as

$$u(\zeta) = a_0 + \sum_{i=-M}^{M} a_i \chi^i(\zeta), \tag{2.4}$$

in which a_0 and a_i are constants, and $M \neq 0$ is a natural number and $\chi(\zeta)$satisfies

$$\chi'(\zeta) = A + B\chi(\zeta) + C\chi^2(\zeta), \tag{2.5}$$

where, $\chi'(\zeta) = \frac{d\chi}{d\zeta}$ and $A,\ B,\ C$ are parameters.

In order to solve Eq (1.4) via F−expansion method, equating $u_{\xi\xi}$ with u^3 yields $M = 1$. Hence, Eq (2.4) reads

$$u(\zeta) = a_0 + a_1\chi(\zeta) + \frac{a_{-1}}{\chi(\zeta)}, \tag{2.6}$$

in which a_0, a_1 and $a_{-1}$are constants. Inserting Eq (2.6) into the reduced Eq (1.4) yields:

Case (1.1)**:** $a_{-1} = 0,\ a_1 = 1,\ a_0 = -1,\ \alpha = \alpha$ and $\omega = -\alpha$. Using the transformation (1.2), the corresponding solution in terms of the original coordinates is as follows

$$u_1(x,t) = -\frac{1}{2} + \frac{1}{4}\tanh\left(x - \int_0^t g(t')\,dt'\right) \tag{2.7}$$

where $g(t)$ is an arbitrary function.

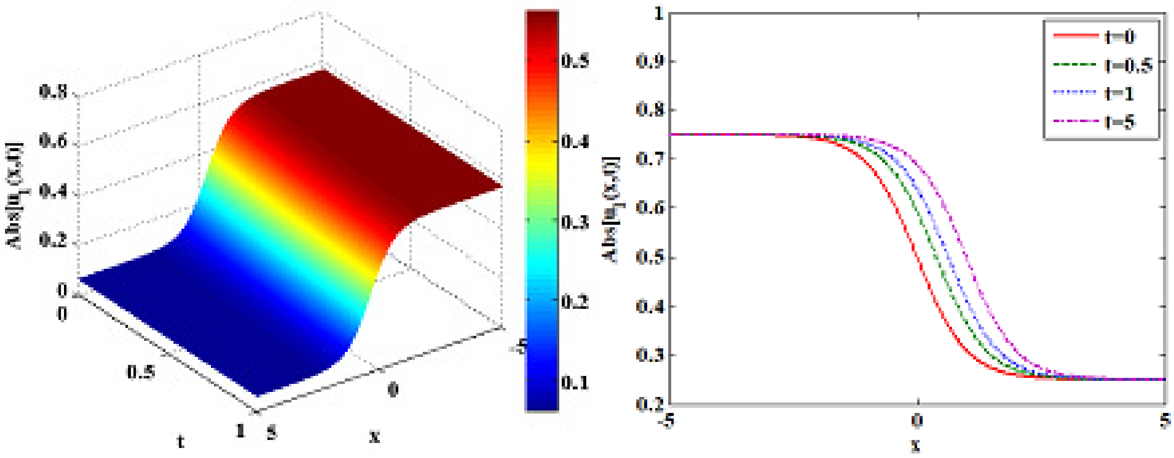

Figure 1. (a) Three-dimensional mesh of $Abs[u_1(x,t)]$ versus t and x, (b) variation $Abs[u_1(x,t)]$ with the normalized propagation position x for different values of the time.

Case (1.2)**:** $a_{-1} = 0,\ a_1 = -1,\ a_0 = \frac{1}{2}$ and $\omega = -\frac{\alpha}{4}$. Using the transformation (1.2), the corresponding solution in terms of the original coordinates is as follows

$$u_2(x,t) = \frac{1}{4}\coth\left(x - \frac{1}{4}\int_0^t g(t')\,dt'\right), \tag{2.8}$$

where $g(t)$ is an arbitrary function.

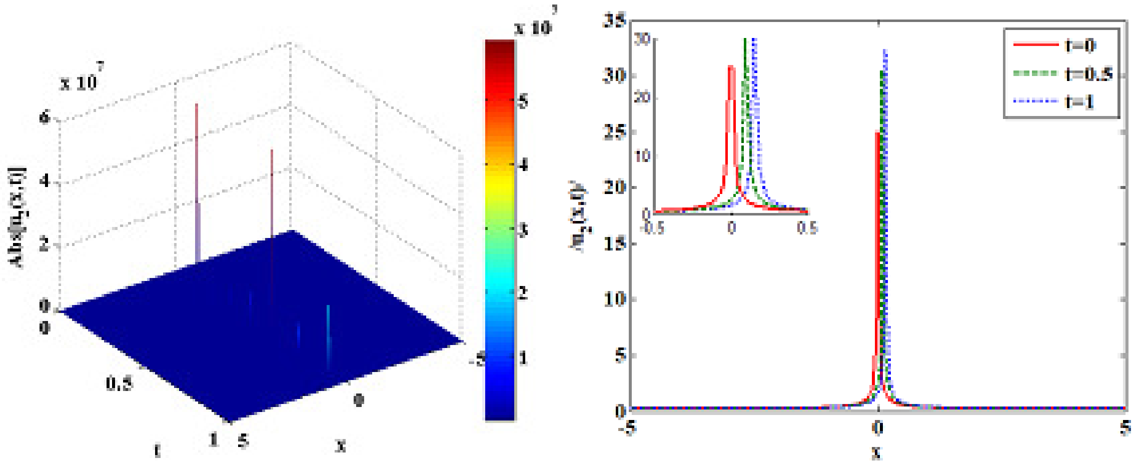

Figure 2. (a) Three-dimensional mesh of $Abs[u_2(x,t)]$ versus t and x, (b) variation $Abs[u_2(x,t)]$ with the normalized propagation position x for different values of the time.

Case (1.3)**:** $a_{-1} = \frac{1}{2}$, $a_1 = \frac{1}{2}$, $a_0 = -1$, $\alpha = \alpha$ and $\omega = -4\alpha$. Using the transformation (1.2), the corresponding solutions in terms of the original coordinates is

$$u_3(x,t) = -1 + \frac{1}{2\left[\coth\left(x-4\int_0^t g(t')dt'\right) \pm \csc h\left(x-4\int_0^t g(t')dt'\right)\right]} + \frac{1}{2}\left[\coth\left(x-4\int_0^t g(t')\,dt'\right) \pm \csc h\left(x-4\int_0^t g(t')\,dt'\right)\right], \tag{2.9}$$

$$u_4(x,t) = -1 + \frac{1}{2\left[\tanh\left(x-4\int_0^t g(t')dt'\right) \pm i\sec h\left(x-4\int_0^t g(t')dt'\right)\right]} + \frac{1}{2}\left[\tanh\left(x-4\int_0^t g(t')\,dt'\right) \pm i\sec h\left(x-4\int_0^t g(t')\,dt'\right)\right], \tag{2.10}$$

where $g(t)$ is an arbitrary function.

Case (1.4)**:** $a_{-1} = 1$, $a_1 = 1$, $a_0 = -2$, $\alpha = \alpha$ and $\omega = -16\alpha$. From the transformation (1.2), the corresponding solution in terms of the original coordinates is as follows

$$u_5(x,t) = -2 + \coth\left(x - 16\int_0^t g(t')\,dt'\right) + \tanh\left(x - 16\int_0^t g(t')\,dt'\right), \tag{2.11}$$

where $g(t)$ is an arbitrary function.

Case (2.1)**:** $a_{-1} = 1$, $a_1 = -1$, $a_0 = 2i$, $\alpha = \alpha$ and $\omega = 16\alpha$. Using the transformation (1.2), the corresponding solution in terms of the original coordinates is taken as

$$u_6(x,t) = 2i + \cot\left(x + 16\int_0^t g(t')\,dt'\right) - \tanh\left(x + 16\int_0^t g(t')\,dt'\right), \tag{2.12}$$

where $g(t)$ is an arbitrary function.

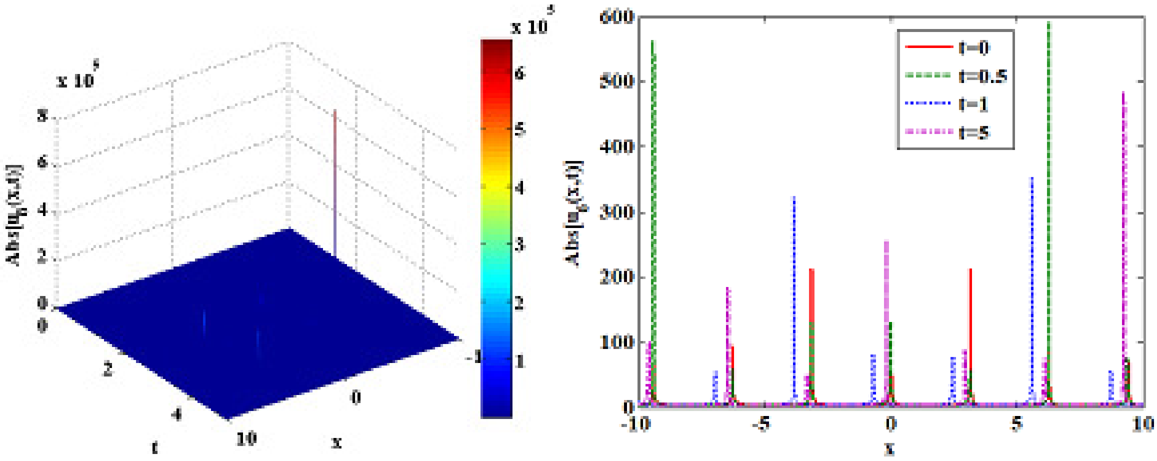

Figure 3. (a) Three-dimensional mesh of $Abs[u_6(x,t)]$ versus t and x, (b) variation $Abs[u_6(x,t)]$ with the normalized propagation position x for different values of the time.

Case (2.2)**:** $a_{-1} = A$, $a_1 = 0$, $a_0 = \frac{B}{2}$, $\alpha = \alpha$ and $\omega = -\frac{B^2\alpha}{4}$. By means of Eq (1.2), the corresponding solution in terms of the original coordinates gives

$$u_7(x,t) = \frac{B}{2} + \frac{AB}{\exp\left\{B\left[x - \frac{B^2}{4}\int_0^t g(t')\,dt'\right]\right\} - A}, \tag{2.13}$$

where $g(t)$ is an arbitrary function.

Case (2.3)**:** $a_{-1} = -1$, $a_1 = 1$, $a_0 = 2i$, $\alpha = \alpha$ and $\omega = 16\alpha$. Using the transformation (1.2), the corresponding solution in terms of the original coordinates admits to

$$u_8(x,t) = 2i - \frac{1}{\cot\left(x + 16\int_0^t g(t')\,dt'\right)} + \cot\left(x + 16\int_0^t g(t')\,dt'\right), \tag{2.14}$$

where $g(t)$ is an arbitrary function.

Case (2.4)**:** $a_{-1} = \frac{1}{2}$, $a_1 = -\frac{1}{2}$, $a_0 = \pm i$, $\alpha = \alpha$ and $\omega = 4\alpha$. Making use the transformation (1.2), the corresponding solutions in terms of the original coordinates yields

$$\begin{aligned} u_9(x,t) = \pm i &+ \frac{1}{2\left[\sec\left(x+4\int_0^t g(t')dt'\right)+\tan\left(x+4\int_0^t g(t')dt'\right)\right]} \\ &- \frac{1}{2}\left[\sec\left(x + 4\int_0^t g(t')\,dt'\right) + \tan\left(x + 4\int_0^t g(t')\,dt'\right)\right], \end{aligned} \tag{2.15}$$

$$\begin{aligned} u_{10}(x,t) = \pm i &+ \frac{1}{2\left[\csc\left(x+4\int_0^t g(t')dt'\right)-\cot\left(x+4\int_0^t g(t')dt'\right)\right]} \\ &- \frac{1}{2}\left[\csc\left(x + 4\int_0^t g(t')\,dt'\right) - \cot\left(x + 4\int_0^t g(t')\,dt'\right)\right], \end{aligned} \tag{2.16}$$

where $g(t)$ is an arbitrary function.

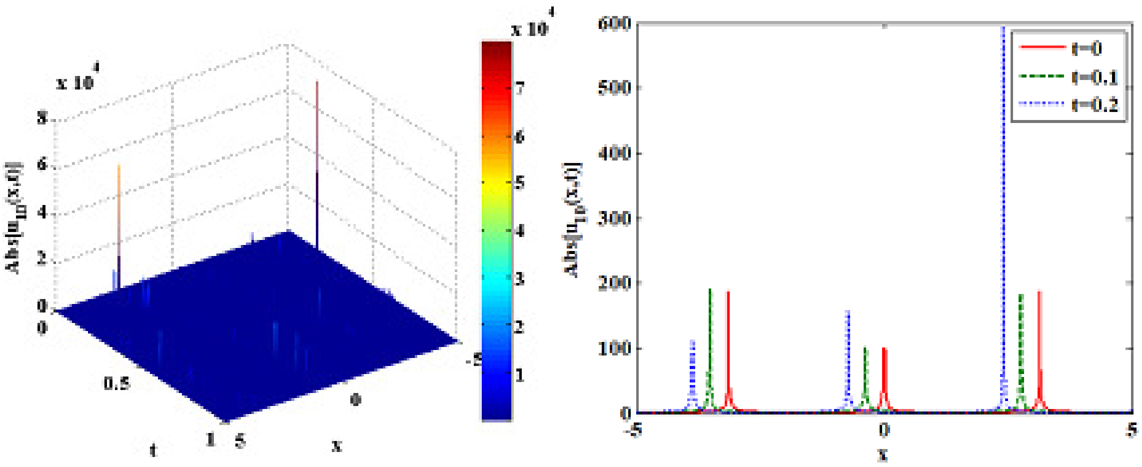

Figure 4. (a) Three-dimensional mesh of $Abs[u_{10}(x,t)]$ versus t and x, (b) variation $Abs[u_{10}(x,t)]$ with the normalized propagation position x for different values of the time.

2.2. *New sub-equation method*

In view this method (MAE) [24], affirms the general solution as the form as

$$u(\varsigma) = a_0 + \sum_{j=1}^{N} a_j A^{jf(\varsigma)} + \sum_{j=1}^{N} b_j A^{-jf(\varsigma)}, \tag{2.17}$$

The parametersa_j, $b_j$are arbitrary constants and $f(\varsigma)$satisfy the following auxiliary equation

$$f'(\varsigma) = \frac{\alpha + \beta A^{-f(\varsigma)} + \sigma A^{f(\varsigma)}}{\ln(A)}, \tag{2.18}$$

in which $\alpha, \beta, \sigma$are arbitrary constants and $A > 0$, $A \neq 1$.

To solve Eq (1.4), we employ Eq (2.17) to get solutions taking into consideration the homogeneous balance between u^3 and u''in Eq (1.4) that results N=1. Set N=1 in Eq (2.17), we get

$$u(\varsigma) = a_0 + a_1 A^{f(\varsigma)} + b_1 A^{-f(\varsigma)}, \tag{2.19}$$

According to (MAE) method, writing Eq (2.19) in Eq (1.4) with the help of Eq (2.18), we get
Case 1: $\left\{w = \left(4\alpha\sigma - \beta^2\right)\delta,\ \delta = \delta,\ a_0 = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma},\ a_1 = -\sigma,\ b_1 = 0\right\}$
Case 2: $\left\{w = -\frac{1}{4}\delta\beta^2 + \delta\alpha\sigma,\ \delta = \delta,\ a_0 = \frac{1}{2}\beta,\ \ a_1 = 0,\ b_1 = \alpha\right\}$
Case 3: $\left\{w = -\delta\beta^2 + 4\delta\alpha\sigma,\ \delta = \delta,\ a_0 = \beta,\ \ a_1 = 0,\ b_1 = 2\alpha\right\}$
Case 4: $\left\{w = \left(4\alpha\sigma - \beta^2\right)\delta,\ \delta = \delta,\ a_0 = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma},\ a_1 = 0,\ b_1 = \alpha\right\}$
Case 5: $\left\{w = -\delta\beta^2 + 4\delta\alpha\sigma,\ \delta = \delta,\ a_0 = 0,\ \ a_1 = -\sigma,\ b_1 = \alpha\right\}$

In view of case [1], exact solutions of Eq (1.1) are given a
when $\beta^2 - 4\alpha\sigma < 0$, and $\sigma \neq 0$,

$$u_1(\varsigma) = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} - \sigma\frac{-\beta + \sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\zeta\right)}{2\sigma},$$

$$u_2(\varsigma) = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} + \sigma\frac{\beta + \sqrt{4\alpha\sigma - \beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\zeta\right)}{2\sigma}. \tag{2.20}$$

If $\beta^2 - 4\alpha\sigma > 0$, and $\sigma \neq 0$, we have

$$u_3(\varsigma) = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} + \sigma\frac{\beta + \sqrt{\beta^2 - 4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\zeta\right)}{2\sigma},$$

$$u_4(\varsigma) = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} + \sigma\frac{\beta + \sqrt{\beta^2 - 4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\zeta\right)}{2\sigma}. \tag{2.21}$$

If $\beta^2 - 4\alpha\sigma = 0$, and $\sigma \neq 0$,

$$u_5(\varsigma) = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} + \sigma\frac{2 + \beta\zeta}{2\sigma\zeta}. \tag{2.22}$$

Similarly as before, according to case [4], new exact solution s of Eq (1.1) is:
As long as $\beta^2 - 4\alpha\sigma < 0$, and $\sigma \neq 0$, we have

$$u_6(\varsigma) = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} + \alpha\frac{2\sigma}{-\beta + \sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\zeta\right)},$$

$$u_7(\varsigma) = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} - \alpha\frac{2\sigma}{\beta + \sqrt{4\alpha\sigma - \beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\zeta\right)}. \tag{2.23}$$

if $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, admits to

$$u_8(\varsigma) = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} - \alpha\frac{2\sigma}{\beta + \sqrt{\beta^2 - 4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\zeta\right)},$$

$$u_9(\varsigma) = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} - \alpha\frac{2\sigma}{\beta + \sqrt{\beta^2 - 4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\zeta\right)}. \tag{2.24}$$

if $\beta^2 - 4\alpha\sigma = 0$ and $\sigma \neq 0$,

$$u_{10}(\varsigma) = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma} - \alpha\frac{2\sigma\zeta}{2 + \beta\zeta},$$

$$\zeta = x + \frac{\omega}{\alpha}\int_0^t g(t')\,dt'. \tag{2.25}$$

3. Generalized Kudryashov expansion method

In view this method [25], suppose that the solution of Eq (1.4) is written as:

$$u(\zeta) = \sum_{i=-N}^{N} a_i\psi^i(\zeta), \tag{3.1}$$

where a_i are constants to be calculated afterward and verifies:

$$\psi'(\varsigma) = \ln(A)\left[\alpha + \beta\psi(\varsigma) + \gamma\psi^2(\varsigma)\right], \tag{3.2}$$

where A, α, β and γ are constants.

Equating $u''(\varsigma)$ and $u^3(\varsigma)$, we get N = 1,thus Eq (3.1) leads to:

$$u(\varsigma) = a_0 + a_1\psi(\varsigma) + \frac{a_{-1}}{\psi(\varsigma)}, \tag{3.3}$$

Now, we have:
Case [1]: $a_0 = 0$, $a_1 = -\sigma\ln(A)$, $a_{-1} = \alpha\ln(A)$
Case [2]: $a_0 = \left(-\frac{\beta}{2} + \frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\right)\ln(A)$,$a_1 = -\sigma\ln(A)$, $a_{-1} = 0$

In view of case [1], new exact travelling wave solutions of Eq (1.1) are

$$u_i(\zeta) = -\sigma\ln(A)\,\psi_i(\zeta) + \frac{\alpha\ln(A)}{\psi_i(\zeta)}. \tag{3.4}$$

According to case [2], exact solutions of Eq (1.1) are:

$$u_i(\zeta) = \left(-\frac{\beta}{2} + \frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\right)\ln(A) - \sigma\ln(A)\,\psi_i(\zeta), \tag{3.5}$$

where $\psi_i(\varsigma)$ is:
Family 1. In case of $\Delta = \beta^2 - 4\alpha\sigma < 0$, $\sigma \neq 0$, $\psi_i(\varsigma)$ reads

$$\psi_1(\varsigma) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma}\tan_A\left(\frac{\sqrt{-\Delta}}{2}\varsigma\right), \tag{3.6}$$

$$\psi_2(\varsigma) = -\frac{\beta}{2\sigma} - \frac{\sqrt{-\Delta}}{2\sigma}\cot_A\left(\frac{\sqrt{-\Delta}}{2}\varsigma\right), \tag{3.7}$$

$$\psi_3(\varsigma) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{4\sigma}\tan_A\left(\frac{\sqrt{-\Delta}}{4}\varsigma\right) - \frac{\sqrt{-\Delta}}{4\sigma}\cot_A\left(\frac{\sqrt{-\Delta}}{4}\varsigma\right), \tag{3.8}$$

Family 2. In case of $\Delta = \beta^2 - 4\alpha\sigma > 0$, $\sigma \neq 0$, $\psi_i(\varsigma)$ reads

$$\psi_4(\varsigma) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma}\tanh_A\left(\frac{\sqrt{\Delta}}{2}\varsigma\right), \tag{3.9}$$

$$\psi_5(\varsigma) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma}\coth_A\left(\frac{\sqrt{\Delta}}{2}\varsigma\right), \tag{3.10}$$

$$\psi_6(\varsigma) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma}\coth_A\left(\sqrt{\Delta}\varsigma\right) \pm \frac{\sqrt{pq\Delta}}{2\sigma}\csc h_A\left(\sqrt{\Delta}\varsigma\right), \tag{3.11}$$

$$\psi_7(\varsigma) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma}\tanh_A\left(\sqrt{\Delta}\varsigma\right) \pm i\frac{\sqrt{pq\Delta}2\sigma}{S}\sec h_A\left(\sqrt{\Delta}\varsigma\right), \tag{3.12}$$

Family 3. In the limiting cas if $\alpha\sigma > 0, \beta = 0$, then

$$\psi_8(\varsigma) = \sqrt{\frac{\alpha}{\sigma}} \tan_A\left(\sqrt{\alpha\sigma}\varsigma\right), \tag{3.13}$$

$$\psi_9(\varsigma) = -\sqrt{\frac{\alpha}{\sigma}} \cot_A\left(\sqrt{\alpha\sigma}\varsigma\right), \tag{3.14}$$

$$\psi_{10}(\varsigma) = \sqrt{\frac{\alpha}{\sigma}} \tan_A\left(2\sqrt{\alpha\sigma}\varsigma\right) \pm \sqrt{pq\frac{\alpha}{\sigma}} \sec_A\left(2\sqrt{\alpha\sigma}\varsigma\right), \tag{3.15}$$

$$\psi_{11}(\varsigma) = -\sqrt{\frac{\alpha}{\sigma}} \cot_A\left(2\sqrt{\alpha\sigma}\varsigma\right) \pm \sqrt{pq\frac{\alpha}{\sigma}} \csc_A\left(2\sqrt{\alpha\sigma}\varsigma\right), \tag{3.16}$$

Family 4. when $\sigma = -\alpha, \beta = 0$, then

$$\psi_{12}(\varsigma) = -\tanh_A(\alpha\varsigma), \tag{3.17}$$

$$\psi_{13}(\varsigma) = -\coth_A(\alpha\varsigma), \tag{3.18}$$

$$\psi_{14}(\varsigma) = -\tanh_A(2\alpha\varsigma) \pm i\sqrt{pq}\sec h_A(2\alpha\varsigma), \tag{3.19}$$

Family 5. when $\beta = k, \sigma = mk$, $\beta = \alpha = 0, \beta = k, \alpha = mk, \sigma = 0$, then

$$\psi_{16}(\varsigma) = \frac{pA^{k\varsigma}}{q - mpA^{k\varsigma}}, \tag{3.20}$$

$$\psi_{17}(\varsigma) = \frac{-1}{\sigma\varsigma\ln(A)}, \tag{3.21}$$

$$\psi_{18}(\varsigma) = A^{k\varsigma} - m, \tag{3.22}$$

$$\zeta = x + \frac{\omega}{\alpha}\int_0^t g(t')\,dt', \tag{3.23}$$

where $\sinh_A(\varsigma) = \frac{pA^{\varsigma}-qA^{-\varsigma}}{2}$, $\cosh_A(\varsigma) = \frac{pA^{\varsigma}+qA^{-\varsigma}}{2}$, $\tanh_A(\varsigma) = \frac{pA^{\varsigma}-qA^{-\varsigma}}{pA^{\varsigma}+qA^{-\varsigma}}$,

$$\coth_A(\varsigma) = \frac{pA^{\varsigma} + qA^{-\varsigma}}{pA^{\varsigma} - qA^{-\varsigma}}, \tan_A(\varsigma) = -i\frac{pA^{i\varsigma} - qA^{-i\varsigma}}{pA^{i\varsigma} + qA^{-i\varsigma}}, \cot_A(\varsigma) = i\frac{pA^{i\varsigma} + qA^{-i\varsigma}}{pA^{i\varsigma} - qA^{-i\varsigma}}. \tag{3.24}$$

4. Conclusions

Methods of the extended sub-equation, direct algebraic and F-expansion have been successfully applied to solve the variable coefficient STO equation with its fission and fusion. Using the F-expansion method, one may able to classify ten types of solutions in terms of the arbitrary function $g(t)$. The advantage of the presence of that arbitrary function $g(t)$, enable us to construct a wide range classes of solutions according to the different choices of $g(t)$and any initial condition may be persuaded.

On the other hand, using different mathematical methods may lead us to another type of solutions. For example, applying the improved tanh method, one obtains the following type of solution

$$u(x,t) = \pm\left(\sec\left(x + \alpha\int_0^t g(t')\,dt'\right) \pm \tan\left(x + \alpha\int_0^t g(t')\,dt'\right)\right), \tag{4.1}$$

that maps to the triangular periodic solution where $\omega = \alpha$. In addition, one may also obtain the numerous soliton like solutions,

$$u(x,t) = \pm\frac{1}{2}\frac{\tanh\left(x+\alpha\int_0^t g(t')\,dt'\right)}{1 \pm \sec h\left(x+\alpha\int_0^t g(t')\,dt'\right)}. \tag{4.2}$$

where $\omega = -\alpha$ and $g(t)$ is an arbitrary function of t.

Application of these methods to fractal order PDEs may be seen in, e.g. [25–27, 45–53]. We will investigate the applicability of these methods to fractional stochastic differential equations in a future work.

Appendix

(A, B, C) values and $F(\xi)$ in $F' = A + BF(\xi) + CF^2(\xi)$.

A	B	C	$\chi(\zeta)$
0	1	-1	$\chi(\zeta) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\zeta}{2}\right)$
0	-1	1	$\chi(\zeta) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\zeta}{2}\right)$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\chi(\zeta) = \coth(\zeta) \pm \csc h(\zeta),\ \tan(\zeta) \pm i\sec h(\zeta)$
1	0	-1	$\chi(\zeta) = \tanh(\zeta), \coth(\zeta)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\chi(\zeta) = \sec(\zeta)+\tan(\zeta), \csc(\zeta)-\cot(\zeta)$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\chi(\zeta) = \sec(\zeta)-\tan(\zeta), \csc(\zeta)+\cot(\zeta)$
$1(-1)$	0	$1(-1)$	$\chi(\zeta) = \tan(\zeta), \cot(\zeta)$
0	0	$\neq 0$	$\chi(\zeta) = \frac{-1}{C\zeta+\lambda}$
Constant	0	0	$\chi(\zeta) = A\zeta$
Constant	$\neq 0$	0	$\chi(\zeta) = \frac{\exp(B\zeta)-A}{B}$

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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