

Research article

An application of theory of distributions to the family of λ -generalized gamma function

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Abstract: Gamma function and its generalizations always have played a basic role in various disciplines. The aim of present study is to investigate a new representation of the λ -generalized gamma function. This representation is developed by using different modified forms of delta function. This development explores their extended use as generalized functions (distributions), which are meaningful to exist over some particular space of test functions. Further to this a discussion is presented for the suitable applications of this new representation.

Keywords: λ -generalized gamma function; Fourier transformation; slowly increasing test functions; generalized functions (distributions); delta function

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1. Introduction

Fundamental importance of Hurwitz-Lerch zeta function has its roots in analytic number theory. More recently, a new class of Hurwitz-Lerch zeta function has been introduced and investigated by Srivastava [1]. Following this investigation, various new studies with diverse themes can be found in the literature [2–11]. By taking motivation from these researches, Tassaddiq [12] has investigated a series representation for this class of Hurwitz-Lerch zeta functions by introducing λ -generalized gamma function. The original gamma function was first generalized by Chaudhry and Zubair [13] which proved very useful for the solution of heat conduction problems. After that some other researchers have introduced and investigated different generalizations of gamma function. For review of such generalizations, the interested reader is referred to [14,15] and references there in.

More recently, Mubeen et al [14] have reviewed all previous extensions and used the approach of Chaudhry and Zubair [13] to present some extensions of k -gamma and k -beta functions. The literature review for gamma function and its generalizations have not only motivated to mathematicians for the development of modern theories but their applications in miscellaneous subjects are central. The purpose of current study is to find a novel series representation of λ -generalized gamma function in relation with delta function. Recent investigations [16–24] are mentionable to achieve the goals of this paper. As a result, one can analytically compute various new integrals of products of special functions which are not the part of existing literature [25,26].

Plan of this paper is as follows: essential preliminaries related to the family of λ -generalized gamma function as well as test functions spaces are given in Sections 2.1 and 2.2. Organization of the remaining part is given as: Section 3.1 includes new series form related with λ -generalized gamma function. Section 3.2 consists of the criteria about the existence as well as uses of the novel series. Validation of these outcomes is given in Section 3.3. Further results are a part of Sections 3.4 and 3.5. Section 4 highlights and concludes the present as well as future work.

2. Materials and method

2.1. λ -generalized gamma function

Commonly used symbols are stated as follows

$$\mathbb{Z}^+ = \mathbb{N} := \{1, 2, \dots\}; \mathbb{N}_0 := \{0\} \cup \mathbb{N}; \mathbb{Z}^- := \{-1, -2, \dots\}; \mathbb{Z}_0^- := \{0\} \cup \mathbb{Z}^-.$$

Here \mathbb{N} denotes the set of natural numbers whereas the sets of positive and negative integers are symbolized by \mathbb{Z}^+ and \mathbb{Z}^- respectively. Moreover, \mathbb{C} denotes the set of complex numbers and the set of real is denoted by \mathbb{R} .

Gamma function as a generalization of factorial has its integral representation [13]

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt; \Re(s) > 0. \quad (1)$$

Diaz and Pariguan [15] studied its generalization in the following integral form known as k -gamma function

$$\Gamma_k(s) = \int_0^\infty t^{s-1} e^{-\frac{t^k}{k}} dt \quad (k \geq 0), \quad (2)$$

and one can notice that $\Gamma_1(s) = \Gamma(s)$ and

$$\Gamma_2(s) = \int_0^\infty t^{s-1} e^{-\frac{t^2}{2}} dt \quad (3)$$

is an integral of Gaussian function, which has fundamental applications. These types of gamma function are also important to express other basic notions such as Pochhammer symbols

$$(\lambda)_\rho = \frac{\Gamma(\lambda + \rho)}{\Gamma(\lambda)} = \begin{cases} 1 & (\rho = 0, \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \dots (\lambda + \rho - 1) & (\rho = k \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \quad (4)$$

and

$$(\lambda)_{\kappa,\rho} = \frac{\Gamma_k(\lambda + \kappa\rho)}{\Gamma_k(\lambda)}. \quad (5)$$

The focus point of this paper is a newly studied special function namely λ -generalized gamma function as defined in [12]

$$\Gamma_b^\lambda(s; a) = \int_0^\infty t^{s-1} \exp\left(-at - \frac{b}{t^\lambda}\right) dt; \quad (\lambda \geq 0; \Re(b) \geq 0; \min[\Re(s), \Re(a)] > 0). \quad (6)$$

The λ -generalized gamma functions satisfy certain useful relations as investigated in [12] such as the generalized difference equation

$$\Gamma_{a,b}^\lambda(s+1) = \frac{s}{a} \Gamma_{a,b}^\lambda(s) + \frac{b\lambda}{a} \Gamma_{a,b}^\lambda(s-\lambda), \quad (b \geq 0), \quad (7)$$

and the following inequality known as log-convex property

$$\Gamma_{a,b}^\lambda\left(\frac{s}{p} + \frac{u}{q}\right) \leq \left(\Gamma_{a,b}^\lambda(s)\right)^{\frac{1}{p}} \left(\Gamma_{a,b}^\lambda(u)\right)^{\frac{1}{q}}; \quad (s, u \in \mathbb{R}; 1 < p < \infty; \frac{1}{p} + \frac{1}{q} = 1). \quad (8)$$

For, $\lambda = 1$, (6) reduces to the following generalization of $\Gamma(s)$ as defined in [13]

$$\Gamma_b(s) = \int_0^\infty t^{s-1} e^{-t - \frac{b}{t}} dx, \quad (\Re(s) > 0, b \geq 0,). \quad (9)$$

Further comprehensive details and new developments about gamma function can be found in recent important works [27–36] and references therein

2.2. Distributions and test functions

Corresponding to each space of test functions there is a dual space known as space of distributions (or generalized functions). Consideration of such functions is vital due to their important property of representing the singular functions. In this way, one can apply different operations of calculus as in the case of classical functions. For the requirements of this investigation we need to mention about delta function, which is a commonly used singular function given by

$$\langle \delta(s - \omega), \varphi(s) \rangle = \varphi(\omega) \quad (\forall \varphi \in D, \omega \in \mathbb{R}) \quad (10)$$

and

$$\delta(-s) = \delta(s); \quad \delta(\omega s) = \frac{\delta(s)}{|\omega|}, \text{ where } \omega \neq 0. \quad (11)$$

An ample discussion and explanation of distributions (or generalized functions) has been presented in five different volumes by Gelfand and Shilov [37]. Functions having compact support and infinitely differentiable as well as fast decaying are commonly used test functions. The spaces containing such functions are denoted by D and S respectively. Obviously, corresponding duals are the spaces D' and S' . A mentionable fact about such spaces is that D and D' do not hold the closeness property with respect to Fourier transform but S and S' do. In this way it is remarkable that the elements of D' have Fourier transforms that form distributions for entire functions space Z whose Fourier transforms belong to D [38]. Further to this explanation, it is noticeable that as the entire function is nonzero for a particular range $\omega_1 < s < \omega_2$, but zero otherwise so the following

inclusion of above mentioned spaces holds

$$Z \cap D \equiv 0; Z \subset S \subset S' \subset Z'; D \subset S \subset S' \subset D'. \quad (12)$$

More specifically, space Z comprise of entire and analytic functions sustaining the subsequent criteria

$$|s^q \wp(s)| \leq C_q e^{\eta|s|}; \quad (q \in \mathbb{N}_0). \quad (13)$$

Here and what follows, the numbers η and C_q are dependent on \wp . The following identities ([37], Vol 1, p. 169, Eq (8)), ([38], (p. 159), Eq (4)), see also ([40], p. 201, Eq (9)) will be used in the proof of our main result

$$\mathcal{F}[e^{\alpha t}; \theta] = 2\pi\delta(\theta - i\alpha) \quad (14)$$

$$g(s+b) = \sum_{j=0}^{\infty} g^{(j)}(s) \frac{b^j}{j!}, \quad \forall g \in Z' \quad (15)$$

$$\delta(s+b) = \sum_{j=0}^{\infty} \delta^{(j)}(s) \frac{b^j}{j!}; \text{ where } \langle \delta^{(j)}(s), \wp(s) \rangle = (-1)^j \wp^{(j)}(0). \quad (16)$$

$$\delta(\omega_1 - s)\delta(s - \omega_2) = \delta(\omega_1 - \omega_2). \quad (17)$$

Further such examples are $\sin(t), \cos(t), \sinh t$ and $\cosh t$ whose Fourier transformations are delta (singular) functions. The relevant detailed discussions about such spaces can be found in [37–41].

Throughout in this paper, except if mentioned particularly the conditions for the involved parameters are taken as stated in Sections (2.1) and (2.2).

3. Results

3.1. New Representation of λ -generalized gamma function

In this section, computation of λ -generalized gamma function is given as a series of complex delta function but the discussion about its rigorous use as a generalized function over a space of test functions is a part of the next section.

Theorem 1. λ -generalized gamma function has the subsequent series representation

$$\Gamma_b^\lambda(s; a) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \delta(\theta - i(\nu + n - \lambda r)). \quad (18)$$

Proof. A replacement of $t = e^x$ and $s = \nu + i\theta$ in the integral representation of λ -generalized gamma function as given in (6) yields the following

$$\Gamma_b^\lambda(s; a) = \int_{-\infty}^{\infty} e^{x(\nu+i\theta)} \exp(-ae^x) \exp(-be^{-\lambda x}) dx. \quad (19)$$

Then the involved exponential function can be represented as

$$\exp(-ae^x) \exp(-be^{-\lambda x}) = \sum_{n=0}^{\infty} \frac{(-ae^x)^n}{n!} \sum_{r=0}^{\infty} \frac{(-be^{-\lambda x})^r}{r!}. \quad (20)$$

Next, combining the expressions (19) and (20) leads to the following

$$\Gamma_b^\lambda(s; a) = \int_{-\infty}^{\infty} e^{ix\theta} \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} e^{(\nu+n-\lambda r)x} dx, \quad (21)$$

which gives

$$\Gamma_b^\lambda(s; a) = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \int_{-\infty}^{\infty} e^{ix\theta} e^{(\nu+n-\lambda r)x} dx. \quad (22)$$

The actions of summation and integration are exchangeable because the involved integral is uniformly convergent. An application of identity (14) produces the following

$$\int_{-\infty}^{\infty} e^{i\theta x} e^{(\nu+n-\lambda r)x} dx = \mathcal{F} [e^{(\nu+n-\lambda r)x}; \theta] = 2\pi \delta(\theta - i(\nu + n - \lambda r)). \quad (23)$$

A combination of these Eqs (22) and (23) yields the required result (18). \square

Corollary 1 λ -generalized gamma function has the following series form

$$\Gamma_b^\lambda(s; a) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (-i(\nu + n - \lambda r))^p}{n! r! p!} \delta^{(p)}(\theta) \quad (24)$$

Proof. Eq (24) can be obtained by considering the following combination of Eq (16) as well as Eq (23)

$$\delta(\theta - i(\nu + n - \lambda r)) = \sum_{p=0}^{\infty} \frac{(-i(\nu + n - \lambda r))^p}{p!} \delta^{(p)}(\theta) \quad (25)$$

Next, by making use of this relation in (18) leads to the required form. \square

Corollary 2 λ -generalized gamma function has the following series form

$$\Gamma_b^\lambda(s; a) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \delta(s + n - \lambda r). \quad (26)$$

Proof. Eq (23) can be rewritten as follows

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\theta x} e^{(\nu+n-\lambda r)x} dx &= \mathcal{F} [e^{(\nu+n-\lambda r)x}; \theta] = 2\pi \delta(\theta - i(\nu + n - \lambda r)) = 2\pi \delta \left[\frac{1}{i} (i\theta + (\nu + n - \lambda r)) \right] \\ &= 2\pi |i| \delta(\nu + i\theta + n - \lambda r) = 2\pi \delta(s + n - \lambda r) \end{aligned} \quad (27)$$

Next, by making use of this relation in (18) leads to the required form. \square

Corollary 3 λ -generalized gamma function has the following series form

$$\Gamma_b^\lambda(s; a) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^p}{n! r! p!} \delta^{(p)}(s). \quad (28)$$

Proof. A suitable combination of Eqs (16) and (26) gives

$$\delta(s + n - \lambda r) = \sum_{p=0}^{\infty} \frac{(n - \lambda r)^p}{p!} \delta^{(p)}(s); \quad \langle \delta^{(p)}(s), \delta^{(p)}(s) \rangle = (-1)^p \delta^{(p)}(0), \quad (29)$$

which is a key to the required form.

Remark 1. It is to be remarked that the following results are straightforward from the above corollaries for $\lambda = 1$

$$\Gamma_b(s) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (-b)^r}{n! r!} \delta(\theta - i(\nu + n - r)); \quad (30)$$

$$\Gamma_b(s) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-1)^n (-b)^r (-i(\nu + n - r))^p}{n! r! p!} \delta^{(p)}(\theta); \quad (31)$$

$$\Gamma_b(s) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (b)^r}{n! r!} \delta(s + n - r); \quad (32)$$

$$\Gamma_b(s) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - r)^p}{n! r! p!} \delta^{(p)}(s). \quad (33)$$

Now, by putting $b = 0$ leads to the following [24]

$$\Gamma(s) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(\theta - i(\nu + n)) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (-i(\nu + n))^r}{n! r!} \delta^{(r)}(\theta); \quad (34)$$

$$\Gamma(s) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(s + n); \quad (35)$$

$$\Gamma(s) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n n^r}{n! r!} \delta^{(r)}(s). \quad (36)$$

It is noticeable that the above series representations are given in the form of delta function. Such functions make sense only if defined as distributions (generalized functions) over a space of test functions as discussed in Section (1.2). Consequently, one needs to be very careful to choose a suitable function for which this representation holds true. As an illustration, one can put $b = 0$ in

identity (26) and multiply it by $\frac{1}{\Gamma_0^\lambda(s; a)}$ to get the following

$$1 = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n! \Gamma_0^\lambda(s; a)} \delta(s + n). \quad (37)$$

Therefore, singular points of delta function at $s = -n$ are canceled with the zeros of $\Gamma_0^\lambda(s; a)$ in this expression i.e $\lim_{s \rightarrow -n} \frac{\delta(s+n)}{\Gamma_0^\lambda(-n; a)} = \lim_{s \rightarrow -n} \frac{\frac{1}{s+n}}{\frac{1}{s+n}} = \lim_{s \rightarrow -n} \frac{s+n}{s+n} = 1$. Hence, by making use of

$$\delta(t) = \begin{cases} \infty & (t = 0) \\ 0 & (t \neq 0), \end{cases} \quad (38)$$

in the above statement (37), one can get the following

$$1 = \begin{cases} 2\pi \exp(-a) & (s = -n) \\ 0 & (s \in \mathbb{C} \setminus \{-n\}), \end{cases} \quad (39)$$

which is false or inconsistent. At the same time, a consideration of the following special product

$$\langle \Gamma_0^\lambda(s; a), \frac{1}{\Gamma_0^\lambda(s; a)} \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \langle \delta(s + n), \frac{1}{\Gamma_0^\lambda(s; a)} \rangle \quad (40)$$

gives the following

$$\int_{s \in \mathbb{C}} 1 \, ds = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n! \Gamma_0^\lambda(-n; a)}. \quad (41)$$

Since $\frac{1}{\Gamma_0^\lambda(-n; a)} = 0$ due to the poles of gamma function and we get

$$\begin{aligned} \int_{s \in \mathbb{C}} 1 \, ds &= 0 \\ \int_{s \in \mathbb{C}} 1 \, ds &= \int_{-\infty}^{+\infty} 1 \, ds = 0 \\ &\Rightarrow \infty = 0. \end{aligned} \quad (42)$$

Therefore, one needs to be very careful in making a choice of function to analyse the behavior of new series representation that is discussed in the next subsection.

3.2. Analysis of the behavior of new representation

λ -generalized gamma function $\Gamma_b^\lambda(s; a)$ is expressed in a new form involving singular distributions namely delta function. Therefore, it is proved in the subsequent theorem that this new form of $\Gamma_b^\lambda(s; a)$ is a generalized function (distribution) over Z (space of entire test functions).

Theorem 2 Prove that $\Gamma_b^\lambda(s; a)$ acts as a generalized function (distribution) over Z .

Proof. For each $\phi_1(s), \phi_2(s) \in Z$ and $c_1, c_2 \in \mathbb{C}$

$$\langle \Gamma_b^\lambda(s; a), c_1 \wp_1(s) + c_2 \wp_2(s) \rangle \quad (43)$$

$$= \langle 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \delta(s + n - \lambda r), c_1 \wp_1(s) + c_2 \wp_2(s) \rangle$$

$$\Rightarrow \langle \Gamma_b^\lambda(s; a), c_1 \wp_1(s) + c_2 \wp_2(s) \rangle = c_1 \langle \Gamma_b^\lambda(s; a), \wp_1(s) \rangle + c_2 \langle \Gamma_b^\lambda(s; a), \wp_2(s) \rangle. \quad (44)$$

Then, for any sequence $\{\wp_k\}_{k=1}^{\infty}$ in Z converging to zero one can assume that $\{\langle \delta(s + n - \lambda r), \wp_k \rangle\}_{k=1}^{\infty} \rightarrow 0$ due to the continuity of $\delta(s)$

$$\Rightarrow \{\langle \Gamma_b^\lambda(s; a), \wp_k(s) \rangle\}_{k=1}^{\infty} = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \{\langle \delta(s + n - \lambda r), \wp_k(s) \rangle\}_{k=1}^{\infty} \rightarrow 0 \quad (45)$$

Henceforth, λ -generalized gamma function is a generalized function (distribution) over test function space Z due to the convergence of its new form (26) explored below

$$\begin{aligned} \langle \Gamma_b^\lambda(s; a), \wp(s) \rangle &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \delta(s + n - \lambda r), \wp(s) \rangle; (\forall \wp(s) \in Z) \\ &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \wp(\lambda r - n), \end{aligned} \quad (46)$$

whereas,

$$\langle \delta(s + n - \lambda r), \wp(s) \rangle = \wp(\lambda r - n). \quad (47)$$

One can observe that $\forall \wp \in Z$; $\wp(\lambda r - n)$ are functions of slow growth as well as

$$\text{sum over the coefficients} = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} = \exp(-a - b) \quad (48)$$

exists and is rapidly decreasing. Consequently, for $\forall \wp(s) \in Z$; $\langle \Gamma_b^\lambda(s; a), \wp(s) \rangle$ as a product of the functions of slow growth and rapid decay is convergent. Similarly, other special cases as given in (30–36) are also meaningful in the sense of distributions. This fact is also obvious by making use of basic Abel theorem. \square

Hence the behavior of this new series is discussed for the functions of slow growth but it is mentionable that this new series may converge for a larger class of functions. Consequently, new integrals of products of different functions in view of this new form of $\Gamma_b^\lambda(s; a)$ are obtained. For example, start with a basic illustration i.e. $\wp(s) = \tau^{s\xi}$ ($\xi > 0$; $s \in \mathbb{C}$). Hence by considering (26) and shifting property of delta function the inner product $\langle \Gamma_b^\lambda(s; a), \wp(s) \rangle$ yields

$$\begin{aligned} \int_{z \in \mathbb{C}} \tau^{s\xi} \Gamma_b^\lambda(s; a) ds &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \tau^{-n\xi + \lambda r\xi} = 2\pi \sum_{n=0}^{\infty} \frac{(-a\tau^{-\xi})^n}{n!} \frac{(-b\tau^{\lambda\xi})^r}{r!} \\ &= 2\pi \exp(-a\tau^{-\xi} - b\tau^{\lambda\xi}). \end{aligned} \quad (49)$$

Similarly, by considering the distributional form of generalized gamma function as given in (32), we obtain the following specific form of (49) with $\lambda = a = 1$

$$\begin{aligned} \int_{s \in \mathbb{C}} \tau^{s\xi} \Gamma_b^1(s; 1) ds &= \int_{s \in \mathbb{C}} \tau^{s\xi} \Gamma_b(s) ds \\ &= 2\pi \sum_{n=0}^{\infty} \frac{(-\tau^{-\xi})^n}{n!} \frac{(-b\tau^{\xi})^r}{r!} = 2\pi \exp(-\tau^{-\xi} - b\tau^{\xi}). \end{aligned} \quad (50)$$

Remark 2. Sequences as well as sums of delta function have significant importance in diverse engineering problems, for example these are used as an electromotive force in electrical engineering. This is noticeable that if one multiplies $\{\delta(s + n - \lambda r)\}_{n=0}^{\infty}$ with $2\pi \exp(-a - b)$ then it will produce the distributional representation of λ -generalized gamma function. Furthermore, if one takes $a = 1 = \lambda; b = 0$, then related outcome do hold for special cases as well. This discussion illustrates the possibility of further important identities. For instance if one considers $\tau = e^{-1}$ in (49) then it will compute Laplace transform of $\Gamma_b^{\lambda}(s; a)$. Therefore it becomes more important to check the validation of such results that is discussed in the following section.

3.3. Validation of the results obtained by new representation

Considering $t = e^x$ as well as $s = v + i\xi$ in (6), the λ -generalized gamma function can be expressed as a Fourier transform given below

$$\Gamma_b^{\lambda}(v + i\theta; a) = \sqrt{2\pi} \mathcal{F}[e^{v^x} \exp(-ae^x - be^{-\lambda x}); \xi] \quad (b > 0), \quad (51)$$

and considering $\lambda = 1$, the generalized gamma function can be expressed as

$$\Gamma_{b,1}(v + i\theta) = \Gamma_{b,1}(v + i\theta) = \sqrt{2\pi} \mathcal{F}[e^{v^x} \exp(-e^x - be^{-x}); \xi]. \quad (52)$$

Fourier transform of an arbitrary function $u(t)$, satisfy the following

$$\mathcal{F}[\sqrt{2\pi} \mathcal{F}[u(t); \theta]; \xi] = 2\pi u(-\xi). \quad (53)$$

Hence, by applying this on identities (51–52), will lead to the following

$$\begin{aligned} \mathcal{F}\{\Gamma_b^{\lambda}(v + i\theta; a); \xi\} &= \mathcal{F}[\sqrt{2\pi} \mathcal{F}[e^{v^x} \exp(-ae^x - be^{-\lambda x})]; \xi] \\ &= f(-\xi) = 2\pi e^{-v\xi} \exp(-ae^{-\xi} - be^{\lambda\xi}), \end{aligned} \quad (54)$$

equivalently,

$$\int_{-\infty}^{+\infty} e^{i\theta\xi} \Gamma_b^{\lambda}(v + i\theta; a) d\theta = 2\pi e^{-v\xi} \exp(-ae^{-\xi} - be^{\lambda\xi}), \quad (55)$$

which is also obtainable as a specific case of our main result (49) by substituting $\tau = e; s = v + i\theta$. Furthermore, a substitution $\xi = 0$ in (55), leads to the following

$$\int_{-\infty}^{+\infty} \Gamma_b^{\lambda}(v + i\theta; a) d\theta = 2\pi \exp(-a - b), \quad (56)$$

which is also attainable as a precise case of our main result (49). Hence it is testified that the new representation of λ -generalized gamma function produces novel identities, which are unattainable by known techniques but specific forms of new identities are trustworthy with the known methods. Some interesting special cases are for $a = 1 = \lambda$

$$\int_{-\infty}^{+\infty} e^{i\theta\xi} \Gamma_b(v + i\theta) d\theta = 2\pi e^{-1-b} \quad (57)$$

$$\text{and } \xi = 0 = b$$

$$\int_{-\infty}^{+\infty} \Gamma(v + i\theta) d\theta = \frac{2\pi}{e}. \quad (58)$$

Remark 3. It is noticeable that the new obtained integrals contribute only the sum over residues due to the existing poles or singular points in the integrand, which is consistent with the basic result of complex analysis.

Next, an application of Parseval's identity of Fourier transform in (54), leads to the following new results about λ -generalized gamma functions $\Gamma_b^\lambda(s; a)$

$$\begin{aligned} \int_{-\infty}^{+\infty} \Gamma_b^\lambda(v + i\theta; a) \overline{\Gamma_b^\lambda(\mu + i\theta; a)} d\theta &= 2\pi \int_0^\infty t^{v+\mu-1} e^{-2at-2bt^{-\lambda}} dt \\ &= \pi 2^{1-(v+\mu)} \Gamma_{2\lambda+1b}^\lambda(v + \mu; a). \end{aligned} \quad (59)$$

A substitution $a = 1$ in (59) leads to the following

$$\begin{aligned} \int_{-\infty}^{+\infty} \Gamma_b^\lambda(v + i\theta; 1) \overline{\Gamma_b^\lambda(\mu + i\theta; 1)} d\theta &= 2\pi \int_0^\infty t^{v+\mu-1} e^{-2t-2\frac{b}{t}} dt \\ &= \pi 2^{1-(v+\mu)} \Gamma_{4b}(v + \mu) \end{aligned} \quad (60)$$

and $b = 0$ leads to the following known result [16,17]

$$\int_{-\infty}^{+\infty} |\Gamma(v + i\theta)|^2 d\tau = \int_0^\infty t^{2v-1} e^{-2t} dt = \pi 2^{1-2v} \Gamma(2v). \quad (61)$$

3.4. Further properties of the λ -generalized gamma function as a distribution

Here, by taking motivation from [38, Chapter 7], a list of basic properties of the λ -generalized gamma functions are stated and proved.

Theorem 3 λ -generalized gamma function holds the subsequent properties as a distribution

- (i) $\langle \Gamma_b^\lambda(s; a), \wp_1(s) + \wp_2(s) \rangle = \langle \Gamma_b^\lambda(s; a), \wp_1(s) \rangle + \langle \Gamma_b^\lambda(s; a), \wp_2(s) \rangle; \quad \forall \wp(s) \in Z$
- (ii) $\langle c_1 \Gamma_b^\lambda(s; a), \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), c_1 \wp(s) \rangle; \quad \forall \wp(s) \in Z$
- (iii) $\langle \Gamma_b^\lambda(s - \gamma; a), \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), \wp(s + \gamma) \rangle; \quad \forall \wp(s) \in Z$
- (iv) $\langle \Gamma_b^\lambda(c_1 s; a), \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), \frac{1}{c_1} \wp\left(\frac{s}{c_1}\right) \rangle; \quad \forall \wp(s) \in Z$
- (v) $\langle \Gamma_b^\lambda(c_1 s - \gamma; a), \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), \frac{1}{c_1} \wp\left(\frac{s}{c_1} + \gamma\right) \rangle; \quad \forall \wp(s) \in Z$
- (vi) $\psi(s) \Gamma_b^\lambda \in Z$ is a distribution over Z for any regular distribution $\psi(z)$.
- (vii) $\Gamma_0^\lambda(s + 1) = s \Gamma_0^\lambda(s)$ iff $\wp(s - 1) = s \wp(s)$ where $\wp \in Z$
- (viii) $\langle \Gamma_b^\lambda(s; a)^{(m)}(s), \wp(s) \rangle = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} (-1)^m \wp^m(-n + \lambda r); \quad \forall \wp(s) \in Z$

$$(ix) \quad \Gamma_b^\lambda(\omega_1 - s; a) \Gamma_b^\lambda(s - \omega_2; a) = (2\pi \exp(-a - b))^2 \delta(\omega_1 - \omega_2); \quad \forall \wp(s) \in Z$$

$$(x) \quad \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), \mathcal{F}[\wp(s)] \rangle; \quad \forall \wp(s) \in Z$$

$$(xi) \quad \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \mathcal{F}[\wp(s)] \rangle = 2\pi \langle \Gamma_b^\lambda(s; a), \wp(-s) \rangle; \quad \forall \wp(s) \in Z$$

$$(xii) \quad \overline{\langle \Gamma_b^\lambda(s; a), \mathcal{F}[\wp(s)] \rangle} = 2\pi \langle \Gamma_b^\lambda(s; a), \wp^T(s) \rangle, \text{ where } \overline{\wp(-s)} = \wp^T(s); \quad \forall \wp(s) \in Z$$

$$(xiii) \quad \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle \Gamma_b^\lambda(s; a), \wp^T(s) \rangle; \quad \forall \wp(s) \in Z$$

$$(xiv) \quad \overline{\langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \overline{\mathcal{F}[\wp(s)]} \rangle} = 2\pi \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \mathcal{F}[\wp(s)] \rangle; \quad \forall \wp(s) \in Z$$

$$(xv) \quad \mathcal{F}[\Gamma_b^{\lambda(m)}(s; a)] = [(-it)^m \Gamma_b^\lambda(s; a)]; \quad \forall \wp(s) \in Z$$

$$(xvi) \quad \Gamma_b^\lambda(s + c_1; a) = \sum_{n=0}^{\infty} \frac{(c_1)^n}{n!} \Gamma_b^{\lambda(n)}(s; a) \quad \forall \wp(s) \in Z$$

where c_1 , γ and c_2 are arbitrary real or complex constants.

Proof. It can be checked that the methodology to prove (i–vi) is trivial that can be achieved by using the properties of delta function. Therefore, we start proving (vii)

$$\begin{aligned} \langle \Gamma_0^\lambda(s + 1; a), \wp(s) \rangle &= \langle \Gamma_0^\lambda(s; a), \wp(s - 1) \rangle, \\ &\Leftrightarrow \langle s \Gamma_0^\lambda(s; a), \wp(s) \rangle = \langle \Gamma_0^\lambda(s; a), \wp(s - 1) \rangle, \\ &\Leftrightarrow \langle \Gamma_0^\lambda(s; a), s \wp(s) \rangle = \langle \Gamma_0^\lambda(s; a), \wp(s - 1) \rangle, \end{aligned}$$

as required.

Next we prove result (viii) by making use of Eq (16) (see Section 2.1) and we get

$$\langle \Gamma_b^{\lambda(m)}(s; a), \wp(s) \rangle = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} (-1)^m \wp^m(-n + \lambda r),$$

which is meaningful and finite as a product of fastly decaying as well as slow growth functions. Result (ix) is proved here in view of relation (17) (see Section 2.1),

$$\begin{aligned} \langle \Gamma_b^\lambda(\omega_1 - s; a) \Gamma_b^\lambda(s - \omega_2; a), \wp(s) \rangle &= \left(2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \right)^2 \langle \delta(\omega_1 - \omega_2), \wp(s) \rangle \\ &= (2\pi \exp(-a - b))^2 \langle \delta(\omega_1 - \omega_2), \wp(s) \rangle. \end{aligned}$$

Identities (x)–(xv) can also be proved in view of different properties of delta function. Let us start proving (x)

$$\begin{aligned} \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \wp(s) \rangle &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \mathcal{F}[\delta(s + n - \lambda r)], \wp(s) \rangle \\ &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \delta(s + n - \lambda r), \mathcal{F}[\wp(s)] \rangle = \langle \Gamma_b^\lambda(s; a), \mathcal{F}[\wp(s)] \rangle. \end{aligned}$$

Next result (xi–xii) are proved as follows

$$\langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \mathcal{F}[\wp(s)] \rangle = \langle \Gamma_b^\lambda(s; a), \mathcal{F}[\mathcal{F}[\wp(s)]] \rangle = 2\pi \langle \Gamma_b^\lambda(s; a), \wp(-s) \rangle,$$

$$\begin{aligned} \langle \overline{\mathcal{F}[\Gamma_b^\lambda(s; a)]}, \mathcal{F}[\wp(s)] \rangle &= 2\pi \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \overline{\wp(-s)} \rangle \\ &= 2\pi \langle \Gamma_b^\lambda(s; a), \wp^T(s) \rangle, \end{aligned}$$

whereas the transpose of \wp is denoted by \wp^T . Proof of the results (xiii)–(xiv) are

$$\langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle \Gamma_b^\lambda(s; a), \overline{\wp(-s)} \rangle = 2\pi \langle \Gamma_b^\lambda(s; a), \wp^T(s) \rangle$$

$$\langle \overline{\mathcal{F}[\Gamma_b^\lambda(s; a)]}, \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \overline{\overline{\mathcal{F}[\wp(s)]}} \rangle = 2\pi \langle \mathcal{F}[\Gamma_b^\lambda(s; a)], \mathcal{F}[\wp(s)] \rangle,$$

whereas the last line follows in view of Parseval's formula of Fourier transform. The proof of (xv) is as follows

$$\langle \mathcal{F}[\Gamma_b^{\lambda(1)}(s; a)], \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), \mathcal{F}[\wp^{(1)}(s)] \rangle$$

$$\langle \mathcal{F}[\Gamma_b^{\lambda(1)}(s; a)], \wp(s) \rangle = \langle \Gamma_b^\lambda(s; a), (-it)\wp(s)e^{-ist} \rangle$$

$$\langle \mathcal{F}[\Gamma_b^{\lambda(1)}(s; a)], \wp(s) \rangle = \langle (-it)\mathcal{F}[\Gamma_b^\lambda(s; a)], \wp(s) \rangle$$

$$\langle \mathcal{F}[\Gamma_b^{\lambda(1)}(s; a)], \wp(s) \rangle = \langle (-it)\mathcal{F}[\Gamma_b^\lambda(s; a)], \wp(s) \rangle.$$

and so on, we get

$$\langle \mathcal{F}[\Gamma_b^{\lambda(m)}(s; a)], \wp(s) \rangle = \langle (-it)^m \mathcal{F}[\Gamma_b^\lambda(s; a)], \wp(s) \rangle,$$

as the requirement of (xv). The last result (xvi) is true in view of the statement mentioned in [38, p. 201], “Suppose $f \in Z'$ and Δ is a complex constant then the translation of the function f by the quantity $-\Delta$ is represented by $f(z + \Delta) = \sum_{n=0}^{\infty} \frac{(\Delta)^n}{n!} f^{(n)}(z)$.” Consequently, we get

$$\begin{aligned} \langle \Gamma_b^\lambda(s + c_1; a), \wp(s) \rangle &= \langle \Gamma_b^\lambda(s; a), \wp(s - c_1) \rangle = \lim_{v \rightarrow \infty} \langle \Gamma_b^\lambda(s; a), \sum_{n=0}^v \frac{(-c_1)^n}{n!} \wp^{(n)}(s) \rangle \\ &= \lim_{v \rightarrow \infty} \langle \sum_{n=0}^v \frac{(c_1)^n}{n!} \Gamma_b^{\lambda(n)}(s; a), \wp(s) \rangle, \end{aligned}$$

as required. \square

Remark 4. Space of generalized functions denoted by D' is mapped onto Z' with the help of Fourier transformation and similarly this mapping can be inverted from Z' onto D' [38, p. 203]. Both ways, it is a continuous linear mapping. Therefore, (54) explores that $2\pi e^{-v\xi} \exp(-ae^\xi - be^{-\lambda\xi}) \in D'$. In the same way if one considers (55) and invert it by Fourier transform then

$$\mathcal{F}\{\Gamma_b^\lambda(s; a)\} \in D'.$$

3.5. Further Discussion of the class of validity of new representation

Being a singular generalized function, delta function is a linear mapping that maps every function to its value at zero. Due to this property, this new representation has the power to calculate the integrals, which are divergent in the classical sense.

Let us consider (28) and restrict the variable $s = t$, to real numbers then we have

$$\Gamma_b^\lambda(t; a) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^p}{n! r! p!} \delta^{(p)}(t) \quad (62)$$

that can be defined over S , that means it is a distribution in S' because it is convergent for rapidly decreasing and infinitely differentiable functions at 0, such that

$$\begin{aligned} \langle \Gamma_b^\lambda(t; a), \varphi(t) \rangle &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^p}{n! r! p!} \langle \delta^{(p)}(t), \varphi(t) \rangle \\ &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^p}{n! r! p!} (-1)^p \varphi^{(p)}(0). \end{aligned} \quad (63)$$

Next, we take a wider space of infinitely differentiable functions whose derivatives of all order at 0 exist and release the condition of rapidly decreasing. Here we consider some examples

Example 1. Let $\varphi(t) = e^{ct}$ then $\varphi^{(p)}(0) = c^p$; $p = 0, 1, 2, 3 \dots$

$$\begin{aligned} \langle \Gamma_b^\lambda(t; a), e^{ct} \rangle &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^p}{n! r! p!} (-1)^p c^p \\ &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} e^{-cn + \lambda rc} \\ &= 2\pi \exp(-ae^{-c} - be^{\lambda c}) \end{aligned} \quad (64)$$

Example 2. Let $\varphi(t) = \text{sinc } t$ then $\varphi^{(2p+1)}(0) = (-1)^p c^{2p+1}$; $\varphi^{(2p)}(0) = 0$

$$\begin{aligned} \langle \Gamma_b^\lambda(t; a), \varphi(t) \rangle &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^{2p+1}}{n! r! (2p+1)!} (-1)^{2p+1} (-1)^p c^{2p+1} \\ &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r \text{sinc}(\lambda r - n)}{n! r!} \\ &= \text{IMG} \left(2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r e^{i(c(\lambda r - n))}}{n! r!} \right) \end{aligned}$$

$$\begin{aligned}
&= \text{IMG} \left(2\pi \sum_{n,r=0}^{\infty} \frac{(-ae^{-ic})^n (-be^{ic\lambda})^r}{n! r!} \right) \\
&= \text{IMG} \left(2\pi \exp(-ae^{-ic} - be^{ic\lambda}) \right)
\end{aligned} \tag{65}$$

Similarly, $\wp(t) = \cos ct$ then

$$\begin{aligned}
\wp^{(2p)}(0) &= (-1)^p c^{2p}; \wp^{(2p+1)}(0) = 0 \\
\langle \Gamma_b^\lambda(t; a), \wp(t) \rangle &= \text{Re} \left(2\pi \exp(-ae^{-ic} - be^{ic\lambda}) \right)
\end{aligned} \tag{66}$$

Example 3. Let $\wp(t) = \frac{1}{1-t}$ then $\wp^{(p)}(0) = p!$

$$\begin{aligned}
\langle \Gamma_b^\lambda(t; a), \frac{1}{1-t} \rangle &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (-1)^p (n - \lambda r)^p}{n! r! p!} p! \\
&= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r! (1 + n - \lambda r)}
\end{aligned} \tag{67}$$

Example 4. Let $\wp(t) = \ln(1+t)$ then $\wp^{(p)}(0) = (-1)^{p+1}(p-1)!$

$$\begin{aligned}
\langle \Gamma_b^\lambda(t; a), \ln(1+t) \rangle &= 2\pi \sum_{n,r=0; p=1}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^p}{n! r! p!} (-1)^{2p+1} (p-1)! \\
&= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r \ln(1 - n + \lambda r)}{n! r!}
\end{aligned} \tag{68}$$

Example 5. Let $\wp(t) = \arctant$ then $\wp^{(2p+1)}(0) = -(2p)!$ and $\wp^{(2p)}(0) = 0$;

$$\begin{aligned}
\langle \Gamma_b^\lambda(t; a), \arctant \rangle &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (n - \lambda r)^{2p+1} (-1)^{2p+1}}{n! r! (2p+1)!} - (2p)! \\
&= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r \arctan(\lambda r - n)}{n! r!}
\end{aligned} \tag{69}$$

These examples show that new representation of the λ -generalized gamma function is meaningful for all those functions who have derivatives of all orders at 0. This statement can also be generalized as “The new representation of the λ -generalized gamma functions is valid for complex analytic functions at $s = 0$ ”. It is also convergent for all complex analytic functions (who have derivatives of all orders at 0) that also means that example 1–5 are consistent if we consider complex

s instead of real t . Similar results hold for the special cases of the λ -generalized gamma functions i.e, extended gamma, and gamma functions given by Eqs (28), (32) and (36).

As already stated as a distribution, the Dirac delta function is a linear functional that maps every function to its value at zero. Due to this property, new representation has the power to calculate the integrals, which cannot be calculated by using classical method. For example, let $\wp(t) = e^{ct^k}$ then,

$$\langle \Gamma_0^\lambda(t; a), \wp(t) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \langle \delta(t+n), \wp(t) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \wp(-n) \quad (70)$$

$$\langle \Gamma_0^\lambda(t; a), e^{ct^N} \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \langle \delta(t+n), \wp(t) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \exp(-c(n)^N)$$

It is to be remarked that new representation is convergent for rapidly increasing functions. The integral of rapidly increasing functions is always a challenge nevertheless; this generalized extension of the function has the capacity to do so and it can be defined over the space of rapidly increasing functions. The integral of gamma function is finite so multiplying it with rapidly decreasing function is always convergent. That is trivial to prove. Next, we discuss some further special cases by considering [38, p. 55, problem 10]

$$t^N \delta^{(r)}(t) = \begin{cases} 0 & r < N \\ (-1)^n N! \delta(t) & r = N \\ (-1)^n \frac{N!}{(r-N)!} \delta^{(r-N)}(t) & r > N \end{cases} \quad (71)$$

Therefore,

$$\begin{aligned} t^N \Gamma(t) &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n n^r}{n! r!} t^N \delta^{(r)}(t) \\ &= 0 + 2\pi \sum_n \frac{(-1)^n n^N}{n! N!} (-1)^n N! \delta(t) \\ &\quad + 2\pi \sum_{n,r=N+1}^{\infty} \frac{(-1)^n n^N}{n! N!} (-1)^n \frac{N!}{(r-N)!} \delta^{(r-N)}(t) \\ \langle t^N \Gamma(t), \wp(t) \rangle &= 2\pi \sum_n \frac{(-1)^n n^N}{n! N!} (-1)^n N! \langle \delta(t), \wp(t) \rangle \\ &\quad + 2\pi \sum_{n,r=N+1}^{\infty} \frac{(-1)^n n^N}{n! N!} (-1)^n \frac{N!}{(r-N)!} \langle \delta^{(r-N)}(t), \wp(t) \rangle \\ &= 2\pi \sum_n \frac{(-1)^n n^N}{n! N!} (-1)^n N! \wp(0) + 2\pi \sum_{n,r=N+1}^{\infty} \frac{(-1)^n n^N}{n! N!} (-1)^n \frac{N!}{(r-N)!} \wp^{(r-N)}(0) \end{aligned} \quad (72)$$

It is meaningful for a class of functions that have derivatives of all orders at point $t = 0$. By using these new representations obtained for the family of gamma functions, it can be observed that all the results that hold for the Laplace transform of delta function, similarly hold for the family of gamma functions, for example

$$L\{\delta^{(r)}(s)\} = z^p \quad (73)$$

Therefore,

$$\begin{aligned} L\left(\Gamma_b^\lambda(s; a)\right) &= L\left(2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n(-b)^r(n-\lambda r)^p}{n!r!p!} \delta^{(p)}(s)\right) \\ L\left(\Gamma_b^\lambda(s; a)\right) &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n(-b)^r(n-\lambda r)^p}{n!r!p!} L\left(\delta^{(p)}(s)\right) \\ &= 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n(-b)^r(n-\lambda r)^p}{n!r!p!} z^p = 2\pi \exp(-ae^z - be^{-\lambda z}). \end{aligned} \quad (74)$$

This gives

$$\left. \begin{aligned} L(\Gamma_b(s)) &= 2\pi \exp(-e^z - be^{-\lambda z}) \\ L\{\Gamma(s)\} &= 2\pi \exp(-e^z) \end{aligned} \right\} \quad (75)$$

That yields further,

$$\left. \begin{aligned} L(\Gamma_b(s-c)) &= 2\pi e^{-zc} \exp(-e^z - be^{-\lambda z}) \\ L\{\Gamma(s-c)\} &= 2\pi e^{-zc} \exp(-e^z) \end{aligned} \right\} \quad (76)$$

It can be remarked that all the results that hold for delta function can be applied to the family of gamma functions by using this new representation. It is due to the reason that the sum over the coefficients of the new representation is finite and well defined as given in (51).

By considering the classical theory of the family of gamma function, for example Eqs (2)–(6), we can note that gamma function has poles at $s = -n$ but λ -generalized gamma function extends the definition because the exponential factor in the integrand involves parameter $b > 0$. Same fact holds for our new representation, that can be easily proved by taking

$$\delta(-n + n - \lambda r) = \delta(-\lambda r) = \delta(\lambda r) = 0; (r, \lambda \neq 0)$$

That means for $b > 0$, our new representation is meaningful at $s = -n$

$$\Gamma_b^\lambda(-n; a) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \delta(\lambda r) \quad b > 0 \quad (77)$$

$$\langle \Gamma_b^\lambda(-n; a), \varphi(\lambda r) \rangle = 2\pi \sum_{r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \langle \delta(\lambda r), \varphi(\lambda r) \rangle$$

$$= 2\pi \sum_{r=1}^{\infty} \frac{(-a)^n (-b)^r}{r!} \wp(0); \quad b > 0.$$

By assuming $\wp(0) = 1$, the above equation implies that

$$\langle \Gamma_b^\lambda(-n; a), \wp(\lambda r) \rangle = 2\pi \exp(-a - b); \quad b > 0.$$

Nevertheless when $b = 0$ then the terms involving λ disappear and at $s = -n$, we get

$$\Gamma_0^\lambda(-n; a) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-0)^r}{n! r!} \delta(0) = \infty$$

that is undefined similar to as classical representation of gamma functions. For $a = 1; b = 0$ we get the generalized representation of original gamma function that has singularities at $s = -n$. The similar fact holds in classical theory.

4. Summary and Forthcoming Directions

The combination of distribution theory with different integral transforms is well explored for the analysis of partial differential equations (PDE). Numerous practical questions are impossible to be answered by applying the known techniques but became possible by using this combination. In this paper, a new form of the λ -generalized gamma function is discussed by using delta function so that a new definition of these functions is established for a particular set of test functions. Extensive results are obtained by exploring the details of distributional concepts for λ -generalized gamma function and enlightening their applications for the solution of new problems. As an illustration, we consider the famous Riemann zeta function for the interval $0 < \Re(s) < 1$, as follows

$$\langle \Gamma_b^\lambda(s; a), \zeta(s) \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \delta(s + n - \lambda r), \zeta(s) \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \zeta(-n + \lambda r),$$

and for $\lambda = 2$, we have an integral of extended Gaussian function

$$\langle \Gamma_b^2(s; a), \zeta(s) \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \delta(s + n - 2r), \zeta(s) \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \zeta(-n + 2r)$$

and for $a = 1; b = 0$, it yields the following

$$\langle \Gamma(s), \zeta(s) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \zeta(-n) = \frac{2\pi}{e-1} - 2\pi.$$

λ -generalized gamma function precisely specifies the original gamma function and therefore led to novel outcomes involving different special cases of gamma function. The λ -generalized gamma function and its different special cases are fundamental in different disciplines such as engineering, astronomy and related sciences. Method of computing the new identities involves the desired simplicity. Here we presented only a small number of examples. Further, it is expected that the results obtained in this study will prove significant for further development of λ -generalized gamma function in future work.

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Conflict of interest

The author declares no conflict of interest.

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