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Research article

Incorporating a tensor in the effective viscosity model of turbulence and the Reynolds stress

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Abstract: The mean field model of turbulence proposed by the author describes interaction among mean velocity and effective viscosity. In this paper, the model is extended to incorporate a tensor field by keeping invariance under Galilei transformation and rotation. It is found that, when the form and the strengths of interactions among fields are appropriately chosen, the symmetric components of the tensor for steady channel turbulence exhibit fair correspondence with the observed Reynolds stress.

Keywords: Navier-Stokes equation; turbulence; variational principle; eddy viscosity; Reynolds stress
Mathematics Subject Classification: 35Q30, 70H30, 76D05

1. Introduction

Turbulence is a most prominent and ubiquitous phenomenon of fluid motion. Obstacles in understanding its nature arise from its extreme complexity, which, thanks to development in simulation method, has been overcome gradually to yield rich visual as well as numerical outcomes.

Along with experiments in laboratory and numerical simulations on computer, we need constructing a model that accounts for complex phenomena from a view point of universal concepts abstracted from observational data. Specifically, Richardson-Kolmogorov scaling and eddy viscosity have been established as the most fundamental notions, on which so-called eddy viscosity models (EVMs) are based. EVM is a truncated set of originally infinite number of equations for moments of velocity field derived from the Navier-Stokes equation. EVMs are phenomenology because of the truncation that requires physically plausible but more or less arbitrary assumptions. For a comprehensive review of turbulence in incompressible fluid, see, e.g., [1].

In the dynamical effective viscosity model (DEVM) proposed by Takahashi [2], the equations of mean turbulent flow are derived by the variational principle on a pseudo-action that does not involve the non-holonomic condition [3]. DEVM is constructed in terms of a general complex two by two matrix field and its Hermitian conjugate so as to fulfill invariance under Galilei transformation and rotation. The independent four variables are interpreted as a complex mean velocity and a complex effective viscosity, which are to be finally set real. When applied to channel and pipe flows, the equations in the simplest model, i.e., the minimal DEVM, reproduce the flow profiles fairly well, which suggests that the symmetry structure of the model must be most important. In addition, this success may be attributed to action-reaction correlations among elements of turbulence incorporated automatically into the model through the variational principle that renders the model free from the so-called closure problem.

One important objective of research of turbulence is to understand the spatial variation of Reynolds stress. The DEVM involves a scalar and a vector, but not a tensor and therefore says nothing about the Reynolds stress. In this paper, we extend the minimal DEVM so as to incorporate a tensor in invariant ways and explore if the tensor bears the property of the Reynolds stress in incompressible turbulent flow.

In Section 2, we elaborate how the minimal DEVM is extended to include a tensor. In Sec. 3, the model constructed in Sec. 2 is applied to turbulence in channel flow. Sec. 4 is devoted to a concluding remark.

2. Incorporating tensor in the minimal DEVM

We first give a review of the minimal DEVM. We introduce a traceless scalar matrix Φ by

$$\Phi = \boldsymbol{u} \cdot \boldsymbol{\sigma}\,, \tag{2.1}$$

where σ_i, $i = x, y, z,$ are Pauli's spin matrices and $\boldsymbol{u}$ is a complex velocity field. Next, we define a following quantity

$$\begin{aligned} A_{\rm NS} &= \int L_{\rm NS}\,dt \equiv \int \mathcal{L}_{\rm NS}\,d\boldsymbol{r}dt, \\ \mathcal{L}_{\rm NS} &= \frac{\mathrm{i}}{2}\mathrm{Tr}\left(\Phi^\dagger\dot{\Phi} + \frac{1}{4}\Phi^\dagger\boldsymbol{\sigma}\Phi^\dagger\cdot\boldsymbol{\nabla}\Phi - \frac{1}{4}\boldsymbol{\nabla}\Phi^\dagger\cdot\Phi\boldsymbol{\sigma}\Phi + \frac{\nu}{2}(\boldsymbol{\nabla}\Phi^\dagger)^2 - \frac{\nu}{2}(\boldsymbol{\nabla}\Phi)^2 - \Phi^\dagger\mathbf{F} + \mathbf{F}^\dagger\Phi\right). \end{aligned} \tag{2.2}$$

Here, the dot stands for a partial derivative in time. ν is the kinematic viscosity. $\mathbf{F}$ is the force given by

$$\mathbf{F} \equiv \tilde{\boldsymbol{f}} \cdot \boldsymbol{\sigma} = \left(-\frac{\nabla p}{\rho} + \boldsymbol{f}_{\rm ext}\right)\cdot\boldsymbol{\sigma} \tag{2.3}$$

with p, ρ and $\boldsymbol{f}_{\rm ext}$ being the pressure, density and external body force that acts on a unit mass. The Navier-Stokes equation is derived by the variation of $A_{\rm NS}$. We call the quantity like $A_{\rm NS}$ the *pseudo*-action (PA) because, although the correct equation of motion is derived by the variational principle, PA does not have a canonical structure of the kinetic energy subtracted by the potential energy and therefore its Legendre transformation does not give the system's energy.

The first three terms in L_{NS} yield the Lagrange derivative terms. One can construct the minimal DEVM by extending Φ by giving it the center of $GL(2,C)$ as

$$\Phi = \varphi + \boldsymbol{u} \cdot \boldsymbol{\sigma} \tag{2.4}$$

where φ is a complex scalar and by introducing into A_{NS} a minimal number of other terms invariant under $SU(2) \sim O(3)$ transformations. Taking variations, we have

$$\begin{aligned}
&\dot{u}_{R,i} + \boldsymbol{u}_R \cdot \boldsymbol{\nabla} u_{R,i} + \varphi_R \partial_i \varphi_R = \nu \boldsymbol{\nabla}^2 u_{R,i} + f_{R,i} - \boldsymbol{u}_I \cdot \partial_i \boldsymbol{u}_I + u_{I,i} \boldsymbol{\nabla} \cdot \boldsymbol{u}_I, \\
&\dot{\varphi}_R + \boldsymbol{\nabla}(\varphi_R \boldsymbol{u}_R) = \nu \boldsymbol{\nabla}^2 \varphi_R, \\
&\dot{u}_{I,i} + \boldsymbol{u}_R \cdot \boldsymbol{\nabla} u_{I,i} + \varphi_R \partial_i \varphi_I = -\nu \boldsymbol{\nabla}^2 u_{I,i} + f_{I,i} + \boldsymbol{u}_I \cdot \partial_i \boldsymbol{u}_R - u_{I,i} \boldsymbol{\nabla} \cdot \boldsymbol{u}_R, \\
&\dot{\varphi}_I + \boldsymbol{u}_R \cdot \boldsymbol{\nabla} \varphi_I + \varphi_R \boldsymbol{\nabla} \cdot \boldsymbol{u}_I = -\nu \boldsymbol{\nabla}^2 \varphi_I,
\end{aligned} \tag{2.5}$$

where suffixes R and I denote real and imaginary part, respectively. We obtain the equations of motion for the minimal DEVM by setting φ and $\boldsymbol{u}$ real, or Φ be Hermitian. Higher order interactions among $\boldsymbol{u}$ and φ are introduced by incorporating higher order terms in $\boldsymbol{u}$ and φ. After such an extension, φ comes to play the role of the effective or eddy viscosity [2].

We now incorporate a tensor into the model. Let R_{ij}, $i, j = 1 \sim 3$, be the (i, j) component of a tensor R, which is assumed to be invariant under the translation and Galilei transformation. At the beginning of constructing a model of turbulence, as in the minimal DEVM, a traceless complex vector matrix, which is an element of $GL(2, C)$, is considered

$$\mathrm{R}_i = R_{ij} \sigma_j \,. \tag{2.6}$$

($\tilde{\mathrm{R}}_i \equiv \sigma_j R_{ji}$ will do, too.) Repetition of indices implies summation. In (2.6), for simplicity, we omit the center of the group and express the PA mainly in terms of R_{ij}. We envisage that R_{ij}, like φ, is some physical entity which is conveyed by the flow and dissipates by the frictional force. This means that time derivative in the equation of motion is always accompanied with advection term. The relation between R_{ij} and the observed Reynolds stress will be clarified after solving the equations of motion.

To begin with, we notice that the Lagrange derivative $\dot{R}_{ij} + \boldsymbol{u} \cdot \boldsymbol{\nabla} R_{ij}$ of R_{ij} is derived by variation of R_{ij}^* in

$$R_{ij}^* \dot{R}_{ij} + \frac{1}{4} R_{ij}^* (\boldsymbol{u} + \boldsymbol{u}^*) \cdot \boldsymbol{\nabla} R_{ij} - \frac{1}{4} \boldsymbol{\nabla} R_{ij}^* \cdot (\boldsymbol{u} + \boldsymbol{u}^*) R_{ij} \tag{2.7}$$

and subsequently by setting all quantities real. Thus, we are lead to consider the following $SU(2)$ invariant PA

$$\begin{aligned}
A_{\mathrm{R,Ld}} &= i \int \left(R_{ij}^* \dot{R}_{ij} + \frac{1}{4} R_{ij}^* (\boldsymbol{u} + \boldsymbol{u}^*) \cdot \boldsymbol{\nabla} R_{ij} - \frac{1}{4} \boldsymbol{\nabla} R_{ij}^* \cdot (\boldsymbol{u} + \boldsymbol{u}^*) R_{ij} \right) d\tau \\
&= i \int \mathrm{Tr} \left(\frac{1}{2} \mathrm{R}_i^{\dagger} \dot{\mathrm{R}}_i + \frac{1}{8} \mathrm{R}_i^{\dagger} \{\boldsymbol{\sigma}, \Phi + \Phi^{\dagger}\} \cdot \boldsymbol{\nabla} \mathrm{R}_i - \frac{1}{8} \boldsymbol{\nabla} \mathrm{R}_i^{\dagger} \cdot \{\boldsymbol{\sigma}, \Phi + \Phi^{\dagger}\} \mathrm{R}_i \right) d\tau
\end{aligned} \tag{2.8}$$

where Φ and R_i are traceless and $d\tau \equiv d\boldsymbol{r}dt$. Variation of $A_{\mathrm{R,Ld}}$ in R_{ij}^* gives

$$\dot{R}_{ij} + \boldsymbol{u}_R \cdot \boldsymbol{\nabla} R_{ij} + \frac{1}{2} R_{ij} \boldsymbol{\nabla} \cdot \boldsymbol{u}_R \,. \tag{2.9}$$

These terms altogether indeed coincide with the Lagrange derivative of R_{ij} for incompressible fluid that fulfils $\nabla\cdot\boldsymbol{u}_R = 0$. The last term in (2.9) that gives rise to an effect in compressible flow is the necessary outcome in our formulation.

We know that the minimal DEVM without tensor, when applied to turbulent channel and pipe flows, reproduces the profiles of the mean velocity quite well. We do not wish to spoil this favorable feature of the model by incorporating the tensor. The above $A_{\mathrm{R,Ld}}$ in fact does not affect the equation of motion for $\boldsymbol{u}$. It is because only the real part of $\boldsymbol{u}$ appears in $A_{\mathrm{R,Ld}}$ so that the net effect of variation of $\boldsymbol{u}^*$ in $A_{\mathrm{R,Ld}}$ vanishes, i.e., $(\mathrm{i}/4)(R_{ij}^*\nabla R_{ij} - \nabla R_{ij}^* R_{ij}) = 0$ for $\mathrm{Im}\, R_{ij} = 0$. This property of $A_{\mathrm{R,Ld}}$ can be shared by any other PAs involving R_{ij} if the real part of $\boldsymbol{u}$ solely appear in them. In the followings, we will construct invariant PAs involving tensor by employing the real part of $\boldsymbol{u}$ only. Henceforth the symbol $\boldsymbol{u}$ stands for real velocity.

If R_{ij} is to somehow express the Reynolds stress, $\overline{\delta u_i \delta u_j}$, the Reynolds stress equation tells us that the mean velocity and R_{ij} interact via. $\partial_k u_j R_{ik} + \partial_k u_i R_{kj}$ that originates from the advection term in the Navier-Stokes equation. See, e.g., [1,4]. The PA that yields such terms will be given by

$$A_{\mathrm{R,adv}} = \frac{\mathrm{i}}{2}\int (R_{ij}^* - R_{ij})\left(\partial_k u_j (R_{ik} + R_{ik}^*) + \partial_k u_i (R_{kj} + R_{kj}^*)\right) d\tau \, . \tag{2.10}$$

In terms of the Hermitian matrix field $\underline{\Phi} \equiv (\Phi + \Phi^\dagger)/2 = \mathrm{Re}\,\varphi + \boldsymbol{u}\cdot\boldsymbol{\sigma}$, the above $A_{\mathrm{R,adv}}$ can be rewritten as

$$A_{\mathrm{R,adv}} = \frac{\mathrm{i}}{4}\mathrm{Tr}\int \left(R_{ij}^* \left(\sigma_j \partial_k \underline{\Phi}(R_{ik} + R_{ik}^*) + \sigma_i \partial_k \underline{\Phi}(R_{kj} + R_{kj}^*)\right) - h.c.\right) d\tau \, . \tag{2.11}$$

Expressions in terms of the matrices R_i are also possible. They will be useful when we intend to extend the model so as to include the center of the group. In this paper, for simplicity, we restrict ourselves to the traceless R_i.

Let us construct remaining interaction terms in PA that meet the invariance requirements. The diffusion of R_i is given by

$$A_{\mathrm{R,dif}} = \frac{\mathrm{i}}{4}\int \left(\mathrm{Tr}(\eta\underline{\Phi} + \lambda)((\partial_k R_{ij}^*)^2 - (\partial_k R_{ij})^2)\right) d\tau \, . \tag{2.12}$$

We list other possible lower order invariant interactions below. (Note the identities $\mathrm{Tr}(\sigma_i \mathrm{R}_i) = 2R_{kk}$, $\mathrm{Tr}(\mathrm{R}_k \mathrm{R}_k) = 2R_{kl}R_{kl}$ etc.)

$$A_{\mathrm{R}}^{(1)} = \frac{\mathrm{i}}{2}\int \left(g_0 (R_{kk}^*)^2 + g_1 (R_{ij}^*)^2 - c.c.\right) d\tau \, ,$$

$$A_{\mathrm{R}}^{(2)} = \mathrm{i}\int \left(g_2 \mathrm{Tr}(\mathbf{F}\partial_i \underline{\Phi}\sigma_j) R_{ij}^* - c.c.\right) d\tau = \mathrm{i}\int \left(g_2 \tilde{f}_j \partial_i \,\mathrm{Re}\,\varphi R_{ij}^* - c.c.\right) d\tau \, ,$$

$$A_{\mathrm{R}}^{(3)} = \frac{\mathrm{i}}{4}\int \left((g_3 \mathrm{Tr}\underline{\Phi} + 2g_3')\mathrm{Tr}(\partial_i \underline{\Phi}\sigma_j) R_{ij}^* - c.c.\right) d\tau = \mathrm{i}\int \left((g_3 \,\mathrm{Re}\,\varphi + g_3')\partial_i u_j R_{ij}^* - c.c.\right) d\tau \, , \tag{2.13}$$

$$A_{\mathrm{R}}^{(4)} = \mathrm{i}\int \left(M(\nabla\underline{\Phi}) R_{kk}^* - c.c\right) d\tau = \mathrm{i}\int \left((g_4((\nabla \,\mathrm{Re}\,\varphi)^2 + (\nabla \boldsymbol{u})^2) + g_4'(\nabla \,\mathrm{Re}\,\varphi)^2) R_{kk}^* - c.c\right) d\tau \, ,$$

$$A_{\mathrm{R}}^{(5)} = \frac{\mathrm{i}}{2}\int g_5 \mathrm{Tr}\left(\partial_i \underline{\Phi}\partial_j \underline{\Phi} R_{ij}^* - c.c\right) d\tau = \mathrm{i}\int g_5 \left((\partial_i \,\mathrm{Re}\,\varphi \partial_j \,\mathrm{Re}\,\varphi + \partial_i \boldsymbol{u}\cdot\partial_j \boldsymbol{u}) R_{ij}^* - c.c\right) d\tau \, .$$

Note that, instead of employing the matrix $\underline{\Phi} = \Phi$, we can express the pseudo-action in terms of ϕ and $\partial_i \boldsymbol{u}$ by using $2\,\mathrm{Re}\phi = \mathrm{Tr}\Phi$ and $2\partial_i \boldsymbol{u} = \partial_i \mathrm{Tr}(\Phi\boldsymbol{\sigma})$.

$A_{\mathrm{R}}^{(1)}$ represents the inertia of the R-field. $A_{\mathrm{R}}^{(2)}$ describes generation or destruction of the R-field by the force or pressure gradient and the effective viscosity. Other interactions between Φ and R are given in $A_{\mathrm{R}}^{(3)} \sim A_{\mathrm{R}}^{(5)}$, which involve up to the third order interactions. M in $A_{\mathrm{R}}^{(4)}$ is a polynomial of $\boldsymbol{\nabla}\underline{\Phi}$ and acts isotropically to R . Here, we have adopted the simplest one

$$M(\boldsymbol{\nabla}\Phi) = \frac{g_4}{2}\mathrm{Tr}(\boldsymbol{\nabla}\Phi)^2 = g_4\left((\boldsymbol{\nabla}\,\mathrm{Re}\phi)^2 + (\boldsymbol{\nabla}\boldsymbol{u})^2\right). \tag{2.14}$$

The rotational asymmetry that possibly emerge from $A_{\mathrm{R}}^{(5)}$ implies a dependence of interaction on the direction relative to the mean flow and will give rise to the effect of boundary.

We sum up $A_{\mathrm{R,Ld}}$, $A_{\mathrm{R,dif}}$, $A_{\mathrm{R}}^{(1)} \sim A_{\mathrm{R}}^{(5)}$ to obtain the total PA, A_{R}. Taking variations of R_{ij}^* followed by letting all physical quantities be real leads to the equations for R_{ij}:

$$\begin{aligned}\dot{R}_{ij} + \boldsymbol{u}\cdot\boldsymbol{\nabla}R_{ij} &= -\partial_k u_j R_{ik} - \partial_k u_i R_{kj} + \boldsymbol{\nabla}\left((\eta\varphi+\lambda)\boldsymbol{\nabla}R_{ij}\right) - g_0\delta_{ij}R_{kk} - g_1 R_{ij} - g_2\tilde{f}_j\partial_i\varphi \\ &\quad -(g_3\varphi + g_3')\partial_i u_j - \delta_{ij}M(\boldsymbol{\nabla}\Phi) - g_5(\partial_i\varphi\partial_j\varphi + \partial_i\boldsymbol{u}\cdot\partial_j\boldsymbol{u}).\end{aligned} \tag{2.15}$$

The above ‘R-equation’ is not symmetric in i and *j*. The symmetric and anti-symmetric components $S_{ij} \equiv (R_{ij} + R_{ji})/2$ and $A_{ij} \equiv (R_{ij} - R_{ji})/2$ obey the equations

$$\begin{aligned}\dot{S}_{ij} + \boldsymbol{u}\cdot\boldsymbol{\nabla}S_{ij} &= -\partial_k u_j S_{ik} - \partial_k u_i S_{jk} + \boldsymbol{\nabla}\cdot\left((\eta\varphi+\lambda)\boldsymbol{\nabla}S_{ij}\right) - g_0\delta_{ij}S_{kk} - g_1 S_{ij} \\ &\quad -\frac{g_2}{2}(\tilde{f}_i\partial_j\varphi + \tilde{f}_j\partial_i\varphi) - \frac{1}{2}(g_3\varphi + g_3')(\partial_i u_j + \partial_j u_i) - \delta_{ij}M(\boldsymbol{\nabla}\Phi) - g_5(\partial_i\varphi\partial_j\varphi + \partial_i\boldsymbol{u}\cdot\partial_j\boldsymbol{u}),\end{aligned} \tag{2.16a}$$

$$\begin{aligned}\dot{A}_{ij} + \boldsymbol{u}\cdot\boldsymbol{\nabla}A_{ij} &= -\partial_k u_j A_{ik} + \partial_k u_i A_{jk} + \boldsymbol{\nabla}\left((\eta\varphi+\lambda)\boldsymbol{\nabla}A_{ij}\right) - g_1 A_{ij} \\ &\quad +\frac{g_2}{2}(\tilde{f}_i\partial_j\varphi - \tilde{f}_j\partial_i\varphi) - \frac{1}{2}(g_3\varphi + g_3')(\partial_i u_j - \partial_j u_i).\end{aligned} \tag{2.16b}$$

Assuming a steady turbulence with sufficiently weak fields, from (2.16a), we have

$$S_{ij} \approx -\frac{1}{2g_1}(g_3\varphi + g_3')(\partial_i u_j + \partial_j u_i) - \frac{g_0}{g_1}S_{kk}\delta_{ij} - \frac{g_2}{2g_1}(\tilde{f}_i\partial_j\varphi + \tilde{f}_j\partial_i\varphi)\,. \tag{2.17}$$

This relation is compared with the Boussinesq hypothesis employed in some eddy viscosity models,

$$-\tau_{ij} = \nu_t\left(\partial_i u_j + \partial_j u_i\right) - \frac{1}{3}\tau_{kk}\delta_{ij}\,, \tag{2.18}$$

where $\tau_{ij} = \overline{\delta u_i \delta u_j}$ is the Reynolds stress, ν_t the eddy viscosity and $\tau_{kk}/2$ the average turbulent kinetic energy. For locally homogeneous turbulent flow with zero average velocity gradient, (2.18) leads to $\tau_{xx} = \tau_{yy} = \tau_{zz}$, which is in contradiction to experiments (see, e.g., sec.4.1.4 in [1]). By contrast, (2.17) predicts the diagonal component S_{ii} along the direction of the average flow to differ from the orthogonal components if $\partial_i\varphi$ is non-vanishing. However, we have to remember that (2.17) hold only approximately and the full equation (2.16a) has to generally be solved.

We obtain the equation for the would-be turbulent kinetic energy $K \equiv \sum_i S_{ii}/2$ by setting $i = j$ and summing over the indices in the equations for S_{ij}. Having the homogeneous turbulence in mind, let us further assume that S_{ij} are spatially constant and the mean velocity and the viscosity field take the forms $u_i = w_{ij} r_j$ and $\varphi = s_i r_i$. We then have

$$\dot{K} = -S_{ij} w_{ij} - (3g_0 + g_1)K - \frac{g_2}{2}\tilde{\boldsymbol{f}} \cdot \boldsymbol{s} - \frac{3}{2} M(\nabla\Phi) - \frac{g_5}{2}\left(\boldsymbol{s}^2 + w_{ij}^2\right). \tag{2.19}$$

The corresponding equation for the turbulent kinetic energy derived from the Reynolds equation is

$$\dot{K}^{(\mathrm{R})} = -\overline{\delta u_i \delta u_j} w_{ij} - \nu \overline{\left(\nabla \delta \boldsymbol{u}\right)^2} + \overline{\delta \boldsymbol{f} \cdot \delta \boldsymbol{u}}, \tag{2.20}$$

where the superscript (R) and the symbol δ stand for the Reynolds equation and the fluctuation, respectively. Comparing these equations, we expect that the first term in K-equation will represent the production of the turbulent kinetic energy due to shear flow. The dissipative term in $K^{(\mathrm{R})}$-equation is intensively represented by the second term in K-equation provided that $3g_0 + g_1 > 0$. The remaining terms in the K-equation are new ones peculiar to our modeling.

The substantial difference between our R-equation and the Reynolds equations lies in that the latter do not close because of the third moments, while the former does in the sense of the fulfillment of the variational principle. If the success of the minimal DEVM is due to the closure nature of its dynamical system, we may hope the R-equation to work as well.

3. Channel turbulent flow

In this section, we apply the *S*-equations (2.16a) in the previous section to a channel turbulent flow. We are interested in the relation between calculated S_{ij} and the observed Reynolds stress $\overline{\delta u_i \delta u_j}$. The experiment of the diagonal Reynolds stress for channel flow has been known as is given in, e.g., [5]. Many works on the direct numerical simulation (See, e.g., [6–8]) have reported results consistent with the experiment. The check of the consistency is important because experiments on turbulence in laboratory sometimes exhibit discrepancy [6]. In the followings, we refer to the results on the Reynolds stress reported in [5] for diagonal components and in [9] for an off-diagonal component. They are shown in Figure 1.

For a channel flow $\bar{\boldsymbol{u}} = (u_x, 0, 0)$ with a unit half channel width, we choose the *y*-axis to be perpendicular to the walls so that it is sufficient to consider $0 \le y \le 1$ because of the symmetry. One of the walls is at $y = 0$. Taking the symmetry of the Reynolds stress into account, we set

$$S_{xz} = S_{zx} = S_{yz} = S_{zy} = 0. \tag{3.1}$$

Assuming that S_{ij} has y-dependences only, the remaining quantities in (2.16) obey the equations (the prime stands for derivative in y)

$$\left((\eta\varphi + \lambda) S'_{xx}\right)' - 2u'_x S_{xy} - g_0 \Sigma - g_1 S_{xx} - M(\nabla\Phi) = 0, \tag{3.2a}$$

$$\left((\eta\varphi + \lambda) S'_{yy}\right)' - g_0 \Sigma - g_1 S_{yy} - M(\nabla\Phi) - g_5(\varphi'^2 + u_x'^2) = 0, \tag{3.2b}$$

$$\left((\eta\varphi+\lambda)S'_{zz}\right)' - g_0\Sigma - g_1 S_{zz} - M(\nabla\Phi) = 0\,, \tag{3.2c}$$

$$\left((\eta\varphi+\lambda)S'_{xy}\right)' - u'_x S_{yy} - g_1 S_{xy} - \frac{g_2}{2}\tilde{f}_x\varphi' - \frac{1}{2}(g_3\varphi+g'_3)u'_x = 0\,, \tag{3.2d}$$

$$\left((\eta\varphi+\lambda)A'_{xy}\right)' - g_1 A_{xy} + \frac{g_2}{2}\tilde{f}_x\varphi' + \frac{g_3}{2}\varphi u'_x = 0. \tag{3.2e}$$

Here, $S_{xx}+S_{yy}+S_{zz}\equiv\Sigma$. Summing (3.2a) ~ (3.2c) yields the equation for Σ

$$\left((\eta\varphi+\lambda)\Sigma'\right)' - 2u'_x S_{xy} - (3g_0+g_1)\Sigma - 3M(\nabla\Phi) - g_5(\varphi'^2+u'^2_x) = 0\,. \tag{3.3}$$

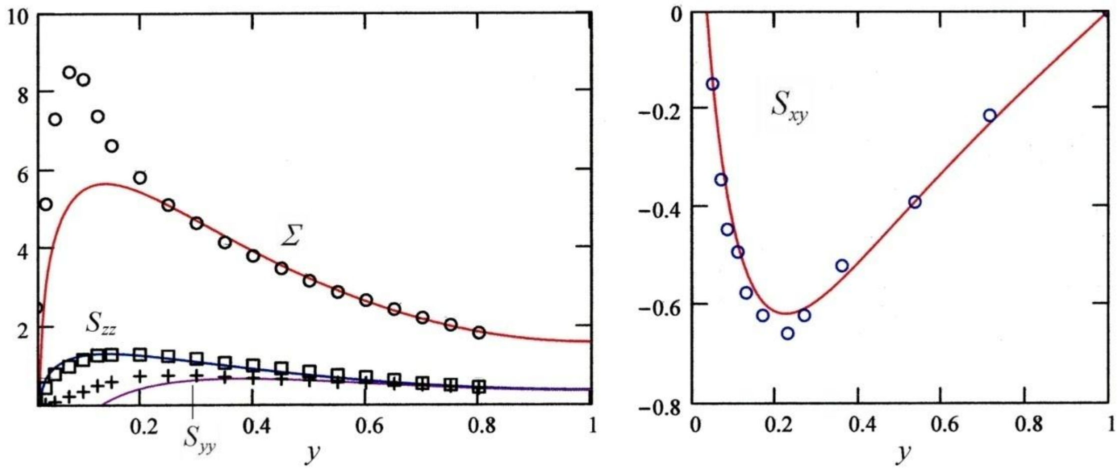

Figure1. Example of solutions to (3.3) are depicted by solid curves. Parameters are $\bar{g}_0=-2/3$, $\bar{g}_1=50$, $\bar{g}_2\tilde{f}_x=-118$, $\bar{g}_3=-40$, $\bar{g}'_3=39$, $\bar{g}_4=-0.9$, $\bar{g}_5=-1$, $\bar{\lambda}=5.2$, $\eta=1$. Boundary values at $y=1$ are $S_{yy}=0.37$, $S_{zz}=0.38$, $\Sigma=1.6$, $S'_{yy}=S'_{zz}=\Sigma'=0$, $S_{xy}=0$, $S'_{xy}=0.78$. Left panel: Symbols denote the data of $\mathrm{Re}_c=3755$ for $\overline{\delta\boldsymbol{u}^2}$ (circles), $\overline{\delta u_z^2}$ (squares) and $\overline{\delta u_y^2}$ (crosses) adapted from [5]. Right panel: Circles denote the data of $\mathrm{Re}_c=2970$ for $\overline{\delta u_x\delta u_y}$ adapted from [9].

We would like to find out whether there exist values of parameters with which these equations reproduce the experimental results for the Reynolds stress [5–8]. Unfortunately, these equations have too many parameters to handle directly. In order to minimize the unruly effects from varying these model parameters, we consider only the equations for S_{yy}, S_{zz}, S_{xy} and Σ.

In the previous work [2], a dimensionless field — viscosity function — was introduced by $\varphi/\xi_0\to\varphi$ with a parameter ξ_0 that has the dimension of velocity. ξ_0 is the position where the potential of original dimensionful φ takes the local minimum, thereby restricting the range of the dimensionless φ as $\varphi<1$. For convenience, we also rescale the mean velocity field as

$$u_x\to\xi_0 u_x$$

and divide the equations by $\eta\xi_0$ to express the equations (3.3), (3.2b) and (3.2d) as

$$\begin{aligned}
&((\varphi+\bar{\lambda})\Sigma')' - \frac{2}{\eta}u_x' S_{xy} - (3\bar{g}_0+\bar{g}_1)\Sigma - 3\bar{g}_4(\varphi'^2+u_x'^2) - \bar{g}_5(\varphi'^2+u_x'^2) = 0,\\
&((\varphi+\bar{\lambda})S_{yy}')' - \bar{g}_0\Sigma - \bar{g}_1 S_{yy} - \bar{g}_4(\varphi'^2+u_x'^2) - \bar{g}_5(\varphi'^2+u_x'^2) = 0,\\
&((\varphi+\bar{\lambda})S_{xy}')' - \frac{1}{\eta}u_x' S_{yy} - \bar{g}_1 S_{xy} - \frac{\bar{g}_2}{2}\tilde{f}_x\varphi' - \frac{1}{2}(\bar{g}_3\varphi+\bar{g}_3')u_x' = 0,
\end{aligned} \tag{3.4}$$

where new constants were introduced by

$$\bar{\lambda} = \lambda/(\eta\xi_0),\ \bar{g}_{0,1} = g_{0,1}/(\eta\xi_0),\ \bar{g}_2 = g_2/\eta,\ \bar{g}_3 = g_3\xi_0/\eta,\ \ \bar{g}_3' = g_3'/\eta,\ \bar{g}_{4,5} = g_{4,5}\xi_0/\eta\ .$$

Note that the equations for Σ, S_{yy} and S_{xy} are closed.

$u_x(y)$ has been known from experiments [5,10] or by direct numerical calculation [6–8]. Its approximate functional form is given in Appendix. $\varphi(y)$ in minimal DEVM has been numerically calculated by Takahashi [2], according to which, $\varphi(y)$ approximately linearly increases near the wall and gradually approaches a constant value. For the present calculation for flows with the Reynolds number of a few thousands, we adopt an approximation

$$\varphi(y) \approx 0.6\sin(\pi y/2) \tag{3.5}$$

for the sake of simplicity in numerical calculations. The Σ obtained by solving (3.4) is employed in solving (3.2c) for S_{zz}. The results are shown in Fig.1 together with the experimental data. The model parameters are found by trial and error. The “boundary” condition is given at $y=1$, i.e., at the center of the channel. The calculation shows fairly good agreements with experiments in the region $y>0.3$ for the choice of the model parameters given in the caption. Other set of values may be also possible.

Marked deficits in Σ and S_{yy} from the experimentally known Reynolds stress are observed in $y<0.3$. This indicates that our model lacks some terms that should be effective near the wall. From Fig.1, it is anticipated that addition of interactions of S_{xx} and S_{yy} with $\nabla\underline{\Phi}$ will remedy the disagreement for $\nabla\underline{\Phi}$ has relatively large values in the vicinity of the wall. The simplest candidates may consist of terms $\mathrm{i}F(\nabla\underline{\Phi})R_{ij}^*\partial_k u_i\partial_k u_j$, $\mathrm{i}G(\nabla\underline{\Phi})R_{ij}^*\partial_i\phi\partial_j\phi$, $\mathrm{i}H(\nabla\underline{\Phi})R_{ij}^*\partial_i u_k\partial_j u_k$ and their complex conjugates. For the sake of simplicity, we try the first two terms with

$$F = g_6(\nabla\varphi)^2,\quad G = g_7(\nabla\varphi)^2\,, \tag{3.6}$$

which respectively give new contribution to the equation for S_{xx} and S_{yy} near $y=0$. Accordingly, the equations for Σ and S_{yy} are altered as

$$\begin{aligned}
&((\varphi+\bar{\lambda})\Sigma')' - \frac{2}{\eta}u_x' S_{xy} - (3\bar{g}_0+\bar{g}_1)\Sigma - 3\bar{g}_4(\varphi'^2+u_x'^2) - \bar{g}_5(\varphi'^2+u_x'^2) - \bar{g}_6\varphi'^2 u_x'^2 - \bar{g}_7\varphi'^4 = 0,\\
&((\varphi+\bar{\lambda})S_{yy}')' - \bar{g}_0\Sigma - \bar{g}_1 S_{yy} - \bar{g}_4(\varphi'^2+u_x'^2) - \bar{g}_5(\varphi'^2+u_x'^2) - \bar{g}_7\varphi'^4 = 0,
\end{aligned} \tag{3.7}$$

where $\bar{g}_{6,7} = g_{6,7}\xi_0^3/\eta$. These equations together with the one for S_{xy} are solved with the result depicted in Figure 2. As was expected, the region of y with better fitting to data has been extended toward the wall. S_{zz} and S_{xy} are not visibly changed by the above modification of the equations.

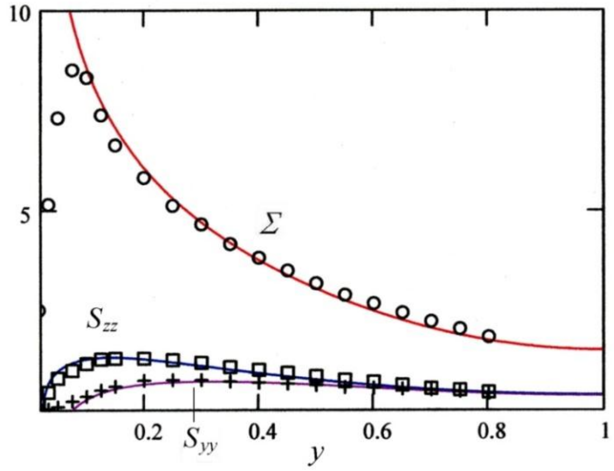

Figure 2. Solutions to (3.2) added by new terms (3.6). The meanings of curves and symbols are same as in Fig.1. Parameters are $\bar{g}_0 = -2/3$, $\bar{g}_1 = 50$, $\bar{g}_2 \tilde{f}_x = -118$, $\bar{g}_3 = -40$, $\bar{g}_3' = 39$, $\bar{g}_4 = -0.9$, $\bar{g}_5 = -1$, $\bar{g}_6 = 4$, $\bar{g}_7 = 90$, $\bar{\lambda} = 5.2$, $\eta = 1$. Same boundary values as in Fig.1 are adopted except that $\Sigma = 1.5$.

We notice that the components of the symmetric tensor agree semi-quantitatively with the experimental results for the Reynolds stress in the central and upper half of the logarithmic region.

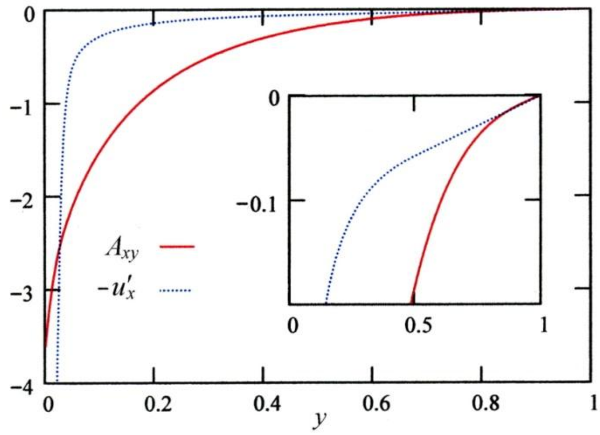

Figure 3. Example of A_{xy} as a solution to (3.2) for the same parameters as in Fig.1. Red solid curve: A_{xy}. Blue dotted curve: $-u_x'$ (vorticity). The inset depicts the same curves with small ordinate values being enlarged.

Finally, in Figure 3, we show the result for the antisymmetric component A_{xy} which is obtained by solving the A-equation (3.2e) with a boundary condition $A_{xy} = 0, A_{xy}' = u_x'$ at $y = 1$. Although A is antisymmetric, it substantially differs from the vorticity. The physical meaning of A is unclear.

4. Concluding remarks

We constructed a conventional model that incorporates a tensor into the minimal DEVM in an invariant way and explored if the symmetric part of the tensor shows some correspondence to the

Reynolds stress observed in channel turbulent flow. The new tensor terms were added to the minimal DEVM so as not to affect the mean velocity which had already been well reproduced within the minimal DEVM [2]. We found that a correspondence to the Reynolds stress in fact exists off the wall and is improved further by adding higher order interaction terms. A new form of the Boussinesq hypothesis adapted to the model was also found. The preliminary result reported in this paper suggests that tensor model seems promising in describing the Reynolds stress.

A relative improvement of the model was achieved by incorporating higher order terms which are not small as compared with the lower order terms. Indeed, our fitting gives

$$\max(|\bar{g}_6|,|\bar{g}_7|)/\max(|\bar{g}_0|,|\bar{g}_1|,|\bar{g}_2\tilde{f}_x|,|\bar{g}_3|,|\bar{g}_3'|,|\bar{g}_4|,|\bar{g}_5|)=0.75\,.$$

Higher order terms may not be safely neglected.

The DEVM gives the action-reaction relations among the elements of fluid through the variational principle as was manifested in [2]. Taking advantage of this property of the DEVM, we also want to know what the reaction of the Reynolds stress to the mean flow is. It is desirable to find a more sophisticated method for constructing a model.

From an aesthetic point of view, it is preferable to treat tensor and vector equally. For this purpose, it may be worthwhile to consider integrating the components of the tensor as a vector matrix by

$$\mathrm{R}_i = R_{ij}\sigma_j \quad \text{or} \quad \tilde{\mathrm{R}}_i = \sigma_j R_{ji}$$

and explicitly express the pseudo-action in an $SU(2)$ invariant way. In order to close multiplications in $GL(2, C)$, we add the center as

$$\mathrm{R}_i = v_i + R_{ij}\sigma_j \quad \text{or} \quad \tilde{\mathrm{R}}_i = v_i + \sigma_j R_{ji}$$

Here, $\boldsymbol{v}$ can be a vector or an axial-vector. Such an extension of R_i is analogous to the one in the minimal DEVM, wherein the effective viscosity was introduced by requiring the scalar matrices Φ and $\Phi^\dagger$ to form a set closed under multiplication. It is interesting to note that, if we require these matrices to have a definite parity, $\boldsymbol{v}$ will be an axial-vector. One candidate is the vorticity. The role of the antisymmetric components of the tensor R, if any, will be reexamined in such a model that embodies the aim of the original DEVM. Further study along this line will be intriguing.

Appendix: An empirical formula for the mean velocity of turbulent channel flow

The interval of the dimensionless coordinate ζ is divided into three regions.

i) Viscous sublayer + buffer region $\zeta \le \zeta_1$:

$u(\zeta) = A\sin(\zeta / A).$

ii) Buffer region + logarithmic region $\zeta_1 \le \zeta \le \zeta_2$:

$u(\zeta) = (2.4\ln\zeta + 5.5)[1 - B\exp(-C(\zeta - \zeta_1))].$

iii) Central region $\zeta_2 \le \zeta \le \zeta_{\max}$:

$u(\zeta) = D - E(\zeta_{\max} - \zeta)^2.$

Imposing a condition that $u(\zeta)$ belongs to the C^1 class yields following relations among constants

$$B(A) = 1 - A\sin(\zeta_1 / A) / Fl(\zeta_1),$$

$$C(A) = \frac{\cos(\zeta_1 / A) - (2.4/\zeta_1)(1 - B(A))}{B(A)Fl(\zeta_1)},$$

$$D(A) = Fl(\zeta_2)\big(1 - B(A)\exp[-C(A)(\zeta_2 - \zeta_1)]\big) + E(A)(\zeta_{\max} - \zeta_2)^2,$$

$$E(A) = \frac{1}{2(\zeta_{\max} - \zeta_2)}\left[\frac{2.4}{\zeta_2} + B(A)\left(Fl(\zeta_2)C(A) - \frac{2.4}{\zeta_2}\exp[-C(A)(\zeta_2 - \zeta_1)]\right)\right],$$

where

$$Fl(\zeta) = 2.4\ln\zeta + 5.5.$$

In the text, following values are employed

$$\zeta_1 = 10,\ \ \zeta_2 = \zeta_{\max}/2,\ \ \zeta_{\max} = 450,\ \ A = 15.$$

Conflict of interest

The authors declare no conflict of interest.

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