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Research article

L^p-analysis of one-dimensional repulsive Hamiltonian with a class of perturbations

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Abstract: The spectrum of one-dimensional repulsive Hamiltonian with a class of perturbations $H_p = -\frac{d^2}{dx^2} - x^2 + V(x)$ in $L^p(\mathbb{R})$ $(1 < p < \infty)$ is explicitly given. It is also proved that the domain of H_p is embedded into weighted L^q-spaces for some $q > p$. Additionally, non-existence of related Schrödinger (C_0-)semigroup in $L^p(\mathbb{R})$ is shown when $V(x) \equiv 0$.

Keywords: repulsive Hamiltonian; WKB methods
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1. Introduction

In this paper we consider

$$H := -\frac{d^2}{dx^2} - x^2 + V(x) \tag{1}$$

in $L^p(\mathbb{R})$, where $V \in C(\mathbb{R})$ is a real-valued and satisfies $V(x) \geq -a(1 + x^2)$ for some constant $a \geq 0$ and

$$\int_{\mathbb{R}} \frac{|V(x)|}{\sqrt{1 + x^2}}\, dx < \infty. \tag{2}$$

The operator (1) describes the quantum particle affected by a strong repulsive force from the origin. In fact, in the classical sense the corresponding Hamiltonian (functional) is given by $\hat{H}(x, p) = p^2 - x^2$ and then the particle satisfying $\dot{x} = \partial_p \hat{H}$ and $\dot{p} = -\partial_x \hat{H}$ goes away much faster than that for the free Hamiltonian $\hat{H}_0(x, p) = p^2$.

In the case where $p = 2$, the essential selfadjointness of H, endowed with the domain $C_0^\infty(\Omega)$, has been discussed by Ikebe and Kato [7]. After that several properties of H is found out in a mount of subsequent papers (for studies of scattering theory e.g., Bony et al. [2], Nicoleau [10] and also Ishida [8]).

In contrast, if p is different from 2, then the situation becomes complicated. Actually, papers which deals with the properties of H is quite few because of absence of good properties like symmetricity. In the L^p-framework, it is quite useful to consider the accretivity and sectoriality of the second-order differential operators. In fact, the case $-\frac{d^2}{dx^2} + V(x)$ with a nonnegative potential V is formally sectorial in L^p, and therefore one can find many literature even N-dimensional case (e.g., Kato [9], Goldstein [6], Tanabe [14], Engel-Nagel [5]). However, it seems quite difficult to describe such a kind of non-accretive operators in a certain unified theory in the literature.

The present paper is in a primary position to make a contribution for theory of non-accretive operators in L^p as mentioned above. The aim of this paper is to give a spectral properties of $H = -\frac{d^2}{dx^2} - x^2 + V(x)$ for the case where $V(x)$ can be regarded as a perturbation of the leading part $-\frac{d^2}{dx^2} - x^2$; note that if $V(x) = [\log(e + |x|)]^{-\alpha}$ ($\alpha \in \mathbb{R}$), then $\alpha < 1$ is admissible, which is same threshold as in the short range potential for $-\frac{d^2}{dx^2} - x^2$ stated in Bony [2] and also Ishida [8].

Here we define the minimal realization $H_{p,\min}$ of H in $L^p = L^p(\mathbb{R})$ as

$$\begin{cases} D(H_{p,\min}) := C_0^\infty(\mathbb{R}), \\ H_{p,\min}u(x) := -u''(x) - x^2u(x) + V(x)u(x). \end{cases} \tag{3}$$

Theorem 1.1. *For every $1 < p < \infty$, $H_{p,\min}$ is closable and the spectrum of the closure H_p is explicitly given as*

$$\sigma(H_p) = \left\{\lambda \in \mathbb{C}\ ;\ |\mathrm{Im}\,\lambda| \le \left|1 - \frac{2}{p}\right|\right\}.$$

Moreover, for every $1 < p < q < \infty$, one has consistence of the resolvent operators:

$$(\lambda + H_p)^{-1} f = (\lambda + H_q)^{-1} f \ \textit{a.e. on}\ \mathbb{R} \quad \forall\, \lambda \in \rho(H_p) \cap \rho(H_q), \quad \forall\, f \in L^p \cap L^q.$$

Remark 1.1. If $p = 2$, then our assertion is nothing new. The crucial part is the case $p \neq 2$ which is the case where the symmetricity of H breaks down. The similar consideration for $-\frac{d^2}{dx^2} + V$ (but in L^2-setting) can be found in Dollard-Friedman [4].

This paper is organized follows: In Section 2, we prepare two preliminary results. In Section 3, we consider the fundamental systems of $\lambda u + Hu = 0$, and estimate the behavior of their solutions. By virtue of that estimates, we will describe the resolvent set of H_p in Section 4. In section 5, we prove never to be generated C_0-semigroups by $\pm iH_p$ under the condition $V = 0$.

2. Preliminary results

First we state well-known results for the essentially selfadjointness of Schrödinger operators in L^2 which is firstly described in [7]. We would like to refer also Okazawa [12].

Theorem 2.1 (Okazawa [12, Corollary 6.11])**.** *Let $V(x)$ be locally in $L^2(\mathbb{R})$ and assume that $V(x) \ge -c_1 - c_2|x|^2$, where $c_1, c_2 \ge 0$ are constants. Then $H_{2,\min}$ is essentially selfadjoint.*

Next we note the asymptotic behavior of solutions to second-order linear ordinary differential equations of the form

$$y''(x) = (\Phi(x) + \Psi(x))y(x)$$

in which the term $\Psi(x)y(x)$ can be treated as a perturbation of the leading part $\Phi(x)y(x)$.

Theorem 2.2 (Olver [13, Theorem 6.2.2 (p.196)])**.** *In a given finite or infinite interval (a_1, a_2), let $a \in (a_1, a_2)$, $\Psi(x)$ a positive, real, twice continuously differentiable function, $\Psi(x)$ a continuous real or complex function, and*

$$F(x) = \int \left\{ \frac{1}{\Phi(x)^{1/4}} \frac{d^2}{dx^2} \left(\frac{1}{\Phi(x)^{1/4}} \right) - \frac{\Psi(x)}{\Phi(x)^{1/2}} \right\} dx.$$

Then in this interval the differential equation

$$\frac{d^2 w}{dx^2} = \{\Phi(x) + \Psi(x)\} w$$

has twice continuously differential solutions

$$\begin{aligned} w_1(x) =& \frac{1}{\Phi(x)^{1/4}} \exp \left\{ i \int \Phi(x)^{1/2}\, dx \right\} (1 + \varepsilon_1(x)), \\ w_2(x) =& \frac{1}{\Phi(x)^{1/4}} \exp \left\{ -i \int \Phi(x)^{1/2}\, dx \right\} (1 + \varepsilon_2(x)), \end{aligned}$$

such that

$$|\varepsilon_j(x)|,\ \frac{1}{\Phi(x)^{1/2}} |\varepsilon_j(x)| \leq \exp \left\{ \frac{1}{2} \mathcal{V}_{a_j,x}(F) \right\} - 1 \quad (j = 1,\, 2)$$

provided that $\mathcal{V}_{a_j,x}(F) < \infty$ (where $\mathcal{V}_{a_j,x}(F) = \int |F'(t)|\, dt$ is the total variation of F). If $\Psi(x)$ is real, then the solutions $w_1(x)$ and $w_2(x)$ are complex conjugates.

For the above theorem, see also Beals-Wong [1, 10.12, p.355].

3. Fundamental systems of $\lambda u - u'' - x^2 u + V u = 0$

3.1. The case $\lambda \in \mathbb{R}$

We consider the behavior of solutions to

$$\lambda u(x) - u''(x) - x^2 u(x) + V(x)u(x) = 0, \quad x \in \mathbb{R}, \tag{4}$$

where $\lambda \in \mathbb{R}$.

Proposition 3.1. *There exist solutions $u_{\lambda,1}, u_{\lambda,2}$ of* (4) *such that $u_{\lambda,1}$ and $u_{\lambda,2}$ are linearly independent and satisfy*

$$\begin{aligned} |u_{\lambda,1}(x)| \leq C_\lambda (1 + |x|)^{-\frac{1}{2}}, \quad & |u_{\lambda,2}(x)| \leq C_\lambda (1 + |x|)^{-\frac{1}{2}} \quad \forall x \in \mathbb{R}, \\ |u_{\lambda,1}(x)| \geq \frac{1}{2} (1 + |x|)^{-\frac{1}{2}}, \quad & |u_{\lambda,2}(x)| \geq \frac{1}{2} (1 + |x|)^{-\frac{1}{2}} \quad \forall x \geq R_\lambda \end{aligned}$$

for some constants $C_\lambda, R_\lambda > 0$ independent of x. In particular, $u_{\lambda,1}$, $u_{\lambda,2} \in L^p(\mathbb{R})$ if and only if $2 < p < \infty$.

Proof. First we consider (4) for $x > 0$. Using the Liouville transform

$$v(y) := (2y)^{\frac{1}{4}} u\left((2y)^{\frac{1}{2}}\right), \qquad \text{or equivalently,} \qquad u(x) = x^{-\frac{1}{2}} v\left(\frac{x^2}{2}\right),$$

we have

$$(\lambda - x^2)x^{-\frac{1}{2}} v\left(\frac{x^2}{2}\right) = u''(x) - V(x)u(x) = x^{\frac{3}{2}} v''\left(\frac{x^2}{2}\right) + \frac{3}{4} x^{-\frac{5}{2}} v\left(\frac{x^2}{2}\right) - x^{-\frac{1}{2}} V(x) v\left(\frac{x^2}{2}\right).$$

Therefore noting that $y = x^2/2$, we see that

$$v''(y) = \left[-\left(1 - \frac{\lambda}{4y}\right)^2 + \frac{\lambda^2 - 3}{16y^2} + \frac{V((2y)^{\frac{1}{2}})}{2y}\right] v(y) = (\Phi(y) + \Psi(y))v(y). \tag{5}$$

Here we have put for $y > 0$,

$$\Phi(y) := -\left(1 - \frac{\lambda}{4y}\right)^2, \quad \Psi(y) := \frac{\lambda^2 - 3}{16y^2} + \frac{V((2y)^{\frac{1}{2}})}{2y}.$$

Let

$$\Pi(y) := |\Phi(y)|^{-\frac{1}{4}} \left(-\frac{d^2}{dx^2} + \Psi(y)\right) |\Phi(y)|^{-\frac{1}{4}}, \quad y \geq \lambda_+ := \max\{\lambda, 0\}.$$

Then we see that for every $y \geq \lambda_+$,

$$|\Pi(y)| \leq \left(1 - \frac{\lambda}{4y}\right)^{-3} \frac{3\lambda^2}{64y^2} + \left(1 - \frac{\lambda}{4y}\right)^{-2} \frac{\lambda}{4y^3} + \left(1 - \frac{\lambda}{4y}\right)^{-1} \frac{|\lambda^2 - 3|}{16y^2} + \frac{|V((2y)^{\frac{1}{2}})|}{2y} \leq \frac{M_\lambda}{y^2} + \frac{|V((2y)^{\frac{1}{2}})|}{2y},$$

where M_λ is a positive constant depending only on λ. Therefore

$$\int_{\lambda_+}^{\infty} |\Pi(y)|\, dy \leq M_\lambda \int_{\lambda_+}^{\infty} \frac{1}{y^2}\, dy + \int_{\sqrt{2\lambda_+}}^{\infty} \frac{|V(x)|}{x}\, dx < \infty.$$

Thus $\Pi \in L^1((\lambda_+, \infty))$. By Theorem 2.2, we obtain that there exists a fundamental system $(v_{\lambda,1}, v_{\lambda,2})$ of (5) such that

$$v_{\lambda,1}(y) y^{i\frac{\lambda}{4}} e^{-iy} \to 1, \quad v_{\lambda,2}(y) y^{-i\frac{\lambda}{4}} e^{iy} \to 1 \quad \text{as } y \to \infty$$

(see also [11]). Taking $u_{\lambda,j}(x) = x^{-\frac{1}{2}} v_{\lambda,j}(x^2/2)$ for $j = 1, 2$, we obtain that $(u_{\lambda,1}, u_{\lambda,2})$ is a fundamental system of (4) on (λ_+, ∞) and

$$u_{\lambda,1}(y) x^{\frac{1}{2} + i\frac{\lambda}{2}} e^{-i\frac{x^2}{2}} \to 2^{-i\frac{\lambda}{4}}, \quad u_{\lambda,2}(x) x^{\frac{1}{2} - i\frac{\lambda}{2}} e^{i\frac{x^2}{2}} \to 2^{i\frac{\lambda}{4}},$$

as $x \to \infty$. The above fact implies that there exists a constant $R_\lambda > \lambda_+$ such that

$$\frac{1}{2} x^{-\frac{1}{2}} \leq |u_{\lambda,j}(x)| \leq \frac{3}{2} x^{-\frac{1}{2}}, \quad x \geq R_\lambda, \quad j = 1, 2.$$

We can extend $(u_{\lambda,1}, u_{\lambda,2})$ as a fundamental system on $\mathbb{R}$. By applying the same argument as above to (4) for $x < 0$, we can construct a different fundamental system $(\tilde{u}_{\lambda,1}, \tilde{u}_{\lambda,2})$ on $\mathbb{R}$ satisfying

$$\frac{1}{2}|x|^{-\frac{1}{2}} \leq |\tilde{u}_{\lambda,j}(x)| \leq \frac{3}{2}|x|^{-\frac{1}{2}}, \quad x \leq -\tilde{R}_\lambda, \quad j = 1, 2.$$

By definition of fundamental system, $u_{\lambda,j}$ can be rewritten as

$$u_{\lambda,1}(x) = c_{11}\tilde{u}_{\lambda,1}(x) + c_{12}\tilde{u}_{\lambda,2}(x), \quad u_{\lambda,2}(x) = c_{21}\tilde{u}_{\lambda,1}(x) + c_{22}\tilde{u}_{\lambda,2}(x).$$

Hence we have the upper and lower estimates of $u_{\lambda,j}$ ($j = 1, 2$), respectively. □

3.2. The case $\lambda \in \mathbb{C} \setminus \mathbb{R}$

We consider the behavior of solutions to

$$\lambda u(x) - u''(x) - x^2 u(x) + V(x)u(x) = 0, \tag{6}$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Im}\lambda > 0$. The case $\operatorname{Im}\lambda < 0$ can be reduced to the problem $\operatorname{Im}\lambda > 0$ via complex conjugation.

3.2.1. Properties of solutions to an auxiliary problem

We start with the following function φ_λ:

$$\varphi_\lambda(x) := x^{-\frac{1+\lambda i}{2}} e^{i\frac{x^2}{2}}, \quad x > 0. \tag{7}$$

Then by a direct computation we have

Lemma 3.2. *φ_λ satisfies*

$$\lambda\varphi_\lambda - \varphi_\lambda'' - x^2\varphi_\lambda + g_\lambda\varphi_\lambda = 0, \quad x \in (0, \infty), \tag{8}$$

where $g_\lambda(x) := \frac{(1+\lambda i)(3+\lambda i)}{4x^2}$, $x > 0$.

Remark 3.1. If $\lambda = i$ or $\lambda = 3i$, then φ_λ is nothing but a solution of the original equation (6) with $V = 0$.

Next we construct another solution of (8) which is linearly independent of φ_λ. Before construction, we prepare the following lemma.

Lemma 3.3. *Let λ satisfy $\operatorname{Im}\lambda > 0$ and let φ_λ be given in (7). Then for every $a > 0$, there exists $F_a^\lambda \in \mathbb{C}$ such that*

$$\int_a^x \varphi_\lambda(t)^{-2}\, dt \to F_a^\lambda \quad \text{as } x \to \infty$$

and then $x \mapsto \int_a^x \varphi_\lambda(t)^{-2}\, dt - F_a^\lambda$ is independent of a. Moreover, for every $x > 0$,

$$\left| \int_a^x \varphi_\lambda(t)^{-2}\, dt - F_a^\lambda - \frac{i}{2} x^{\lambda i} e^{-ix^2} \right| \le C_\lambda x^{-\operatorname{Im}\lambda - 2},$$

where $C_\lambda := \dfrac{|\lambda|}{4}\left(1 + \sqrt{1 + \left(\dfrac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda + 2}\right)^2}\right)$.

Remark 3.2. If $a = 0$ and $\lambda = i$, then F_0^i gives the Fresnel integral $\lim_{x\to\infty} \int_0^x e^{-it^2}\, dt$. Hence $F_0^i = \sqrt{\pi/8}(1 - i)$.

Proof. By integration by part, we have

$$\int_a^x t^{1+\lambda i} e^{-it^2}\, dt = \left(\frac{i}{2} x^{\lambda i} e^{-ix^2} - \frac{i}{2} a^{\lambda i} e^{-ia^2}\right) + \frac{\lambda i}{4}\left(x^{\lambda i - 2} e^{-ix^2} - a^{\lambda i - 2} e^{-ia^2}\right) - \frac{\lambda i(\lambda i - 2)}{4} \int_a^x t^{\lambda i - 3} e^{-it^2}\, dt.$$

Noting that $t^{\lambda i-3}e^{-it^2}$ is integrable in (a,∞), we have

$$\int_a^x t^{1+\lambda i}e^{-it^2}\,dt \to -\frac{i}{2}a^{\lambda i}e^{-ia^2} - \frac{\lambda i}{4}a^{\lambda i-2}e^{-ia^2} - \frac{\lambda i(\lambda i-2)}{4}\int_a^\infty t^{\lambda i-3}e^{-it^2}\,dt =: F_a^\lambda$$

as $x\to\infty$. And therefore $\int_a^x t^{1+\lambda i}e^{-it^2}\,dt - F_a^\lambda$ is independent of a and

$$\left|\int_a^x t^{1+\lambda i}e^{-it^2}\,dt - F_a^\lambda - \frac{i}{2}x^{\lambda i}e^{-ix^2}\right| = \left|\frac{\lambda}{4}x^{-\lambda-2}e^{-ix^2} + \frac{\lambda i(\lambda i-2)}{4}\int_x^\infty t^{\lambda i-3}e^{-it^2}\,dt\right| \le C_\lambda x^{-\operatorname{Im}\lambda-2}.$$

This is nothing but the desired inequality. □

Lemma 3.4. *Let φ_λ be as in* (7) *and define ψ_λ as*

$$\psi_\lambda(x) := \varphi_\lambda(x)\int_a^x \frac{1}{\varphi_\lambda(t)^2}\,dt - F_a^\lambda\varphi_\lambda(x), \quad x>0. \tag{9}$$

Then ψ_λ is independent of a and $(\varphi_\lambda,\psi_\lambda)$ is a fundamental system of (8)*. Moreover, there exists $a_0>0$ such that*

$$\frac{1}{3}x^{-\frac{\operatorname{Im}\lambda+1}{2}} \le |\psi_\lambda(x)| \le x^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x\in[a_0,\infty).$$

Proof. From Lemma 3.3 we have

$$x^{\frac{\operatorname{Im}\lambda+1}{2}}\left|\psi_\lambda(x) - \frac{i}{2}x^{-\frac{1-\lambda i}{2}}e^{-i\frac{x^2}{2}}\right| = x^{\frac{\operatorname{Im}\lambda+1}{2}}|\varphi_\lambda(x)|\left|\int_a^x \frac{1}{\varphi_\lambda(t)^2}\,dt - F_a^\lambda - \frac{i}{2}x^{\lambda i}e^{-ix^2}\right| \le C_\lambda x^{-2}.$$

Putting $a_0 = (6C_\lambda)^{\frac{1}{2}}$, we deduce the desired assertion. □

3.2.2. Fundamental system of the original problem

Next we consider

$$\lambda w - w'' - x^2 w + g_\lambda w = \tilde{g}_\lambda h, \quad x>0 \tag{10}$$

with a given function h, where g_λ is given as in Lemma 3.2 and $\tilde{g}_\lambda := g_\lambda - V$. To construct solutions of (6), we will define two types of solution maps $h\mapsto w$ and consider their fixed points.

First we construct a solution of (6) which behaves like ψ_λ at infinity.

Definition 3.5. *For $b>0$, define*

$$Uh(x) := \psi_\lambda(x) - \psi_\lambda(x)\int_b^x \varphi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds - \varphi_\lambda(x)\int_x^\infty \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds, \quad x\in[b,\infty)$$

for h belonging to a Banach space

$$X_\lambda(b) := \left\{h\in C([b,\infty))\ ;\ \sup_{x\in[b,\infty)}\left(x^{\frac{\operatorname{Im}\lambda+1}{2}}|h(x)|\right)<\infty\right\}, \quad \|h\|_{X_\lambda(b)} := \sup_{x\in[b,\infty)}\left(x^{\frac{\operatorname{Im}\lambda+1}{2}}|h(x)|\right).$$

Remark 3.3. For arbitrary fixed $b > 0$, all solutions of (10) can be described as follows:

$$w_{c_1,c_2}(x) = c_1\varphi_\lambda(x) + c_2\psi_\lambda(x) + \int_b^x \Big(\varphi_\lambda(x)\psi_\lambda(s) - \varphi_\lambda(s)\psi_\lambda(x)\Big)\tilde{g}_\lambda(s)h(s)\,ds,$$

where $c_1, c_2 \in \mathbb{C}$. Suppose that $h \in C_0^\infty((b,\infty))$ with $\operatorname{supp} h \subset [b_1, b_2]$. Then $w_{c_1,c_2} \in C([b,\infty))$. In particular, for $x \geq b_2$,

$$w_{c_1,c_2}(x) = \left(c_1 + \int_{b_1}^{b_2} \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds\right)\varphi_\lambda(x) + \left(c_2 - \int_{b_1}^{b_2} \varphi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds\right)\psi_\lambda(x).$$

Therefore w_{c_1,c_2} behaves like ψ_λ (that is, $w_{c_1,c_2} \in X_\lambda(b)$) only when

$$c_1 = -\int_{b_1}^{b_2} \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds = -\int_b^\infty \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds.$$

In Definition 3.5 we deal with such a solution with $c_2 = 1$.

Well-definedness of U in Definition 3.5 and its contractivity are proved in next lemma.

Lemma 3.6. *The following assertions hold*:

(i) *for every $b > 0$, the map $U : X_\lambda(b) \to X_\lambda(b)$ is well-defined*;

(ii) *there exists $b_\lambda > 0$ such that U is contractive in $X_\lambda(b_\lambda)$ with*

$$\|Uh_1 - Uh_2\|_{X_\lambda(b)} \leq \frac{1}{5}\|h_1 - h_2\|_{X_\lambda(b)}, \quad h_1, h_2 \in X_\lambda(b_\lambda)$$

and then U has a unique fixed point $w_1 \in X_\lambda(b_\lambda)$;

(iii) *w_1 can be extended to a solution of* (6) *in $\mathbb{R}$ satisfying*

$$\frac{1}{12}x^{-\frac{\operatorname{Im}\lambda+1}{2}} \leq |w_1(x)| \leq 2x^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \in [b_\lambda, \infty).$$

Proof. **(i)** By Lemma 3.4 we have $\psi_\lambda \in X_\lambda(b)$. Therefore to prove well-definedness of U, it suffices to show that the second term in the definition of U belongs to $X_\lambda(b)$.

Let $h \in X_\lambda(b)$. Then for $x \in [b,\infty)$,

$$x^{\frac{\operatorname{Im}\lambda+1}{2}}\left|\varphi_\lambda(x)\int_x^\infty \psi_\lambda(s)\tilde{g}(s)h(s)\,ds\right| \leq x^{\operatorname{Im}\lambda}\|h\|_X \int_x^\infty s^{-\operatorname{Im}\lambda-1}|\tilde{g}_\lambda(s)|\,ds \leq \|h\|_X\|s^{-1}\tilde{g}_\lambda\|_{L^1(b,\infty)}$$

and

$$x^{\frac{\operatorname{Im}\lambda+1}{2}}\left|\psi_\lambda(x)\int_b^x \varphi_\lambda(s)\tilde{g}(s)h(s)\,ds\right| \leq \|h\|_X \int_b^x s^{-1}|\tilde{g}_\lambda(s)|\,ds \leq \|h\|_X\|s^{-1}\tilde{g}_\lambda\|_{L^1(b,\infty)}.$$

Hence we have $Uh \in C([b,\infty))$ and therefore $Uh \in X_\lambda(b)$, that is, $U : X_\lambda(b) \to X_\lambda(b)$ is well-defined.

(ii) Let $h_1, h_2 \in X_\lambda(b)$. Then we have

$$Uh_1(x) - Uh_2(x) = -\psi_\lambda(x)\int_b^x \varphi_\lambda(s)\tilde{g}_\lambda(s)(h_1(s) - h_2(s))\,ds - \varphi_\lambda(x)\int_x^\infty \psi_\lambda(s)\tilde{g}_\lambda(s)(h_1(s) - h_2(s))\,ds.$$

Proceeding the same computation as above, we deduce

$$\|Uh_1 - Uh_2\|_{X_\lambda(b)} \le 2\|s^{-1}\tilde{g}_\lambda\|_{L^1(b,\infty)}\|h_1 - h_2\|_{X_\lambda(b)}.$$

Choosing b large enough, we obtain $\|Uh_1 - Uh_2\|_{X_\lambda(b)} \le 5^{-1}\|h_1 - h_2\|_{X_\lambda(b)}$, that is U is contractive in $X_\lambda(b)$. By contraction mapping principle, we obtain that U has a unique fixed point $w_1 \in X_\lambda(b)$.

(iii) Since w_1 satisfies (10) with $h = w_1$, w_1 is a solution of the original equation (6) in $[b, \infty)$. As in the last part of the proof of Proposition 3.1, we can extend w_1 as a solution of (6) in $\mathbb{R}$. Since $Uw_1 = w_1$ and $U0 = \psi_\lambda$, it follows from the contractivity of U that

$$\|w_1 - \psi_\lambda\|_X = \|Uw_1 - U0\|_X \le \frac{1}{5}\|w_1\|_X \le \frac{1}{5}\|w_1 - \psi_\lambda\|_X + \frac{1}{5}\|\psi_\lambda\|_X.$$

Consequently, we have $\|w_1 - \psi_\lambda\|_X \le 4^{-1}\|\psi_\lambda\|_X \le 4^{-1}$ and then for $x \ge b$,

$$|w_1(x)| \ge |\psi_\lambda(x)| - |w_1(x) - \psi_\lambda(x)| \ge \left(\frac{1}{3} - \|w_1 - \psi_\lambda\|_X\right)x^{-\frac{\mathrm{Im}\lambda+1}{2}} \ge \frac{1}{12}x^{-\frac{\mathrm{Im}\lambda+1}{2}}.$$

□

Next we construct another solution of (6) which behaves like φ_λ at infinity.

Definition 3.7. *Let $b > 0$ be large enough. Define*

$$\widetilde{U}h(x) := \varphi_\lambda(x) + \int_b^x \Big(\varphi_\lambda(x)\psi_\lambda(s) - \varphi_\lambda(s)\psi_\lambda(x)\Big)\tilde{g}_\lambda(s)h(s)\,ds$$

for h belonging to a Banach space

$$Y_\lambda(b) := \left\{h \in C([b,\infty))\ ;\ \sup_{x\in[b,\infty)}\Big(x^{-\frac{\mathrm{Im}\lambda-1}{2}}|h(x)|\Big) < \infty\right\},\quad \|h\|_{Y_\lambda(b)} := \sup_{x\in[b,\infty)}\Big(x^{-\frac{\mathrm{Im}\lambda-1}{2}}|h(x)|\Big).$$

Lemma 3.8. *The following assertions hold*:

(i) *for every $b > 0$, the map $\widetilde{U} : Y_\lambda(b) \to Y_\lambda(b)$ is well-defined*;

(ii) *there exists $b_\lambda > 0$ such that $\widetilde{U}$ is contractive in $Y_\lambda(b_\lambda)$ with*

$$\|\widetilde{U}h_1 - \widetilde{U}h_2\|_{Y_\lambda(b)} \le \frac{1}{5}\|h_1 - h_2\|_{Y_\lambda(b)},\quad h_1, h_2 \in Y_\lambda(b_\lambda)$$

and then $\widetilde{U}$ has a unique fixed point $\tilde{w}_1 \in Y_\lambda(b_\lambda)$;

(iii) *$\tilde{w}_1$ can be extended to a solution of (6) in $\mathbb{R}$ satisfying*

$$\frac{1}{2}x^{\frac{\mathrm{Im}\,\lambda-1}{2}} \le |\tilde{w}_1(x)| \le 2x^{\frac{\mathrm{Im}\,\lambda-1}{2}},\quad x \in [b_\lambda, \infty).$$

Proof. The proof is similar to the one of Lemma 3.6. □

Considering the equation (6) for $x < 0$, we also obtain the following lemma.

Lemma 3.9. *For every $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$, there exist a fundamental system (w_1, w_2) of (6) and positive constants $c_\lambda, C_\lambda, R_\lambda$ such that*

$$|w_1(x)| \le C_\lambda(1+|x|)^{\frac{\operatorname{Im}\lambda-1}{2}}, \quad x \le 0, \qquad |w_1(x)| \le C_\lambda(1+|x|)^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \ge 0, \tag{11}$$

$$|w_2(x)| \le C_\lambda(1+|x|)^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \le 0, \qquad |w_2(x)| \le C_\lambda(1+|x|)^{\frac{\operatorname{Im}\lambda-1}{2}}, \quad x \ge 0 \tag{12}$$

and

$$|w_1(x)| \ge c_\lambda(1+|x|)^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \ge R_\lambda, \quad |w_2(x)| \ge c_\lambda(1+|x|)^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \le -R_\lambda. \tag{13}$$

Proof. In view of Lemma 3.6, it suffices to find w_2 satisfying the conditions above.

Let w_* and $\tilde{w}_*$ be given as in Lemmas 3.6 and 3.8 with $V(x)$ replaced with $V(-x)$. Noting that w_1 can be rewritten as $w_1(x) = c_1 w_*(-x) + c_2\tilde{w}_*(-x)$, we see from Lemma 3.6 and 3.8 that (11) and the first half of (13) are satisfied. Set $w_2(x) = w_*(-x)$ for $x \in \mathbb{R}$. As in the same way, we can verify (12).

Finally, we prove the last half of (13). Since $H_{2,\min}$ is essentially selfadjoint in $L^2(\mathbb{R})$, λ belongs to the resolvent set of H_2, that is, $N(\lambda + H_2) = \{0\}$. This implies that $w_2 \notin L^2(\mathbb{R})$. Noting that $w_2 \in L^2((-\infty, 0))$, we have $w_2 \notin L^2((0,\infty))$. Now using the representation

$$w_2(x) = c_1 w_1(x) + c_2\tilde{w}_1(x), \quad x \in \mathbb{R},$$

we deduce that $c_2 \neq 0$. Therefore using Lemma 3.6 **(iii)** and Lemma 3.8 **(iii)**, we have

$$|w_2(x)| \ge |c_2|\,|\tilde{w}_1(x)| - |c_1|\,|w_1(x)| \ge \frac{|c_2|}{2}x^{\frac{\operatorname{Im}\lambda-1}{2}} - 2|c_1|\,x^{-\frac{\operatorname{Im}\lambda+1}{2}} \ge \frac{|c_2|}{4}x^{\frac{\operatorname{Im}\lambda-1}{2}}$$

for x large enough. □

4. Resolvent estimates in L^p

The following lemma, verified by the variation of parameters, gives a possibility of representation of the Green function for resolvent operator H in L^p.

Lemma 4.1. *Assume that $\lambda \in \rho(\widetilde{H})$ in L^p, where $\widetilde{H}$ is a realization of H in L^p. Then for every $u \in C_0^\infty(\mathbb{R})$,*

$$u(x) = \frac{w_1(x)}{W_\lambda}\int_{-\infty}^{x} w_2(s)f(s)\,ds + \frac{w_2(x)}{W_\lambda}\int_{x}^{\infty} w_1(s)f(s)\,ds, \quad x \in \mathbb{R},$$

where $f := \lambda u - u'' - x^2u + Vu \in C_0^\infty(\mathbb{R})$ and $W_\lambda \neq 0$ is the Wronskian of (w_1, w_2).

Proposition 4.2. *Let $1 < p < \infty$. If $|1 - \frac{2}{p}| < \operatorname{Im}\lambda$, then the operator defined as*

$$R(\lambda)f(x) := \frac{w_1(x)}{W_\lambda}\int_{-\infty}^{x} w_2(s)f(s)\,ds + \frac{w_2(x)}{W_\lambda}\int_{x}^{\infty} w_1(s)f(s)\,ds, \quad f \in C_0^\infty(\mathbb{R})$$

can be extended to a bounded operator on L^p. More precisely, there exists $M_\lambda > 0$ such that

$$\|R(\lambda)f\|_{L^p} \le M_\lambda\left[|\operatorname{Im}\lambda|^2 - \left(1 - \frac{2}{p}\right)^2\right]^{-1}\|f\|_{L^p}, \quad f \in L^p(\mathbb{R}). \tag{14}$$

In particular, $H_{p,\min}$ is closable and its closure H_p satisfies

$$\left\{\lambda \in \mathbb{C}\ ;\ |\mathrm{Im}\,\lambda| > \left|1 - \frac{2}{p}\right|\right\} \subset \rho(H_p).$$

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Set

$$u_1(x) := w_1(x)\int_{-\infty}^{x} w_2(s)f(s)\,ds, \quad u_2(x) := w_1(x)\int_{x}^{\infty} w_1(s)f(s)\,ds.$$

We divide the proof of $u_1 \in L^p(\mathbb{R})$ into two cases $x \geq 0$ and $x < 0$; since the proof of $u_2 \in L^p(\mathbb{R})$ is similar, this part is omitted.

The case u_1 for $x \geq 0$, it follows from Lemma 3.9 and Hölder inequality that

$$\begin{aligned}
|u_1(x)| &\leq C_\lambda^2(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}\left[\int_{-\infty}^{0}(1+|s|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}|f(s)|\,ds + \int_0^x (1+|s|)^{\frac{\mathrm{Im}\,\lambda-1}{2}}|f(s)|\,ds\right]\\
&\leq C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}_-)}(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}\\
&\quad + C_\lambda^2(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}\left(\int_0^x (1+|s|)^{\frac{\mathrm{Im}\,\lambda-1}{2}p'-\alpha p'}\,ds\right)^{\frac{1}{p'}}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)^{\frac{1}{p}}\\
&\leq C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}_-)}(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}\\
&\quad + C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{1}{p'}}(1+|x|)^{-\frac{1}{p}-\alpha}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)^{\frac{1}{p}} \qquad (15)
\end{aligned}$$

with $0 < \alpha < \frac{\mathrm{Im}\lambda+1}{2} + 1/p'$. By the triangle inequality we have

$$\|u_1\|_{L^p(\mathbb{R}_+)} \leq C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\left(\frac{\mathrm{Im}\,\lambda+1}{2}p-1\right)^{-\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_-)} + \mathcal{I}_1(\alpha)$$

and

$$\begin{aligned}
(\mathcal{I}_1(\alpha))^p &= C_\lambda^{2p}\left(\frac{\mathrm{Im}\,\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{p}{p'}}\int_0^\infty (1+|x|)^{-1-\alpha p}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)dx\\
&= C_\lambda^{2p}\left(\frac{\mathrm{Im}\,\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{p}{p'}}(\alpha p)^{-1}\int_0^\infty |f(s)|^p\,ds.
\end{aligned}$$

Choosing $\alpha = \frac{1}{pp'}\left(\frac{\mathrm{Im}\,\lambda-1}{2}p'+1\right)$, we obtain

$$\|u_1\|_{L^p(\mathbb{R}_+)} \leq C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\left(\frac{\mathrm{Im}\,\lambda+1}{2}p-1\right)^{-\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_-)} + C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda-1}{2}+\frac{1}{p'}\right)^{-1}\|f\|_{L^p(\mathbb{R}_+)}.$$

The case u_1 for $x < 0$, by the same way as the case $x > 0$, we have

$$|u_1(x)|^p \leq C_\lambda^{2p}\left(\frac{\mathrm{Im}\lambda+1}{2}p'-\beta p'-1\right)^{-\frac{p}{p'}}(1+|x|)^{-1+\beta p}\int_{-\infty}^{x}(1+|s|)^{-\beta p}|f(s)|^p\,ds, \qquad (16)$$

where $0 < \beta < \frac{\mathrm{Im}\lambda+1}{2} - \frac{1}{p'}$. Taking $\beta = \frac{1}{pp'}\left(\frac{\mathrm{Im}\lambda+1}{2}p' - 1\right)$, we have

$$\|u_1\|_{L^p(\mathbb{R}_-)} \leq C_\lambda^2 \left(\frac{\mathrm{Im}\lambda + 1}{2} - \frac{1}{p'}\right)^{-1} \|f\|_{L^p(\mathbb{R}_-)}.$$

Proceeding the same argument for u_2 and combining the estimates for u_1 and u_2, we obtain (14). □

Corollary 4.3. *Let $\mathcal{R}(\lambda)$ be as in Proposition 4.2. Then for every $f \in L^p(\mathbb{R})$, $\mathcal{R}(\lambda)f \in C(\mathbb{R})$ and*

$$\sup_{x\in\mathbb{R}}\Big((1+|x|)^{\frac{1}{p}}|\mathcal{R}(\lambda)f(x)|\Big) \leq \tilde{C}_\lambda\|f\|_{L^p}. \tag{17}$$

Proof. Let $f \in C_0^\infty(\mathbb{R})$ and set u_1 and u_2 as in the proof of Proposition 4.2. Since the proof for u_1 and u_2 are similar, we only show the estimate of u_1. From (15), we have for $x \geq 0$,

$$\begin{aligned}
(1+|x|)^{\frac{1}{p}}|u_1(x)| &\leq C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda+1}{2}p' - 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_-)}(1+|x|)^{-\frac{\mathrm{Im}\,\lambda}{2}+\frac{1}{p}-\frac{1}{2}} \\
&\quad + C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda-1}{2}p' - \alpha p' + 1\right)^{-\frac{1}{p'}} (1+|x|)^{-\alpha}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leq C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda+1}{2}p' - 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_-)} + C_\lambda^2\left(\frac{\mathrm{Im}\,\lambda-1}{2}p' - \alpha p' + 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_+)},
\end{aligned}$$

where $0 < \alpha < \frac{\mathrm{Im}\lambda+1}{2} + \frac{1}{p'}$. This implies (17) for $x \geq 0$. If $x \leq 0$, then from (16) we can obtain

$$(1+|x|)^{\frac{1}{p}}|u_1(x)| \leq C_\lambda^2\left(\frac{\mathrm{Im}\lambda+1}{2}p' - \beta p' - 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_-)},$$

where $0 < \beta < \frac{\mathrm{Im}\lambda+1}{2} - \frac{1}{p'}$. This yields (17) for $x \leq 0$. The proof is completed. □

By interpolation inequality, we deduce the following assertion.

Proposition 4.4. *Let $1 < p < \infty$ and $p \leq q \leq \infty$. Then*

$$D(H_p) \subset \left\{w \in C(\mathbb{R})\ ;\ \langle x\rangle^{\frac{1}{p}-\frac{1}{q}}w \in L^q\right\}.$$

More precisely, there exists a constant $C_{p,q} > 0$ such that

$$\left\|\langle x\rangle^{\frac{1}{p}-\frac{1}{q}}u\right\|_{L^q} \leq C_{p,q}\Big(\|H_p u\|_{L^p} + \|u\|_{L^p}\Big), \quad u \in D(H_p).$$

Proof. The assertion follows from Proposition 4.2 and Corollary 4.3. □

Proposition 4.5. **(i)** *If $2 < p < \infty$ and $0 < |\mathrm{Im}\,\lambda| < 1 - \frac{2}{p}$, then $N(\lambda + H_p) \neq \{0\}$, and then*

$$\left\{\lambda \in \mathbb{C}\ ;\ |\mathrm{Im}\,\lambda| \leq 1 - \frac{2}{p}\right\} \subset \sigma(H_p);$$

(ii) *If $1 < p < 2$ and $0 < |\mathrm{Im}\,\lambda| < \frac{2}{p} - 1$, then $\overline{N(\lambda + H_p)} \subsetneq L^p$, and then*

$$\left\{\lambda \in \mathbb{C}\ ;\ |\mathrm{Im}\,\lambda| \leq \frac{2}{p} - 1\right\} \subset \sigma(H_p).$$

Proof. **(i)** ($2 < p \le \infty$, $\mathrm{Im}\,\lambda < 1 - \frac{2}{p}$) Noting that

$$\frac{\mathrm{Im}\,\lambda + 1}{2} > \frac{1}{p}, \quad -\frac{\mathrm{Im}\,\lambda - 1}{2} > \frac{1}{p},$$

we have by (11),

$$\begin{aligned}\int_{-\infty}^{\infty} |w_1(x)|^p \, dx &\le C_\lambda \left(\int_{-\infty}^{0} (1+|s|)^{\frac{\mathrm{Im}\,\lambda - 1}{2} p} \, ds + \int_0^\infty (1+|s|)^{-\frac{\mathrm{Im}\,\lambda+1}{2} p} \, ds \right) \\ &\le C_\lambda \left[\left(\frac{1 - \mathrm{Im}\,\lambda}{2} p - 1 \right)^{-1} + \left(\frac{\mathrm{Im}\,\lambda + 1}{2} p - 1 \right)^{-1} \right] < \infty.\end{aligned}$$

This means that $w_1, w_2 \in N(\lambda + H_p)$.

(ii) ($1 < p < 2, \mathrm{Im}\,\lambda < \frac{2}{p} - 1$) Note that H_p is the adjoint operator of $H_{p'}$. Since $w_1 \in D(H_{p'})$ for every $u \in C_0^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} (\lambda u + H_p u) w_1 \, dx = \int_{-\infty}^{\infty} u(\lambda w_1 + H_{p'} w_1) \, dx = 0,$$

the closure of $R(\lambda + H_p)$ does not coincide with L^p, that is, $\overline{R(\lambda + H_p)} \subsetneq L^p$.

Since $\sigma(H_p)$ is closed in $\mathbb{C}$ and we can argue the same assertion for $\mathrm{Im}\lambda < 0$ via complex conjugation, we obtain the assertion. □

Combining the assertions above, we finally obtain Theorem 1.1.

5. Absence of C_0-semigroups on L^p ($p \neq 2$, $V = 0$)

In Theorem 1.1, we do not prove any assertions related to generation of C_0-semigroups by $\pm iH_p$. In this subsection we prove

Theorem 5.1. *Neither iH_p nor $-iH_p$ generates C_0-semigroup on L^p.*

Proof. We argue by a contradiction. Assume that iH_p generates a C_0-semigroup $T(t)$ on L^p. Then it follows from Theorem 1.1 (the coincidence of resolvent operators) that we have $T(t)f = S(t)f$ for every $t > 0$ and $f \in L^2 \cap L^p$, where $S(t)$ is the C_0-group generated by the skew-adjoint operator iH_2.

Fix $f_0 \in L^2 \cap L^p$ such that $\mathcal{F} f_0 \notin L^p$ ($\mathcal{F}$ is the Fourier transform). Then by the Mehler's formula (see e.g., Cazenave [3, Remark 9.2.5]), we see that

$$[S(t)]f(x) = \left(\frac{1}{2\pi \sinh(2t)} \right)^{\frac{N}{2}} e^{-i \frac{1}{2\tanh(2t)} |x|^2} \int_{-\infty}^{\infty} e^{-\frac{i}{\sinh(2t)} x \cdot y} e^{-i \frac{1}{2\tanh(2t)} |y|^2} f(y) \, dy.$$

In other words, using the operators

$$M_\tau g(x) := e^{-i \frac{|x|^2}{2\tau}} g(x), \quad D_\tau g(x) := \tau^{-\frac{N}{2}} g(\tau^{-1} x),$$

we can rewrite $S(t)$ as the following form $S(t)f = M_{\tanh(2t)} \mathcal{F} D_{\sinh(2t)} M_{\tanh(2t)} f$. Taking $f_{t_0} = M_{\tanh(2t_0)}^{-1} D_{\sinh(2t_0)}^{-1} f_0 \in L^p$, we have

$$S(t_0) f_{t_0} = M_{\tanh(2t)} \mathcal{F} f_0 \notin L^p.$$

This contradicts the fact $T(t_0) f_{t_0} \in L^p$. This completes the proof. □

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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