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Research article

On the integrality of the first and second elementary symmetric functions of $1, 1/2^{s_2}, ..., 1/n^{s_n}$

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Abstract: It is well known that the harmonic sum $H_n(1) = \sum_{1\le k\le n} \frac{1}{k}$ is never an integer for $n > 1$. Erdös and Niven proved in 1946 that the multiple harmonic sum $H_n(\{1\}^r) = \sum_{1\le k_1<\cdots<k_r\le n} \frac{1}{k_1\cdots k_r}$ can take integer values for at most finite many integers n. In 2012, Chen and Tang refined this result by showing that $H_n(\{1\}^r)$ is an integer only for $(n,r) = (1,1)$ and $(n,r) = (3,2)$. In this paper, we consider the integrality problem for the first and second elementary symmetric function of $1, 1/2^{s_2}, ..., 1/n^{s_n}$, we show that none of them is an integer with some natural exceptions.

Keywords: elementary symmetric function; integrality; Bertrand's postulate; p-adic valuation
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1. Introduction

A well-known result in number theory states that the harmonic sum

$$H_n(1) := \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n > 1$. The first published proof went back to 1915 by Leopold Theisinger. In 1946, Erdös and Niven proved that the multiple harmonic sum

$$H_n(\{1\}^r) = \sum_{1\le k_1<\cdots<k_r\le n} \frac{1}{k_1\cdots k_r}$$

is not an integer with finite exceptions. In 2012, Chen and Tang showed a stronger result stating that $H_n(\{1\}^r)$ is an integer only for $(n,r) = (1,1)$ and $(n,r) = (3,2)$.

For an n-tuple vector $s = (s_1, s_2, \cdots, s_n)$ with $s_i \in \mathbb{Z}^+$ and $n \geq 2$, we define *the first ordinary multiple harmonic sum* $H_n^{(1)}(s_1, \ldots, s_n)$ as

$$H_n^{(1)}(s_1, \ldots, s_n) := \sum_{i=1}^{n} \frac{1}{i^{s_i}},$$

and *the second ordinary multiple harmonic sum* $H_n^{(2)}(s_1, \ldots, s_n)$ as

$$H_n^{(2)}(s_1, \ldots, s_n) := \sum_{1 \leq i < j \leq n} \frac{1}{i^{s_i} j^{s_j}},$$

and *the second star multiple harmonic sum* $H_n^{*(2)}(s_1, \ldots, s_n)$ as

$$H_n^{*(2)}(s_1, \ldots, s_n) = \sum_{1 \leq i \leq j \leq n} \frac{1}{i^{s_i} j^{s_j}}.$$

We shall prove that $H_n^{(1)}(s_1, \ldots, s_n)$, $H_n^{(2)}(s_1, \ldots, s_n)$ and $H_n^{*(2)}(s_1, \ldots, s_n)$ are not integers except some special cases.

Let p be a prime and $v_p(q)$ be the p-adic valuation of rational number q, that is, if $q = \frac{ap^n}{bp^m}$ with $\gcd(a, p) = \gcd(b, p) = 1$ and $m, n \in \mathbb{Z}^+$, then $v_p(\frac{ap^n}{bp^m}) = n - m$. It is well known that the following two statements are true:

(1). For any $x, y \in \mathbb{Q}$, one has

$$v_p(x + y) \geq \min(v_p(x), v_p(y)),$$

and the equality holds if $v_p(x) \neq v_p(y)$.

(2). For any $x_1, x_2, \ldots, x_n \in \mathbb{Q}$, one has

$$v_p(x_1 + x_2 + \cdots + x_n) \geq \min\left(v_p(x_1), v_p(x_2), \ldots, v_p(x_n)\right),$$

and the equality holds if there exists an i such that $v_p(x_i) < v_p(x_j)$ for all integers j with $j \neq i$.

To express concisely, we take the following abbreviations: $H_n^{(2)} := H_n^{(2)}(s_1, \ldots, s_n)$, $H_n^{*(2)} := H_n^{*(2)}(s_1, \ldots, s_n)$. We denote the sum $H_n^{(2)}$ when s_i is fixed and $s_j \to \infty$ by $H_n^{(2)}(s_i = k)$ and $H_n^{(2)}(s_j \to \infty)$ respectively.

Now we state our main results.

Theorem 1.1. *Let n be an integer with $n \geq 2$ and $s_i \in \mathbb{Z}^+$ for $1 \leq i \leq n$. Then $H_n^{(1)}$ is never an integer.*

Theorem 1.2. *Let n be an integer with $n \geq 2$ and $s_i \in \mathbb{Z}^+$ for $1 \leq i \leq n$. Then each of the following is true:*

(i). *$H_n^{*(2)}$ is never an integer.*

(ii). *$H_n^{(2)}$ is never an integer except that $n = 3, s_2 = s_3 = 1$, in which case, $H_n^{(2)}$ is an integer.*

Remark 1.1. We can get an approximating value by the following expansion

$$H_n^{(2)} = \frac{1}{2}\Big(\Big(\sum_{i=1}^{n} \frac{1}{i^{s_i}}\Big)^2 - \sum_{i=1}^{n} \frac{1}{i^{2s_i}}\Big).$$

Actually, if we fix the valuation of n and $s_1, \ldots, s_n$, we can get an approximate value by hand from the expansion above.

In the next two sections, we first show that when $n = 9$ and $n = 21$, $H_n^{(2)}$ is not an integer, i.e. $H_9^{(2)}$ and $H_{21}^{(2)}$ are not integers. And the remaining cases will be treated in the final section. The proofs of Theorems 1.1 and 1.2 are also given in the final section.

2. The case $n = 9$

In this section, we show the fact that $H_9^{(2)}$ is not an integer. At first, we have following lemmas.

Lemma 2.1. *For a fixed prime p and a positive integer n with $p < n$, let $k := [\frac{n}{p}]$,*

$$T_p := \sum_{\substack{1\le i<j\le n,\\ v_p(i)v_p(j)\ge 1}} \frac{1}{i^{s_i} j^{s_j}}$$

and

$$m := \min\big(- v_p(p^{s_p}), -v_p((2p)^{s_{2p}}), \ldots, -v_p((kp)^{s_{kp}})\big).$$

If $v_p(T_p) < m$, then $H_n^{(2)} \notin \mathbb{Z}^+$.

Proof. Since $v_p(\frac{1}{i^{s_i} j^{s_j}}) \ge m$ for any $1 \le i \ne j \le n$ with $v_p(i)v_p(j) = 0$, it then follows that

$$v_p(H_n^{(2)} - T_p) \ge m.$$

On the other hand, since $v_p(T_p) < m$, we have $v_p(T_p) < m \le v_p(H_n^{(2)} - T_p)$. Thus

$$v_p(H_n^{(2)}) = v_p(T_p + (H_n^{(2)} - T_p)) = v_p(T_p) < 0,$$

the last inequality is true because $v_p(T_p) < m < 0$. So $H_n^{(2)}$ is not an integer. □

Remark 2.1. Let $t_p = T_p p^{s_p}(2p)^{s_{2p}} \cdots (kp)^{s_{kp}}$. Then

$$v_p(t_p) = v_p(T_p) + \sum_{i=1}^{k} v_p((ip)^{s_{ip}}).$$

So $v_p(T_p) < m$ if and only if

$$v_p(t_p) < m + \sum_{i=1}^{k} v_p((ip)^{s_{ip}}). \tag{2.1}$$

Remark 2.2. For a fixed k with $2 \le k \le n$, if $s_k < s'_k$, then

$$H_n^{(2)}(s_1, \ldots, s_k, \ldots, s_n) > H_n^{(2)}(s_1, \ldots, s'_k, \ldots, s_n),$$

Moreover, for any t_k satisfies $s_k \ge t_k$, we denote all $H_n^{(2)}(t_k)$ as $H_n^{(2)}(s_k \ge t_k)$, then

$$H_n^{(2)}(s_k \to \infty) < H_n^{(2)}(s_k \ge t_k) \le H_n^{(2)}(s_k = t_k).$$

Lemma 2.2. *We have that $H_9^{(2)}$ is not an integer.*

Proof. By Lemma 2.1 and Remark 2.1, we consider the cases satisfying (2.1) for $p = 3$. Namely,

$$v_3(t_3) < m + s_3 + s_6 + 2s_9 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9), \tag{2.2}$$

where $m = \min(-s_3, -s_6, -2s_9)$, $t_3 = 3^{s_3} + 6^{s_6} + 9^{s_9}$. Next, we consider the following cases.

Case 1. $s_3 = s_6 = 2s_9$. Since $s_3 = 2s_9 \geq 2$, so $2 + 2^{s_3} < 3^{s_3}$.Then we have $t_3 = 3^{s_3} + 6^{s_3} + 3^{2s_9} = 3^{s_3}(2 + 2^{s_3}) < 3^{2s_3}$, which implies that

$$v_3(t_3) < 2s_3 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9)$$

Case 2. There is only one of $s_3, s_6, 2s_9$ equals to $\min(s_3, s_6, 2s_9)$. Hence it is obvious that $v_3(t_3) = \min(s_3, s_6, 2s_9) < \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9)$.

Case 3. $s_6 = 2s_9 < s_3$. Then one has $t_3 = 3^{s_3} + 3^{2s_9}(1 + 2^{2s_9})$. But $1 + 2^{2s_9} = 1 + 4^{s_9} \equiv 2 \pmod 3$, so we deduce that

$$\begin{aligned} v_3(t_3) &= v_3(3^{s_3} + 3^{2s_9}(1 + 2^{2s_9})) \\ &= v_3(3^{2s_9}(1 + 2^{2s_9})) = 2s_9 \\ &< 4s_9 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9) \end{aligned}$$

Case 4. $s_3 = 2s_9 < s_6$. Thus $t_3 = 2 \times 3^{2s_9} + 6^{s_6} = 2 \times 3^{2s_9} + 2^{s_6}3^{s_6}$, since 3 does not divide 2 or 2^{s_6}, then it follows that

$$v_3(t_3) = v_3(2 \times 3^{2s_9}) = 2s_9 < 4s_9 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9).$$

Case 5. $s_3 = s_6 < 2s_9$. We divide the proof into the following subcases.

Case 5.1. $2 \leq s_3 < 2s_9 < 2s_3$. Then we have $t_3 = 3^{s_3} + 6^{s_3} + 3^{2s_9} \leq 3^{s_3}(1 + 2^{s_3}) + 3^{2s_3-2} = 3^{s_3}(1 + 2^{s_3} + 3^{s_3-2})$. Since $s_3 \geq 2$, so $1 + 2^{s_3} + 3^{s_3-2} < 3^{s_3}$, which implies that

$$v_3(t_3) < 2s_3 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9)$$

as excepted.

Case 5.2. $2 \leq s_3 < 2s_3 \leq 2s_9$. One has

$$\begin{aligned} v_3(t_3) &= v_3(3^{s_3}(1 + 2^{s_3}) + 3^{2s_9}) \\ &= v_3(3^{s_3}(1 + 2^{s_3})) \\ &< 2s_3 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9). \end{aligned}$$

Case 5.3. $1 = s_3 = s_6 < 2s_9$. In this case, we consider $H_9^{(2)}(s_2 = 4, s_3 = s_4 = s_5 = s_6 = s_7 = s_8 = s_9 = 1)$. Then it is easy to see that

$$v_3(t_3) = 2 = \min(s_3 + s_6, s_3 + 2s_9, s_6 + 2s_9),$$

i.e. (2.2) does not hold any more. Thus we have to investigate (2.1) for another prime. Actually, we choose $p = 2$, that is, we will prove the following inequality

$$v_2(t_2) < \min(s_2 + 2s_4 + 1, s_2 + 1 + 3s_8, 2s_4 + 1 + 3s_8, s_2 + 2s_4 + 3s_8).$$

Notice that $3s_8, s_2, 2s_4 \geq 1$, then we only need to show

$$v_2(t_2) < \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1. \tag{2.3}$$

Subcase 5.3.1. $s_2 = 1$. Then we deduce that $t_2 = 2^2 \times 3 + 2^{2s_4+3} + 2^{3+3s_8} + 2^{2s_4+3s_8}$ which implies that

$$v_2(t_2) = 2 < 2 + \min(2s_4, 3s_8) = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

Subcase 5.3.2. $s_2 > 1$, and there's only one of $s_2, 2s_4, 3s_8$ equals to $\min(s_2, 2s_4, 3s_8)$. Just like Case 2 stated, $v_2(t_2) = \min(s_2, 2s_4, 3s_8) + 1$, so

$$v_2(t_2) < \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

Subcase 5.3.3. $3s_8 = 2s_4 < s_2$. Then we have $t_2 = 2^{s_2+2s_4+1} + 2^{4s_4} + 3 \times 2^{s_2+1} + 3 \times 2^{2s_4+2}$. Obviously, we only need to compare $2s_4 + 2$ with $s_2 + 2s_4$.

(1) $2s_4 + 2 \neq s_2 + 1$. Then

$$v_2(t_2) = \min(2s_4 + 2, s_2 + 1) \leq 2 + 2s_4 < 4s_4 + 1 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1,$$

(2) $2s_4 + 2 = s_2 + 1$. We deduce that

$$v_2(t_2) = 2s_4 + 3 < 4s_4 + 1 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

Subcase 5.3.4. $1 < s_2 = 2s_4 < 3s_8$. It infers that

$$t_2 = 2^{2s_4+3s_8+1} + 2^{4s_4} + 3 \times 2^{3s_8+1} + 3 \times 2^{2s_4+2}.$$

Moreover, it's easy to see $2s_4 + 3s_8 + 1 > \max(4s_4, 3s_8 + 1, 2s_4 + 2)$.

(1) $s_4 \geq 3$, so $s_2 \geq 6$. Then we have $2s_4 + 2 \leq 3s_8 + 1$ and $2s_4 + 2 < 4s_4$, which implies that if $2s_4 + 2 = 3s_8 + 1$, then

$$\begin{aligned} v_2(t_2) &= v_2(2^{2s_4+3s_8+1} + 2^{4s_4} + 3 \times 2^{2s_4+3}) \\ &= 2s_4 + 3 \\ &< 1 + 4s_4 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1. \end{aligned}$$

and if $2s_4 + 2 < 3s_8 + 1$ we have

$$v_2(t_2) = 2 + 2s_4 < 1 + 4s_4 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

(2) $s_4 = 2$, so $s_2 = 4$. Then we have $2s_4 + 2 < \min(4s_4, 3s_8 + 1)$, which infers that

$$v_2(t_2) = 2s_4 + 2 < 1 + 4s_4 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

(3) $s_4 = 1$ and $s_8 = 1$, so $s_2 = 2$. Then $t_2 = 2^7 + 3 \times 2^4$, and we derive that

$$v_2(t_2) = 4 < 7 = \min(s_2 + 2s_4 + 1, s_2 + 1 + 3s_8, 2s_4 + 1 + 3s_8).$$

(4) $s_4 = 1$, $s_8 \geq 2$,so $s_2 = 2$.

In this subcase, the inequality (2.3) does not hold. In fact, we have $t_2 = 2^{3s_8+3} + 2^6 + 3 \times 2^{3s_8+1}$ and $\min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1 = 1 + \min(4, 2 + 2s_8)$. Then it follows that $v_2(t_2) = \min(6, 3s_8) > 1 + \min(4, 2 + 2s_8)$ $(s_8 \geq 2)$.

But fortunately, we can calculate the approximate value. Recall that $s_3 = s_4 = s_6 = 1$ and $s_8 \geq 2$.

If $s_5 \geq 3$, then by the expansion in Remark 1.1 and Remark 2.2 we have

$$\begin{aligned}1 &< H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = 2, s_5, s_7, s_8, s_9 \to \infty)\\ &< H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = 2, s_5 \geq 3, s_8 \geq 2)\\ &\leq H_9^{(2)}(s_3 = s_4 = s_6 = s_7 = s_9 = 1, s_2 = s_8 = 2, s_5 = 3) < 2.\end{aligned}$$

Hence $H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = 2, s_5 \geq 3, s_8 \geq 2)$ is not an integer.

If $s_5 = 2$, we have to classify the case by s_7, s_9. Indeed, one has

$$1 < H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = s_5 = 2, s_8 \geq 2, s_7 = 1, s_9 \geq 2) < 2$$

and

$$1 < H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = s_5 = 2, s_8 \geq 2, s_7 \geq 2, s_9 = 1) < 2.$$

Moreover $H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = s_5 = 2, s_8 \geq 2, s_7 = s_9 = 1)$ is a decreasing function with the single variable s_8. Thus we derive that

$$H_9^{(2)}(s_3 = s_4 = s_6 = s_7 = s_9 = 1, s_2 = s_5 = s_8 = 2) \approx 2.02,$$

and by Remark 2.2,

$$1 < H_9^{(2)}(s_3 = s_4 = s_6 = s_7 = s_9 = 1, s_2 = s_5 = 2, s_8 \geq 3) < 1.99,$$

which yields that $H_9^{(2)}(s_3 = s_4 = s_6 = 1, s_2 = s_5 = 2, s_8 \geq 2)$ is not an integer.

if $s_5 = 1$, as the discussion above, we get that

$$2 < H_9^{(2)}(s_3 = s_4 = s_6 = s_5 = 1, s_2 = 2, s_8 \geq 2, s_7 = s_9 = 1) < 3,$$

$$\begin{aligned}2 &< H_9^{(2)}(s_3 = s_4 = s_6 = s_5 = 1, s_2 = 2, s_8 \geq 2, s_7 = 1, s_9 > 1) < 3,\\ 2 &< H_9^{(2)}(s_3 = s_4 = s_6 = s_5 = 1, s_2 = 2, s_8 \geq 2, s_9 = 1, s_7 > 1) < 3,\\ 1 &< H_9^{(2)}(s_3 = s_4 = s_6 = s_5 = 1, s_2 = 2, s_8 \geq 2, s_7 > 1, s_9 > 1) < 2.\end{aligned}$$

so $H_9^{(2)}(s_3 = s_4 = s_6 = s_5 = 1, s_2 = 2, s_8 \geq 2)$ is not an integer.

Subcase 5.3.5. $1 < s_2 = 3s_8 < 2s_4$. Then

$$t_2 = 2^{2s_4+3s_8+1} + 2^{6s_8} + 3 \times 2^{2s_4+1} + 3 \times 2^{3s_8+2}.$$

(1) $s_4 = 2$. We can get an approximate value in the same way.

i.e. $1 < H_9^{(2)}(s_3 = s_6 = s_8 = 1, s_2 = 3, s_4 = 2) < 2$. It implies that $H_9^{(2)}(s_3 = s_6 = s_8 = 1, s_2 = 3, s_4 = 2)$ is not an integer.

(2) $s_4 \geq 3$ and $2s_4 > 3s_8 + 1$. Then

$$v_2(t_2) = 3s_8 + 2 < 1 + 6s_8 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

(3) $s_4 \geq 3$ and $2s_4 = 3s_8 + 1$. Then we have $t = 2^{6s_8+2} + 2^{6s_8} + 3 \times 2^{3s_8+3}$, which implies that

$$v_2(t_2) = 3s_8 + 3 < 1 + 6s_8 = \min(s_2 + 2s_4, s_2 + 3s_8, 2s_4 + 3s_8) + 1.$$

Thus we finish the proof of Lemma 2.2. □

3. The case $n = 21$

In this section, we take the same results of Lemma 2.1 and Remark 2.1 in proof. And we have the following lemma. This is also needed in the proof of Theorem 1.2.

Lemma 3.1. *We have that $v_7(H_{21}^{(2)}) < 0$. So $H_{21}^{(2)}$ is not an integer.*

Proof. By Lemma 2.1 and Remark 2.1, we consider the cases satisfying (2.1) for $p = 7$. That is,

$$v_7(t_7) < m + s_7 + s_{14} + s_{21} = \min(s_7 + s_{14}, s_7 + s_{14}, s_{14} + s_{21}) \tag{3.1}$$

where $m = \min(-s_7, -s_{14}, -s_{21})$, $t_3 = 7^{s_7} + 14^{s_{14}} + 21^{s_{21}}$.

Case 1. $s_7 = s_{14} = s_{21}$. Then we have $t_7 = 7^{s_7}(1 + 2^{s_7} + 3^{s_7}) < 7^{s_7}7^{s_7} = 7^{2s_7}$ and it implies $v_7(t_7) < 2s_7 = \min(s_7 + s_{14}, s_7 + s_{14}, s_{14} + s_{21})$.

Case 2. There's only one of s_7, s_{14}, s_{21} equals to $\min(s_7, s_{14}, s_{21})$. Just like the process we took before, we have

$$v_7(t_7) = \min(7^{s_7}, 14^{s_{14}}, 21^{s_{21}}) < \min(s_7 + s_{14}, s_7 + s_{14}, s_{14} + s_{21}).$$

Case 3. $s_7 = s_{14} < s_{21}$. We can simplify $t_7 = 7^{s_7}(1 + 2^{s_7}) + 7^{s_{21}}3^{s_{21}}$. Since 7 does not divide $1 + 2^{s_7}$, then

$$v_7(t_7) = \min(s_7, s_{21}) = s_7 < 2s_7 = \min(s_7 + s_{14}, s_7 + s_{14}, s_{14} + s_{21}).$$

Case 4. $s_{14} = s_{21} < s_7$. Consider the following cases:

(1) $s_{14} < 2s_{14} \le s_7$. Thus $2^{s_{14}} + 3^{s_{14}} < 7^{s_{14}}$. So we have $7^{s_{14}}(2^{s_{14}} + 3^{s_{14}}) < 7^{2s_{14}} < 7^{s_7}$. It means

$$\begin{aligned} v_7(t_7) &= v_7(7^{s_{14}}(2^{s_{14}} + 3^{s_{14}}) + 7^{s_7}) \\ &= v_7(7^{s_{14}}(2^{s_{14}} + 3^{s_{14}})) \\ &< 2s_{14} = \min(s_7 + s_{14}, s_7 + s_{14}, s_{14} + s_{21}). \end{aligned}$$

(2) $s_{14} < s_7 < 2s_{14}$. Then $t_7 = 7^{s_{14}}(7^{s_7 - s_{14}} + 2^{s_{14}} + 3^{s_{14}}) < 7^{s_{14}}(7^{s_{14}-1} + 5^{s_{14}}) < 7^{2s_{14}}$. Hence $v_7(t_7) < 2s_{14} = \min(s_7 + s_{14}, s_7 + s_{21}, s_{14} + s_{21})$.

Case 5. $s_7 = s_{21} < s_{14}$. Then $t_7 = 7^{s_7}(1 + 3^{s_7}) + 7^{s_{14}}2^{s_{14}}$. Since 7 divides $1 + 3^{s_7}$ sometimes, let $v := v_7(1 + 3^{s_7})$.

Case 5.1. $v \neq s_{14} - s_7$. Then $v_7(t_7) = \min(s_7 + v, s_{14}) \le s_7 + v$. Since $1 + 3^{s_7} < 7^{s_7}$, so $v < s_7$, which implies that

$$v_7(t_7) \le s_7 + v < 2s_7 = \min(s_7 + s_{14}, s_7 + s_{21}, s_{14} + s_{21}).$$

Case 5.2. $v = s_{14} - s_7$. Since $3^{s_7} \equiv -1 \pmod{7^v}$, then $3^{2s_7} \equiv 1 \pmod{7^v}$.

On the other hand, it is easy to see 3 is a primitive root module 7, which yields that the order of 3 modulo 7^v is $(7-1)7^{v-1} = \varphi(7^v)$ (see [6], Theorem 3.6), that is, 3 is a primitive root module 7^v. Then $2s_7 \ge \varphi(7^v) = 6 \times 7^{v-1}$. Hence $s_7 \ge 3 \times 7^{v-1} \ge 3v$, so we have $\frac{s_7+v}{2} \le s_7 - v$. Then it follows that

$$\begin{aligned} a = 7^{s_7} + 14^{s_{14}} + 21^{s_{21}} &= 7^{s_7} + 14^{s_7+v} + 21^{s_7} \\ &= 7^{s_7+v}\left(\frac{1 + 3^{s_7}}{7^v} + 4^{\frac{s_7+v}{2}}\right) \\ &< 7^{s_7+v}(3^{s_7-v} + 4^{s_7-v}) \end{aligned}$$

$$< 7^{s_7+v}7^{s_7-v} = 7^{2s_7}.$$

Thus $v_7(t_7) < 2s_7 = \min(s_7 + s_{14}, s_7 + s_{14}, s_{14} + s_{21})$.

Therefore, Lemma 3.1 is proved. □

4. Proofs of Theorems 1.1 and 1.2

In this section, we present the proofs of Theorems 1.1 and 1.2. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. If $n \geq 2$, then by Bertrand's postulate, there exists one prime p such that $\frac{n}{2} < p \leq n$. Then $p \leq n < 2p$. Thus one has

$$H_n^{(1)} = \sum_{\substack{1\leq i\leq n,\\ i=p}} \frac{1}{i^{s_i}} + \sum_{\substack{1\leq i\leq n,\\ i\neq p}} \frac{1}{i^{s_i}} = \frac{1}{p^{s_p}} + \sum_{\substack{1\leq i\leq n,\\ i\neq p}} \frac{1}{i^{s_i}}.$$

Since $v_p(\frac{1}{i^{s_i}}) = 0 > v_p(\frac{1}{p^{s_p}}) = -s_p$ for any $1 \leq i \leq n$ with $i \neq p$, then

$$v_p(H_n^{(1)}) = v_p(\frac{1}{p^{s_p}}) = -s_p < 0.$$

i.e. for $n \geq 2$, $H_n^{(1)}$ is never an integer. □

In order to prove Theorem 1.2, we need the following lemmas.

Lemma 4.1. *Let $n \geq 2$. If there is a prime p and $p \in (\frac{n}{3}, \frac{n}{2}]$, then $H_n^{(2)}$ is not an integer.*

Proof. Since $p \in (\frac{n}{3}, \frac{n}{2}]$, then $2p \leq n < 3p$. Hence

$$H_n^{(2)} = \sum_{\substack{1\leq i<j\leq n,\\ v_p(i)v_p(j)>0}} \frac{1}{i^{s_i} j^{s_j}} + \sum_{\substack{1\leq i<j\leq n,\\ v_p(i)v_p(j)=0}} \frac{1}{i^{s_i} j^{s_j}} = \frac{1}{2^{s_{2p}} p^{s_p+s_{2p}}} + \sum_{\substack{1\leq i<j\leq n,\\ v_p(i)v_p(j)=0}} \frac{1}{i^{s_i} j^{s_j}},$$

Since for any $1 \leq i \neq j \leq n$ with $v_p(i)v_p(j) = 0$, we have

$$v_p(\frac{1}{i^{s_i} j^{s_j}}) \geq \min(-s_p, -s_{2p}) > -(s_p + s_{2p}) \geq v_p(\frac{1}{2^{s_{2p}} p^{s_p+s_{2p}}}).$$

The last inequality holds for the case of $p = 2$. Then

$$v_p(H_n^{(2)}) = v_p(\frac{1}{2^{s_{2p}} p^{s_p+s_{2p}}}) \leq -(s_p + s_{2p}) < 0.$$

So if $n \geq 2$ and there is a prime p such that $p \in (\frac{n}{3}, \frac{n}{2}]$, then $H_n^{(2)}$ is not an integer. □

Let

$$H = \bigcup_{p\in\mathcal{P}} [2p, 3p)$$

where $\mathcal{P}$ is the set consisting of all primes. Then for any integer n, it is easy to see that $n \in H$ if and only if there exists one prime p, such that $p \in (\frac{n}{3}, \frac{n}{2}]$. Moreover, we have the following lemma.

Lemma 4.2. ([3]) *The set* $\mathbb{Z}^+ \backslash H$ *is finite and* $\max(\mathbb{Z}^+ \backslash H) = 21$.

We can now give the proof of Theorem 1.2 as the conclusion of this paper.

Proof of Theorem 1.2. We divide the proof into two parts. First of all, we prove that $H_n^{(2)}$ is never an integer except that $n = 3$, $s_2 = s_3 = 1$.

Case 1. $n \in H$. Then by Lemma 4.1, we have $H_n^{(2)}$ is not an integer.

Case 2. $n \notin H$. By Lemma 4.2, we have $\mathbb{Z}^+ \backslash H$ is finite and $\max(\mathbb{Z}^+ \backslash H) = 21$. So if $n > 21$, then $n \in H$. Furthermore, we have that

$$H = [4, 6) \bigcup [6, 9) \bigcup [10, 15) \bigcup [14, 21) \bigcup_{p>7, p\in\mathcal{P}} [2p, 3p)$$

Thus it follows that if $n \notin H$, n equals to one of $2, 3, 9, 21$. Next, we will prove that none of $H_2^{(2)}, H_3^{(2)}, H_9^{(2)}$ and $H_{21}^{(2)}$ is an integer.

Case 2.1. $n = 2$. Then it is obvious that

$$H_2^{(2)} = \frac{1}{1^{s_1} 2^{s_2}} \notin \mathbb{Z},$$

so $H_2^{(2)}$ is not an integer.

Case 2.2. $n = 3$. Then we have

$$H_3^{(2)} = \frac{1}{1^{s_1} 2^{s_2}} + \frac{1}{1^{s_1} 3^{s_3}} + \frac{1}{2^{s_2} 3^{s_3}} = \frac{1^{s_1} + 2^{s_2} + 3^{s_3}}{1^{s_1} 2^{s_2} 3^{s_3}}.$$

Since $(2^{s_2} - 1)(3^{s_3} - 1) \geq 2$, we have $2^{s_2} 3^{s_3} \geq 2^{s_2} + 3^{s_3} + 1$. Thus we get that $H_3^{(2)} \leq 1$, and $H_3^{(2)} = 1$ if and only if $s_2 = s_3 = 1$.

Case 2.3. $n = 9$ and $n = 21$. Then Lemma 2.2 and Lemma 3.1 give us the desired result.

Secondly, we give a brief proof to the fact that when $n \geq 2$, $H_n^{*(2)}$ is never an integer. This will finish the proof of Theorem 1.2.

If $n \geq 2$, then by Bertrand's postulate, there is at least one prime p such that $\frac{n}{2} < p \leq n$. Then $p \leq n < 2p$, and $v_p(i) = 0$ or 1 for any integer i with $1 \leq i \leq n$. Thus one has

$$H_n^{*(2)} = \sum_{\substack{1 \leq i \leq j \leq n, \\ v_p(i)=v_p(j)=1}} \frac{1}{i^{s_i} j^{s_j}} + \sum_{\substack{1 \leq i \leq j \leq n, \\ v_p(i)v_p(j)=0}} \frac{1}{i^{s_i} j^{s_j}} = \frac{1}{p^{2s_p}} + \sum_{\substack{1 \leq i \leq j \leq n, \\ v_p(i)v_p(j)=0}} \frac{1}{i^{s_i} j^{s_j}}.$$

Since $0 \geq v_p(\frac{1}{i^{s_i} j^{s_j}}) > v_p(\frac{1}{p^{2s_p}}) = -2s_p$ for any $1 \leq i, j \leq n$ with $v_p(i)v_p(j) = 0$, then

$$v_p(H_n^{*(2)}) = v_p(\frac{1}{p^{2s_p}}) = -2s_p < 0.$$

So when $n \geq 2$, $H_n^{*(2)}$ is never an integer.

Therefore Theorem 1.2 is proved. □

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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