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Research article

A note on derivations and Jordan ideals of prime rings

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Abstract: Let $F : R \to R$ be a generalized derivation of a 2-torsion free prime ring R together with a derivation d. In this paper, we show that a nonzero Jordan ideal J of R contains a nonzero ideal of R. Further, we use this result to prove that if $F([x, y]) \in Z(R)$ for all $x, y \in J$, then R is commutative. Consequently, it extends a result of Oukhtite, Mamouni and Ashraf.

Keywords: Prime rings; Jordan ideals; Generalized derivations; Martindale ring of quotients; Generalized polynomial identities (GPIs)
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1. Introduction

In everything that follows, R denotes an associative ring with center $Z(R)$. Let Q and Q_{mr} stands for the two-sided Martindale quotient ring and right Utumi quotient ring (also known as maximal right ring of quotients) of R respectively. The center of Q_{mr} is called extended centroid of R and is denoted by C (i.e. $C = Z(Q_{mr})$). For the basic idea of these objects we refer the reader to [12]. For any $a, b \in R$, a ring R is called prime ring if $aRb = (0)$ implies $a = 0$ or $b = 0$ and is called semi-prime ring if $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For some fixed $a \in R$, the mapping $I_a : R \to R$ such that $x \mapsto [a, x]$ for all $x \in R$, is a well-known example of a derivation. Specifically, I_a is called the inner derivation of R induced by the element a. In 1991, Brešar [13] introduced a generalized notion of a derivation, called generalized derivation. A generalized derivation of a ring R is an additive mapping $F : R \to R$ uniquely determined by a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for any $x, y \in R$. Clearly, every derivation is a generalized derivation but the converse is not always true. For any $a, b \in R$, $F(x) = ax + xb$ and $F(x) = ax$ are the most natural examples of a generalized derivation of R associated with $d = I_b$ and $d = 0$ respectively. In [14], Lee extended the concept of a generalized derivation. Accordingly, let I be a dense right ideal of R and $\delta : I \to Q_{mr}$ be a derivation. A generalized derivation is an additive mapping $F : I \to Q_{mr}$ such that $F(xy) = F(x)y + x\delta(y)$ holds for all $x, y \in I$. Further, in this paper

Lee also showed that F can be uniquely extended to a generalized derivation of Q_{mr} and defined as $F(x) = ax + \delta(x)$ for some $a \in Q_{mr}$ (see [14], Theorem 3.)

Recall that, a nonempty set J, which is an additive subgroup of R is said to be a Jordan ideal of R if $J \circ R \subseteq J$. The following are some well known facts about Jordan ideals: if J be a nonzero Jordan ideal of a ring and $u \in J$, then

I. $2J[R, R] \subseteq J, 2[R, R]J \subseteq J$ ([1], Lemma 2.4)
II. $4u^2R \subseteq J, 4Ru^2 \subseteq J$ and $2[u^2, R] \subseteq J$ ([15], proof of Lemma 3)
III. $4uRu \subseteq J$ ([15], proof of Theorem 3)
IV. $4[u, R]u \subseteq J$ and $4u[u, R] \subseteq J$

A classical result of Herstein [16] states that if a prime ring R with char(R) $\neq 2$ admits a derivation d such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, then R is commutative. Motivated by this situation, Bell and Daif [17] without any restriction on the char (R), obtained the same conclusion from the identity $d(xy) = d(yx)$ (i.e. $d([x, y]) = 0$) where x, y varies over a nonzero ideal of R. In an addition to this, recently Oukhtite et al. [3] proved the following theorem: *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If R admits a nonzero derivation d such that $d([x, y]) \in Z(R)$ for all $x, y \in J$, then R is commutative.* In this paper, we intend to prove this result for generalized derivations.

2. Main Results

Lemma 2.1. *Let R be a ring and J be a Jordan ideal of R. Then $[[J, J], R] \subseteq J$.*

Proof. For any $r \in R$ and $x \in J$, we have $x \circ r \in J$. That is, $xr + rx \in J$. For any $y \in J$, we have $xyr + yrx + yxr - yxr = [x, y]r + y(x \circ r) \in J$. Again we have $ryx + xry - rxy + rxy = -r[x, y] + (x \circ r)y \in J$. On combining these two expressions, we obtain $[[x, y], r] + y \circ (x \circ r) \in J$ for any $x, y \in J$ and $r \in R$. Clearly, $y \circ (x \circ r) \in J$. Therefore, we have $[[x, y], r] \in J$ for all $x, y \in J$ and $r \in R$. □

Lemma 2.2. *Let R be a 2-torsion free semi-prime ring and $J \not\subseteq Z(R)$ be a Jordan ideal of R. Then J contains a nonzero ideal of R.*

Proof. By Lemma 2.1, we have $[[x, y], r] \in J$ for any $x, y \in J$ and $r \in R$. For some $z \in J$, we find $[x, y]zr - zr[x, y] = [x, y]zr - z[x, y]r + z[x, y]r - zr[x, y] = [[x, y], z]r + z[[x, y], r] = [[x, y], z]r + z[xy, r] - z[yx, r] \in J$. By Lemma 2.4 in [1], we have $2z[xy, r] \in J$ and $2z[yx, r] \in J$ for all $x, y, z \in J$ and $r \in R$. On combining these expressions, we obtain $2[[x, y], z]r \in J$ for all $x, y, z \in J$ and $r \in R$. Again, it gives $2[[x, y], z]rs + 2s[[x, y], z]r \in J$, where $x, y, z \in J$ and $r, s \in R$. It implies that $2R[[J, J], J]R \subseteq J$. Further, if $2R[[J, J], J]R = (0)$ i.e. $R[[J, J], J]R = (0)$ it forces that $(R[[J, J], J])^2 = (0)$, which contradicts the semi-primeness of R. Hence, J contains a nonzero ideal of R. □

The following lemma is may be of independent interest.

Lemma 2.3. *Let R be a ring and J be a Jordan ideal of R. Then $2R[J^2, J]R \subseteq J$.*

Proof. It is well known that $2[x^2, r] \in J$ for any $x \in J$ and $r \in R$. For some $y \in J$, we replace r by yr and get $2(x^2yr - yrx^2) \in J$. That means, $2(x^2y - yx^2)r + 2y(x^2r - rx^2) \in J$, where $x, y \in J$ and $r \in R$. Since $2y[x^2, r] \in J$, we must have $2(x^2y - yx^2)r \in J$. Therefore, $2((x^2y - yx^2)r)s + 2s(x^2y - yx^2)r \in J$ for any $x, y \in J$ and $r, s \in R$. Hence, we obtain $R[2J^2, J]R \subseteq J$. □

Lemma 2.4. *Let R be a 2-torsion prime ring. Let $J \nsubseteq Z(R)$ be a Jordan ideal of R and $d : R \to R$ be a derivation of R. If $x^2 d(x^2) = 0$ for all $x \in J$, then $d = 0$.*

Proof. By hypothesis, we have $x^2(d(x)\circ x) = 0$ for any $x \in J$. By Lemma 2.2, J contains a nonzero ideal I of R i.e. $I \subseteq J$, where $2R[[J, J], J]R = I$. That gives, $x^2(d(x) \circ x) = 0$ for all $x \in I$. By Kharchenko's theory [6] of differential identities, we have the following two cases:

Case 1: If d is a Q-outer derivation, then I satisfies the polynomial identity

$$x^2(y \circ x) = 0,$$

for all $x, y \in I$. On replacing y by $2x$, we have $(2x^2)^2 = 0$ for all $x \in I$. Which is a contradiction by Xu [11].

Case 2: Suppose d is a Q-inner derivation induced by some $q \in Q$ i.e $d(r) = [q, r]$ for all $r \in R$. For any $x \in I$, we have

$$x^2([q, x] \circ x) = 0.$$

In view of Theorem 1 in [7], Q and I satisfy same GPIs. Therefore, we have

$$u^2([q, u] \circ u) = 0,$$

for all $u \in Q$. By Theorem 2.5 and 3.5 in [8], Q and $Q \bigotimes_C \overline{C}$ both are prime and centrally closed. So, we may replace R by Q or $Q \bigotimes_C \overline{C}$ according as C is finite or infinite. In case, Q has infinite center C, we have $u^2([q, u] \circ u) = 0$ for any $u \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ stands for algebraic closure of extended centroid C. Thus, we may assume that R is centrally closed over C (i.e. $RC = R$) which is either finite or algebraically closed and

$$u^2([q, u] \circ u) = 0, \tag{1}$$

for all $u \in R$. By Theorem 3 of Martindale [9], RC (and so R) is a primitive ring having nonzero socle $\mho$ with associated division ring D. Now, by a result of Jacobson [[10], pg. 75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over D and $\mho$ contains the linear transformation of R with finite rank. If $\mathcal{V}$ is finite dimensional over D, the density of R on $\mathcal{V}$ implies that $R \cong M_h(D)$, where $h = dim_D(\mathcal{V})$. Let us suppose that $dim_D(\mathcal{V}) \geq 2$, otherwise we are done.

Next, for any $v \in \mathcal{V}$, we claim that $\{v, qv\}$ is a linearly $D-$dependent set. If $qv = 0$, then there is nothing to prove. Let $qv \neq 0$. If possible, we assume that v and qv are linearly independent over D. By the density of R, we can find some $x \in R$ such that

$$xv = 0, \qquad xqv = qv$$

The equation (1) forces that

$$0 = (u^2([q, u] \circ u))v = -qv,$$

which is a contradiction. Thus, $\{v, qv\}$ must be linearly dependent over D for all $v \in \mathcal{V}$. That means, we can find some $\beta \in D$ such that $qv = v\beta$. Next, we shall show that β is independent of the choice of v. Let us choose linearly independent u and v in $\mathcal{V}$. By above process, we can find β_u, β_v and β_{u+v} in D such that

$$qu = u\beta_u, qv = v\beta_v \text{ and } q(u + v) = \beta_{u+v}.$$

Further, $u\beta_u + v\beta_v = (u+v)\beta_{u+v}$. It implies $u(\beta_u - \beta_{u+v}) + v(\beta_v - \beta_{u+v}) = 0$. Hence, $\beta_u = \beta_v = \beta_{u+v}$, as u and v are chosen to be linearly independent.

Now, for any $r \in R$ and $v \in \mathcal{V}$, we have $qv = v\beta, r(qv) = r(v\beta)$ and $q(rv) = rv\beta$. It implies that $[q, r]v = 0$ for all $v \in \mathcal{V}$. But $\mathcal{V}$ is left faithful irreducible $R-$module, hence $[q, r] = 0$ i.e. $q \in Z(R)$ i.e. $d = 0$. □

Theorem 2.5. *Let R be a 2-torsion free prime ring. Let J be a nonzero Jordan ideal of R and $F : R \to R$ be a generalized derivation of R associated with a nonzero derivation d. If $F([J, J]) \in Z(R)$, then R is commutative.*

Proof. We divide the proof into the following two cases:
Case 1: If $J \subseteq Z(R)$. With the aid of Lemma 3 of [2], R is commutative.
Case 2: If $J \nsubseteq Z(R)$. Firstly, we claim that $Z(R) \cap J \neq (0)$. Let us assume that $Z(R) \cap J = (0)$. By our hypothesis, $F([u, v]) \in Z(R)$ for all $u, v \in J$. We replace u and v by $2u^2$ and $2vu^2$ respectively in order to get $4F([u^2, vu^2]) \in Z(R)$. It is easy to see that $4F([u^2, vu^2]) = 4F([u^2, v])u^2 + 4[u^2, v](d(u) \circ u) \in J$. Therefore, we find $4F([u^2, v]u^2) = 0$ for all $u, v \in J$. That gives

$$F([u^2, v])u^2 + [u^2, v]d(u^2) = 0 \tag{2}$$

On replacing v by $2vu^2$ in (2), we get

$$F[u^2, v]u^4 + [u^2, v]d(u^2)u^2 + [u^2, v]u^2d(u^2) = 0 \tag{3}$$

On combining Eq. (2) and Eq. (3), we get $[u^2, v]u^2d(u^2) = 0$. Substitute $v = 2[r, s]v$, we get $[u^2, [r, s]]Ju^2d(u^2) = (0)$. Primeness of J implies that either $[u^2, [r, s]] = 0$ or $u^2d(u^2) = 0$. Let us assume that $u^2d(u^2) = 0$ for all $u \in J$. It leads to a contradiction with the aid of Lemma 2.4. In the latter case, we have $[u^2, [r, s]] = 0$ for any $u \in J$ and $r, s \in R$. Putting $r = sr$, we get $[u^2, s][r, s] = 0$. It implies that $[u^2, s]R[r, s] = (0)$. It forces that $u^2 \in Z(R)$. From the proof of Lemma 5 in [3], $J \subseteq Z(R)$, again a contradiction.

Therefore, we must have $0 \neq w \in Z(R) \cap J$. By our hypothesis, we have $F([u, v]) \in Z(R)$ for all $u, v \in J$. Replace v by $2v^2w$, we get $F([u, 2v^2])w + [u, 2v^2]d(w) \in Z(R)$. Since $F([u, 2v^2])$ and w are in $Z(R)$, so we find $[[u, 2v^2], r]d(w) = 0$ for all $u, v \in J$. It implies that $[[u, 2v^2], r]Rd(w) = (0)$. Therefore, either $[[u, 2v^2], r] = 0$ or $d(w) = 0$. Let us consider $[[u, 2v^2], r] = 0$. Put $v = v + w$, we get $[[u, 2vw], r] = 0$, since $w \in Z(R)$. It implies that $[[x, y], r]w = 0 \Rightarrow [[x, y], r]Rw = (0)$. But $w \neq 0$, so only possibility is $[[x, y], r] = 0$, where $x, y \in J$ and $r \in R$. That is $[J, J] \subseteq Z(R)$. Hence, $J \subseteq Z(R)$ by Lemma 3 of [4], which is not possible.

On other side if $d(w) = 0$. For some $r \in R$, we substitute $2rw$ in the place of u in the equation $F([u, v]) \in Z(R)$, we get $F([r, v])w \in Z(R)$. It implies that $F([r, v]) \in Z(R)$. Replacing y by $2sw$, by the same reasons we get $F([r, s]) \in Z(R)$ for all $r, s \in R$. Let $\zeta(r, s) = rs - sr$, a multilinear polynomial in R. Then we have $F(\zeta(r, s)) \in Z(R)$ i.e. $[F(\zeta(r, s)), \zeta(r, s)] = 0$. By Theorem 2 in [5], either $\zeta(r, s)$ is central valued or $F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$. In case, $F(x) = \lambda x$, our hypothesis yields that $\lambda[r, s] \in Z(R)$. Since $F \neq 0$ so $\lambda \neq 0$ and hence $[r, s] \in Z(R)$. It implies that R is commutative. □

It is trivial that, if F is a generalized derivation of R associated with a derivation d, then so is $F \pm I$, where I is the identity map on R.

Corollary 2.1. *Let R be a 2-torsion free prime ring. Let J be a nonzero Jordan ideal of R and $F : R \to R$ be a generalized derivation of R associated with a nonzero derivation d. If any one of the following:*

1. $F([x, y]) + [x, y] \in Z(R)$
2. $F([x, y]) - [x, y] \in Z(R)$

holds on J, then R is commutative.

We conclude with the following remark, which shows that our main result can't be extended to the class of semiprime rings.

Remark 2.6. Let R^1 be any noncommutative semiprime ring and S^1 be any commutative integral domain. Evidently, $R = S^1 \times R^1$ is a semiprime ring and $J = S^1 \times \{0\}$ is a nonzero Jordan ideal of R. Let $F : R^1 \to R^1$ be a generalized derivation of R^1 associated with a derivation d. We define a mapping $\mathcal{F} : R \to R$ as $(s, r) \mapsto (0, F(r))$ and a mapping $\delta : R \to R$ as $(s, r) \mapsto (0, d(r))$. Note that, $\mathcal{F}$ is a generalized derivation of R associated with derivation δ. Now, it is easy to check that $\mathcal{F}([J, J]) \in Z(R)$, but R is not commutative.

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Conflict of Interest

No potential conflict of interest was reported by the authors.

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