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Research article

Monotonicity of eigenvalues of Witten-Laplace operator along the Ricci-Bourguignon flow

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Abstract: In this article we will investigate monotonicity for the first eigenvalue problem of the Witten-Laplace operator acting on the space of functions along the Ricci-Bourguignon flow on closed manifolds. We find the first variation formula for the eigenvalues of Witten-Laplacian on a closed manifold evolving by the Ricci-Bourguignoni flow and construct various monotonic quantities. At the end we find some applications in 2-dimensional and 3-dimensional manifolds and give an example.

Keywords: Eigenvalue; Laplace; Ricci-Bourguignon flow; Riemannian manifold; Eigenvector

1. Introduction

Let $(M, g(t))$ be a closed Riemannian manifold. Studying the eigenvalues of geometric operators is a very powerful tool for the understanding Riemannian manifolds. It is well known that the spectrum of p-Laplacian and other geometric operators on a compact Riemannian manifold M is an important analytic invariant and has important geometric meanings. There are many mathematicians who investigate properties of the spectrum of geometric operators and estimate the spectrum in terms of the other geometric quantities of M. In [12], Perelman showed that the functional

$$F = \int_M (R + |\nabla f|^2)e^{-f}\, dv$$

is nondecreasing along the Ricci flow coupled to a backward heat-type equation, where R is the scalar curvature with respect to the metric $g(t)$ and dv denotes the volume form of the metric $g = g(t)$. The nondecreasing of the functional F implies that the lowest eigenvalue of the geometric operator $-4\Delta + R$ is nondecreasing along the Ricci flow. As an application, Perelman shown that there are no nontrivial steady or expanding breathers on compact manifolds. Then, Li [11] and Cao [3] extended the geometric operator $-4\Delta + R$ to the operator $-\Delta + cR$ and both them proved that the first eigenvalue of the geometric operator $-\Delta + cR$ for $c \geq \frac{1}{4}$ is nondecreasing along the Ricci flow. Zeng and et 'al [15] studied the monotonicity of eigenvalues of the operator $-\Delta + cR$ along the Ricci-Bourguignon flow. In [8] and [13] have been studied the evolution for the first eigenvalue of geometric operator $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow and Ricci flow, respectively, where $-\Delta_\phi$ is the Witten-Laplacian operator, $\phi \in C^2(M)$, and constructed some monotonic quantities under this flow. For the other recent research in this direction, see [5, 6, 7, 9, 10, 14].

Also, over the last few years the Ricci flow and other geometric flows as the Ricci-Bourguignon flow have been a topic of active research interest in both mathematics and physics. A geometric flow is an evolution of a geometric structure under a differential equation related to a functional on a manifold, usually associated with some curvature. They are all related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold.

Let M be an n-dimensional manifold with a Riemannian metric g_0, the family $g(t)$ of Riemannian metrics on M is called a Ricci-Bourguignon flow when it satisfies the equations

$$\frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), \;\; g(0) = g_0 \tag{1.1}$$

where Ric is the Ricci tensor of $g(t)$, R is the scalar curvature and ρ is a real constant. In fact the Ricci-Bourguignon flow is a system of partial differential equations which was introduced by Bourguignon for the first time in 1981 (see [2]). Short time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ have been shown by Catino and et 'al in [4] for $\rho < \frac{1}{2(n-1)}$. When $\rho = 0$, the Ricci-Bourguignon flow is the Ricci flow.

Motivated by the above works, in this paper we will study the first eigenvalue of the Witten-Laplacian operator whose metric satisfies the Ricci-Bourguignon flow (1.1).

2. Preliminaries

In this section, we will first give the definitions for the first eigenvalue of the Witten-Laplace operator Δ_ϕ then we will find the formula for the evolution of the first eigenvalue of the Witten-Laplace operator under the Ricci-Bourguignon flow on a closed manifold. Let $(M, g(t))$ be a compact Riemannian manifold, and $(M, g(t))$ be a smooth solution to the Ricci-Bourguignon flow (1.1) for $t \in [0, T)$. Let ∇ be the Levi-Civita connection on $(M, g(t))$ and $f : M \to \mathbb{R}$ be a smooth function on M or $f \in W^{1,2}(M)$ where $W^{1,2}(M)$ is the Sobolev space. The Laplacian of f is defined as

$$\Delta f = div(\nabla f) = g^{ij}(\partial_i \partial_j f - \Gamma^k_{ij}\partial_k f). \tag{2.1}$$

Assume that dv the Riemannian volume measure, and $d\mu$ the weight volume measure on $(M, g(t))$ related to function ϕ; i.e.

$$d\mu = e^{-\phi(x)}dv \tag{2.2}$$

where $\phi \in C^2(M)$. The Witten-Laplacian is defined by

$$\Delta_\phi = \Delta - \nabla\phi.\nabla \tag{2.3}$$

which is a symmetric operator on $L^2(M, \mu)$ and satisfies the following integration by part formula:

$$\int_M < \nabla u, \nabla v > d\mu = -\int_M v\Delta_\phi u \, d\mu = -\int_M u\Delta_\phi v \, d\mu \qquad\qquad \forall u, v \in C^\infty(M),$$

The Witten-Laplacian is generalize of Laplacian operator, for example, when ϕ is a constant function, the Witten-Laplacian operator is just the Laplace-Belterami operator.
We say that $\lambda_1(t)$ is an eigenvalue of the Witten-Laplace operator Δ_ϕ at time $t \in [0, T)$ whenever for some $f \in W^{1,2}(M)$,

$$-\Delta_\phi f = \lambda_1(t) f, \tag{2.4}$$

or equivalently

$$\int_M < \nabla f, \nabla h > d\mu = \lambda_1 \int_M f\, h\, d\mu \qquad \forall h \in C^\infty(M), \tag{2.5}$$

hence

$$\lambda_1 = \frac{\int_M |\nabla f|^2 d\mu}{\int_M f^2\, d\mu},$$

the first eigenvalue of the Witten-Laplace operator defined as

$$\lambda = \min_{f \neq 0} \left\{ \int_M |\nabla f|^2 d\mu : \ f \in C^\infty(M), \ \int_M f^2 d\mu = 1 \right\}.$$

Lemma 2.1. *If g_1 and g_2 are two metrics on Riemannian manifold M which satisfy*

$$\frac{1}{1+\epsilon} g_1 \le g_2 \le (1+\epsilon) g_1,$$

then

$$\lambda(g_2) - \lambda(g_1) \le \left((1+\epsilon)^{\frac{n}{2}+1} - (1+\epsilon)^{-\frac{n}{2}} \right) (1+\epsilon)^{\frac{n}{2}} \lambda(g_1).$$

In particular, λ is a continuous function respect to the C^2-topology.

Proof. The proof is straightforward. We have

$$(1+\epsilon)^{-\frac{n}{2}} d\mu_{g_1} \le d\mu_{g_2} \le (1+\epsilon)^{\frac{n}{2}} d\mu_{g_1}.$$

Let

$$\mathcal{G}(g, f) = \int_M |\nabla f|_g^2 d\mu_g,$$

then

$$\begin{aligned}
& \int_M f^2 d\mu_{g_1} \mathcal{G}(g_2, f) - \int_M f^2 d\mu_{g_2} \mathcal{G}(g_1, f) \\
= \ & \int_M f^2 d\mu_{g_1} \int_M |\nabla f|_{g_2}^2 d\mu_{g_2} - \int_M f^2 d\mu_{g_2} \int_M |\nabla f|_{g_1}^2 d\mu_{g_1} \\
= \ & \int_M f^2 d\mu_{g_1} \left(\int_M |\nabla f|_{g_2}^2 d\mu_{g_2} - \int_M |\nabla f|_{g_1}^2 d\mu_{g_1} \right) \\
& + \left(\int_M f^2 d\mu_{g_1} - \int_M f^2 d\mu_{g_2} \right) \int_M |\nabla f|_{g_1}^2 d\mu_{g_1} \\
\le \ & \left((1+\epsilon)^{\frac{n}{2}+1} - 1 \right) \int_M f^2 d\mu_{g_1} \int_M |\nabla f|_{g_1}^2 d\mu_{g_1}
\end{aligned}$$

$$+\left(1-(1+\epsilon)^{-\frac{n}{2}}\right)\int_M f^2 d\mu_{g_1}\int_M |\nabla f|^2_{g_1} d\mu_{g_1},$$

so that

$$\begin{aligned}&\int_M f^2 d\mu_{g_1}\int_M |\nabla f|^2_{g_1} d\mu_{g_1}\left(\frac{\mathcal{G}(g_2,f)}{\int_M f^2 d\mu_{g_2}}-\frac{\mathcal{G}(g_1,f)}{\int_M f^2 d\mu_{g_1}}\right)\\ \le\ &\left((1+\epsilon)^{\frac{n}{2}+1}-(1+\epsilon)^{-\frac{n}{2}}\right)\int_M f^2 d\mu_{g_1}\int_M |\nabla f|^2_{g_1} d\mu_{g_1},\end{aligned}$$

it implie that

$$\frac{\mathcal{G}(g_2,f)}{\int_M f^2 d\mu_{g_2}}-\frac{\mathcal{G}(g_1,f)}{\int_M f^2 d\mu_{g_1}}\le\left((1+\epsilon)^{\frac{n}{2}+1}-(1+\epsilon)^{-\frac{n}{2}}\right)\frac{\int_M |\nabla f|^2_{g_1} d\mu_{g_1}}{\int_M f^2 d\mu_{g_2}},$$

hence

$$\lambda(g_2)-\lambda(g_1)\le\left((1+\epsilon)^{\frac{n}{2}+1}-(1+\epsilon)^{-\frac{n}{2}}\right)(1+\epsilon)^{\frac{n}{2}}\lambda(g_1),$$

this completes the proof of lemma. □

If $\lambda=\int_M |\nabla f|^2 d\mu$ then f is eigenfunction corresponding to λ. Normalized eigenfuctions are defined as $\int_M f^2 d\mu = 1$. At time $t_0\in[0,T)$, we first let $f_0=f(t_0)$ be the eigenfunction for the eigenvalue $\lambda(t_0)$ of Witten-Laplacian. We consider the following smooth function

$$h(t)=f_0\left[\frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))}\right]^{\frac{1}{2}}$$

along the Ricci-Bourguignon flow. We assume that

$$f(t)=\frac{h(t)}{\left(\int_M (h(t))^2 d\mu\right)^{\frac{1}{2}}}$$

which $f(t)$ is smooth function under the Ricci-Bourguignon flow, satisfies $\int_M f^2 d\mu = 1$ and at time t_0, f is the eigenfunction for λ of Witten-Laplacian. Now we define a smooth eigenvalue function

$$\lambda(f,t):=\int_M |\nabla f|^2 d\mu \tag{2.6}$$

where $\lambda(f(t_0),t_0)=\lambda(t_0)$, f is smooth function and satisfies

$$\int_M f^2\, d\mu = 1. \tag{2.7}$$

3. Variation of $\lambda(t)$

In this section, we will give some useful evolution formulas for $\lambda(t)$ under the Ricci-Bourguignon flow. Now, we give a useful proposition about the variation of eigenvalues of Witten-Laplacian under the Ricci-Bourguignon flow.

Proposition 3.1. Let $(M^n, g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$. If $\lambda(t)$ denotes the evolution of the first eigenvalue under the Ricci-Bourguignon flow, then

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = (1-n\rho)\lambda(t_0)\int_M Rf^2 d\mu + ((n-2)\rho - 1)\int_M R|\nabla f|^2 d\mu + 2\int_M Ric(\nabla f, \nabla f)d\mu. \quad (3.1)$$

Proof. $\lambda(f,t)$ is a smooth function and by derivating (2.6) we have

$$\frac{d}{dt}\lambda(f,t) = \int_M \frac{d}{dt}(|\nabla f|^2)d\mu + \int_M |\nabla f|^2 \frac{d}{dt}(d\mu). \quad (3.2)$$

On the other hand, we have

$$\frac{d}{dt}(d\mu_t) = \frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu, \quad (3.3)$$

and

$$\begin{aligned} \frac{d}{dt}\Big(|\nabla f|^2\Big) &= \frac{d}{dt}\Big(g^{ij}\nabla_i f \nabla_j f\Big) = \frac{\partial}{\partial t}(g^{ij})\nabla_i f \nabla_j f + 2g^{ij}\nabla_i f' \nabla_j f \quad (3.4) \\ &= -g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i f \nabla_j f + 2 < \nabla f', \nabla f > . \end{aligned}$$

Replace (3.3) and (3.4) in (3.2), then

$$\begin{aligned} \frac{d}{dt}\lambda(f,t) &= \int_M \left\{-g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i f \nabla_j f + 2 < \nabla f', \nabla f >\right\} d\mu \quad (3.5) \\ &\quad + \int_M |\nabla f|^2 \frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu. \end{aligned}$$

From (1.1), we can then write

$$\begin{aligned} \frac{d}{dt}\lambda(f,t) &= 2\int_M \left\{-g^{il}g^{jk}(-Ric_{lk} + \rho R g_{lk})\nabla_i f \nabla_j f + < \nabla f', \nabla f >\right\} d\mu \quad (3.6) \\ &\quad + \int_M |\nabla f|^2 (n\rho - 1)R d\mu \\ &= 2\int_M Ric(\nabla f, \nabla f)d\mu + 2\int_M < \nabla f', \nabla f >\} d\mu \\ &\quad + ((n-2)\rho - 1)\int_M |\nabla f|^2 R d\mu. \end{aligned}$$

Now, using (2.7), from the condition

$$\int_M f^2 d\mu = 1,$$

and the time derivative, we can get

$$2\int_M f' f d\mu = (1-n\rho)\int_M f^2 R d\mu, \quad (3.7)$$

(2.5) and (3.7) imply that

$$\int_M < \nabla f', \nabla f > d\mu = \lambda(t_0)\int_M f' f d\mu = \frac{\lambda(t_0)}{2}(1-n\rho)\int_M f^2 R d\mu. \tag{3.8}$$

Replacing (3.8) in (3.6), we obtain

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = (1-n\rho)\lambda(t_0)\int_M R f^2 d\mu + ((n-2)\rho - 1)\int_M R|\nabla f|^2 d\mu + 2\int_M Ric(\nabla f, \nabla f) d\mu.$$

□

Theorem 3.2. *Let $g(t)$, $t \in [0,T)$, be a solution of the Ricci-Bourguignon flow (1.1) on a closed manifold M^n, $\rho < \frac{1}{2(n-1)}$ and $\lambda(t)$ be the first eigenvalue of the Witten-Laplace operator of $g(t)$. If $c = \min\limits_{x \in M} R(0)$ and*

$$R_{ij} - \frac{1-(n-2)\rho}{2} R g_{ij} \geq 0 \quad \text{in } \ M^n \times [0,T)$$

then the quantity $\lambda(t)(n - 2(1-n\rho)ct)^{\frac{n}{2}}$ is strictly increasing along the Ricci-Bourguignon flow.

Proof. According to (3.1) of Proposition 3.1, we have

$$\begin{aligned}\frac{d}{dt}\lambda(\lambda(f,t)|_{t=t_0} &= (1-n\rho)\lambda(t_0)\int_M R f^2 d\mu + \int_M (2R_{ij} - (1-(n-2)\rho)R g_{ij})\nabla_i f \nabla_j f \, d\mu \\ &\geq (1-n\rho)\lambda(t_0)\int_M R f^2 d\mu, \end{aligned} \tag{3.9}$$

on the other hand, the scalar curvature under the Ricci-Bourguignon flow evolve by

$$\frac{\partial R}{\partial t} = (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2$$

and inequality $|Ric|^2 \geq \frac{R^2}{n}$ yields

$$\frac{\partial R}{\partial t} \geq (1-2(n-1)\rho)\Delta R + 2(\frac{1}{n} - \rho)R^2. \tag{3.10}$$

Since the solution to the corresponding ODE $y' = 2(\frac{1}{n} - \rho)$ with initial value $c = \min\limits_{x \in M} R(0)$ is

$$\sigma(t) = \frac{nc}{n - 2(1-n\rho)ct} \quad \text{on } \ [0,T).$$

using the maximum principle to (3.10), we get $R_{g(t)} \geq \sigma(t)$. Therefore (3.9) becomes $\frac{d}{dt}\lambda(f,t)|_{t=t_0} \geq (1-n\rho)\lambda(t_0)\sigma(t_0)$, this results that in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{d}{dt}\lambda(f,t) \geq (1-n\rho)\lambda(f,t)\sigma(t).$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln\frac{\lambda(f(t_0), t_0)}{\lambda(f(t_1), t_1)} > \ln(\frac{n - 2(1-n\rho)ct_1}{n - 2(1-n\rho)ct_0})^{-\frac{n}{2}}.$$

Since $\lambda(f(t_0), t_0) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$ we conclude that

$$\ln \frac{\lambda(t_0)}{\lambda(t_1)} > \ln\left(\frac{n - 2(1 - n\rho)ct_1}{n - 2(1 - n\rho)ct_0}\right)^{-\frac{n}{2}},$$

that is the quantity $\lambda(t)(n - 2(1 - n\rho)ct)^{\frac{n}{2}}$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)(n - 2(1 - n\rho)ct)^{\frac{n}{2}}$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. □

Remark 3.3. If $\rho < 0$ and $c > 0$ then function $(n - 2(1 - n\rho)ct)^{\frac{n}{2}}$ is decrasing in t-variable, thus Theorem 3.2, implies that $\lambda(t)$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$.

Corollary 3.4. Let $g(t)$ and $\lambda(t)$ be the same as in Theorem 3.2 where assume $n = 3$ and $\frac{1}{6} < \rho < \frac{1}{4}$. If

$$R_{ij} > \frac{1 - \rho}{2} R g_{ij} \quad \text{in} \;\; M^n \times \{0\}$$

then the conclusion of Theorem 3.2 is also true.

Proof. The pinching inequality $R_{ij} > \frac{1-\rho}{2} R g_{ij}$ is preserved along the Ricci-Bourguignon flow, therefore, for $t \in [0, T)$ we have $R_{ij} - \frac{1-\rho}{2} R g_{ij} > 0$, which Theorem 3.2 implies that the quantity $\lambda(t)(3 - 2(1 - 3\rho)ct)^{\frac{3}{2}}$ is strictly increasing. □

Theorem 3.5. *Let $g(t)$, $t \in [0, T)$, be a solution of the Ricci-Bourguignon flow (1.1) on a closed manifold M^n and $\lambda(t)$ be the first eigenvalue of the Witten-Laplace operator of $g(t)$. If $C = \max_{x \in M} R(0)$ and*

$$0 \leq R_{ij} < \frac{1 - (n - 2)\rho}{2} R g_{ij} \quad \textit{in} \;\; M^n \times [0, T)$$

then the quantity $\lambda(t)(1 - CAt)^{\frac{n\rho - 1}{A}}$ is strictly decreasing along the Ricci-Bourguignon flow on $[0, T')$ where $T' = \min\{T, \frac{1}{C\alpha}\}$ and $A = 2(n(\frac{1-(n-2)\rho}{2})^2 - \rho)$.

Proof. The proof is similar to that of Theorem 3.2 with the difference that we need to estimate the upper bound of the right hand (3.1). Note that $R_{ij} < \frac{1-(n-2)\rho}{2} R g_{ij}$ implies that $|Ric|^2 < n(\frac{1-(n-2)\rho}{2})^2 R^2$. So the evolution of the scalar curvature under the Ricci-Bourguignon flow evolve by

$$\frac{\partial R}{\partial t} = (1 - 2(n - 1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2$$

yields

$$\frac{\partial R}{\partial t} \leq (1 - 2(n - 1)\rho)\Delta R + 2\left(n\left(\frac{1 - (n - 2)\rho}{2}\right)^2 - \rho\right)R^2. \tag{3.11}$$

Applying the maximum principle to (3.11) we have $0 \leq R_{g(t)} \leq \gamma(t)$ where

$$\gamma(t) = \left[C^{-1} - 2\left(n\left(\frac{1 - (n - 2)\rho}{2}\right)^2 - \rho\right)t\right]^{-1} = \frac{C}{1 - CAt} \quad \text{on} \;\; [0, T').$$

Substituting $0 \leq R_{g(t)} \leq \gamma(t)$ and $R_{ij} < \frac{1-(n-1)}{2} R g_{ij}$ into equation (3.1) we obtain $\frac{d}{dt}\lambda(f(t), t) \leq \frac{(1-n\rho)C}{1-CAt}\lambda(f(t), t)$ in any sufficiently small neighborhood of t_0, hence the quantity $\lambda(t)(1 - CAt)^{\frac{n\rho-1}{A}}$ is strictly decreasing. □

Theorem 3.6. *Let $(M, g(t))$, $t \in [0, T)$ be a solution of the Ricci-Bourguignon flow (1.1) on a closed manifold M^n and $\rho < \frac{1}{2(n-1)}$. Let $\lambda(t)$ be the first eigenvalue of the Witten-Laplace operator of the metric $g(t)$. If there is a non-negative constant a such that*

$$R_{ij} - \frac{1-(n-2)\rho}{2} R g_{ij} \geq -a g_{ij} \quad in \;\; M^n \times [0, T) \tag{3.12}$$

and

$$R \geq \frac{2a}{1-n\rho} \quad in \;\; M^n \times \{0\} \tag{3.13}$$

then $\lambda(t)$ is strictly monotone increasing along the Ricci-Bourguignon flow.

Proof. By Proposition 3.1, we have

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = (1-n\rho)\lambda(t_0)\int_M R f^2 d\mu + \int_M (2R_{ij} - (1-(n-2)\rho)Rg_{ij})\nabla_i f \nabla_j f \, d\mu \tag{3.14}$$

combining (3.12), (3.13) and (3.14), we arrive at $\frac{d}{dt}\lambda(f(t), t) > 0$ in any sufficiently small neighborhood of t_0, then $\lambda(f(t_1), t_1) < \lambda(f(t_0), t_0)$ on $[t_1, t_0]$. Since $\lambda(f(t_0), t_0) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$ we conclude that $\lambda(t_1) < \lambda(t_0)$ which show that $\lambda(t)$ is strictly monotone increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. □

Theorem 3.7. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution of the Ricci-Bourguignon flow (1.1) on a closed manifold M^n with positive curvature operator and $\rho < \frac{1}{2(n-1)}$. Let $\lambda(t)$ be the first nonzero eigenvalue of the Witten-Laplace operator of the metric $g(t)$. Then $\lambda(t) \to +\infty$ in finite time, where $R_{ij} + \nabla^2\phi \geq aRg_{ij}$ in $M^n \times [0, T)$ and a is a constant positive real number.*

Proof. In [1], Bakry and Emery proved that on a closed manifold M^n, for any smooth function f,

$$\frac{1}{2}\Delta_\phi |\nabla f|^2 - < \nabla f, \nabla \Delta_\phi f > = |\nabla^2 f|^2 + (Ric + \nabla^2\phi)(\nabla f, \nabla f)$$

then by integration of both above equation, we obtain

$$\int_M \left((\Delta_\phi f)^2 - |\nabla^2 f|^2\right) d\mu = \int_M (Ric + \nabla^2\phi)(\nabla f, \nabla f) d\mu. \tag{3.15}$$

We easily get the following inequality

$$(\Delta f)^2 = \left(\Delta_\phi f + \nabla\phi.\nabla f\right)^2 \geq \frac{(\Delta_\phi f)^2}{2} - |\nabla\phi.\nabla f|^2. \tag{3.16}$$

By Cauchy-Schwartz inequality, we obtain

$$|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2 \geq \frac{(\Delta_\phi f)^2}{2n} - \frac{|\nabla\phi.\nabla f|^2}{n}, \tag{3.17}$$

and $|\nabla\phi.\nabla f|^2 \leq |\nabla\phi|^2|\nabla f|^2$. On the other hand $\phi \in C^2(M)$, then $|\nabla\phi|^2$ is uniformly bounded, we assume that exist a constan real number $b > 0$ such that $|\nabla\phi|^2 < b$. Hence $|\nabla\phi.\nabla f|^2 \leq b|\nabla f|^2$, this yields

$$|\nabla^2 f|^2 \geq \frac{(\Delta_\phi f)^2}{2n} - \frac{b|\nabla f|^2}{n}. \tag{3.18}$$

Recall that $\Delta_\phi f = -\lambda f$, which implies

$$\int_M (\Delta_\phi f)^2 d\mu = \lambda^2 \int_M f^2 d\mu = \lambda^2 \tag{3.19}$$

Combinnig (3.18) and (3.19), we get

$$\int_M \left((\Delta_\phi f)^2 - |\nabla^2 f|^2\right) d\mu \leq \frac{2n-1}{2n}\lambda^2 + \frac{b}{n}\lambda. \tag{3.20}$$

Putting (3.20) into (3.15) results that

$$\int_M (Ric + \nabla^2\phi)(\nabla f, \nabla f) d\mu = \int_M \left((\Delta_\phi f)^2 - |\nabla^2 f|^2\right) d\mu \leq \frac{2n-1}{2n}\lambda^2 + \frac{b}{n}\lambda. \tag{3.21}$$

The inequality $R_{ij} + \nabla^2\phi \geq aRg_{ij}$ leads to

$$\frac{2n-1}{2n}(\lambda(t))^2 + \frac{b}{n}\lambda(t) \geq a\int_M R|\nabla f|^2 d\mu \geq aR_{\min}(t)\lambda(t), \tag{3.22}$$

then

$$\lambda(t) \geq \frac{2n}{2n-1}aR_{\min}(t) + \frac{2b}{2n-1}. \tag{3.23}$$

Since $R_{\min}(t) \to +\infty$ in finite time $T_0 = \frac{n}{2(1-n\rho)\alpha}$ where $\alpha = \min_{x\in M} R(0)$ (see [4], Proposition 4.1) then $\lambda(t) \to +\infty$ in finite time. □

3.1. *Variation of $\lambda(t)$ on a surface*

Now, we write Proposition 3.1 in some remarkable particular cases.

Corollary 3.8. Let $(M^2, g(t))$, $t \in [0, T)$ be a solution of the Ricci-Bourguignon flow on a closed surface (M^2, g_0) for $\rho < \frac{1}{2}$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of the Witten-Laplace operator under the Ricci-Bourguignon flow, then

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = (1-2\rho)\lambda(t_0)\int_M R f^2 d\mu. \tag{3.24}$$

Proof. In dimension $n = 2$, we have $Ric = \frac{1}{2}Rg$, then (3.1) implies that

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} &= (1-2\rho)\lambda(t_0)\int_M R f^2 d\mu - \int_M R|\nabla f|^2 d\mu + \int_M R|\nabla f|^2 d\mu \\ &= (1-2\rho)\lambda(t_0)\int_M R f^2 d\mu.\end{aligned}$$

□

Lemma 3.9. *Let $(M^2, g(t))$, $t \in [0, T)$ be a solution of the Ricci-Bourguignon flow on a closed surface (M^2, g_0) with nonnegative scalar curvature for $\rho < \frac{1}{2}$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of the Witten-Laplace operator under the Ricci-Bourguignon flow, then*

$$\frac{\lambda(0)}{1 - c(1-2\rho)t} \leq \lambda(t)$$

on $(0, T')$ where $c = \min_{x\in M} R(0)$ and $T' = \min\{T, \frac{1}{C(1-2\rho)}\}$.

Proof. In dimension two we have $Ric = \frac{1}{2}Rg$, and the evolution of the scalar curvature R on a closed surface M under the Ricci-Bourguignon flow is

$$\frac{\partial R}{\partial t} = (1-2\rho)(\Delta R + R^2). \tag{3.25}$$

The minimum of R satisfies the differential inequality

$$\frac{d}{dt}R_{\min} \geq (1-2\rho)R^2_{\min}, \qquad c = \min_{x\in M} R(0) \tag{3.26}$$

and this inequality yields $R_{\min} \geq \frac{c}{1-c(1-2\rho)t}$. Therefore

$$\frac{c}{1-c(1-2\rho)t} \leq R, \qquad \text{on} \quad [0, T') \tag{3.27}$$

where $T' = \min\{T, \frac{1}{C(1-2\rho)}\}$. According to (3.24) and $\int_M f^2 d\mu = 1$ we have

$$\frac{c(1-2\rho)\lambda(f,t)}{1-c(1-2\rho)t} \leq \frac{d}{dt}\lambda(f,t), \tag{3.28}$$

in any sufficiently small neighborhood of t_0. Integrating above inequality with respect to time t, we get

$$\frac{\lambda(f(0),0)}{1-c(1-2\rho)t} \leq \lambda(t_0).$$

Since $\lambda(f(0),0) \geq \lambda(0)$, we have $\frac{\lambda(0)}{1-c(1-2\rho)t} \leq \lambda(t_0)$. Since t_0 is arbitrary, then $\frac{\lambda(0)}{1-c(1-2\rho)t} \leq \lambda(t)$ on $(0, T')$. □

Lemma 3.10. *Let (M^2, g_0) be a closed surface with nonnegative scalar curvature, then the eigenvalues of Witten-Laplacian are increasing under the Ricc-Bourguignoni flow for $\rho < \frac{1}{2}$.*

Proof. From [4], under the Ricci-Bourguignoni flow on a surface, we have

$$\frac{\partial R}{\partial t} = (1-2\rho)(\Delta R + R^2)$$

by the scalar maximum principle, the nonnegativity of the scalar curvature is preserved along the Ricci-Bourguignoni flow. Then (3.24) implies that $\frac{d}{dt}\lambda(f,t)|_{t=t_0} > 0$, this results that in any sufficiently small neighborhood of t_0 as I_0, we get $\frac{d}{dt}\lambda(f,t) > 0$. On interval $[t_1, t_0] \subset I_0$, we have $\lambda(f(t_1), t_1) \leq \lambda(f(t_0), t_0)$. Since $\lambda(f(t_0), t_0) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$ we conclude that $\lambda(t_1) \leq \lambda(t_0)$. that is the quantity $\lambda(t)$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. □

3.2. Variation of $\lambda(t)$ on homogeneous manifolds

In this section, we consider the behavior of the spectrum when we evolve an initial homogeneous metric.

Proposition 3.11. Let $(M^n, g(t))$ be a solution of the un-normalized Ricci flow on the smooth closed homogeneous manifold (M^n, g_0). If $\lambda(t)$ denote the evaluation of an eigenvalue under the Ricci-Bourguignoni flow, then

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = 2\int_M Ric(\nabla f, \nabla f)d\mu - 2\rho R\lambda(t_0). \tag{3.29}$$

Proof. Since the evolving metric remains homogeneous and a homogeneous manifold has constant scalar curvature. Therefore (3.1) implies that

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} &= (1-n\rho)\lambda(t_0)R\int_M f^2 d\mu + ((n-2)\rho - 1)R\int_M |\nabla f|^2 d\mu \\ &\quad +2\int_M Ric(\nabla f, \nabla f)d\mu = 2\int_M Ric(\nabla f, \nabla f)d\mu - 2\rho R\lambda(t_0).\end{aligned}$$

□

3.3. Variation of $\lambda(t)$ on 3-dimensional manifolds

In this section, we consider the behavior of $\lambda(t)$ on 3-dimensional manifolds.

Proposition 3.12. Let $(M^3, g(t))$ be a solution of the Ricci-Borguignon flow (1.1) on a closed manifold M^3 whose Ricci curvature is initially positive and there exists $0 \le \epsilon \le \frac{1}{3}$ such that

$$Ric \ge \epsilon R g$$

then the quantity $e^{-\int_0^t A(\tau)d\tau}\lambda(t)$ is nondcreasing along the Ricci-Borguignon flow (1.1) on closed manifold M^3, where $A(t) = \frac{3\beta(1-3\rho)}{3-2(1-3\rho)\beta t} + (\rho - 1 + 2\epsilon)\left(-2(1-\rho)t + \frac{1}{\alpha}\right)^{-1}$, $\alpha = \max_{x\in M} R(0)$ and $\beta = \min_{x\in M} R(0)$.

Proof. In [4] has been shown that the pinching inequality $Ric \ge \epsilon R g$ and nonnegative scalar curvature are preserved along the Ricci-Borguignon flow (1.1) on closed manifold M^3, then using (3.1) we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} &\ge (1-3\rho)\lambda(t_0)\int_M Rf^2 d\mu + (\rho-1)\int_M R|\nabla f|^2 d\mu + 2\epsilon\int_M R|\nabla f|^2 d\mu \\ &= (1-3\rho)\lambda(t_0)\int_M Rf^2 d\mu + (\rho - 1 + 2\epsilon)\int_M R|\nabla f|^2 d\mu,\end{aligned}$$

on the other hand the scalar curvature under the Ricci-Bourguignon flow evolves by

$$\frac{\partial R}{\partial t} = (1-4\rho)\Delta R + 2|Ric|^2 - 2\rho R^2,$$

by $|Ric|^2 \le R^2$ we have

$$\frac{\partial R}{\partial t} \le (1-4\rho)\Delta R + 2(1-\rho)R^2.$$

Let $\sigma(t)$ be the solution to the ODE $y' = 2(1-\rho)y^2$ with initial value $\alpha = \max_{x\in M} R(0)$. By the maximum principle, we have

$$R(t) \le \sigma(t) = \left(-2(1-\rho)t + \frac{1}{\alpha}\right)^{-1} \tag{3.30}$$

on $[0, T')$, where $T' = \min\{T, \frac{1}{2(1-\rho)\alpha}\}$. Also, the inequality $|Ric|^2 \geq \frac{R^2}{3}$ results that

$$\frac{\partial R}{\partial t} \geq (1-4\rho)\Delta R + 2(\frac{1}{3} - \rho)R^2.$$

we assume that $\gamma(t)$ be the solution to the ODE $y' = 2(\frac{1}{3} - \rho)y^2$ with initial value $\beta = \min_{x\in M} R(0)$. Then the maximum principle implies that

$$R(t) \geq \gamma(t) = \frac{3\beta}{3 - 2(1-3\rho)\beta t} \quad \text{on} \quad [0, T). \tag{3.31}$$

Hence

$$\begin{aligned} \frac{d}{dt}\lambda(f,t)|_{t=t_0} &\geq (1-3\rho)\lambda(t_0)\frac{3\beta}{3-2(1-3\rho)\beta t_0} + (\rho - 1 + 2\epsilon)\lambda(t_0)\left(-2(1-\rho)t_0 + \frac{1}{\alpha}\right)^{-1} \\ &= \lambda(t_0)A(t_0) \end{aligned}$$

this results that in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{d}{dt}\lambda(f,t) \geq \lambda(f,t)A(t).$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln\frac{\lambda(f(t_0), t_0)}{\lambda(f(t_1), t_1)} > \int_{t_1}^{t_0} A(\tau)d\tau.$$

Since $\lambda(f(t_0), t_0) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$ we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \int_{t_1}^{t_0} A(\tau)d\tau.$$

that is the quantity $\lambda(t)e^{-\int_0^t A(\tau)d\tau}$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)e^{-\int_0^t A(\tau)d\tau}$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. □

Proposition 3.13. Let $(M^3, g(t))$ be a solution to the Ricci-Bourguignon flow for $\rho < 0$ on a closed homogeneous 3-manifold whose Ricci curvature is initially nonnegative, then the eigenvalues of the Witten-Laplacian are increasing.

Proof. In dimension three the nonnegativity of the Ricci curvature is preserved under the Ricci-Bourguignon flow [4]. From (3.29), its implies that $\lambda(t)$ is increasing. □

4. Example

In this section, we show that the variational formula is effective to derive some properties of the evolving spectrum of the Witten-Laplace operator and then we find $\lambda(t)$ for some of Riemannian manifolds.

Example 4.1. Let (M^n, g_0) be an Einstein manifold i.e. there exists a constant a such that $Ric(g_0) = ag_0$. Assume that we have a solution to the Ricci-Bourguignon flow which is of the form

$$g(t) = u(t)g_0, \quad u(0) = 1$$

where $u(t)$ is a positive function. We compute

$$\frac{\partial g}{\partial t} = u'(t)g_0, \quad Ric(g(t)) = Ric(g_0) = ag_0 = \frac{a}{u(t)}g(t), \quad R_{g(t)} = \frac{an}{u(t)},$$

for this to be a solution of the Ricci-Bourguignon flow, we require

$$u'(t)g_0 = -2Ric(g(t)) + 2\rho R_{g(t)}g(t) = (-2a + \frac{2\rho an}{u(t)})g_0$$

this shows that

$$u'(t) = -2a + \frac{2\rho an}{u(t)},$$

therefore satisfies

$$e^{2at+u(t)-1}\left(\frac{u(t)-\rho n}{1-\rho n}\right)^{\rho n} = 1,$$

so $g(t)$ is an Einstein metric. Using equation (3.1), we obtain the following relation

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = (1-n\rho)\frac{an}{u(t_0)}\lambda(t_0)\int_M f^2 d\mu + ((n-2)\rho - 1)\frac{an}{u(t_0)}\int_M |\nabla f|^2 d\mu + 2\frac{a}{u(t_0)}\int_M |\nabla f|^2 d\mu.$$

or equivalently

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = \frac{2a(1-n\rho)\lambda(t_0)}{u(t_0)}$$

this results that in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{d}{dt}\lambda(f,t) = \frac{2a(1-n\rho)\lambda(f,t)}{u(t)}$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln\frac{\lambda(f(t_0),t_0)}{\lambda(f(t_1),t_1)} = \int_{t_1}^{t_0}\frac{2a(1-n\rho)}{u(\tau)}d\tau$$

Since $\lambda(f(t_0), t_0) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$ we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \int_{t_1}^{t_0}\frac{2a(1-n\rho)}{u(\tau)}d\tau$$

that is the quantity $\lambda(t)e^{-\int_0^t \frac{2a(1-n\rho)}{u(\tau)}d\tau}$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$.

Conflict of Interest

The author declares no conflicts of interest in this paper.

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