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Research article

Global stability for a class of functional differential equations with distributed delay and non-monotone bistable nonlinearity

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Abstract: The present work is devoted to the global stability analysis for a class of functional differential equations with distributed delay and non-monotone bistable nonlinearity. First, we characterize some subsets of attraction basins of equilibria. Next, by Lyapunov functional and fluctuation method, we obtain a series of criteria for the global stability of equilibria. Finally, we illustrate our results by applying them to a problem with Allee effect.

Keywords: distributed delay; bistable nonlinearity; global stability; Lyapunov functional; Allee effect

1. Introduction

In this paper, we study the following general class of functional differential equations with distributed delay and bistable nonlinearity,

$$\begin{cases} x'(t) = -f(x(t)) + \displaystyle\int_0^\tau h(a)g(x(t-a))da, & t > 0, \\ x(t) = \phi(t), & -\tau \le t \le 0. \end{cases} \tag{1.1}$$

Many mathematical models issued from ecology, population dynamics and other scientific fields take the form of Eq (1.1) (see, e.g., [1–15] and references therein). When a spatial diffusion is considered, many models are studied in the literature (see, e.g., [16–22]).

For the case where $g(x) = f(x)$ has not only the trivial equilibrium but also a unique positive equilibrium, Eq (1.1) is said to be a problem with monostable nonlinearity. In this framework, many authors studied problem (1.1) with various monostable nonlinearities such as Blowflies equations, where $f(x) = \mu x$ and $g(x) = \beta x e^{-\alpha x}$, and Mackey-Glass equations, where $f(x) = \mu x$ and $g(x) = \beta x/(\alpha + x^n)$ (see, e.g., [1, 3, 6, 7, 9–13, 21, 23–27]).

When $g(x)$ allows Eq (1.1) to have two positive equilibria x_1^* and x_2^* in addition to the trivial equilibrium, Eq (1.1) is said to be a problem with bistable nonlinearity. In this case, Huang et al. [5] investigated the following general equation:

$$x'(t) = -f(x(t)) + g(x(t - \tau)).$$

The authors described the basins of attraction of equilibria and obtained a series of invariant intervals using the decomposition domain. Their results were applied to models with Allee effect, that is, $f(x) = \mu x$ and $g(x) = \beta x^2 e^{-\alpha x}$. We point out that the authors in [5] only proved the global stability for $x_2^* \leq M$, where $g(M) = \max_{s \in R^+} g(s)$. In this paper, we are interested in the dynamics of the bistable nonlinearity problem (1.1). More precisely, we will present some attracting intervals which will enable us to give general conditions on f and g that ensure global asymptotic stability of equilibria x_1^* and x_2^* in the both cases $x_2^* < M$ and $x_2^* \geq M$.

The paper is organized as follows: in Section 2, we give some preliminary results including existence, uniqueness and boundedness of the solution as well as a comparison result. We finish this section by proving the global asymptotic stability of the trivial equilibrium. Section 3 is devoted to establish the attractive intervals of solutions and to prove the global asymptotic stability of the positive equilibrium x_2^*. In Section 4, we investigate an application of our results to a model with Allee effect. In Section 5, we perform numerical simulation that supports our theoretical results. Finally, Section 6 is devoted to the conclusion.

In the whole paper, we suppose that the function h is positive and

$$\int_0^\tau h(a)da = 1.$$

We give now some standard assumptions.

(T1) f and g are Lipschitz continuous with $f(0) = g(0) = 0$ and there exists a number $B > 0$ such that $\max_{v \in [0,s]} g(v) < f(s)$ for all $s > B$.

(T2) $f'(s) > 0$ for all $s \geq 0$.

(T3) $g(s) > 0$ for all $s > 0$ and there exists a unique $M > 0$ such that $g'(s) > 0$ for $0 < s < M$, $g'(M) = 0$ and $g'(s) < 0$ for $s > M$.

(T4) There exist two positive constants x_1^* and x_2^* such that $g(x) < f(x)$ if $x \in (0, x_1^*) \cup (x_2^*, \infty)$ and $g(x) > f(x)$ if $x \in (x_1^*, x_2^*)$ and $g'(x_1^*) > f'(x_1^*)$.

Let $C := C([-\tau, 0], \mathbb{R})$ be the Banach space of continuous functions defined in $[-\tau, 0]$ with $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$ and $C_+ = \{\phi \in C; \phi(\theta) \geq 0, -\tau \leq \theta \leq 0\}$ is the positive cone of C. Then, (C, C_+) is a strongly ordered Banach space. That is, for all $\phi, \psi \in C$, we write $\phi \geq \psi$ if $\phi - \psi \in C_+$, $\phi > \psi$ if $\phi - \psi \in C_+ \setminus \{0\}$ and $\phi >> \psi$ if $\phi - \psi \in \mathrm{Int}(C_+)$. We define the following ordered interval:

$$C_{[\phi,\psi]} := \{\xi \in C; \phi \leq \xi \leq \psi\}.$$

For any $\chi \in \mathbb{R}$, we write χ^* for the element of C satisfying $\chi^*(\theta) = \chi$ for all $\theta \in [-\tau, 0]$. The segment $x_t \in C$ of a solution is defined by the relation $x_t(\theta) = x(t + \theta)$, where $\theta \in [-\tau, 0]$ and $t \geq 0$. In particular,

$x_0 = \phi$. The family of maps

$$\Phi : [0, \infty) \times C_+ \to C_+,$$

such that

$$(t, \phi) \to x_t(\phi)$$

defines a continuous semiflow on C_+ [28]. For each $t \geq 0$, the map $\Phi(t, .)$ is defined from C_+ to C_+ which is denoted by Φ_t:

$$\Phi_t(\phi) = \Phi(t, \phi).$$

The set of equilibria of the semiflow, which is generated by (1.1), is given by

$$E = \{\chi^* \in C_+; \chi \in \mathbb{R} \text{ and } g(\chi) = f(\chi)\}.$$

2. Preliminary results

In this section, we first provide existence, uniqueness and boundedness of solution to problem (1.1). We then present a Lyapunov functional and show the global asymptotic stability of the trivial equilibrium. We begin by recalling a useful theorem related to a comparison principle (see Theorem 1.1 in page 78 in [29]).

We consider the following problem:

$$\begin{cases} x'(t) = F(x_t), & t > 0, \\ x(t) = \phi(t), & -\tau \leq t \leq 0, \end{cases} \tag{2.1}$$

where $F : \Omega \to \mathbb{R}$ is continuous on Ω, which is an open subset of C. We write $x(t, \phi, F)$ for the maximal defined solution of problem (2.1). When we need to emphasize the dependence of a solution on initial data, we write $x(t, \phi)$ or $x(\phi)$.

Theorem 2.1. *Let $f_1, f_2 : \Omega \to \mathbb{R}$ be continuous, Lipschitz on each compact subset of Ω, and assume that either f_1 or f_2 is a nondecreasing function, with $f_1(\phi) \leq f_2(\phi)$ for all $\phi \in \Omega$. Then*

$$x(t, \phi, f_1) \leq x(t, \phi, f_2),$$

holds for all $t \geq 0$, for which both are defined.

The following lemma states existence, uniqueness and boundedness of the positive solution to problem (1.1). For the proof, see [15, 30, 31].

Lemma 2.2. *Suppose that (**T1**) holds. For any $\phi \in C_+$, the problem (1.1) has a unique positive solution $x(t) := x(t, \phi)$ on $[0, \infty)$ satisfying $x_0 = \phi$, provided $\phi(0) > 0$. In addition, we have the following estimate:*

$$0 \leq \limsup_{t \to \infty} x(t) \leq B.$$

Moreover, the semiflow Φ_t admits a compact global attractor, which attracts every bounded set in C_+.

The following lemma can be easily proved (see also the proof of Theorem 1.1 in [29]).

Lemma 2.3. *Let $\phi \in C$ be a given initial condition, $x(\phi)$ be the solution of problem (1.1) and $x_\epsilon(\phi)$, $\epsilon > 0$ be the solution of problem (1.1) when replacing f by $f \pm \epsilon$. Then*

$$x_\epsilon(t,\phi) \to x(t,\phi) \;\; as \;\; \epsilon \to 0, \;\; for\ all \;\; t \in [0,\infty).$$

We focus now on the global stability of the trivial equilibrium. For this purpose, we suppose that

$$f(s) > g(s) \text{ for all } s \in (0, x_1^*) \text{ and } f(x_1^*) = g(x_1^*). \tag{2.2}$$

Lemma 2.4. *Assume that **(T1)** and condition (2.2) hold. For a given $\phi \in C_{[0,x_1^*]} \setminus \{x_1^*\}$, the solution $x(\phi)$ of problem (1.1) satisfies*

$$\limsup_{t\to\infty} x(t) < x_1^*, \tag{2.3}$$

provided that one of the following hypotheses holds:

***(i)** g is a nondecreasing function over $(0, x_1^*)$.*

***(ii)** f is a nondecreasing function over $(0, x_1^*)$.*

Proof. Without loss of generality, we assume that $\phi(0) < x_1^*$. Suppose that **(i)** holds. We first claim that $x(t) < x_1^*$ for all $t \geq 0$. Let $x_\epsilon(\phi)$ be the solution of problem (1.1) by replacing f by $f + \epsilon$. We prove that $x_\epsilon(t) := x_\epsilon(t,\phi) \leq x_1^*$ for all $t > 0$. Suppose, on the contrary, that there exists $t_1 > 0$ such that $x_\epsilon(t_1) = x_1^*$, $x_\epsilon(t) \leq x_1^*$ for all $t \leq t_1$ and $x'_\epsilon(t_1) \geq 0$. Then, using the equation of $x_\epsilon(t_1)$, we have

$$\begin{aligned} 0 \leq x'_\epsilon(t_1) &= -f(x_\epsilon(t_1)) - \epsilon + \int_0^\tau h(a)g(x_\epsilon(t_1 - a))da, \\ &< -f(x_1^*) + g(x_1^*) = 0, \end{aligned}$$

which leads to a contradiction. Now, applying Lemma 2.3, we obtain $x(t) \leq x_1^*$ for all $t \geq 0$. Next, set $X(t) = x_1^* - x(t)$. We then have

$$X'(t) = f(x(t)) - f(x_1^*) + \int_0^\tau h(a)(g(x_1^*) - g(x(t-a)))da.$$

Since $x(t) \leq x_1^*$ for all $t \geq 0$ and g is a nondecreasing function, we have, for $t \geq \tau$,

$$X'(t) \geq -LX(t),$$

where L is the Lipschitz constant of f. Consequently, $X(t) \geq X(0)e^{-Lt}$ and the claim is proved. Finally, to prove inequality (2.3), we suppose, on the contrary, that there exist an increasing sequence $(t_n)_n$, $t_n \to \infty$, $t_0 \geq \tau$ and a nondecreasing sequence $(x(t_n))_n$, such that $x(t_n) \to x_1^*$ as $t_n \to \infty$, $x(t_n) < x_1^*$ for some n and $x(t_n) = \max_{t\in[0,t_n]} x(t)$. Then, in view of condition (2.2), the equation of $x(t_n)$ satisfies

$$\begin{aligned} 0 \leq x'(t_n) &= -f(x(t_n)) + \int_0^\tau h(a)g(x(t_n - a))da, \\ &\leq -f(x(t_n)) + g(x(t_n)) < 0, \end{aligned}$$

which is a contradiction.

Suppose now that (**ii**) holds. Let $x_\epsilon(t)$ be the solution of problem (1.1) by replacing $f(s)$ by $f(s)+\epsilon$ and $g(s)$ by $g^+(s) = \max_{\sigma\in[0,s]} g(\sigma)$.

Then, again by contradiction, suppose that there exists $t_1 > 0$ such that $x_\epsilon(t_1) = x_1^*$, $x_\epsilon(t) \le x_1^*$ for all $t \le t_1$ and $x'_\epsilon(t_1) \ge 0$. Then, arguing as above, we obtain

$$\begin{aligned} 0 \le x'_\epsilon(t_1) &= -f(x_\epsilon(t_1)) - \epsilon + \int_0^\tau h(a)g^+(x_\epsilon(t-a))da, \\ &< -f(x_1^*) + g^+(x_1^*). \end{aligned} \tag{2.4}$$

Since f is a nondecreasing function and g^+ is the smallest nondecreasing function that is greater than g, we obtain $f(s) > g^+(s)$ for all $s \in (0, x_1^*)$. This contradicts with inequality (2.4). Further, by combining Theorem 2.1 and Lemma 2.3, we get $x(t) \le x_1^*$ for all $t \ge 0$. Finally, following the same arguments as in the first part of this proof, the lemma is proved. ■

We now prove the global asymptotic stability of the trivial equilibrium.

Theorem 2.5. *Assume that (**T1**) and condition (2.2) hold. Suppose also that* $\phi \in C_{[0,x_1^*]} \setminus \{x_1^*\}$. *The trivial equilibrium is globally asymptotically stable if one of the following hypotheses holds:*

(i) *g is a nondecreasing function over* $(0, x_1^*)$.

(ii) *f is a nondecreasing function over* $(0, x_1^*)$.

Proof. Suppose that (**i**) holds. Let V be the Lyapunov functional defined by

$$V(s) = s + \int_0^\tau \psi(a)g(\phi(-a))da, \tag{2.5}$$

where $\psi(a) = \int_a^\tau h(\sigma)d\sigma$. The derivative of V along the solution of problem (1.1) gives

$$\frac{dV(x_t)}{dt} = g(x(t)) - f(x(t)) \quad \forall t > 0.$$

From condition (2.2) and Lemma 2.4, we have $dV(x_t)/dt < 0$ and thus, the result is reached by classical Lyapunov theorem (see [6]). Next, suppose that (**ii**) holds. Let the function V be defined in (2.5) by replacing $g(s)$ by $g^+(s) := \max_{\sigma\in[0,s]} g(\sigma)$. Since $g^+(s) < f(s)$ for all $s \in (0, x_1^*)$, we can employ the same argument as above to get $dV(x_t)/dt < 0$. Finally the result is obtained by applying Theorem 2.1 and Lemma 2.4. This completes the proof. ■

Now, suppose that

$$f(s) > g(s) \text{ for all } s > 0. \tag{2.6}$$

Using the same proof as in Theorem 2.5, we immediately obtain the following theorem.

Theorem 2.6. *Assume that (**T1**) and condition (2.6) hold. The trivial equilibrium is globally asymptotically stable for all* $\phi \in C_+$.

3. Global stability of the positive equilibrium x_2^*

The following theorem concerns the global stability of x_2^* in the case where g is a nondecreasing function.

Theorem 3.1. *Suppose that $\phi \in C_{[x_1^*, \sup_{s\in[-\tau,0]} \phi]} \setminus \{x_1^*\}$ and g is a nondecreasing function. Assume also that (**T1**) and (**T4**) hold. Then, the positive equilibrium x_2^* is globally asymptotically stable.*

Proof. Without loss of generality, suppose that $\phi(0) > x_1^*$. We first claim that $\liminf_{t\to\infty} x(t) > x_1^*$. For this, let $x_\epsilon(t) := x_\epsilon(t;\phi)$ be the solution of problem (1.1) when replacing f by $f - \epsilon$. To reach the claim, we begin by proving that $x_\epsilon(t) > x_1^*$ for all $t \geq 0$. Otherwise, there exists $t_1 > 0$ such that $x_\epsilon(t_1) = x_1^*$, $x_\epsilon(t) \geq x_1^*$ for all $t \leq t_1$ and $x'_\epsilon(t_1) \leq 0$. Then, the equation of $x_\epsilon(t_1)$ satisfies

$$\begin{aligned} 0 \geq x'_\epsilon(t_1) &= -f(x_\epsilon(t_1)) + \epsilon + \int_0^\tau h(a)g(x_\epsilon(t_1 - a))da \\ &> -f(x_1^*) + g(x_1^*) = 0. \end{aligned}$$

This reaches a contradiction and thus $x_\epsilon(t) > x_1^*$ for all $t > 0$. This result, together with Lemma 2.3, gives $x(t) \geq x_1^*$ for all $t \geq 0$.

Next, for $X(t) = x(t) - x_1^*$ and since $f(x_1^*) = g(x_1^*)$, the equation of $X(t)$ satisfies

$$X'(t) = f(x_1^*) - f(x(t)) + \int_0^\tau h(a)(g(x(t-a)) - g(x_1^*))da,$$

this leads to

$$X'(t) \geq -LX(t),$$

where L is the Lipschitz constant of f. Consequently $x(t) > x_1^*$ for all $t > 0$.

Now, suppose that there exist an increasing sequence $(t_n)_n$, $t_n \to \infty$, $t_0 \geq \tau$ and a nonincreasing sequence $(x(t_n))_n$, such that $x(t_n) \to x_1^*$ as $t_n \to \infty$, $x(t_n) > x_1^*$ for some n and $x(t_n) = \min_{t\in[0,t_n]} x(t)$. In view of (**T4**), the equation of $x(t_n)$ satisfies

$$\begin{aligned} 0 \geq x'(t_n) &= -f(x(t_n)) + \int_0^\tau h(a)g(x(t_n - a))da, \\ &\geq -f(x(t_n)) + g(x(t_n)) > 0, \end{aligned}$$

which is a contradiction. The claim is proved.

To prove that x_2^* is globally asymptotically stable, we consider the following Lyapunov functional

$$V(\phi) = \int_{x_2^*}^{\phi(0)} [g(s) - g(x_2^*)]ds + \frac{1}{2}\int_0^\tau \psi(a)[g(\phi(-a)) - g(x_2^*)]^2 da,$$

with $\psi(a) = \displaystyle\int_a^\tau h(\sigma)d\sigma$. By a straightforward computation, the derivative of V along the solution of problem (1.1) gives, for all $t > 0$,

$$
\begin{aligned}
\frac{dV(x_t)}{dt} &= [g(x(t)) - g(x_2^*)]x'(t) - \frac{1}{2}\int_0^\tau \psi(a)\frac{\partial}{\partial a}[g(x(t-a)) - g(x_2^*)]^2 da \\
&= [g(x(t)) - g(x_2^*)]\left[-f(x(t)) + \int_0^\tau h(a)g(x(t-a))da\right] \\
&\quad -\frac{1}{2}\left\{\psi(\tau)[g(x(t-\tau)) - g(x_2^*)]^2 - \psi(0)[g(x(t)) - g(x_2^*)]^2\right. \\
&\quad \left. - \int_0^\tau \psi'(a)[g(x(t-a)) - g(x_2^*)]^2 da\right\} \\
&= [g(x(t)) - g(x_2^*)]\,[-f(x(t)) + g(x(t))] \\
&\quad +[g(x(t)) - g(x_2^*)]\left[-g(x(t)) + \int_0^\tau h(a)g(x(t-a))da\right] \\
&\quad +\frac{1}{2}\int_0^\tau h(a)da[g(x(t)) - g(x_2^*)]^2 - \frac{1}{2}\int_0^\tau h(a)[g(x(t-a)) - g(x_2^*)]^2 da \\
&= [g(x(t)) - g(x_2^*)]\,[-f(x(t)) + g(x(t))] \\
&\quad +\frac{1}{2}\int_0^\tau h(a)\left\{2[g(x(t)) - g(x_2^*)][-g(x(t)) + g(x(t-a))] + [g(x(t)) - g(x_2^*)]^2\right. \\
&\quad \left. -[g(x(t-a)) - g(x_2^*)]^2\right\} da \\
&= [g(x(t)) - g(x_2^*)]\,[-f(x(t)) + g(x(t))] - \frac{1}{2}\int_0^\tau h(a)\,[g(x(t)) - g(x(t-a))]^2\, da.
\end{aligned}
$$

Note that $\psi(\tau) = 0$. In view of (**T4**), we have $dV(x_t)/dt \leq 0$. If g is an increasing function, then the result is reached by using a classical Lyapunov theorem (see, e.g., [30]). If g is a nondecreasing function, then the result is proved by using the same argument as in the proof of Theorem 2.6 in [31]. This completes the proof. ■

We focus now on the case where g is non-monotone. Suppose that there exists $\hat{G}(x)$ such that $\hat{G}(x) = \hat{g}^{-1}og(x)$, where $\hat{g}(.)$ denotes the restriction of g to the interval $[M, \infty)$. Then, $\hat{G}(x) = x$ for $x \in [M, \infty)$ and $\hat{G}(x) > M > x$ for $x \in [0, M)$.

Lemma 3.2. *Assume that (**T1**)–(**T4**) hold. For a given $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$, let x be the solution of problem (1.1). Then, the following assertions hold:*

(i) *if $x_2^* < M$, then $x_1^* < x(t) < \hat{G}(x_1^*)$ for all $t > 0$.*

(ii) *if $x_2^* \geq M$ and $f(\hat{G}(x_1^*)) > g(M)$, then $x_1^* < x(t) < \hat{G}(x_1^*)$ for all $t > 0$.*

Proof. Without loss of generality, suppose that $x_1^* < \phi(0) \leq \hat{G}(x_1^*)$. First, observe that, for a given $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$, we have $f(\hat{G}(x_1^*)) > f(x_1^*) = g(x_1^*) = g(\hat{G}(x_1^*))$, and thus, there exists $\epsilon > 0$ such that $f(\hat{G}(x_1^*)) - \epsilon > g(\hat{G}(x_1^*))$. We begin by proving that $x_1^* \leq x(t) \leq \hat{G}(x_1^*)$ for all $t > 0$. To this end, in view of Lemma 2.3, we only need to prove that $x_1^* < x_\epsilon(t) := x_\epsilon(t; \phi) \leq \hat{G}(x_1^*)$ for all $t > 0$, where $x_\epsilon(t) := x_\epsilon(t; \phi)$ is the solution of problem (1.1) when replacing f by $f - \epsilon$. Let $y_\epsilon := y_\epsilon(\phi)$ be the solution of

$$
\begin{cases}
y_\epsilon'(t) = -f(y_\epsilon(t)) + \epsilon + \displaystyle\int_0^\tau h(a)g^+(y_\epsilon(t-a))da, & t > 0, \\
y_\epsilon(t) = \phi(t), & -\tau \leq t \leq 0.
\end{cases}
\tag{3.1}
$$

with $g^+(s) = \max_{\sigma\in[0,s]} g(\sigma)$. Since f is an increasing function, we have $f(\hat{G}(x_1^*)) - \epsilon > g^+(\hat{G}(x_1^*))$. Accordingly, the function $\hat{G}(x_1^*)$ is a super-solution of problem (3.1). Finally, in view of Theorem 2.1, we obtain $x_\epsilon(t) \le y_\epsilon(t) \le \hat{G}(x_1^*)$ for all $t \ge 0$.

We now prove that $x_\epsilon(t) > x_1^*$. Suppose, on the contrary, that there exists $t_1 > 0$ such that $x_\epsilon(t_1) = x_1^*$, $x_\epsilon(t) \ge x_1^*$ for all $t \le t_1$, and thus, $x'_\epsilon(t_1) \le 0$. Then, the equation of $x_\epsilon(t_1)$ satisfies

$$\begin{aligned} x'_\epsilon(t_1) &= -f(x_\epsilon(t_1)) + \epsilon + \int_0^\tau h(a)g(x_\epsilon(t_1 - a))da \\ &> -f(x_1^*) + g(x_1^*) = 0, \end{aligned}$$

since $g(s) \ge g(x_1^*)$ for all $s \in [x_1^*, \hat{G}(x_1^*)]$. This reaches a contradiction. Further, from Lemma 2.3, we obtain $x(t) \ge x_1^*$. The claim is proved.

Next, let $X(t) = x(t) - x_1^*$. Since $f(x_1^*) = g(x_1^*)$, the equation of $X(t)$ satisfies

$$X'(t) = f(x_1^*) - f(x(t)) + \int_0^\tau h(a)(g(x(t-a)) - g(x_1^*))da.$$

We know that $g(s) \ge g(x_1^*)$ for all $s \in [x_1^*, \hat{G}(x_1^*)]$. We then have

$$X'(t) \ge -LX(t),$$

where L is the Lipschitz constant of f. Consequently $x(t) > x_1^*$ for all $t > 0$.

Next, we prove that $x(t) < \hat{G}(x_1^*)$ for all $t > 0$. Suppose, on the contrary, that there exists $t_1 > 0$ such that $x(t_1) = \hat{G}(x_1^*)$, $x(t) \le \hat{G}(x_1^*)$ for all $t \le t_1$ and $x'(t_1) \ge 0$. Then

$$\begin{aligned} 0 \le x'(t_1) &= -f(x(t_1)) + \textstyle\int_0^\tau h(a)g(x(t_1 - a))da, \\ &\le -f(\hat{G}(x_1^*)) + g(M). \end{aligned} \tag{3.2}$$

Since f is an increasing function and $\hat{G}(x_1^*) > M$, we have

$$0 \le x'(t_1) \le -f(M) + g(M). \tag{3.3}$$

Now, the assertion $x_2^* < M$ implies that $g(M) < f(M)$, which leads to a contradiction.

When $x_2^* \ge M$, inequality (3.2) gives a contradiction by hypothesis. The Lemma is proved. ■

Using Lemma 3.2, we next prove the following lemma.

Lemma 3.3. *Suppose that $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$. Assume also that **(T1)**–**(T4)** hold. Let x be the solution of problem (1.1). Then, we have*

$$\liminf_{t\to\infty} x(t) > x_1^*,$$

provided that one of the following assertions holds:

(i) *$x_2^* < M$.*

(ii) *$x_2^* \ge M$ and $f(\hat{G}(x_1^*)) > g(M)$.*

Proof. Firstly, from Lemma 3.2, both assertions imply that $x_1^* < x(t) < \hat{G}(x_1^*)$ for all $t \geq 0$. Next, observe that there exists $\epsilon > 0$ such that

$$g(s) > f(x_1^* + \epsilon) \quad \text{for all} \quad s \in [x_1^* + \epsilon, \theta_\epsilon],$$

with $\theta_\epsilon \in (M, \hat{G}(x_1^*))$ and $g(\theta_\epsilon) = g(x_1^* + \epsilon)$. Indeed, in view of **(T3)**, $\min_{\sigma \in [x_1^*+\epsilon, \theta_\epsilon]} g(\sigma) = g(\theta_\epsilon) = g(x_1^* + \epsilon)$ and from **(T4)**, we have $g(x_1^* + \epsilon) > f(x_1^* + \epsilon)$.

Now, consider the following nondecreasing function:

$$\underline{g}(s) = \begin{cases} g(s), & \text{for } \ 0 < s < \bar{m}, \\ f(x_1^* + \epsilon), & \text{for } \ \bar{m} < s < \hat{G}(x_1^*), \end{cases} \tag{3.4}$$

where $x_1^* < \bar{m} < x_1^* + \epsilon$, which is a constant satisfying $g(\bar{m}) = f(x_1^* + \epsilon)$. From (**T2**) and (**T4**), we get $\underline{g}(s) > f(s)$ for $x_1^* < s < x_1^* + \epsilon$ and $\underline{g}(s) < f(s)$ for $s > x_1^* + \epsilon$. Let $y(\phi)$ be the solution of problem (1.1) when replacing g by $\underline{g}$. Then, according to Theorem 2.1, we have $y(t;\phi) \leq x(t;\phi)$. In addition, using Theorem 3.1, we obtain

$$x_1^* < \lim_{t\to\infty} y(t;\phi) = x_1^* + \epsilon \leq \liminf_{t\to\infty} x(t;\phi).$$

The lemma is proved. ■

Under Lemma 3.3, we prove the following theorem on the global asymptotic stability of the positive equilibrium $x_2^* < M$.

Theorem 3.4. *Suppose that $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$ and $x_2^* < M$. Assume also that (**T1**)–(**T4**) hold. Then, the positive equilibrium x_2^* is globally asymptotically stable.*

Proof. We first claim that there exists $T > 0$ such that $x(t) \leq M$ for all $t \geq T$. Indeed, let $x_\epsilon := x_\epsilon(\phi)$ be the solution of problem (1.1) when replacing f by $f + \epsilon$. First, suppose that there exists $T > 0$ such that $x_\epsilon(t) \geq M$ for all $t \geq T$. So, since $x_2^* < M$, we have from (**T4**) that $g(M) < f(M)$. Combining this with (**T2**), the equation of x_ϵ satisfies

$$x'_\epsilon(t) \leq -f(M) - \epsilon + g(M) \leq -\epsilon,$$

which contradicts with $x_\epsilon(t) \geq M$. Hence there exists $T > 0$ such that $x_\epsilon(T) < M$. We show that $x_\epsilon(t) < M$ for all $t \geq T$. In fact, at the contrary, if there exists $t_1 > T$ such that $x_\epsilon(t_1) = M$ and so $x'_\epsilon(t_1) \geq 0$ then,

$$x'_\epsilon(t_1) = -f(M) - \epsilon + g(M) < 0,$$

which is a contradiction. Further, according to Lemma 2.3, there exists $T > 0$ such that $x(t) \leq M$ for all $t \geq T$ and the claim is proved. Finally, since g is a nondecreasing function over $(0, M)$, the global asymptotic stability of x_2^* is proved by applying Theorem 3.1. ■

Remark 3.5. *In the case where $g(s) > g(x_1^*)$ for all $s > x_1^*$, the above theorem holds true. In fact, it suffices to replace $\hat{G}(x_1^*)$ in $C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$ by $\sup_{s \in [-\tau, 0]} \phi(s)$.*

We focus now on the case where $x_2^* \geq M$. To this end, suppose that there exists a unique constant A such that

$$A \geq M \ \text{ and } \ g(A) = f(M). \tag{3.5}$$

Lemma 3.6. *Assume that (**T1**)–(**T4**) hold. Suppose also that $x_2^* \geq M$ and $f(A) \geq g(M)$. For a given $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$ and the solution $x := x(\phi)$ of problem (1.1), there exists $T > 0$ such that*

$$M \leq x(t) \leq A \ \textit{for all} \ \ t \geq T.$$

Proof. We begin by claiming that $x_2^* \leq A$. On the contrary, suppose that $x_2^* > A \geq M$. Then, due to (**T2**) and (**T3**), we have $g(x_2^*) < g(A) = f(M) < f(x_2^*) = g(x_2^*)$, which is a contradiction. The claim is proved.

Next, for a given $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$, let $x_\epsilon := x_\epsilon(\phi)$ be the solution of problem (1.1) when replacing f by $f + \epsilon$. Since x_ϵ converges to x as ϵ tends to zero, we only need to prove that there exists $T > 0$ such that $x_\epsilon(t) < A$ for all $t \geq T$. To this end, let $y_\epsilon := y_\epsilon(\phi)$ be the solution of problem (1.1), when replacing f by $f + \epsilon$ and g by g^+ with $g^+(s) = \max\limits_{\sigma \in [0,s]} g(\sigma)$. By Theorem 2.1, we have $x_\epsilon(t) \leq y_\epsilon(t)$ for all $t \geq 0$, and thus, we only need to show that $y_\epsilon(t) < A$ for all $t \geq T$. On the contrary, we suppose that $y_\epsilon(t) \geq A$ for all $t > 0$. Then, combining the equation of y_ϵ and the fact that $g^+(s) = g(M)$ for all $s \geq M$, we obtain

$$y_\epsilon'(t) \quad \leq \quad g(M) - f(A) - \epsilon < 0,$$

which is a contradiction. Then, there exists $T > 0$ such that $y_\epsilon(T) < A$. We further claim that $y_\epsilon(t) < A$ for all $t \geq T$. Otherwise, there exists $t_1 > T$ such that $y_\epsilon(t_1) = A$ and $y_\epsilon'(t_1) \geq 0$. Substituting $y_\epsilon(t_1)$ in Eq (1.1), we get

$$\begin{aligned} 0 \leq y_\epsilon'(t_1) \quad &= \quad -f(y_\epsilon(t_1)) - \epsilon + \int_0^\tau h(a) g^+(y_\epsilon(t_1 - a)) da, \\ &< \quad -f(A) + g(M) \leq 0, \end{aligned}$$

which is a contradiction. Consequently, by passing to the limit in ϵ, we obtain that $x(t) \leq A$ for all $t \geq T$.

We focus now on the lower bound of x. First, define the function

$$\underline{g}(s) = \begin{cases} g(s), & \text{for } \ 0 < s < \bar{m}, \\ f(M), & \text{for } \ \bar{m} < s \leq A, \end{cases} \tag{3.6}$$

where $x_1^* < \bar{m} < M$, which is the constant satisfying $g(\bar{m}) = f(M)$. Note that, due to (**T2**)–(**T4**) and condition (3.5), the function $\underline{g}$ is nondecreasing over $(0, A)$ and satisfies

$$\begin{cases} \underline{g}(s) \leq g(s), & 0 \leq s \leq A, \\ \underline{g}(s) > f(s), & x_1^* < s < M, \\ \underline{g}(s) < f(s), & M < s \leq A. \end{cases}$$

Let $y(\phi)$ be the solution of problem (1.1) when replacing g by $\underline{g}$. From Theorem 2.1, we have $y(t;\phi) \leq x(t;\phi)$ for all $t > 0$, and from Theorem 3.1, we have $\lim\limits_{t\to\infty} y(t;\phi) = M$. Consequently, $\liminf\limits_{t\to\infty} x(t) \geq M$. In addition, if, for all $T > 0$, there exists $(t_n)_n$ such that $t_n > T$, $y(t_n) = M$, $\bar{m} < y(s) \leq A$ for all $0 < s \leq t_n$ and $y'(t_n) < 0$, then, for $t_n > T + \tau$,

$$y'(t_n) = -f(M) + f(M) = 0,$$

which is a contradiction. The lemma is established. ■

Denote $\bar{f} = f|_{[M,A]}$ the restriction function of f over $[M, A]$ and $G(s) := \bar{f}^{-1}(g(s))$ for $s \in [M, A]$. Now, we are ready to state our main theorem related to $x_2^* \geq M$.

Theorem 3.7. *Under the assumptions of Lemma 3.6, the positive equilibrium x_2^* of problem (1.1) is globally asymptotically stable, provided that one of the following conditions holds:*

(H1) *fg is a nondecreasing function on $[M, A]$.*

(H2) *$f + g$ is a nondecreasing function over $[M, A]$.*

(H3) $\dfrac{(GoG)(s)}{s}$ *is a nonincreasing function over $[M, x_2^*]$,*

(H4) $\dfrac{(GoG)(s)}{s}$ *is a nonincreasing function over $[x_2^*, A]$,*

Proof. Denote $x_\infty := \liminf_{t\to\infty} x(t)$ and $x^\infty := \limsup_{t\to\infty} x(t)$. First, suppose that either $x^\infty \leq x_2^*$ or $x_\infty \geq x_2^*$. For $x^\infty \leq x_2^*$, we introduce the following function:

$$\underline{g}(s) = \begin{cases} \min\limits_{\sigma\in[s,x_2^*]} g(\sigma) & \text{for} \quad x_1^* < s < x_2^*, \\ g(x_2^*) & \text{for} \quad x_2^* \leq s \leq A. \end{cases}$$

Let $y(\phi)$ be the solution of problem (1.1) when replacing g by $\underline{g}$. Since $\underline{g}(s) \leq g(s)$ for all $x_1^* \leq s \leq x_2^*$, we have $y(t) \leq x(t)$ for all $t > 0$. Further, in view of Theorem 3.1, we obtain

$$\lim_{t\to\infty} y(t) = x_2^* \leq \limsup_{t\to\infty} x(t) = x^\infty \leq x_2^*.$$

The local stability is obtained by using the same idea as in the proof of Theorem 2.6 in [31].

Now, for $x_\infty \geq x_2^*$, we introduce the function

$$\bar{g}(s) = \begin{cases} \min\limits_{\sigma\in[s,x_2^*]} g(\sigma) & \text{for} \quad x_1^* < s \leq x_2^*, \\ \max\limits_{\sigma\in[x_2^*,s]} g(\sigma) & \text{for} \quad x_2^* < s \leq A, \end{cases} \tag{3.7}$$

and let $y(\phi)$ be the solution of problem (1.1) when replacing g by $\bar{g}$. As above, we have $x(t) \leq y(t)$ for all $t > 0$ and $y(t)$ converges to x_2^* as t goes to infinity. Next, we suppose that $x_\infty < x_2^* < x^\infty$ and we prove that it is impossible. Indeed, according to Lemma 3.6, we know that, for every solution x of problem (1.1), we have $M \leq x_\infty \leq x^\infty \leq A$.

Now, using the fluctuation method (see [28, 32]), there exist two sequences $t_n \to \infty$ and $s_n \to \infty$ such that

$$\lim_{n\to\infty} x(t_n) = x^\infty, \quad x'(t_n) = 0, \quad \forall n \geq 1,$$

and

$$\lim_{n\to\infty} x(s_n) = x_\infty, \quad x'(s_n) = 0, \quad \forall n \geq 1.$$

Substituting $x(t_n)$ in problem (1.1), it follows that

$$0 = -f(x(t_n)) + \int_0^\tau h(a)g(x(t_n - a))da. \tag{3.8}$$

Since g is nonincreasing over $[M, A]$, we obtain, by passing to the limit in Eq (3.8), that

$$f(x^\infty) \leq g(x_\infty). \tag{3.9}$$

Similarly, we obtain

$$f(x_\infty) \geq g(x^\infty). \tag{3.10}$$

Multiplying the expression (3.9) by $g(x^\infty)$ and combining with inequality (3.10), we get

$$f(x^\infty)g(x^\infty) \leq f(x_\infty)g(x_\infty).$$

This fact, together with the hypothesis $(H1)$, gives $x^\infty \leq x_\infty$, which is a contradiction. In a similar way, we can conclude the contradiction for $(H2)$. Now, suppose that $(H3)$ holds. First, notice that G makes sense, that is, for all $s \in [M, A]$, the range of g is contained in $[\bar{f}(M), \bar{f}(A)]$ since $\bar{f}$ is strictly increasing over $[M, A]$. In fact, for all $s \in [M, A]$ and since g is non-increasing over $[M, A]$, we have $g(A) \leq g(s) \leq g(M)$. Now, using the fact that $f(A) \geq f(M)$, we show that

$$\bar{f}(M) \leq g(s) \leq \bar{f}(A) \quad \text{for all} \quad s \in [M, A].$$

Therefore, the function G is nonincreasing and maps $[M, A]$ to $[M, A]$. In view of inequalities (3.9) and (3.10) and the monotonicity of $\bar{f}$, we arrive at

$$x^\infty \leq G(x_\infty), \tag{3.11}$$

and

$$x_\infty \geq G(x^\infty), \tag{3.12}$$

with $G(s) = \bar{f}^{-1}(g(s))$. Now, applying the function G to inequalities (3.11) and (3.12), we find

$$x_\infty \geq G(x^\infty) \geq (GoG)(x_\infty),$$

which gives

$$\frac{(GoG)(x_\infty)}{x_\infty} \leq 1 = \frac{(GoG)(x_2^*)}{x_2^*}. \tag{3.13}$$

Due to $(H3)$, it ensures that $x_2^* \leq x_\infty$, which is impossible. Using the same arguments as above, we obtain a contradiction for $(H4)$. The theorem is proved. ■

Remark 3.8. *In the case where $g(s) > g(A)$ for all $s > M$, the two above results hold true. In fact, it suffices to replace A in Lemma 3.6 and Theorem 3.7 by B defined in (**T1**).*

For the tangential case where two positive equilibria x_1^* and x_2^* are equal, we have the following theorem

Theorem 3.9. *Suppose that (**T1**)–(**T3**) hold. Suppose that, in addition to the trivial equilibrium, problem (1.1) has a unique positive equilibrium x_1^*. Then*

(i) for $\phi \in C_+$, there exists $T > 0$ such that $0 \le x(t) \le M$ for all $t \ge T$.
(ii) $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$ implies that $x_1^* < x(t) \le M$ for all $t \ge T$.
(iii) if $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]}$, then x_1^* attracts every solution of problem (1.1) and x_1^* is unstable.

Proof. The uniqueness of the positive equilibrium implies that $x_1^* \le M$ and $g(x) < f(x)$ for all $x \ne x_1^*$. For (**i**), suppose that there exists $t_0 > 0$ such that $x(t) \ge M$ for all $t \ge t_0$. Then, by substituting x in Eq (1.1), we get

$$x'(t) \le -f(M) + g(M) < 0.$$

This is impossible and then there exists $T > 0$ such that $x(T) < M$. Next, if there exists $t_1 > T$ such that $x(t_1) = M$ and $x(t) \le M$ for all $t \le t_1$, then

$$0 \le x'(t_1) \le -f(M) + g(M) < 0.$$

Consequently (**i**) holds. We argue as in the proof of Lemma 3.2 (i), to show (**ii**). Concerning (**iii**), we consider the following Lyapunov functional

$$V(\phi) \;=\; \int_{x_1^*}^{\phi(0)} (g(s) - g(x_1^*))ds + \frac{1}{2}\int_0^\tau \psi(a)(g(\phi(-a)) - g(x_1^*))^2 da,$$

where $\psi(a) = \displaystyle\int_a^\tau h(\sigma)d\sigma$. As in the proof of Theorem 3.1, the derivative of V along the solution of problem (1.1) gives

$$\frac{dV(x_t)}{dt} = -\frac{1}{2}\int_0^\tau h(a)[g(x(t)) - g(x(t-a))]^2 da + [g(x(t)) - g(x_1^*)][g(x(t)) - f(x(t))], \quad \forall t > 0.$$

Since $g(s) < f(s)$ for all $s \ne x_1^*$ and $g(s) \ge g(x_1^*)$ for all $s \in [x_1^*, M]$, we have $dV(x_t)/dt \le 0$. By LaSalle invariance theorem, x_1^* attracts every solution $x(\phi)$ of problem (1.1) with $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]}$. From Theorems 2.5 and 3.4, we easily show that x_1 is unstable. ■

4. Application to a model with Allee effect and distributed delay

In this section, we apply our results to the following distributed delay differential equation:

$$x'(t) = -\mu x(t) + \int_0^\tau h(a)\frac{kx^2(t-a)}{1 + 2x^3(t-a)}da, \quad \text{for } t \ge 0, \tag{4.1}$$

where μ, k are positive constants. The variable $x(t)$ stands for the maturated population at time t and $\tau > 0$ is the maximal maturation time of the species under consideration. $h(a)$ is the maturity rate at age a. In this model, the death function $f(x) = \mu x$ and the birth function $g(x) = kx^2/(1 + 2x^3)$ reflect the so called Allee effect. Obviously, the functions f and g satisfy the assumptions (**T1**)–(**T3**) and g reaches the maximum value $k/3$ at the point $M = 1$. The equilibria of Eq (4.1) satisfies the following equation:

$$\mu x = \frac{kx^2}{1 + 2x^3}. \tag{4.2}$$

Analyzing Eq (4.2), we obtain the following proposition.

Proposition 4.1. *Equation (4.1) has a trivial equilibrium $x = 0$. In addition*

(i) if $\mu > \frac{2^{\frac{1}{3}}}{3}k$, then Eq (4.1) has no positive equilibrium;

(ii) if $\mu = \frac{2^{\frac{1}{3}}}{3}k$, then Eq (4.1) has exactly one positive equilibrium x_1^*;

(iii) if $0 < \mu < \frac{2^{\frac{1}{3}}}{3}k$, then Eq (4.1) has exactly two positive equilibria $x_1^* < x_2^*$. Moreover,

(iii)-a if $\frac{k}{3} < \mu < \frac{2^{\frac{1}{3}}}{3}k$, then $0 < x_1^* < x_2^* < 1$;

(iii)-b if $0 < \mu \leq \frac{k}{3}$, then $0 < x_1^* < 1 \leq x_2^*$.

Using Theorem 2.6, we obtain

Theorem 4.2. *If $\mu > \frac{2^{\frac{1}{3}}}{3}k$, then the trivial equilibrium of Eq (4.1) is globally asymptotically stable for all $\phi \in C_+$.*

For the case (iii)-a in Proposition 4.1 we have the following result

Theorem 4.3. *Suppose that $\frac{k}{3} < \mu < \frac{2^{\frac{1}{3}}}{3}k$. Then*

(i) The trivial equilibrium of Eq (4.1) is globally asymptotically stable for all $\phi \in C_{[0,x_1^*]} \setminus \{x_1^*\}$.

(ii) The positive equilibrium x_1^* is unstable and the positive equilibrium x_2^* is globally asymptotically stable for all $\phi \in C_{[x_1^*,\hat{G}(x_1^*)]} \setminus \{x_1^*\}$, where $\hat{G}(x_1^*) \in [1,\infty)$ satisfies $g(\hat{G}(x_1^*)) = g(x_1^*)$.

(iii) There exists two heteroclinic orbits $X^{(1)}$ and $X^{(2)}$ connecting 0 to x_1^* and x_1^* to x_2^*, respectively.

Proof. (i) and (ii) directly follow from Proposition 4.1, (iii)-a and Theorems 2.5 and 3.4.

For (iii), we follow the same proof as in Theorem 4.2 in [5]. See also [18]. For the sake of completeness, we rewrite it. Let $K = \{x_1^*\}$. Clearly, K is an isolated and unstable compact invariant set in $C_{[0,x_1^*]}$ and $C_{[x_1^*,\hat{G}(x_1^*)]}$.

By applying Corollary 2.9 in [33] to $\Phi_{|\mathbf{R}^+ \times C[0,x_1^*]}$ and $\Phi_{|\mathbf{R}^+ \times C[x_1^*,\hat{G}(x_1^*)]}$, respectively, there exist two precompact full orbits $X^{(1)} : \mathbf{R} \to C_{[0,x_1^*]} \setminus \{0\}$ and $X^{(2)} : \mathbf{R} \to C_{[x_1^*,\hat{G}(x_1^*)]} \setminus \{0\}$ such that $\alpha(X^1) = \alpha(X^2) = K$. This together with statements (i) and (ii) gives $\omega(X^{(1)}) = \{0\}$ and $\omega(X^{(2)}) = \{x_2^*\}$. In other words, there exist two heteroclinic orbits $X^{(1)}$ and $X^{(2)}$, which connect 0 to x_1^* and x_1^* to x_2^*, respectively. This completes the proof. ■

For the case (iii)-b in Proposition 4.1, we first show that

$$f(A) \geq g(M) \text{ with } g(A) = f(M) \text{ and } A \geq M. \tag{4.3}$$

Lemma 4.4. *Suppose that $f(s) = \mu s$. Condition (4.3) holds if and only if $M \leq \frac{1}{\mu}g(\frac{1}{\mu}g(M))$.*

Proof. It follows that $f(A) \geq g(M)$ if and only if $A \geq f^{-1}(g(M)) \geq M$. Since g is a decreasing function over $[M,\infty)$, we have $g(A) \leq g(f^{-1}(g(M))) = g(\frac{1}{\mu}(g(M)))$. Moreover, since $g(A) = f(M)$, we have $M \leq \frac{1}{\mu}g(\frac{1}{\mu}g(M))$. The lemma is proved. ■

Lemma 4.5. *For Eq (4.1), condition (4.3) holds if and only if*

$$0 < \mu \leq \frac{k}{3}. \tag{4.4}$$

Proof. By a straightforward computation, we have $f^{-1}(x) = x/\mu$. For $G(x) = g(x)/\mu$, we obtain

$$(GoG)(x) = \frac{x(2px^6 + px^3)}{8x^9 + (12 + 2p)x^6 + 6x^3 + 1},$$

where $p = k^3/\mu^3$. By applying Lemma 4.4, it is readily to see that $1 \leq (GoG)(1)$ holds if and only if

$$1 \leq \frac{3p}{2p + 27},$$

and thus, $0 < \mu \leq k/3$. ■

Theorem 4.6. *Suppose that $0 < \mu \leq \frac{k}{3}$. Then, the positive equilibrium x_2^* of Eq (4.1) is globally asymptotically stable for all $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$, where $\hat{G}(x_1^*) \in [1, \infty)$ satisfies $g(\hat{G}(x_1^*)) = g(x_1^*)$.*

Proof. Observe that, in view of Lemmas 4.4 and 4.5, all hypotheses of Lemma 3.6 are satisfied. Now, in order to apply Theorem 3.7, we only show that $\frac{(GoG)(x)}{x}$ is nonincreasing over $[M, A]$. Indeed, by a simple computation, we have

$$\left(\frac{(GoG)(x)}{x}\right)' = \frac{-48px^{14} - 48px^{11} - 6p^2x^8 + 12px^5 + 3px^2}{[8x^9 + (12 + 2p)x^6 + 6x^3 + 1]^2},$$

where $p = k^3/\mu^3$. Finally, observe that

$$\left(\frac{(GoG)(x)}{x}\right)' < 0,$$

for $x \geq 1$. In this case, both of (H3) and (H4) in Theorem 3.7 hold. This completes the proof. ■

Finally, for the tangential case, we have

Theorem 4.7. *Suppose that $\mu = \frac{2^{\frac{1}{3}}}{3}k$. Then*

(i) The trivial equilibrium of Eq (4.1) is globally asymptotically stable for all $\phi \in C_{[0, x_1^*]} \setminus \{x_1^*\}$.
(ii) If $\phi \in C_{[x_1^*, \hat{G}(x_1^*)]} \setminus \{x_1^*\}$, then the unique positive equilibrium x_1^* is unstable and attracts every solution of Eq (4.1).

Proof. The proof of this theorem follows immediately from Theorems 2.5 and 3.9. ■

5. Numerical simulation

In this section, we perform numerical simulation that supports our theoretical results. As in Section 4, we assume that

$$f(x) = \mu x, \quad g(x) = \frac{kx^2}{1 + 2x^3},$$

and confirm the validity of Theorems 4.2, 4.3, 4.6 and 4.7. In what follows, we fix

$$\tau = 1, \quad k = 1, \quad h(a) = \frac{e^{-(a-0.5\tau)^2\times 10^2}}{\int_0^\tau e^{-(\sigma-0.5\tau)^2\times 10^2} d\sigma} \tag{5.1}$$

and change μ and ϕ. Note that $\frac{2^{\frac{1}{3}}}{3}k \approx 0.42$ and h is a Gaussian-like distribution as shown in Figure 1.

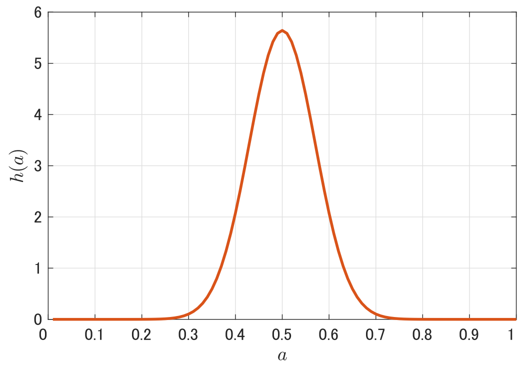
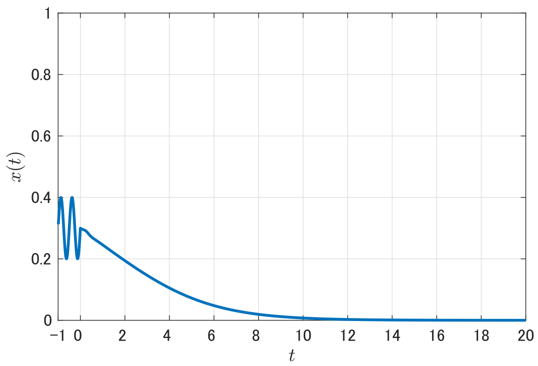
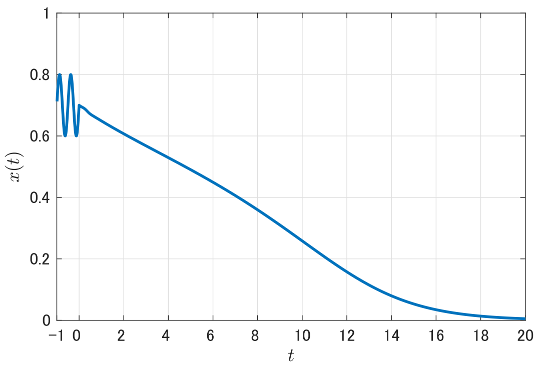


Figure 1. Function h in definition (5.1).

First, let $\mu = 0.5$. In this case, $\mu > \frac{2^{\frac{1}{3}}}{3}k$ and thus, by Theorem 4.2, the trivial equilibrium is globally asymptotically stable. In fact, Figure 2 shows that $x(t)$ converges to 0 as t increases for different initial data.



(a) $\phi(t) = 0.2 + 0.1\,(1 + \sin 4\pi t)$, $-1 \leq t \leq 0$.



(b) $\phi(t) = 0.6 + 0.1\,(1 + \sin 4\pi t)$, $-1 \leq t \leq 0$.

Figure 2. Time variation of function x for $\mu = 0.5 > \frac{2^{\frac{1}{3}}}{3}k \approx 0.42$.

Next, let $\mu = 0.37$. In this case, $\frac{k}{3} < \mu < \frac{2^{\frac{1}{3}}}{3}k$ and $x_1^* \approx 0.43$, $x_2^* \approx 0.89$ and $\hat{G}(x_1^*) \approx 3.09$ (Figure 3). Hence, by Theorem 4.3, we see that the trivial equilibrium is globally asymptotically

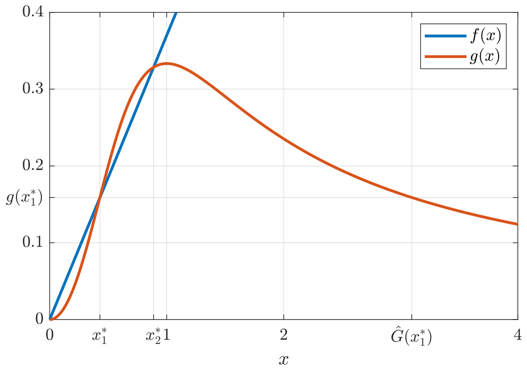
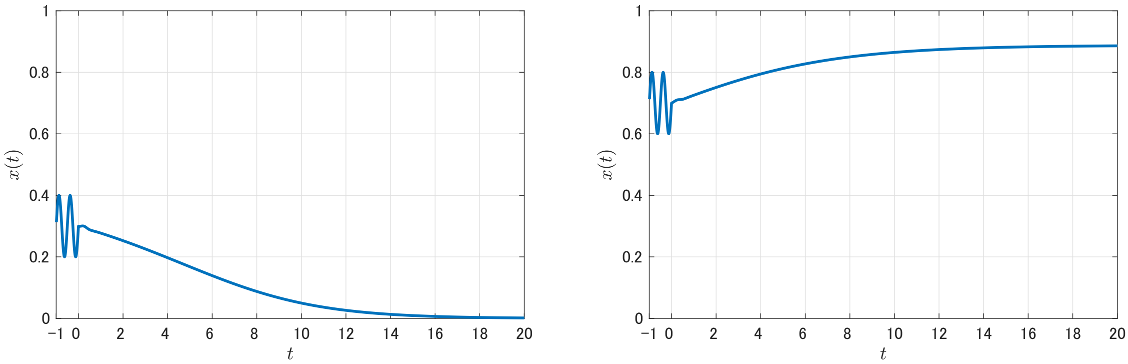


Figure 3. Functions f and g for $\mu = 0.37$.

stable for $\phi \in C_{[0,x_1^*]} \setminus \{x_1^*\}$, whereas the positive equilibrium x_2^* is so for $\phi \in C_{[x_1^*,\hat{G}(x_1^*)]} \setminus \{x_1^*\}$. In fact, Figure 4 shows such a bistable situation. Moreover, heteroclinic orbits $X^{(1)}$ and $X^{(2)}$, which were stated



(a) $\phi(t) = 0.2 + 0.1\,(1 + \sin 4\pi t)$, $-1 \le t \le 0$.

(b) $\phi(t) = 0.6 + 0.1\,(1 + \sin 4\pi t)$, $-1 \le t \le 0$.

Figure 4. Time variation of function x for $\mu = 0.37 \in \left(\frac{k}{3}, \frac{2^{\frac{1}{3}}}{3}k\right) \approx (0.33, 0.42)$.

in Theorem 4.3 (iii), are shown in Figure 5.

Thirdly, let $\mu = 0.27 < \frac{k}{3} \approx 0.33$. In this case, we have $x_1^* \approx 0.28$, $x_2^* \approx 1.2$ and $\hat{G}(x_1^*) \approx 6.52$ (Figure 6). Thus, by Theorem 4.6, we see that the positive equilibrium x_2^* is globally asymptotically stable for $\phi \in C_{[x_1^*,\hat{G}(x_1^*)]} \setminus \{x_1^*\}$. In fact, Figure 7 shows that $x(t)$ converges to x_2^* as t increases for different initial data.

Finally, let $\mu = \frac{2^{\frac{1}{3}}}{3}k \approx 0.42$. In this case, we have $x_1^* \approx 0.63$ and $\hat{G}(x_1^*) \approx 1.72$ (Figure 8). By Theorem 4.7, we see that the trivial equilibrium is globally asymptotically stable for $\phi \in C_{[0,x_1^*]} \setminus \{x_1^*\}$, whereas x_1^* is unstable and attracts all solutions for $\phi \in C_{[x_1^*,\hat{G}(x_1^*)]} \setminus \{x_1^*\}$. In fact, Figure 9 shows such two situations.

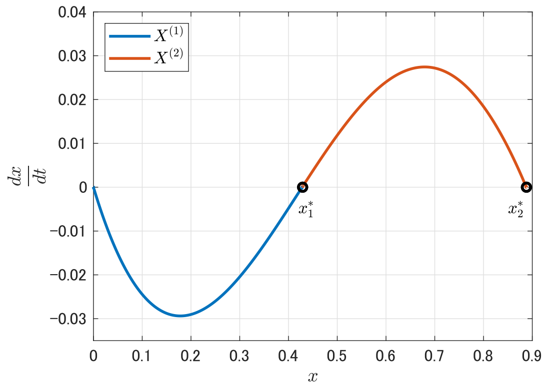


Figure 5. Heteroclinic orbits $X^{(1)}$ and $X^{(2)}$ in the x-x' plane for $\mu = 0.37$.

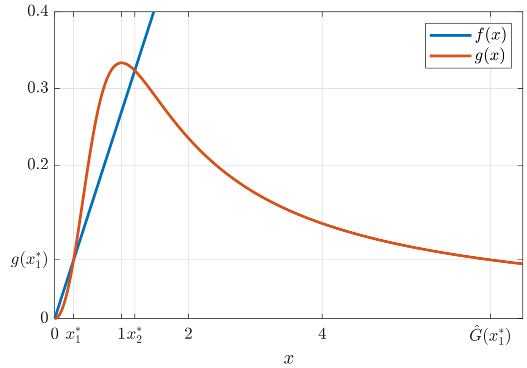
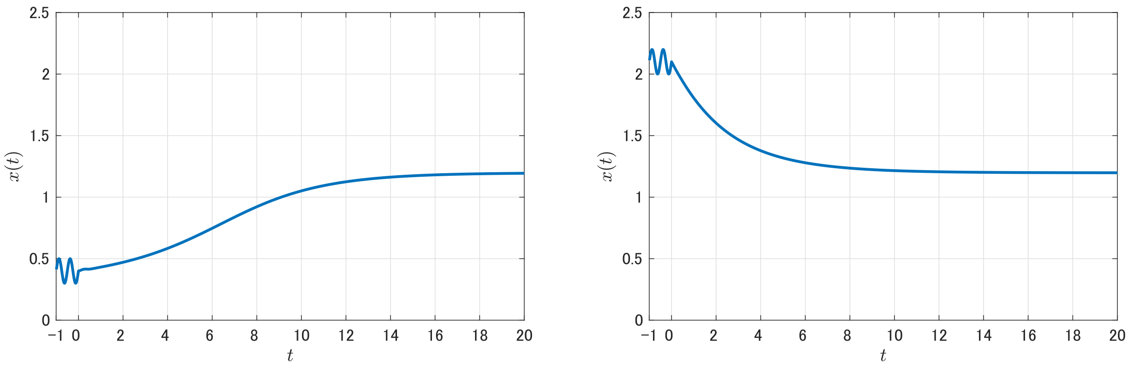


Figure 6. Functions f and g for $\mu = 0.27$.



(a) $\phi(t) = 0.3 + 0.1\,(1 + \sin 4\pi t)$, $-1 \le t \le 0$.

(b) $\phi(t) = 2 + 0.1\,(1 + \sin 4\pi t)$, $-1 \le t \le 0$.

Figure 7. Time variation of function x for $\mu = 0.27 < \dfrac{k}{3} \approx 0.33$.

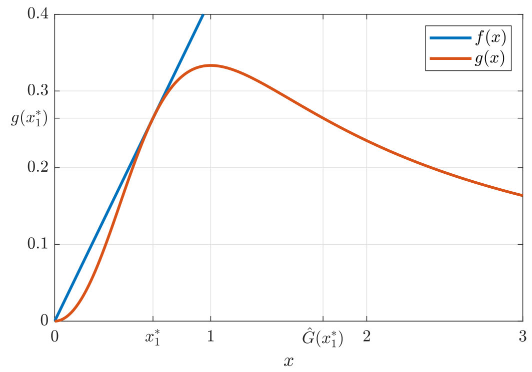
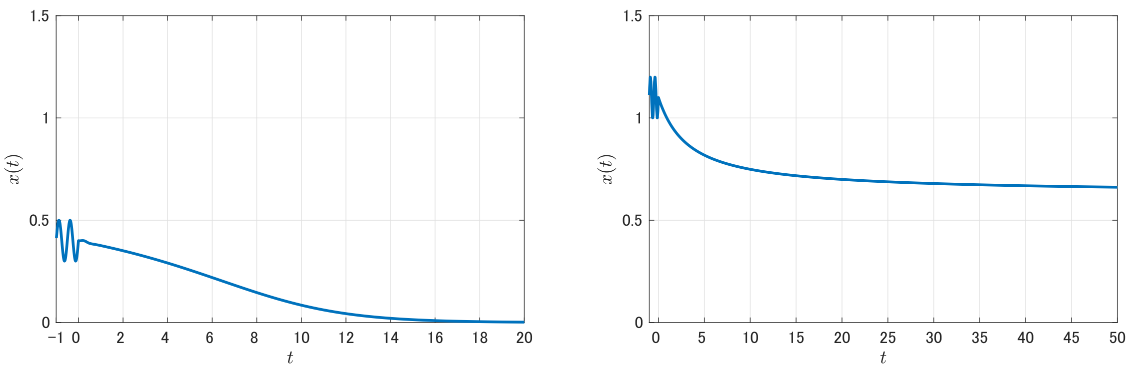


Figure 8. Functions f and g for $\mu = \dfrac{2^{\frac{1}{3}}}{3}k \approx 0.42$.



(a) $\phi(t) = 0.3 + 0.1\,(1 + \sin 4\pi t)$, $-1 \le t \le 0$.

(b) $\phi(t) = 1 + 0.1\,(1 + \sin 4\pi t)$, $-1 \le t \le 0$.

Figure 9. Time variation of function x for $\mu = \dfrac{2^{\frac{1}{3}}}{3}k \approx 0.42$.

6. Conclusions

In this paper, we studied the bistable nonlinearity problem for a general class of functional differential equations with distributed delay, which includes many mathematical models in biology and ecology. In contrast to the previous work [5], we considered both cases $x_2^* < M$ and $x_2^* \ge M$, and obtained sufficient conditions for the global asymptotic stability of each equilibrium. The general results were applied to a model with Allee effect in Section 4, and numerical simulation was performed in Section 5. It should be pointed out that our theoretical results are robust for the variation of the form of the distribution function h. This might suggest us that the distributed delay is not essential for the dynamical system of our model. We conjecture that our results still hold for $\tau = \infty$, and we leave it for a future study. Extension of our results to a reaction-diffusion equation could also be an interesting future problem.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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