Quantitative Finance and Economics

http://www.aimspress.com/journal/QFE

QFE, 6(1): 1–34.
DOI: 10.3934/QFE.2022001
Received: 27 November 2021
Revised: 13 December 2021
Accepted: 04 January 2022
Published: 10 January 2022

Research article

Pricing and hedging bond options and sinking-fund bonds under the CIR model

Manuela Larguinho[1,2], **José Carlos Dias**[3,4,*] **and Carlos A. Braumann**[2,5]

[1] Polytechnic of Coimbra, Coimbra Business School Research Centre — ISCAC, Coimbra, Portugal
[2] Centro de Investigação em Matemática e Aplicações, Instituto de Investigação e Formação Avançada, Universidade de Évora, Évora, Portugal
[3] Iscte - Instituto Universitário de Lisboa, Lisbon, Portugal
[4] Business Research Unit (BRU-IUL), Lisbon, Portugal
[5] Departamento de Matemática, Escola de Ciências e Tecnologia, Universidade de Évora, Évora, Portugal

* **Correspondence:** Email: jose.carlos.dias@iscte.pt; Tel: +351217650531.

Abstract: This article derives simple closed-form solutions for computing Greeks of zero-coupon and coupon-bearing bond options under the CIR interest rate model, which are shown to be accurate, easy to implement, and computationally highly efficient. These novel analytical solutions allow us to extend the literature in two other directions. First, the static hedging portfolio approach is used for pricing and hedging American-style plain-vanilla zero-coupon bond options under the CIR model. Second, we derive analytically the comparative static properties of sinking-fund bonds under the same interest rate modeling setup.

Keywords: CIR model; bond options; Greeks; American options; static hedging; sinking-fund bonds

JEL Codes: G12, G13

1. Introduction

A bond is a contract which pays its holder a known amount, the principal, at a known future date, called the maturity of the contract. The bond may also pay periodically to its holder fixed cash dividends, called the coupons. This type of bonds are known as coupon bonds (sometimes also called coupon-paying or coupon-bearing bonds). If the bond pays no coupons, it is known as a zero-coupon bond (or pure discount bond). Several bonds may contain special clauses or some embedded options. There are also some derivative contracts whose underlying asset is a bond.

This paper offers three contributions to the existent bond option pricing literature. First, we provide closed-form solutions to efficiently and accurately compute sensitivity measures (commonly known as Greeks) of pure discount and coupon-bearing bond options under the Cox et al. (1985) mean-reverting square-root model (hereafter, CIR model), which, to the best of our knowledge, are new in the option pricing literature.[1] Following the insights of Larguinho et al. (2013) and Dias et al. (2020), the obtained closed-form expressions for Greeks under the CIR model are expressed in terms of noncentral chi-square distribution functions, which can be efficiently computed via Benton and Krishnamoorthy (2003, Algorithm 7.3).[2]

These novel Greeks should be valuable to both academics and practitioners. For instance, a dealer of the financial industry should be able not only to price a given option contract but also to hedge it. Knowing and understanding such sensitivity measures is thus pivotal in the design of hedging strategies for a given security or a portfolio of securities, when closing the position is not viable or desirable. Greeks also enjoy many other multiple applications such as market risk measurement, profit and loss attribution, model risk assessment and optimal contract design, and to determine parameter values from market prices. Moreover, the availability of analytical solutions for Greeks reduces substantially the computational burden when dealing with large portfolios of securities that have to be re-evaluated frequently and allows them to be easily coded in any desired computer language.

Armed with these new analytical solutions for Greeks, we can now extend the literature in two other directions.[3] Hence, and as our second contribution, we are able to price (and hedge) American-style option contracts on zero-coupon bonds under the CIR model via the *static hedge portfolio* (hereafter, SHP) approach offered by Chung and Shih (2009) and Ruas et al. (2013) in the context of stock options. It is well-know that the pricing (and hedging) of American-style contingent claims boils down to a boundary value problem in a domain whose boundary is not fully known and, therefore, must be also determined. In other words, the option price and the early exercise boundary must be determined simultaneously as the solution of the same free boundary problem that has been set up by McKean (1965). As for the stock options case, there are no closed-form solutions for pricing American-style options on bonds. Hence, these contracts have been usually evaluated numerically using finite difference, finite volume, and finite element methods—see, for instance, Hull and White (1990), Allegretto et al. (2003), Yang (2004), ShuJin and ShengHong (2006), Zhou et al. (2011), and Thakoor et al. (2012)—, through a binomial or trinomial tree approach—see, for example, Nelson and Ramaswamy (1990), Tian (1992), Tian (1994), and Nawalkha and Beliaeva (2007)—, via the least-squares Monte Carlo scheme of Longstaff and Schwartz (2001), or with the optimal stopping approach proposed by Chesney et al. (1993). More recently, Deng (2015) considers the valuation of American-style put options on zero-coupon bonds in a jump-extended CIR model, Najafi et al. (2018) evaluate the American-style put option on a zero-coupon bond assuming that the interest rate model is governed by a fractional CIR process, whereas Peng and Schellhorn (2018) study the probability distribution of the interest rate under an extended CIR model with time-varying dimension and propose a pricing method for options on zero-coupon bonds.

[1] To simplify the notation and better emphasize our exposition, we focus on the classical constant coefficient CIR model to derive Greeks, but the same line of reasoning can be applied also under the time-varying coefficients version of the CIR model offered by Jamshidian (1995) and Maghsoodi (1996), as well as under the CIR++ model of Brigo and Mercurio (2001).

[2] This algorithm has been used in several option pricing applications involving such distribution, e.g., Ruas et al. (2013), Dias et al. (2015), Nunes et al. (2015), Cruz and Dias (2017), Cruz and Dias (2020), and Dias et al. (2020).

[3] Another possible research direction, outside the scope of the present article, is the comparison of alternative binomial approximation schemes for computing the option hedge ratios in the spirit of Pelsser and Vorst (1994), Chung and Shackleton (2002), Chung and Shackleton (2005), Chung et al. (2011), and Cruz and Dias (2017).

Alternatively, we show how to tackle the valuation of American-style options on pure discount bonds with a distinct approach that has proved to be extremely efficient and accurate in the case of American-style stock options and under different assumptions for the dynamics of the underlying asset price—see Chung and Shih (2009) and Ruas et al. (2013) for more details.[4] Broadly speaking, we use standard European-style zero-coupon bond options with multiple strikes and multiple maturities, because the optimal exercise boundary of such American-style contracts are not known *ex-ante*. This approach creates a static portfolio of European-style options whose values match the payoff of the American-style option being hedged at expiration and along the boundary, by applying the value-matching and smooth-pasting conditions on the early exercise boundary. As for the case of stock options, we show that the SHP methodology is also robust and computationally efficient when dealing with bond options.[5]

As our final contribution, we revisit the Bacinello et al. (1996) work and provide analytic tractable formulae for valuing and analyzing comparative statics of sinking-fund bonds in the CIR framework. We shall note that while Bacinello et al. (1996) have been able to study such issues in closed-form under the Vasicek (1977) model, they analyze numerically the comparative static properties of the sinking-fund bond in the CIR model given the absence of closed-form expressions of Greeks under the CIR modeling setup. Using our novel solutions, we show that the stochastic duration of the sinking-fund bond is between the stochastic duration of the corresponding serial and coupon bonds. Although this issue has been shown already by Bacinello et al. (1996) through numerical differentiation, we are able to establish this property analytically using the proposed closed-form solutions for the CIR sensitivity measures.

The remainder of the paper is organized as follows. Section 2 outlines a brief summary of the CIR interest rate dynamics and the analytical formulae for computing discount bonds, coupon-bearing bonds, and European-style options on discount bonds and coupon-paying bonds in a CIR economy. Section 3 derives analytical tractable solutions of the sensitivity measures of bond options under the same interest rate dynamics setting and presents some numerical examples to enhance the efficiency of our closed-form solutions. Section 4 implements the SHP approach for pricing and hedging American-style options on zero-coupon bonds. Section 5 provides analytically tractable formulae to analyze the comparative statics properties of a sinking-fund bond in the CIR framework. Section 6 presents the concluding remarks. All accessory results are relegated to the Appendix.

2. Model setup and bond option valuation

This section presents a brief remainder of the analytical formulae for computing discount bonds, coupon-bearing bonds, and European-style call and put options on zero-coupon bonds and coupon-paying bonds in a CIR economy that will be required later. Even though these results are well known in the literature they are needed to establish notations and the desire for self-consistency.

[4] The SHP approach has been shown to be useful also for pricing and hedging barrier option contracts, as highlighted in Chung et al. (2010), Chung et al. (2013), Dias et al. (2015), Nunes et al. (2015), Guo and Chang (2020), and Nunes et al. (2020).

[5] Unfortunately, the existence of interim coupons prevents the implementation of the SHP approach for American-style options on coupon-bearing bonds. Nevertheless, the SHP methodology should be useful both in theory and in practice since it provides a fast and accurate method for pricing American-style options on zero-coupon bonds, thus being a viable alternative to the aforementioned schemes available in the literature for these contracts. Moreover, this methodology can also be applied to any other single-factor interest rate model offering closed-form solutions for option prices and hedge ratios.

2.1. CIR interest rate dynamics

Hereafter, we consider a CIR economy in which $\mathbb{E}_t^{\mathbb{Q}}$ denotes the time-t expectation under the martingale (or risk-neutral) probability measure $\mathbb{Q}$, with respect to the risk-adjusted process for the instantaneous interest rate r_t

$$dr_t = \kappa^* (\theta^* - r_t)\, dt + \sigma \sqrt{r_t} dW_t^{\mathbb{Q}}, \tag{1}$$

where $\kappa^* := \kappa + \lambda$ is the risk-neutral parameter that determines the speed of adjustment (reversion rate or reverting rate), $\theta^* := \kappa\theta/\,(\kappa + \lambda)$ is the risk-neutral long-run mean of the instantaneous interest rate (asymptotic interest rate or reverting level), σ is the volatility of the process, λ is the market price of risk parameter, and $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$.[6] It is well known that the $\kappa\theta$ term plays a key role under this diffusion and has important implications for capture of the interest rate process r at a value of zero. The condition $2\kappa\theta \geq \sigma^2$ ensures that the interest rate remains positive.[7]

2.2. Zero-coupon bonds under the CIR model

In a CIR economy, the time-t price of a zero-coupon bond maturing at time s (with $s > t$), $Z(r, t, s)$, is given by

$$Z(r, t, s) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r_u du}\right] = A(t, s)\, e^{-B(t,s)\, r}, \tag{2}$$

with $r = r_t$ at the valuation date t, and where the constants $A(t, s)$, $B(t, s)$, and $\gamma > 0$ are given by

$$A(t, s) := \left[\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(s-t)]/2}}{(\kappa + \lambda + \gamma)(e^{\gamma(s-t)} - 1) + 2\gamma}\right]^{2\kappa\theta/\sigma^2}, \tag{3}$$

$$B(t, s) := \frac{2(e^{\gamma(s-t)} - 1)}{(\kappa + \lambda + \gamma)(e^{\gamma(s-t)} - 1) + 2\gamma}, \tag{4}$$

and

$$\gamma := \left[(\kappa + \lambda)^2 + 2\sigma^2\right]^{1/2}. \tag{5}$$

2.3. Coupon-paying bonds under the CIR model

It is well established in the literature—see, for instance, Jamshidian (1989)—that, for all one-factor term structure models, a coupon-paying bond can be decomposed into a portfolio of zero-coupon bonds of different maturities. Hence, the time-t value of a coupon-bearing bond expiring at time s (with $s > t$), $P(r, t, s)$, can be simply expressed as a weighted sum of zero-coupon bond prices, that is

$$P(r, t, s) = \sum_{i=1}^{N} a_i Z(r, t, s_i), \tag{6}$$

with $s_1, s_2, \cdots, s_N$ (and $s_N = s$) representing the N dates on which payments are made, and each $a_i > 0$ term denoting the amount of the payments made.[8]

---

[6] We recall that, when $\lambda = 0$, we have $\kappa^* = \kappa$ and $\theta^* = \theta$, which implies that the speed of adjustment and the asymptotic interest rate under the physical and risk-neutral measures are the same.

[7] See Feller (1951) for a complete description of the boundary conditions.

[8] For example, consider a 10-year 6% coupon bond with a par value of 100 and semiannual coupon payments. In this case, $N = 20$ since the bond makes 19 semiannual coupon payments of 3 as well as a final payment of 103. Thus, $a_i = 100 \times 6\%/2 = 3$ for $i = 1, 2, \cdots, 19$, $a_{20} = 3 + 100 = 103$, and $s_1 = 0.5, s_2 = 1, \cdots, s_{19} = 9.5$, and $s_{20} = 10$.

2.4. Zero-coupon bond options under the CIR model

Analytic solutions for pricing options on discount bonds have been proposed by Cox et al. (1985). Denote $v^{zc}(r,t,T,s,K;\alpha)$ as the time-t price of a European-style call option (if $\alpha = 1$) or put option (if $\alpha = -1$) with strike price K, expiration date T, written on a zero-coupon bond with maturity date s (with $s > T > t$), and with the instantaneous interest rate at time t given by r.[9] The time-t price of a zero-coupon bond option is given by

$$\begin{aligned} v^{zc}(r,t,T,s,K;\alpha) &= \alpha Z(r,t,s)\, Q\left[x_1(t,T,s,K)\,;a,b_1(r,t,T,s)\,;\alpha\right] \\ &\quad -\alpha K Z(r,t,T)\, Q\left[x_2(t,T,s,K)\,;a,b_2(r,t,T)\,;\alpha\right], \end{aligned} \tag{7}$$

where $Q(x;a,b;\alpha)$ is the distribution function (for $\alpha = 1$) and the complementary distribution function (for $\alpha = -1$) of the noncentral chi-square distribution with a degrees of freedom and non-centrality parameter b,

$$x_1(t,T,s,K) := 2r^*(T,s,K)\left[\phi(t,T) + \psi + B(T,s)\right], \tag{8}$$

$$x_2(t,T,s,K) := 2r^*(T,s,K)\left[\phi(t,T) + \psi\right], \tag{9}$$

$$a := \frac{4\kappa\theta}{\sigma^2}, \tag{10}$$

$$b_1(r,t,T,s) := \frac{2\phi^2(t,T)\, r e^{\gamma(T-t)}}{\phi(t,T) + \psi + B(T,s)}, \tag{11}$$

$$b_2(r,t,T) := \frac{2\phi^2(t,T)\, r e^{\gamma(T-t)}}{\phi(t,T) + \psi}, \tag{12}$$

$$\phi(t,T) := \frac{2\gamma}{\sigma^2(e^{\gamma(T-t)} - 1)}, \tag{13}$$

$$\psi := \frac{\kappa + \lambda + \gamma}{\sigma^2}, \tag{14}$$

and

$$r^*(T,s,K) := \frac{1}{B(T,s)} \ln\left(\frac{A(T,s)}{K}\right), \tag{15}$$

with r^* being the critical interest rate below which exercise will occur, i.e., $K = Z(r^*,T,s)$.

2.5. Coupon-paying bond options under the CIR model

Following the argument of Jamshidian (1989) that an option on a portfolio of zero-coupon bonds decomposes into a portfolio of options on the individual discount bonds in the portfolio, then the time-t price of a European-style call option (if $\alpha = 1$) or put option (if $\alpha = -1$), with strike price K and maturity date T, on a portfolio consisting of N zero-coupon bonds with different expiry dates s_i, is given by

$$v^{cb}(r,t,T,s,K;\alpha) = \sum_{i=1}^{N} a_i v^{zc}(r,t,T,s_i,K_i;\alpha), \tag{16}$$

with $T < s_1 < s_2 < \cdots < s_N = s$, $a_i > 0$, $K_i = Z(r^{**},T,s_i)$, and where r^{**} is the solution to $\sum_{i=1}^{N} a_i Z(r^{**},T,s_i) = K$.[10]

[9] It is well-known that K is restricted to be less than $A(T,s)$, the maximum possible bond price at time T, since otherwise the option would never be exercised and would be worthless—see Cox et al. (1985, Page 396).

[10] Alternatively, we may use the equivalent closed-form expressions offered by Longstaff (1993, Equations 7 and 9).

Remark 1. *Note that the underlying asset for coupon bond options is actually the portfolio of discount bonds expiring after the option's maturity date. However, the value of this portfolio is strictly less than the current price of the coupon bond if the bond pays coupons before the expiry date of the option. As argued by Longstaff (1993, Page 32), the value of the underlying asset for a 5-year option on a 10-year bond is not the current price of a 15-year bond, but the price of a 15-year bond minus the present value of coupon payments to be made during the next 5 years. In other words, the option's payoff—and, hence, the coupon bond option price—does not depend on the payments of the coupon bond to be made before the expiry date of the option.*

3. Greeks of bond options under the CIR model

This section derives closed-form solutions for Greeks under the CIR model, which, to the best of our knowledge, are new in the literature.

3.1. Preliminaries

Let us begin with two important general relations, which will be used for deriving Greeks under the CIR model. Following Johnson et al. (1995, pp. 442-443) or Larguinho et al. (2013, Equations A2a and A2b), we know that

$$\frac{\partial Q[x; a, b; \alpha]}{\partial x} = \alpha\, p(x; a, b), \tag{17}$$

and

$$\frac{\partial Q[x; a, b; \alpha]}{\partial b} = -\alpha\, p(x; a+2, b), \tag{18}$$

where $p(x; a, b)$ is the probability density function of a noncentral chi-square distribution as given by Johnson et al. (1995, Equation 29.4), that is

$$p(x; a, b) = \frac{1}{2}\, e^{-(b+x)/2} \left(\frac{x}{b}\right)^{(a-2)/4} I_{(a-2)/2}(\sqrt{bx}), \quad x > 0, \tag{19}$$

with $I_q(\cdot)$ being the modified Bessel function of the first kind of order q, as defined in Abramowitz and Stegun (1972, Equation 9.6.10). We will also need to use the first derivative of the probability density function (19) with respect to the non-centrality parameter b, which can be computed through the following recurrence relation given by Cohen (1988):

$$\frac{\partial p(x; a, b)}{\partial b} = \frac{1}{2}\left[-p(x; a, b) + p(x; a+2, b)\right]. \tag{20}$$

3.2. Greeks formulas

Next propositions and remarks offer the proposed novel closed-form solutions for computing sensitivity measures of zero-coupon bond options under the CIR model, namely the *rho* (or *interest rate delta*), *interest rate gamma*, *theta*, and *eta* (or *strike delta*).[11] We notice that the corresponding *rho*,

[11] Note that the so-called *vega*—which is the sensitivity of the bond option price with respect to the volatility parameter σ—depends on the degrees of freedom parameter a of the noncentral chi-square distribution function, for which (to the authors knowledge) there is no simple relationship as those given in equations (17) and (18). See Alvarez (2001) who discusses the conditions which determine the sign of the effect of increased volatility on the price of a general interest rate claim under a broad class of interest rate models.

interest rate gamma, and *theta* measures of coupon-bearing bond options arise immediately, because it is possible to apply the decomposition technique of Jamshidian (1989) to these Greeks. For the case of *eta*, however, it is necessary to combine the decomposition technique with the classic chain rule.

3.2.1. Interest rate delta

The rho or interest rate delta can be computed as:

Proposition 1. *Consider the pricing solution of a zero-coupon bond option under the CIR model as defined in equation* (7)*. Then, the rho (or interest rate delta) of a zero-coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) is given by*

$$\begin{aligned} \rho_v^{zc}(.) &:= \frac{\partial v^{zc}(.)}{\partial r} \\ = \; & Z(r,t,s)\left[-\alpha B(t,s)\,Q[x_1(.);a,b_1(.);\alpha] - \frac{b_1(.)}{r}\,p(x_1(.);a+2,b_1(.))\right] \\ & -KZ(r,t,T)\left[-\alpha B(t,T)\,Q[x_2(.);a,b_2(.);\alpha] - \frac{b_2(.)}{r}\,p(x_2(.);a+2,b_2(.))\right]. \end{aligned} \quad (21)$$

Proof. Please see Appendix A.■

Remark 2. *The rho of a coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) arises immediately if one applies the decomposition technique of Jamshidian (1989), that is*

$$\rho_v^{cb}(.) := \frac{\partial v^{cb}(.)}{\partial r} = \sum_{i=1}^{N} a_i \frac{\partial v^{zc}(r,t,T,s_i,K_i;\alpha)}{\partial r} = \sum_{i=1}^{N} a_i \rho_v^{zc}(r,t,T,s_i,K_i;\alpha). \quad (22)$$

The previous remark shows that it is straightforward to compute call and put interest rate deltas in closed-form for coupon-paying bond options under the CIR framework. This will allow us to compare the results calculated by expression (22) with the rho values shown in Wei (1997, Table II), which have been obtained through a numerical integration scheme.

3.2.2. Interest rate gamma

The interest rate gamma can be computed as:

Proposition 2. *Consider the pricing solution of a zero-coupon bond option under the CIR model as defined in equation* (7)*. Then, the interest rate gamma of a zero-coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) is given by*

$$\begin{aligned} \Gamma_{v,r}^{zc}(.) &:= \frac{\partial^2 v^{zc}(.)}{\partial r^2} \\ = \; & Z(r,t,s)\Bigg[\alpha B^2(t,s) Q[x_1(.);a,b_1(.);\alpha] + 2B(t,s)\frac{b_1(.)}{r} p(x_1(.);a+2,b_1(.)) \\ & -\frac{1}{2}\left(\frac{b_1(.)}{r}\right)^2 (-p(x_1(.);a+2,b_1(.)) + p(x_1(.);a+4,b_1(.)))\Bigg] \end{aligned}$$

$$-KZ(r,t,T)\Bigg[\alpha B^2(t,T)Q[x_2(.);a,b_2(.);\alpha]+2B(t,T)\frac{b_2(.)}{r}p(x_2(.);a+2,b_2(.))$$
$$-\frac{1}{2}\left(\frac{b_2(.)}{r}\right)^2\big(-p(x_2(.);a+2,b_2(.))+p(x_2(.);a+4,b_2(.))\big)\Bigg]. \tag{23}$$

Proof. Please see Appendix B.■

Remark 3. *The interest rate gamma of a coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) arises immediately if one applies the decomposition technique of Jamshidian (1989), that is*

$$\Gamma^{cb}_{v,r}(.) := \frac{\partial^2 v^{cb}(.)}{\partial r^2} = \sum_{i=1}^{N} a_i \frac{\partial \rho^{zc}_v(r,t,T,s_i,K_i;\alpha)}{\partial r} = \sum_{i=1}^{N} a_i \Gamma^{zc}_{v,r}(r,t,T,s_i,K_i;\alpha). \tag{24}$$

3.2.3. Theta

The theta can be computed as:

Proposition 3. *Consider the pricing solution of a zero-coupon bond option under the CIR model as defined in equation* (7)*. Then, the theta of a zero-coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) is given by*

$$\theta^{zc}_v(.) := \frac{\partial v^{zc}(.)}{\partial t}$$
$$= Z(r,t,s)\Big[\alpha\zeta_s Q[x_1(.);a,b_1(.);\alpha]+\xi p(x_1(.);a,b_1(.))-\varrho_1 p(x_1(.);a+2,b_1(.))\Big]$$
$$-KZ(r,t,T)\Big[\alpha\zeta_T Q[x_2(.);a,b_2(.);\alpha]+\xi p(x_2(.),a,b_2(.))-\varrho_2 p(x_2(.);a+2,b_2(.))\Big], \tag{25}$$

where

$$\zeta_j = \frac{\kappa\theta}{\sigma^2}\frac{(\kappa+\lambda+\gamma)(e^{\gamma(j-t)}-1)(2\gamma-(\kappa+\lambda+\gamma))}{(\kappa+\lambda+\gamma)(e^{\gamma(j-t)}-1)+2\gamma} + \frac{4r\gamma^2 e^{\gamma(j-t)}}{\big[(\kappa+\lambda+\gamma)(e^{\gamma(j-t)}-1)+2\gamma\big]^2}, \tag{26}$$

for $j \in \{T, s\}$,

$$\xi = \frac{4r^*\gamma^2 e^{\gamma(T-t)}}{\sigma^2(e^{\gamma(T-t)}-1)^2}, \tag{27}$$

$$\varrho_1 = b_1(r,t,T,s)\,\gamma\frac{(\phi(t,T)+\psi+B(T,s))+(\psi+B(T,s))e^{\gamma(T-t)}}{(e^{\gamma(T-t)}-1)(\phi(t,T)+\psi+B(T,s))}, \tag{28}$$

and

$$\varrho_2 = b_2(r,t,T)\,\gamma\frac{(\phi(t,T)+\psi)+\psi e^{\gamma(T-t)}}{(e^{\gamma(T-t)}-1)(\phi(t,T)+\psi)}. \tag{29}$$

Proof. Please see Appendix C.■

Remark 4. *The theta of a coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) arises immediately if one applies the decomposition technique of Jamshidian (1989), that is*

$$\theta^{cb}_v(.) := \frac{\partial v^{cb}(.)}{\partial t} = \sum_{i=1}^{N} a_i \frac{\partial v^{zc}(r,t,T,s_i,K_i;\alpha)}{\partial t} = \sum_{i=1}^{N} a_i \theta^{zc}_v(r,t,T,s_i,K_i;\alpha). \tag{30}$$

3.2.4. Eta

The eta can be computed as:

Proposition 4. *Consider the pricing solution of a zero-coupon bond option under the CIR model as defined in equation* (7)*. Then, the eta of a zero-coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) is given by*

$$\begin{aligned}\eta_v^{zc}(.) := \frac{\partial v^{zc}(.)}{\partial K} &= -2Z(r,t,s)\, p(x_1(.);a,b_1(.))\frac{\phi(t,T)+\psi+B(T,s)}{B(T,s)\,K} \\ &\quad -Z(r,t,T)\left[\alpha Q[(x_2(.);a,b_2(.);\alpha]-2p(x_2(.);a,b_2(.))\frac{\phi(t,T)+\psi}{B(T,s)}\right]. \end{aligned} \quad (31)$$

Proof. Please see Appendix D.■

Proposition 5. *The eta of a coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) arises immediately if one combines the decomposition technique of Jamshidian (1989) and the classic chain rule, obtaining*

$$\begin{aligned}\eta_v^{cb}(.) := \frac{\partial v^{cb}(.)}{\partial K} &= \sum_{i=1}^{N} a_i \frac{\partial v^{zc}(r,t,T,s_i,K_i;\alpha)}{\partial K} \\ &= \frac{\sum_{i=1}^{N} a_i B(T,s_i) Z(r^{**},T,s_i)\eta_v^{zc}(r,t,T,s_i,K_i;\alpha)}{\sum_{j=1}^{N} a_j B(T,s_j) Z(r^{**},T,s_j)}. \end{aligned} \quad (32)$$

Proof. Please see Appendix E.■

3.3. Delta and gamma with respect to the underlying bond price

We recall that in a CIR economy it is the independent variable r that is assumed to be stochastic. However, the underlying asset of a bond option contract is the bond price itself and not the interest rate. Nevertheless, it is still possible to compute the *delta* (or *hedge ratio*) of the bond option with respect to the underlying bond price. This is accomplished because we can apply the classic chain rule for deriving the delta of a zero-coupon bond option. Moreover, and even though it is not possible to apply the decomposition technique of Jamshidian (1989) when computing the *delta* of a coupon-paying bond option, we can still determine its value by simply using, again, the classic chain rule.

Next remark shows the analytical solutions for computing the delta of zero-coupon and coupon-paying bond options under the CIR model.[12]

Remark 5. *The delta (with respect to the underlying bond price) for both zero-coupon and coupon-paying bond options arise immediately if one uses the results obtained in Proposition 1 and Remark 2, that is*

$$\begin{aligned}\Delta_v^{zc}(.) &:= \frac{\partial v^{zc}(r,t,T,s,K;\alpha)}{\partial Z(r,t,s)} = \frac{\partial v^{zc}(r,t,T,s,K;\alpha)}{\partial r}\,\frac{\partial r}{\partial Z(r,t,s)} = \rho_v^{zc}(r,t,T,s,K;\alpha)\,\frac{1}{\frac{\partial Z(r,t,s)}{\partial r}} \\ &= -\frac{\rho_v^{zc}(r,t,T,s,K;\alpha)}{B(t,s)Z(r,t,s)}, \end{aligned} \quad (33)$$

[12] Additional details about the computation of the delta of a zero-coupon bond option will be discussed later in Remark 6.

and

$$\begin{aligned}\Delta_v^{cb}(.) &:= \frac{\partial v^{cb}(r,t,T,s,K;\alpha)}{\partial P(r,t,s)} = \frac{\partial v^{cb}(r,t,T,s,K;\alpha)}{\partial r}\frac{\partial r}{\partial P(r,t,s)} = \rho_v^{cb}(r,t,T,s,K;\alpha)\frac{1}{\frac{\partial P(r,t,s)}{\partial r}} \\ &= -\frac{\rho_v^{cb}(r,t,T,s,K;\alpha)}{\sum_{i=1}^{N} a_i B(t,s_i)Z(r,t,s_i)}. \end{aligned} \tag{34}$$

Interestingly, the gamma (with respect to the underlying bond price) for both zero-coupon and coupon-paying bond options can be expressed analytically in terms of others sensitivity measures, as shown in the following two propositions.

Proposition 6. *The gamma (with respect to the underlying bond price) of a zero-coupon call option (if $\alpha = 1$) or put option (if $\alpha = -1$) can be computed as*

$$\Gamma_{v,Z}^{zc}(.) := \frac{\partial \Delta_v^{zc}(.)}{\partial Z(r,t,s)} = \frac{\Gamma_{v,r}^{zc}(.)}{[B(t,s)Z(r,t,s)]^2} - \frac{\Delta_v^{zc}(.)}{Z(r,t,s)}. \tag{35}$$

Proof. Please see Appendix F.■

Proposition 7. *The gamma (with respect to the underlying bond price) of a coupon-paying call option (if $\alpha = 1$) or put option (if $\alpha = -1$) can be computed as*

$$\Gamma_{v,Z}^{cb}(.) := \frac{\partial \Delta_v^{cb}(.)}{\partial P(r,t,s)} = \frac{\Gamma_{v,r}^{cb}(.) - \Delta_v^{cb}(.)\sum_{i=1}^{N} a_i\,[B(t,s_i)]^2\,Z(r,t,s_i)}{\left[\sum_{i=1}^{N} a_i B(t,s_i)Z(r,t,s_i)\right]^2}. \tag{36}$$

Proof. Please see Appendix G.■

3.4. Numerical examples

This subsection presents some numerical results of the novel closed-form solutions of Greeks of bond options under the CIR model. For completeness, we note that our software programs were implemented in *Matlab R2021b* and run on a personal computer with an Intel Core i9-10900 2.80 GHz processor and 64 GB of ram memory.

3.4.1. Greeks of zero-coupon bond options

Table 1 values 4-year call option prices (if $\alpha = 1$) and put option prices (if $\alpha = -1$), as well as their corresponding rho, interest rate gamma, theta, eta, delta, and gamma sensitivity measures, written on a 10-year zero-coupon bond with face value equal to \$1.0 for different levels of the interest rate (r), strike price $K = \$0.6$, and using the parameter values $\kappa = 0.2339$, $\theta = 0.0808$, $\sigma = 0.0854$, and $\lambda = 0$, borrowed from Chan et al. (1992, Table III). We should mention that the obtained zero-coupon bond prices and the option prices are both expressed as percentages of the face value. For completeness, we shall also mention that the required noncentral chi-square distribution function and the corresponding probability density function have been computed using, respectively, the Benton and Krishnamoorthy (2003) algorithm and the built-in function ncx2pdf available in *Matlab*.

A simple way to check the analytical formulas of our Greeks is to replace the solutions of the price, rho, interest rate gamma, and theta into the CIR partial differential equation (hereafter, pde).

Table 1. Prices and Greeks of European-style options on zero-coupon bonds under the CIR model.

		Call options							
r	$Z(r,t,s)$	$v^{zc}(.;1)$	$\rho_v^{zc}(.;1)$	$\Gamma_{v,r}^{zc}(.;1)$	$\theta_v^{zc}(.;1)$	$\eta_v^{zc}(.;1)$	$\Delta_v^{zc}(.;1)$	$\Gamma_{v,Z}^{zc}(.;1)$	pde test
0.01	59.3183	7.2123	−0.7992	6.3552	0.0137	−0.8185	0.3624	0.6957	1.95E−17
0.02	57.1534	6.4447	−0.7364	6.1884	0.0113	−0.7765	0.3466	0.7642	3.92E−17
0.03	55.0675	5.7389	−0.6755	5.9827	0.0091	−0.7327	0.3299	0.8281	8.67E−18
0.04	53.0577	5.0929	−0.6169	5.7418	0.0071	−0.6878	0.3127	0.8861	9.11E−18
0.05	51.1213	4.5043	−0.5608	5.4707	0.0053	−0.6422	0.2951	0.9372	2.43E−17
0.06	49.2555	3.9703	−0.5075	5.1752	0.0037	−0.5965	0.2772	0.9805	9.97E−18
0.07	47.4578	3.4881	−0.4574	4.8617	0.0024	−0.5512	0.2592	1.0154	1.73E−17
0.08	45.7258	3.0546	−0.4104	4.5365	0.0012	−0.5067	0.2414	1.0417	−1.78E−17
0.09	44.0569	2.6663	−0.3666	4.2056	0.0002	−0.4634	0.2238	1.0594	−2.26E−17
0.10	42.4490	2.3202	−0.3262	3.8744	−0.0006	−0.4218	0.2067	1.0685	−3.04E−18
0.11	40.8997	2.0128	−0.2891	3.5480	−0.0012	−0.3821	0.1901	1.0695	−2.99E−17
0.12	39.4070	1.7408	−0.2553	3.2303	−0.0017	−0.3445	0.1742	1.0627	−5.55E−17
0.13	37.9688	1.5012	−0.2245	2.9249	−0.0020	−0.3093	0.1590	1.0489	−2.52E−17
0.14	36.5830	1.2909	−0.1967	2.6343	−0.0023	−0.2764	0.1446	1.0287	−3.27E−17
0.15	35.2479	1.1069	−0.1717	2.3607	−0.0024	−0.2460	0.1311	1.0027	2.08E−17
		Put options							
r	$Z(r,t,s)$	$v^{zc}(.;-1)$	$\rho_v^{zc}(.;-1)$	$\Gamma_{v,r}^{zc}(.;-1)$	$\theta_v^{zc}(.;-1)$	$\eta_v^{zc}(.;-1)$	$\Delta_v^{zc}(.;-1)$	$\Gamma_{v,Z}^{zc}(.;-1)$	pde test
0.01	59.3183	0.1474	0.0652	1.5974	−0.0011	0.0524	−0.0295	0.3782	1.33E−17
0.02	57.1534	0.2207	0.0814	1.6427	−0.0012	0.0724	−0.0383	0.4308	2.81E−17
0.03	55.0675	0.3103	0.0979	1.6404	−0.0012	0.0946	−0.0478	0.4781	3.42E−18
0.04	53.0577	0.4163	0.1141	1.5944	−0.0012	0.1185	−0.0578	0.5187	−1.29E−17
0.05	51.1213	0.5382	0.1296	1.5102	−0.0009	0.1437	−0.0682	0.5515	2.04E−17
0.06	49.2555	0.6752	0.1442	1.3940	−0.0006	0.1695	−0.0787	0.5755	−4.55E−18
0.07	47.4578	0.8261	0.1574	1.2523	−0.0001	0.1954	−0.0892	0.5902	1.68E−17
0.08	45.7258	0.9896	0.1691	1.0917	0.0004	0.2210	−0.0995	0.5953	−3.66E−17
0.09	44.0569	1.1639	0.1792	0.9186	0.0011	0.2458	−0.1094	0.5907	−1.91E−17
0.10	42.4490	1.3474	0.1875	0.7387	0.0019	0.2695	−0.1188	0.5764	4.12E−18
0.11	40.8997	1.5383	0.1940	0.5571	0.0028	0.2917	−0.1276	0.5528	−2.91E−17
0.12	39.4070	1.7347	0.1986	0.3783	0.0037	0.3122	−0.1356	0.5203	−5.07E−17
0.13	37.9688	1.9350	0.2016	0.2060	0.0047	0.3308	−0.1428	0.4794	−1.65E−17
0.14	36.5830	2.1373	0.2028	0.0429	0.0058	0.3474	−0.1491	0.4308	−4.34E−17
0.15	35.2479	2.3400	0.2025	−0.1087	0.0068	0.3620	−0.1545	0.3750	9.97E−18

This table values 4-year call option prices (if $\alpha = 1$) and put option prices (if $\alpha = -1$), as well as their corresponding rho, interest rate gamma, theta, eta, delta, and gamma sensitivity measures, written on a 10-year zero-coupon bond with face value equal to \$1.0 for different levels of the interest rate (r) and strike price $K = \$0.6$, and assuming a CIR interest rate dynamics. Parameter values borrowed from Chan et al. (1992, Table III): $\kappa = 0.2339$, $\theta = 0.0808$, $\sigma = 0.0854$, and $\lambda = 0$. The zero-coupon bond prices and the option prices are both expressed as percentages of the face value. The last column of the table tests the CIR pde $\frac{1}{2}\sigma^2 r \Gamma_{v,r}^{zc} + [\kappa\theta - (\kappa+\lambda)\, r]\, \rho_v^{zc} + \theta_v^{zc} - r v^{zc} = 0$ (the value on the left-hand-side is displayed to check how close it is to zero). The required noncentral chi-square distribution function and the corresponding probability density function have been computed using, respectively, the Benton and Krishnamoorthy (2003) algorithm and the built-in function ncx2pdf available in *Matlab*.

This exercise allowed us to conclude that the pde is satisfied in all the tested cases. It is noteworthy to recall that while symbolic algebra programs such as *Mathematica* or *Maple* can be used to derive Greeks through elementary differentiation—see, for example, Shaw (1998) who derives Greeks for stock options under the geometric Brownian motion assumption via *Mathematica*—, these novel analytical solutions are important for several reasons. Firstly, as argued by Carr (2001), the derivation of Greeks through symbolic algebra programs cannot replace an intuitive understanding of the role, genesis, and relationships between Greek measures. Secondly, as highlighted in Larguinho et al. (2013), the computation time needed for computing analytic Greeks is much smaller, which is of pivotal importance when a trader needs to design hedging strategies in real time. For instance, it takes only about 0.29 seconds to perform all the computations shown in Table 1. Thirdly, the existence of Greeks in closed-form allows its coding in any desired computer language, e.g., *Matlab*, *Python*, *Fortran*, *R*, or *C*. Lastly, as it will be shown in Section 4, the delta sensitivity measure can be used to price (and hedge) American-style options on zero-coupon bonds under the CIR model via the SHP pricing methodology developed by Chung and Shih (2009) and Ruas et al. (2013) for stock options.

3.4.2. Greeks of coupon-paying bond options

Table 2 adopts the parameters configuration of Wei (1997, Table II) to value 5-year call option prices (if $\alpha = 1$) and put option prices (if $\alpha = -1$), as well as their corresponding rho, interest rate gamma, theta, eta, delta, and gamma sensitivity measures, written on a 15-year 10% coupon bond—with annual payment of the ten coupons to be delivered after the expiry date of the option contract—with face value equal to \$100 for different levels of the interest rate (r), strike price $K = \$100$, $\kappa = 0.25$, $\theta = 0.085$, $\sigma = 0.05$, and $\lambda = 0$. The required noncentral chi-square distribution function has been computed via the Benton and Krishnamoorthy (2003) algorithm and the corresponding probability density function has been computed using the built-in function ncx2pdf available in *Matlab*.

Wei (1997, Table II) reports only prices and rho values for calls. Column 3 of Table 2 reveals that our call option prices are similar to the ones presented in the third and forth columns of Wei (1997, Table II, Panel A).[13] Our interest rate deltas shown in column 4 of Table 2—obtained via equation (22) with $\alpha = 1$—are also similar to the ones presented in the third and forth columns of Wei (1997, Table II, Panel B), which have been computed through a numerical integration scheme, as mentioned in Wei (1997, Footnote 9). For $r \geq 24\%$, however, it seems that there are some text typos in Wei (1997, Table II), because the corresponding absolute values are approximately equal.[14] We recall that Cox et al. (1985) and Longstaff (1993) show that zero-coupon and coupon-paying bond call options are strictly decreasing functions of the interest rate. Thus, the first derivative of bond call options with respect to interest rates (i.e., interest rate deltas) illustrated in the third and fourth columns of Wei (1997, Table II, Panel B) should always be negative, as shown in column 4 of our Table 2.

To further check the analytical formulas of our Greeks, we have replaced the solutions for the price, rho, interest rate gamma, and theta into the pde of the problem, which is satisfied in all the tested cases. As a final testing exercise, we reproduce, in Table 3 (resp., Table 4), the computation of call and put

[13] If we use the Sankaran (1963) approximation for computing the noncentral chi-square distribution function we obtain exactly the same bond option prices reported in his third column.

[14] For example, when using his *accurate* method, Wei (1997, Table II, Panel A) reports positive values for rho of 9.5254, 6.6349, 4.5098, and 2.9933 for r equal to 0.24, 0.26, 0.28, and 0.30, respectively, whereas we obtain the correct negative sign with values equal to −9.4665, −6.6099, −4.5109, and −3.0114, respectively. A similar pattern of wrong positive rho values is observed under his proposed approximate method for $r \geq 0.24$.

Table 2. Prices and Greeks of European-style options on coupon-paying bonds under the CIR model.

		Call options							
r	$P(r,t,s)$	$v^{cb}(.;1)$	$\rho_v^{cb}(.;1)$	$\Gamma_{v,r}^{cb}(.;1)$	$\theta_v^{cb}(.;1)$	$\eta_v^{cb}(.;1)$	$\Delta_v^{cb}(.;1)$	$\Gamma_{v,Z}^{cb}(.;1)$	pde test
0.04	126.1318	9.1833	−92.5420	586.0740	1.3791	−72.8855	30.2879	26.3717	4.38E−17
0.06	118.6380	7.4484	−81.0065	567.9602	0.9106	−67.1640	28.5330	33.5644	5.12E−17
0.08	111.6294	5.9407	−69.8268	549.3779	0.5076	−60.8831	26.4689	42.1126	3.90E−17
0.10	105.0732	4.6525	−59.0753	524.2961	0.1782	−54.0721	24.0988	51.1757	6.07E−18
0.12	98.9389	3.5737	−48.9233	489.0691	−0.0726	−46.9028	21.4769	59.6656	2.95E−17
0.14	93.1981	2.6902	−39.5845	443.1400	−0.2452	−39.6490	18.6999	66.4988	3.08E−17
0.16	87.8244	1.9836	−31.2550	388.6802	−0.3464	−32.6228	15.8884	70.8323	9.11E−18
0.18	82.7931	1.4323	−24.0685	329.5524	−0.3880	−26.1120	13.1658	72.2162	4.34E−19
0.20	78.0814	1.0129	−18.0749	270.1006	−0.3846	−20.3330	10.6390	70.6356	−2.17E−18
0.22	73.6678	0.7016	−13.2408	214.1606	−0.3514	−15.4096	8.3860	66.4531	9.97E−18
0.24	69.5326	0.4762	−9.4665	164.4774	−0.3019	−11.3740	6.4511	60.2875	6.94E−18
0.26	65.6572	0.3168	−6.6099	122.5353	−0.2466	−8.1836	4.8466	52.8721	8.57E−18
0.28	62.0243	0.2067	−4.5109	88.6897	−0.1931	−5.7450	3.5587	44.9264	−8.02E−18
0.30	58.6179	0.1324	−3.0114	62.4610	−0.1456	−3.9390	2.5561	37.0647	1.08E−17
		Put options							
r	$P(r,t,s)$	$v^{cb}(.;-1)$	$\rho_v^{cb}(.;-1)$	$\Gamma_{v,r}^{cb}(.;-1)$	$\theta_v^{cb}(.;-1)$	$\eta_v^{cb}(.;-1)$	$\Delta_v^{cb}(.;-1)$	$\Gamma_{v,Z}^{cb}(.;-1)$	pde test
0.04	126.1318	0.0382	1.7847	63.3286	−0.0217	1.5388	−0.5841	7.4858	−1.29E−17
0.06	118.6380	0.0885	3.3390	91.9180	−0.0225	3.1537	−1.1761	12.9249	1.77E−17
0.08	111.6294	0.1754	5.4324	116.1725	−0.0044	5.5546	−2.0592	19.5579	2.66E−18
0.10	105.0732	0.3084	7.9183	130.3677	0.0442	8.6996	−3.2301	26.5295	−1.80E−17
0.12	98.9389	0.4932	10.5569	131.1408	0.1319	12.4052	−4.6344	32.7354	5.42E−19
0.14	93.1981	0.7299	13.0718	118.1960	0.2612	16.3866	−6.1752	37.0747	0.00E+00
0.16	87.8244	1.0135	15.2090	93.9454	0.4285	20.3208	−7.7314	38.6850	−1.71E−17
0.18	82.7931	1.3345	16.7814	62.4740	0.6247	23.9103	−9.1796	37.0957	−1.56E−17
0.20	78.0814	1.6803	17.6903	28.3308	0.8376	26.9291	−10.4126	32.2691	−2.78E−17
0.22	73.6678	2.0375	17.9239	−4.4597	1.0544	29.2446	−11.3520	24.5430	−2.60E−18
0.24	69.5326	2.3931	17.5407	−32.9780	1.2639	30.8162	−11.9534	14.5092	1.04E−17
0.26	65.6572	2.7357	16.6445	−55.5794	1.4576	31.6787	−12.2043	2.8713	2.60E−18
0.28	62.0243	3.0563	15.3605	−71.7604	1.6297	31.9177	−12.1181	−9.6831	−6.94E−18
0.30	58.6179	3.3484	13.8147	−81.8640	1.7778	31.6455	−11.7259	−22.5749	5.20E−18

This table values 5-year call option prices (if $\alpha = 1$) and put option prices (if $\alpha = -1$), as well as their corresponding rho, interest rate gamma, theta, eta, delta, and gamma sensitivity measures, written on a 15-year 10% coupon bond—with the ten coupons being paid annually—with face value equal to \$100 for different levels of the interest rate (r) and strike price $K = \$100$, and assuming a CIR interest rate dynamics. Parameter values borrowed from Wei (1997, Table II): $\kappa = 0.25$, $\theta = 0.085$, $\sigma = 0.05$, and $\lambda = 0$. We note that the obtained coupon-paying bond prices and the option prices are both expressed as percentages of the face value. All the Greeks reported in this table are 100 times the corresponding partial derivative of the option price, in order to be consistent with the rho values shown in Wei (1997, Table II). The last column of the table tests the CIR pde $\frac{1}{2}\sigma^2 r\Gamma_{v,r}^{cb} + [\kappa\theta - (\kappa+\lambda)\, r]\,\rho_v^{cb} + \theta_v^{cb} - rv^{cb} = 0$ (the value on the left-hand-side is displayed to check how close it is to zero). The required noncentral chi-square distribution function has been computed via the Benton and Krishnamoorthy (2003) algorithm. The corresponding probability density function has been computed using the built-in function ncx2pdf available in *Matlab*.

deltas (resp., gammas) reported in Longstaff (1993, Tables 1 and 2). Again, our results are similar to the ones shown in Longstaff (1993, Tables 1 and 2), which, to the best of our knowledge, have been obtained via numerical methods or through standard symbolic derivation software since no analytical solutions of Greeks have been provided or mentioned in the paper. The use of our closed-form solutions provides accurate values for computing Greeks of coupon-paying bond options and with a very small computational burden. For instance, it takes only about 4.35 seconds to perform all the computations shown in Table 2.

4. SHP approach

The goal now is to show how to implement the SHP approach for valuing American-style options on discount bonds under the CIR model. Even though the underlying asset is the bond, the independent variable is the stochastic interest rate r and there exists an unknown optimal exercise interest rate for which the exercise of the option becomes optimal. However, to provide a better understanding of the SHP method it is preferable to consider the unknown early exercise boundary, E, in terms of the underlying bond price Z. This requires the use of an alternative option pricing solution equivalent to equation (7), but expressed as a function of the underlying bond price Z.

4.1. Alternative option pricing solution

Let us first make a change of variable to express equation (7) as a function of the underlying bond price $Z = Z(r, t, s)$.[15] To accomplish this purpose, we note that

$$r(Z, t, s) = \frac{1}{B(t, s)} \ln\left(\frac{A(t, s)}{Z}\right). \tag{37}$$

Substituting expression (37) into equations (7), (11), and (12), allows us to rewrite $v^{zc}(r, t, T, s, K; \alpha) = \bar{v}^{zc}(Z, t, T, s, K; \alpha)$ as a function $\bar{v}^{zc}$ of Z instead of r and, therefore, rewrite expression (7) as

$$\begin{aligned} \bar{v}^{zc}(Z, t, T, s, K; \alpha) &= \alpha Z Q\left[x_1(t, T, s, K); a, \bar{b}_1(Z, t, T, s); \alpha\right] \\ &\quad -\alpha K A(t, T)\left(\frac{Z}{A(t, s)}\right)^{\frac{B(t,T)}{B(t,s)}} Q\left[x_2(t, T, s, K); a, \bar{b}_2(Z, t, T, s); \alpha\right], \end{aligned} \tag{38}$$

with

$$\bar{b}_1(Z, t, T, s) := \frac{2\phi^2(t, T)\frac{1}{B(t,s)} \ln\frac{A(t,s)}{Z} e^{\gamma(T-t)}}{\phi(t, T) + \psi + B(T, s)}, \tag{39}$$

and

$$\bar{b}_2(Z, t, T, s) := \frac{2\phi^2(t, T)\frac{1}{B(t,s)} \ln\frac{A(t,s)}{Z} e^{\gamma(T-t)}}{\phi(t, T) + \psi}. \tag{40}$$

Armed with the alternative option pricing solution (38), we can now proceed with the derivation of the corresponding hedge ratio, i.e., the delta with respect to the underlying bond price Z.

[15] We note that the presence of interim coupons prevents the use of a similar approach to equation (16) and, therefore, it is not possible to apply the SHP pricing methodology in the case of options on coupon-paying bonds.

Table 3. Deltas of European-style options on coupon-paying bonds under the CIR model.

Panel A: 8% coupon bond in Longstaff (1993, Table 1)						
	Call deltas			**Put deltas**		
r	$K = 960$	$K = 980$	$K = 1000$	$K = 960$	$K = 980$	K = 1,000
0.01	0.0456	0.0269	0.0120	−0.0015	−0.0004	0.0046
0.02	0.0454	0.0267	0.0119	−0.0015	−0.0004	0.0047
0.03	0.0452	0.0265	0.0118	−0.0015	−0.0003	0.0049
0.04	0.0449	0.0263	0.0117	−0.0015	−0.0002	0.0050
0.05	0.0447	0.0261	0.0116	−0.0014	−0.0001	0.0052
0.06	0.0445	0.0259	0.0115	−0.0014	−0.0001	0.0053
0.07	0.0442	0.0257	0.0113	−0.0014	0.0000	0.0055
0.08	0.0440	0.0256	0.0112	−0.0014	0.0001	0.0056
0.09	0.0438	0.0254	0.0111	−0.0013	0.0002	0.0058
0.10	0.0435	0.0252	0.0110	−0.0013	0.0002	0.0060
0.11	0.0433	0.0250	0.0109	−0.0013	0.0003	0.0061
0.12	0.0431	0.0248	0.0108	−0.0013	0.0004	0.0063
0.13	0.0428	0.0247	0.0107	−0.0012	0.0004	0.0065
0.14	0.0426	0.0245	0.0106	−0.0012	0.0006	0.0066
0.15	0.0424	0.0243	0.0105	−0.0012	0.0007	0.0068
Panel B: 14% coupon bond in Longstaff (1993, Table 2)						
	Call deltas			**Put deltas**		
r	$K = 1,340$	$K = 1,360$	$K = 1,380$	$K = 1,340$	$K = 1,360$	$K = 1,380$
0.01	0.0513	0.0373	0.0244	−0.0014	−0.0014	−0.0001
0.02	0.0511	0.0370	0.0242	−0.0014	−0.0013	0.0000
0.03	0.0508	0.0368	0.0240	−0.0014	−0.0013	0.0001
0.04	0.0506	0.0366	0.0239	−0.0014	−0.0013	0.0001
0.05	0.0504	0.0364	0.0237	−0.0014	−0.0012	0.0002
0.06	0.0501	0.0362	0.0235	−0.0014	−0.0012	0.0003
0.07	0.0499	0.0360	0.0233	−0.0014	−0.0011	0.0004
0.08	0.0496	0.0357	0.0232	−0.0014	−0.0011	0.0005
0.09	0.0494	0.0355	0.0230	−0.0014	−0.0011	0.0006
0.10	0.0492	0.0353	0.0228	−0.0013	−0.0010	0.0007
0.11	0.0489	0.0351	0.0226	−0.0013	−0.0010	0.0007
0.12	0.0487	0.0349	0.0225	−0.0013	−0.0009	0.0008
0.13	0.0485	0.0347	0.0223	−0.0013	−0.0009	0.0009
0.14	0.0482	0.0345	0.0221	−0.0013	−0.0008	0.0010
0.15	0.0480	0.0343	0.0220	−0.0013	−0.0008	0.0011

This table values 5-year call and put deltas on a 10-year 8% coupon bond (in Panel A) and 14% coupon bond (in Panel B)—with the ten coupons being paid annually—with par value 1,000 for different levels of the riskless interest rate (r) and strike price (K) assuming a CIR model. Parameter values borrowed from Longstaff (1993, Tables 1 and 2): $\kappa = 0.75$, $\theta = 0.08$, $\sigma^2 = 0.014$, and $\lambda = 0$. The required noncentral chi-square distribution function has been computed via the Benton and Krishnamoorthy (2003) algorithm. The corresponding probability density function has been computed using the built-in function ncx2pdf available in *Matlab*.

Table 4. Gammas of European-style options on coupon-paying bonds under the CIR model.

Panel A: 8% coupon bond in Longstaff (1993, Table 1)						
	Call gammas			**Put gammas**		
r	K = 960	K = 980	K = 1000	K = 960	K = 980	K = 1,000
0.01	0.2482	0.1990	0.1195	−0.0196	−0.0743	−0.1594
0.02	0.2509	0.2007	0.1201	−0.0206	−0.0764	−0.1626
0.03	0.2535	0.2023	0.1207	−0.0216	−0.0785	−0.1658
0.04	0.2562	0.2039	0.1213	−0.0226	−0.0807	−0.1691
0.05	0.2589	0.2056	0.1219	−0.0236	−0.0829	−0.1724
0.06	0.2617	0.2072	0.1225	−0.0247	−0.0851	−0.1758
0.07	0.2644	0.2089	0.1231	−0.0258	−0.0874	−0.1792
0.08	0.2672	0.2106	0.1237	−0.0270	−0.0897	−0.1827
0.09	0.2700	0.2123	0.1244	−0.0281	−0.0921	−0.1862
0.10	0.2728	0.2140	0.1250	−0.0293	−0.0945	−0.1898
0.11	0.2757	0.2157	0.1256	−0.0306	−0.0970	−0.1935
0.12	0.2786	0.2174	0.1262	−0.0318	−0.0995	−0.1972
0.13	0.2815	0.2191	0.1268	−0.0331	−0.1021	−0.2009
0.14	0.2844	0.2208	0.1274	−0.0345	−0.1047	−0.2047
0.15	0.2873	0.2225	0.1280	−0.0358	−0.1073	−0.2086
Panel B: 14% coupon bond in Longstaff (1993, Table 2)						
	Call gammas			**Put gammas**		
r	$K = 1,340$	$K = 1,360$	$K = 1,380$	$K = 1,340$	$K = 1,360$	$K = 1,380$
0.01	0.1800	0.1635	0.1336	−0.0066	−0.0259	−0.0586
0.02	0.1820	0.1650	0.1347	−0.0071	−0.0269	−0.0601
0.03	0.1841	0.1667	0.1357	−0.0076	−0.0279	−0.0617
0.04	0.1861	0.1683	0.1368	−0.0081	−0.0289	−0.0633
0.05	0.1882	0.1699	0.1378	−0.0087	−0.0299	−0.0649
0.06	0.1903	0.1715	0.1389	−0.0092	−0.0310	−0.0666
0.07	0.1925	0.1732	0.1400	−0.0098	−0.0321	−0.0683
0.08	0.1946	0.1748	0.1410	−0.0104	−0.0332	−0.0701
0.09	0.1968	0.1765	0.1421	−0.0110	−0.0343	−0.0718
0.10	0.1990	0.1782	0.1432	−0.0116	−0.0355	−0.0737
0.11	0.2012	0.1799	0.1443	−0.0122	−0.0367	−0.0755
0.12	0.2034	0.1816	0.1454	−0.0129	−0.0379	−0.0774
0.13	0.2056	0.1833	0.1465	−0.0135	−0.0391	−0.0793
0.14	0.2079	0.1851	0.1476	−0.0142	−0.0404	−0.0812
0.15	0.2102	0.1868	0.1487	−0.0149	−0.0417	−0.0832

This table values 5-year call and put gammas on a 10-year 8% coupon bond (in Panel A) and 14% coupon bond (in Panel B)—with the ten coupons being paid annually—with par value 1,000 for different levels of the riskless interest rate (r) and strike price (K) assuming a CIR model. Parameter values borrowed from Longstaff (1993, Tables 1 and 2): $\kappa = 0.75$, $\theta = 0.08$, $\sigma^2 = 0.014$, and $\lambda = 0$. The required noncentral chi-square distribution function has been computed via the Benton and Krishnamoorthy (2003) algorithm. The corresponding probability density function has been computed using the built-in function ncx2pdf available in *Matlab*.

Proposition 8. *Consider the pricing solution of a zero-coupon bond option under the CIR model as defined in equation* (38)*. Then, the delta (with respect to the underlying bond price Z) of a zero-coupon bond call option (if* $\alpha = 1$*) or put option (if* $\alpha = -1$*) is given by*

$$\begin{aligned} \Delta_{\bar{v}}^{zc}(.) &:= \frac{\partial \bar{v}^{zc}(Z,t,T,s,K;\alpha)}{\partial Z} \\ &= \alpha Q[x_1(.);a,\bar{b}_1(.);\alpha] + 2p(x_1(.);a+2,\bar{b}_1(.))\frac{\phi^2(t,T)\,e^{\gamma(T-t)}}{B(t,s)\left[\phi(t,T)+\psi+B(T,s)\right]} \\ &\quad -\alpha K\frac{A(t,T)B(t,T)}{ZB(t,s)}\left(\frac{Z}{A(t,s)}\right)^{\frac{B(t,T)}{B(t,s)}} Q[x_2(.);a,\bar{b}_2(.);\alpha] \\ &\quad -2K\frac{A(t,T)}{ZB(t,s)}\left(\frac{Z}{A(t,s)}\right)^{\frac{B(t,T)}{B(t,s)}} p(x_2(.);a+2,\bar{b}_2(.))\frac{\phi^2(t,T)\,e^{\gamma(T-t)}}{\phi(t,T)+\psi}. \end{aligned} \tag{41}$$

Proof. Please see Appendix H.■

Remark 6. *As expected,*

$$\bar{v}^{zc}(Z,t,T,s,K;\alpha) = v^{zc}(r,t,T,s,K;\alpha), \tag{42}$$

and

$$\Delta_{\bar{v}}^{zc}(Z,t,T,s,K;\alpha) = \Delta_{v}^{zc}(r,t,T,s,K;\alpha), \tag{43}$$

due to the relation between r and Z given by equation (2) *or, equivalently, by equation* (37)*. Note that this relation depends on t and s, but not on T and K. Hence, if we fix the values of t and s, the relation between r and Z is bijective and, therefore, working with one or the other to determine the price and the delta of the zero-coupon bond option is simply a (non-linear) scale change issue.*[16] *Note also that r and Z are both random variables having exactly the same information, i.e., the same* σ*-algebra* $\mathcal{F}_t$*. Therefore, by fixing t and s, Z is only a function of r (and, vice-versa, r is only a function of Z) so that we are able to use the classic chain rule for univariate functions in Remark 5. These arguments explain why we obtain the same values for prices and deltas of zero-coupon bond options using different (but equivalent) formulas.*

4.2. SHP scheme

Let us define by $\bar{V}^{zc}(Z,t,T,s,K;\alpha)$ the time-t price of an American-style call (if $\alpha = 1$) or put (if $\alpha = -1$) on the asset price Z (i.e., the underlying zero-coupon bond), with strike K, and maturity at time T ($\geq t$).[17] Let us denote the first passage time of the underlying asset price to its time-dependent exercise boundary $\{E_u, t \leq u \leq T\}$ by

$$\tau^* := \inf\{u > t : Z_u = E_u\}. \tag{44}$$

Note that the critical asset price Z_{τ^*} implies the existence of a critical interest rate r_{τ^*}.

Following Chung and Shih (2009) and Ruas et al. (2013), we use two well-known conditions on the early exercise boundary of the American option to solve this pricing problem: the *value-matching* and

[16] For the SHP approach, however, it is convenient to choose Z as the variable characterizing the underlying asset of the option.

[17] Notice that the valuation of American-style call options on discount bonds can be performed via expressions (7) and (38), since such contracts, given the absence of interim coupons, will never be exercised before maturity—see Cox et al. (1985, Footnote 12). Nevertheless, for completeness, we will describe the SHP methodology for the general case.

smooth-pasting conditions. At the maturity date T, if the American-style option has not been exercised earlier, its terminal condition is exactly the same as the corresponding European-style option. Therefore, we start at the maturity date of the American-style zero-coupon option and proceed backwards until the valuation date $t \equiv t_0$. More specifically, at time T, we start our static hedge portfolio with one unit of the European-style zero-coupon bond option (38) with strike K, and expiry date at time T. In addition, we divide the time to maturity of the option contract into n evenly-spaced time points such that $\Delta t := (T - t_0)/n$. At each time $t_i := t_0 + i\Delta t$ (for $i = n-1, \ldots, 1, 0$), the unknown early exercise boundary E_i is matched by adding w_i units of a standard European-style option with strike equal to E_i, and maturity at time t_{i+1}. For each time step, the unknowns E_i and w_i are found by solving simultaneously the following two value-matching and smooth-pasting recurrence conditions:

$$\alpha E_{n-i} - \alpha K = \bar{v}^{zc}\left(E_{n-i}, t_{n-i}, T, s, K; \alpha\right) + \sum_{j=1}^{i} w_{n-j}\, \bar{v}^{zc}\left(E_{n-i}, t_{n-i}, t_{n-j+1}, s, E_{n-j}; \alpha\right), \tag{45}$$

and

$$\alpha = \Delta_{\bar{v}}^{zc}\left(E_{n-i}, t_{n-i}, T, s, K; \alpha\right) + \sum_{j=1}^{i} w_{n-j}\, \Delta_{\bar{v}}^{zc}\left(E_{n-i}, t_{n-i}, t_{n-j+1}, s, E_{n-j}; \alpha\right), \tag{46}$$

for $i = 1, 2, ..., n$, and with $\bar{v}^{zc}(.)$ and $\Delta_{\bar{v}}^{zc}(.)$ being given by expressions (38) and (41), respectively.

After solving for all the unknowns E_i and w_i (for $i = n-1, \ldots, 1, 0$), the time-t_0 SHP price of the American-style zero-coupon bond option, under the CIR model, is finally given by

$$\bar{V}^{zc}(Z_0, t_0, T, s, K; \alpha) := \bar{v}^{zc}\left(Z_0, t_0, T, s, K; \alpha\right) + \sum_{j=1}^{n} w_{n-j}\, \bar{v}^{zc}\left(Z_0, t_0, t_{n-j+1}, s, E_{n-j}; \alpha\right). \tag{47}$$

4.3. Numerical examples

Table 5 adopts the constellation of parameters used in Thakoor et al. (2012, Table 7), that is we consider a 5-year American-style put option on a 10-year zero-coupon bond with face value \$100 for different levels of the interest rate (r) and an exercise price of \$60. The CIR parameters are: $\kappa = 0.50$, $\theta = 0.08$, $\sigma = 0.10$, and $\lambda = 0$. The third and fourth columns of the table report the European-style put prices computed via equations (7) and (38), respectively. Similarly, the sixth and seventh columns of the table show the results of put deltas obtained through equations (33) and (41), respectively. Columns 5 and 8 compute the corresponding differences and validate (numerically) the analytical equivalence demonstrated in Remark 6.

Table 5. Prices and deltas of European-style put options and prices of American-style put options on zero-coupon bonds under the CIR model.

		European puts			European put deltas			American puts	
r	$Z(r,t,s)$	$v^{zc}(.;-1)$	$\bar{v}^{zc}(.;-1)$	Diff.	$\Delta_v^{zc}(.;-1)$	$\Delta_{\bar{v}}^{zc}(.;-1)$	Diff	$n=2$	$n=100$
0.01	52.0729	0.0149	0.0149	−8.67E−19	−0.0014	−0.0014	−2.10E−16	7.9271	7.9271
0.02	51.0671	0.0163	0.0163	1.73E−18	−0.0015	−0.0015	6.94E−18	8.9329	8.9329
0.03	50.0807	0.0178	0.0178	8.67E−19	−0.0016	−0.0016	−2.67E−16	9.9193	9.9193
0.04	49.1134	0.0194	0.0194	−1.73E−18	−0.0017	−0.0017	1.24E−16	10.8866	10.8866
0.05	48.1647	0.0211	0.0211	8.67E−19	−0.0018	−0.0018	−2.36E−16	11.8353	11.8353
0.06	47.2344	0.0228	0.0228	0.00E+00	−0.0019	−0.0019	8.52E−17	12.7656	12.7656
0.07	46.3220	0.0246	0.0246	1.73E−18	−0.0020	−0.0020	5.12E−17	13.6780	13.6780
0.08	45.4273	0.0265	0.0265	5.20E−18	−0.0022	−0.0022	−1.40E−16	14.5727	14.5727
0.09	44.5499	0.0284	0.0284	−1.73E−18	−0.0023	−0.0023	−2.60E−17	15.4501	15.4501
0.10	43.6893	0.0304	0.0304	0.00E+00	−0.0024	−0.0024	3.37E−16	16.3107	16.3107
0.11	42.8455	0.0325	0.0325	1.73E−18	−0.0026	−0.0026	−2.75E−16	17.1545	17.1545
0.12	42.0179	0.0347	0.0347	1.73E−18	−0.0027	−0.0027	−2.91E−16	17.9821	17.9821
0.13	41.2063	0.0369	0.0369	5.20E−18	−0.0028	−0.0028	3.17E−17	18.7937	18.7937
0.14	40.4104	0.0392	0.0392	−5.20E−18	−0.0030	−0.0030	−1.80E−16	19.5896	19.5896
0.15	39.6298	0.0416	0.0416	−5.20E−18	−0.0031	−0.0031	−1.28E−16	20.3702	20.3702

This table adopts the constellation of parameters used in Thakoor et al. (2012, Table 7), that is we consider a 5-year put option contract on a 10-year zero-coupon bond with face value \$100 for different levels of the interest rate (r) and an exercise price of \$60. The CIR parameters are: $\kappa = 0.50$, $\theta = 0.08$, $\sigma = 0.10$, and $\lambda = 0$. The required noncentral chi-square distribution function has been computed via the Benton and Krishnamoorthy (2003) algorithm. The corresponding probability density function has been computed using the built-in function ncx2pdf available in *Matlab*.

Table 6. American-style put options on zero-coupon bonds under the CIR model.

							American puts	
κ	θ	σ	$T-t$	$s-t$	K	$Z(r,t,s)$	$n=2$	$n=100$
0.4	0.08	0.10	5.00	10.00	60	49.0169	10.9831	10.9831
0.5	0.08	0.10	5.00	10.00	60	48.1647	11.8353	11.8353
0.6	0.08	0.10	5.00	10.00	60	47.5871	12.4129	12.4129
0.5	0.06	0.10	5.00	10.00	60	56.4233	3.5767	3.5767
0.5	0.07	0.10	5.00	10.00	60	52.1307	7.8693	7.8693
0.5	0.09	0.10	5.00	10.00	60	44.5005	15.4995	15.4995
0.5	0.08	0.15	5.00	10.00	60	48.7285	11.2715	11.2715
0.5	0.08	0.20	5.00	10.00	60	49.4724	10.5276	10.5276
0.5	0.08	0.25	5.00	10.00	60	50.3605	9.6395	9.6395
0.5	0.08	0.10	4.75	9.75	60	49.1168	10.8832	10.8832
0.5	0.08	0.10	4.50	9.50	60	50.0874	9.9126	9.9126
0.5	0.08	0.10	4.25	9.25	60	51.0768	8.9232	8.9232
0.5	0.08	0.10	5.00	10.00	70	48.1647	21.8353	21.8353
0.5	0.08	0.10	5.00	10.00	80	48.1647	31.8353	31.8353
0.5	0.08	0.10	5.00	10.00	90	48.1647	41.8353	41.8353

This table adopts the base case parameters considered in Table 5 and with $r_t = 0.05$. The CIR parameters κ, θ, and σ, the time to maturity of the put option ($T-t$), the time to maturity of the underlying bond ($s-t$), and the strike price of the put (K) are changed as shown in columns 1-6, respectivelly. The required noncentral chi-square distribution function has been computed via the Benton and Krishnamoorthy (2003) algorithm. The corresponding probability density function has been computed using the built-in function ncx2pdf available in *Matlab*.

Regarding the American-style puts, we observe that the use of only two time-steps (i.e., $n = 2$ in the SHP scheme) allow us to obtain the same price that is determined when using $n = 100$. This implies that the static hedge portfolio replicating the American-style put requires only two European-style put options—at least for this combination of parameters—to hedge the American-style put. Finally, we note that, when $r_t = 0.08$, we are able to reproduce exactly the same put price (14.5727) reported in Thakoor et al. (2012, Table 7) when using both the Crank-Nicolson and the Jain's high-order compact schemes. In summary, the SHP approach can be viewed as a viable alternative to accurately and efficiently price American-style zero-coupon bond options under the CIR model.

To further test the robustness of the proposed SHP scheme, Table 6 reports some additional numerical results by changing the parameters considered in Table 5. We observe, again, that the use of only two time-steps (i.e., $n = 2$ in the SHP scheme) allow us to obtain the same price that is determined when using $n = 100$. As already discussed in Cox et al. (1985) and Longstaff (1993), most of the comparative statics are indeterminate since changes in the interest rate parameters have complex effects on the relative values of bonds and options with different maturities. We note, however, that the observation that bond put prices can be decreasing functions of volatility is consistent with the counter effects arguments explained in Longstaff (1993, Pages 37–38).

As expected, there might be some configurations of parameters where more time-steps are required. For instance, in Figure 1 we use the SHP method with $n = 8$ and the following base case parameters borrowed from Yang (2004): a 1-year American-style put option on a 5-year zero-coupon bond with face value \$100 for different levels of the interest rate (r) and an exercise price of \$70. The CIR parameters are: $\kappa = 0.40$, $\theta = 0.08$, $\sigma = 0.20$, and $\lambda = 0$. Then, we perform some numerical experiments by changing σ, t, κ, and θ. The graphs displayed in Figure 1 reveal the patterns that are expected under the CIR interest rate model and that are similar to the ones reported in Yang (2004, Figures 1 and 2). Hence, these additional results reported in Table 6 and Figure 1 corroborate the previous conclusion that the SHP approach is suitable for valuing American-style zero-coupon bond options under the CIR model and that a small number of time-steps in the SHP scheme is generally sufficient to obtain accurate values.

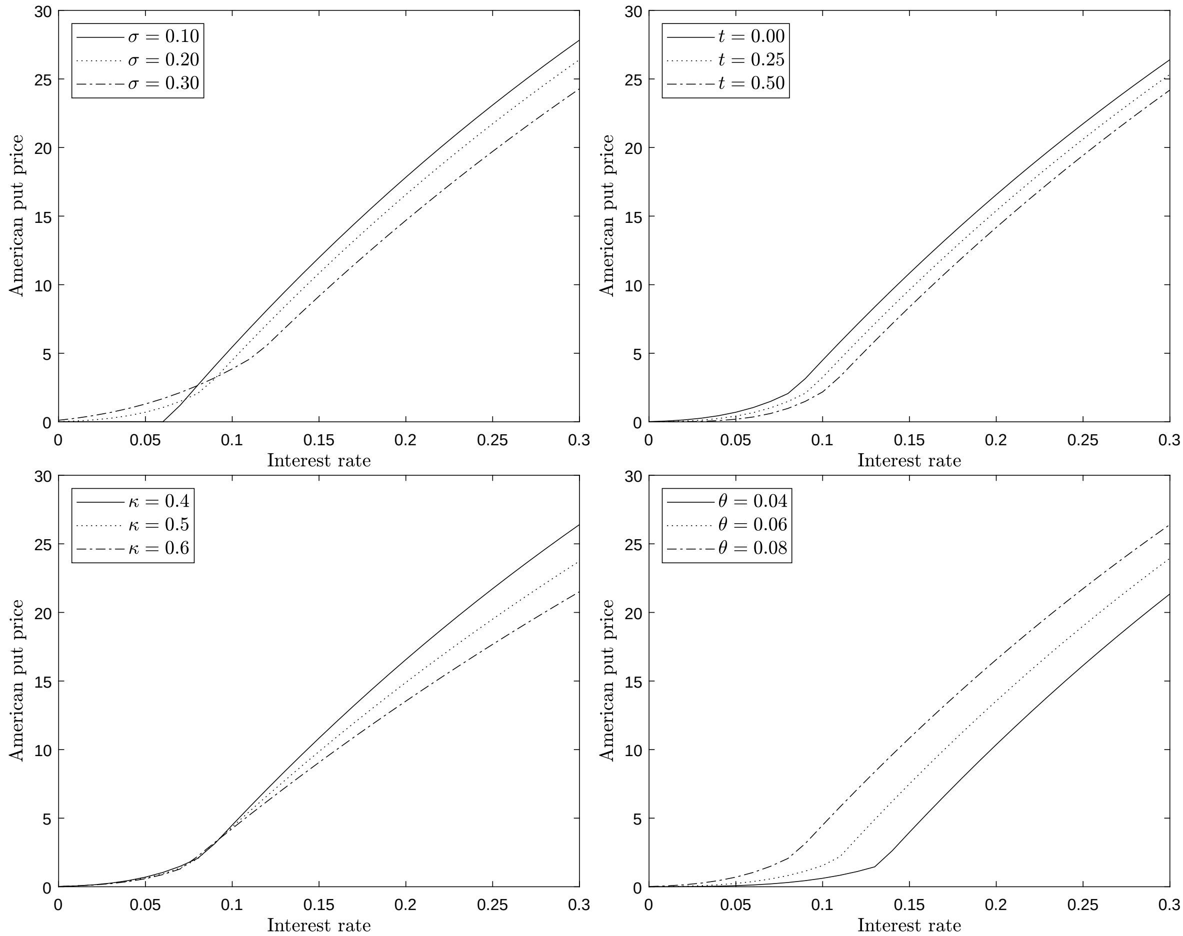

Figure 1. This figure plots 1-year American-style put options on a 5-year zero-coupon bond with face value \$100 for different levels of the interest rate (r) and an exercise price of \$70 borrowed from Yang (2004), using the proposed SHP method with $n = 8$. CIR base case parameters: $\kappa = 0.40$, $\theta = 0.08$, $\sigma = 0.20$, and $\lambda = 0$.

5. Valuation and comparative statics of sinking-fund bonds

Bonds are said to have embedded sinking-fund provisions when the issuer is required to retire portions of the bond issue before maturity, according to a pre-specified amortization schedule. The *delivery option* associated to this clause allows the issuer to retire the portions of the issue either by (i) calling the bonds by lottery at a pre-determinate value, usually at par, or (ii) buying back the bonds at the prevailing market value.

Bacinello et al. (1996) provide an elegant framework for analyzing the delivery option embedded in the sinking-fund bond provision (with only one sinking-fund date before maturity) under the one-dimensional stochastic term structure interest rate models of Vasicek (1977) and Cox et al. (1985). Bacinello et al. (1996) were able to analyze the comparative statics properties of the sinking-fund bond in the Vasicek (1977) framework analytically, but they use a numerical approach for the Cox et al. (1985) model. Thus, the main purpose of this section is to extend the Bacinello et al. (1996) approach by analyzing, in closed-form, the comparative statics properties of a default-free sinking-fund bond in the CIR framework.

Following Bacinello et al. (1996), a sinking-fund bond is characterized by a coupon rate i_c and an amortization schedule $\{(t_j, C_j)\}$, where $C_j > 0$ is the principal that the issuer is required to retire at time t_j. We also assume that $j = 1, 2$ and, without loss of generality, $C_1 + C_2 = 1$, i.e., the sinking-fund bond is issued with a normalized principal, retired in two dates only. Letting t_0 denote the time of issuance of the bond, its coupon payments, I_j, are then assumed to be given by $I_1 = (1 + i_c)^{(t_1 - t_0)} - 1$, and $I_2 = C_2[(1 + i_c)^{(t_2 - t_1)} - 1]$. At time t_1 the issuer has the (delivery) option to retire the fraction C_1 of the principal either by calling it by lottery at par value, or by buying it back at the market value.

Bacinello et al. (1996, Proposition 2.1) show that the time-t price of the sinking-fund bond, $B^{sf}(r, t)$, can be expressed either in terms of the corresponding serial bond and a bond put option, or in terms of the corresponding coupon bond and a bond call option, that is

$$B^{sf}(r,t) = B^{s}(r,t) - C_1(1+i_c)^{(t_2-t_1)} v^{zc}\left(r, t, t_1, t_2, (1+i_c)^{-(t_2-t_1)}; -1\right), \quad (48)$$

or

$$B^{sf}(r,t) = B^{cb}(r,t) - C_1(1+i_c)^{(t_2-t_1)} v^{zc}\left(r, t, t_1, t_2, (1+i_c)^{-(t_2-t_1)}; 1\right), \quad (49)$$

where $B^{s}(r, t)$ and $B^{cb}(r, t)$ represent, respectively, the time-t price of the corresponding serial and coupon bonds as given by Bacinello et al. (1996, Expressions 2.2 and 2.3).

Let us now assume that $t_2 - t_1 = t_1 - t_0 = 1$. Following the same line of reasoning applied by Bacinello et al. (1996) for the Vasicek (1977) framework, we substitute the relations given by Bacinello et al. (1996, Expressions 2.1 and 2.3) and the bond option pricing formula (7), with $\alpha = 1$, in expression (49). We then obtain, for $t < t_1$,

$$B^{sf}(r,t) = Z(r,t,t_1)\left[i_c + C_1 Q[x_2(.); a, b_2(.); 1]\right] + (1+i_c) Z(r,t,t_2)\left[1 - C_1 Q[x_1(.); a, b_1(.); 1]\right], \quad (50)$$

with $x_1(.)$ and $x_2(.)$ defined as in equations (8) and (9), but with $K = (1 + i_c)^{-1}$ in expression (15). Thus, the sinking-fund bond is shown to depend explicitly on the fraction C_1 of outstanding capital to be retired at t_1, the coupon rate i_c, the spot rate r prevailing on the market, and the CIR parameters κ, θ, σ, and λ. We are now able to extend the analytical results provided by Bacinello et al. (1996) under the Vasicek (1977) framework for the CIR model case.

The sinking-fund bond under the CIR model is an increasing function of the coupon rate. To establish this fact, take the derivative of (50) with respect to i_c, and observe that the relation

$$\begin{aligned} & Z(r,t,t_2) p(x_1(.); a, b_1(.))(\phi(t,t_1) + \psi + B(t_1,t_2)) \\ = \; & Z(r,t,t_1) p(x_2(.); a, b_2(.))(\phi(t,t_1) + \psi)(1+i_c)^{-1} \end{aligned} \quad (51)$$

holds as an identity, so that, after some algebraic manipulations, we have

$$\frac{\partial B^{sf}(.)}{\partial i_c} = Z(r,t,t_1) + Z(r,t,t_2)\left(1 - C_1 Q[x_1(.); a, b_1(.); 1]\right) > 0, \quad (52)$$

where the strict positivity follows from the fact that, by assumption, $0 < C_1 < 1$. Considering now the premiums $B^{cb}(.) - B^{sf}(.)$ and $B^{s}(.) - B^{sf}(.)$ of the corresponding coupon and serial bonds over the sinking-fund bond, and using respectively expressions (49) and (48), coupled with $t_2 - t_1 = t_1 - t_0 = 1$, we obtain

$$\begin{aligned}&\frac{\partial(B^{cb}(.) - B^{sf}(.))}{\partial i_c}\\ =\;& -\left(\frac{K}{v^{zc}(.;1)}\frac{\partial v^{zc}(.;1)}{\partial K} - 1\right)C_1 v^{zc}(.;1) = C_1 Z(r,t,t_2) Q[x_1(.);a,b_1(.);1] > 0,\end{aligned} \tag{53}$$

and

$$\begin{aligned}&\frac{\partial(B^{s}(.) - B^{sf}(.))}{\partial i_c}\\ =\;& -\left(\frac{K}{v^{zc}(.;-1)}\frac{\partial v^{zc}(.;-1)}{\partial K} - 1\right)C_1 v^{zc}(.;-1) = -C_1 Z(r,t,t_2) Q[x_1(.);a,b_1(.);-1] < 0,\end{aligned} \tag{54}$$

so that the higher the coupon rate, the larger is the premium demanded by the corresponding coupon bond over the sinking-fund bond, and the smaller is the premium determined by the corresponding serial bond over the sinking-fund bond. Note that the sign of the above derivatives depends entirely on the elasticity of the option prices to the strike price, in particular on the fact that such elasticity is negative for the call and exceeds 1 for the put.

We can also explicitly analyze the comparative statics properties of the sinking-fund bond with respect to the spot rate r (rho) and time t (theta). The first sensitivity measure is given by

$$\begin{aligned}\rho_B^{sf} :=\;& \frac{\partial B^{sf}(.)}{\partial r}\\ =\;& \frac{\partial Z(r,t,t_1)}{\partial r}(i_c + C_1 Q[x_2(.);a,b_2(.);1]) + \frac{\partial Z(r,t,t_2)}{\partial r}(1+i_c)(1 - C_1 Q[x_1(.);a,b_1(.);1])\\ & + C_1(1+i_c) Z(r,t,t_2) p(x_1(.);a+2,b_1(.)) \frac{2\phi^2(t,t_1)e^{\gamma(t-t_1)}}{\phi(t,t_1)+\psi+B(t_1,t_2)}\\ & - C_1 Z(r,t,t_1) p(x_2(.);a+2,b_2(.)) \frac{2\phi^2(t,t_1)e^{\gamma(t-t_1)}}{\phi(t,t_1)+\psi},\end{aligned} \tag{55}$$

with $\partial Z(r,t,t_i)/\partial r$, for $i = t_1, t_2$, given by equation (A.1). The effect on the premiums $B^{cb}(.) - B^{sf}(.)$ and $B^{s}(.) - B^{sf}(.)$ of an infinitesimal change in the spot interest rate r can be stated as

$$\frac{\partial(B^{cb}(.) - B^{sf}(.))}{\partial r} = C_1(1+i_c)\frac{\partial v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};1)}{\partial r}, \tag{56}$$

$$\frac{\partial(B^{s}(.) - B^{sf}(.))}{\partial r} = C_1(1+i_c)\frac{\partial v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};-1)}{\partial r}, \tag{57}$$

where $\partial v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};\alpha)/\partial r$ is given by expression (21), with $K = (1+i_c)^{-1}$.

The effect on $B^{sf}(.)$ of an infinitesimal change in t can be obtained explicitly as

$$\theta_B^{sf} := \frac{\partial B^{sf}(.)}{\partial t}$$

$$= \frac{\partial Z(r,t,t_1)}{\partial t}\Big(i_c + C_1 Q[x_2(.);a,b_2(.);1]\Big) + \frac{\partial Z(r,t,t_2)}{\partial t}(1+i_c)\Big(1 - C_1 Q[x_1(.);a,b_1(.);1]\Big)$$
$$-C_1(1+i_c)Z(r,t,t_2)\Big(p(x_1(.);a,b_1(.))\xi - p(x_1(.);a+2,b_1(.))\varrho_1\Big)$$
$$+C_1 Z(r,t,t_1)\Big(p(x_2(.);a,b_2(.))\xi - p(x_2(.);a+2,b_2(.))\varrho_2\Big), \tag{58}$$

where $\partial Z(r,t,t_i)/\partial t$, with $i = t_1, t_2$, is given by equation (C.3). As for the influence of the parameter t on the premiums $B^{cb}(.) - B^{sf}(.)$ and $B^{s}(.) - B^{sf}(.)$, we have

$$\frac{\partial(B^{cb}(.) - B^{sf}(.))}{\partial t} = C_1(1+i_c)\frac{\partial v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};1)}{\partial t}, \tag{59}$$
$$\frac{\partial(B^{s}(.) - B^{sf}(.))}{\partial t} = C_1(1+i_c)\frac{\partial v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};-1)}{\partial t}, \tag{60}$$

where $\partial v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};\alpha)/\partial t$ is given by (25), with $K = (1+i_c)^{-1}$.

Now we want to prove a result that compares the stochastic durations of the sinking-fund bond with those of the corresponding serial and coupon bonds in the CIR model. Following Cox et al. (1979), the relative basis risk of a zero-coupon bond (under the CIR model), with maturity $\tau := s - t$, is given by $g(\tau) = 2(e^{\gamma\tau} - 1)/((\kappa + \lambda + \gamma)(e^{\gamma\tau} - 1) + 2\gamma) = B(t,s) = B(\tau)$, a function that is strictly increasing ($\partial B(\tau)/\partial\tau > 0$) and continuous on all positive reals, with the inverse function given by $g^{-1}(\tau) = (1/\gamma)\ln(1 - 2\gamma\tau/((\kappa+\lambda+\gamma)\tau - 2))$, and defined on the interval $]0, 2/(\kappa+\lambda+\tau)[$. Moreover, the stochastic duration of any interest rate sensitive instrument with price $f(r,t)$ is given by

$$D^{f} = g^{-1}(x), \tag{61}$$

where $x = -(\partial f(r,t)/\partial r)/f(r,t)$ is the basis risk of f. Next proposition explicitly relates the stochastic durations of the sinking-fund, corresponding coupon and corresponding serial bonds under the CIR framework, thus extending the analytical results provided by Bacinello et al. (1996, Proposition 4.1), but for the Vasicek (1977) model.

Proposition 9. *For any set of parameters, the stochastic durations $D^{sf}(r,t)$, $D^{cb}(r,t)$, and $D^{s}(r,t)$ of the sinking-fund, corresponding coupon and corresponding serial bonds under the CIR model satisfy the relation*

$$D^{s}(r,t) < D^{sf}(r,t) < D^{cb}(r,t). \tag{62}$$

Proof. Please see Appendix I.■

Using the same set of parameters as in Bacinello et al. (1996), Figure 2 highlights that the stochastic duration of the sinking-fund bond is between the stochastic duration of the corresponding serial and coupon bonds. While this issue has been shown already by Bacinello et al. (1996, Figure 13) through numerical differentiation, we have now established this property analytically via Proposition 9 using the aforementioned novel closed-form solutions for the CIR Greeks.

6. Conclusions

In this paper, we derive closed-form expressions for determining sensitivity measures of pure discount and coupon-paying bond options under the CIR framework, which are shown to be accurate,

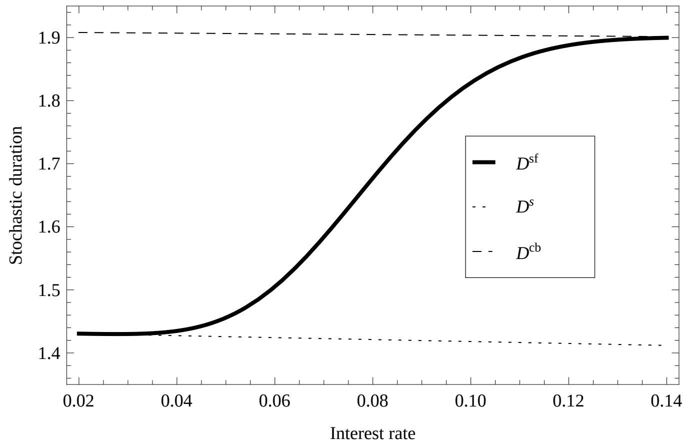

Figure 2. Stochastic duration of a sinking-fund bond under the CIR framework, for the same set of parameters as in Bacinello et al. (1996).

easy to implement, and computationally very efficient. The proposed hedge ratio allow us to evaluate American-style options on zero-coupon bonds through the static hedging approach. Moreover, we offer closed-form tractable expressions to analyze the comparative statics properties of a sinking-fund bond under the same interest rate dynamics setting.

Acknowledgments

We thank two anonymous referees for their valuable comments and suggestions. Manuela Larguinho and Carlos A. Braumann belong to the research center CIMA (Centro de Investigação em Matemática e Aplicações, Instituto de Investigação e Formação Avançada, Universidade de Évora), supported by FCT (Fundação para a Ciência e a Tecnologia, Portugal), project UID/04674/2020. José Carlos Dias belongs to the Business Research Unit (BRU-IUL) and acknowledges the support provided by FCT [grant number UIDB/00315/2020].

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

Abramowitz M, Stegun IA (1972) *Handbook of Mathematical Functions,* (Dover, New York).

Allegretto W, Lin Y, Yang H (2003) Numerical Pricing of American Put Options on Zero-Coupon Bonds. *Appl Numer Math* 46: 113–134. https://doi.org/10.1016/S0168-9274(03)00034-5

Alvarez LHR (2001) On the Form and Risk-Sensitivity of Zero Coupon Bonds for a Class of Interest Rate Models. *Insur Math Econ* 28: 83–90. https://doi.org/10.1016/S0167-6687(00)00068-8

Bacinello AR, Ortu F, Stucchi P (1996) Valuation of Sinking-Fund Bonds in the Vasicek and CIR Frameworks. *Appl Math Financ* 3: 269–294. https://doi.org/10.1080/13504869600000013

Benton D, Krishnamoorthy K (2003) Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral *t* and the Distribution of the Square of the Sample Multiple Correlation Coefficient. *Comput Stat Data Anal* 43: 249–267. https://doi.org/10.1016/S0167-9473(02)00283-9

Brigo D, Mercurio F (2001) A Deterministic-Shift Extension of Analytically-Tractable and Time-Homogeneous Short-Rate Models. *Financ Stoch* 5: 369–387.

Carr P (2001) Deriving Derivatives of Derivative Securities. *J Comput Financ* 4: 5–29. https://10.1109/CIFER.2000.844609

Chan KC, Karolyi GA, Longstaff FA, et al. (1992) An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. *J Financ* 47: 1209–1227. https://doi.org/10.1111/j.1540-6261.1992.tb04011.x

Chesney M, Elliott RJ, Gibson R (1993) Analytical Solutions for the Pricing of American Bond and Yield Options. *Math Financ* 3: 277–294. https://doi.org/10.1111/j.1467-9965.1993.tb00045.x

Chung SL, Shackleton M (2002) The Binomial Black-Scholes Model and the Greeks. *J Futures Mark* 22: 143–153. https://doi.org/10.1002/fut.2211

Chung SL, Shackleton M (2005) On the Errors and Comparison of Vega Estimation Methods. *J Futures Mark* 25: 21–38. https://doi.org/10.1002/fut.20127

Chung SL, Shih PT (2009) Static Hedging and Pricing American Options. *J Bank Financ* 33: 2140–2149. https://doi.org/10.1016/j.jbankfin.2009.05.016

Chung SL, Shih PT, Tsai WC (2010) A Modified Static Hedging Method for Continuous Barrier Options. *J Futures Mark* 30: 1150–1166. https://doi.org/10.1002/fut.20451

Chung SL, Shih PT, Tsai WC (2013) Static Hedging and Pricing American Knock-In Put Options. *J Bank Financ* 37: 191–205. https://doi.org/10.1016/j.jbankfin.2012.08.019

Chung SL, Hung W, Lee HH, et al. (2011) On the Rate of Convergence of Binomial Greeks. *J Futures Mark* 31: 562–597. https://doi.org/10.1002/fut.20484

Cohen JD (1988) Noncentral Chi-Square: Some Observations on Recurrence. *Ame Stat* 42: 120–122. https://10.1080/00031305.1988.10475540

Cox JC, Ingersoll Jr JE, Ross SA (1979) Duration and the Measurement of Basis Risk. *J Bus* 52: 51–61. http://www.jstor.org/stable/2352663

Cox JC, Ingersoll Jr JE, Ross SA (1985) A Theory of the Term Structure of Interest Rates. *Econometrica* 53: 385–408. https://doi.org/10.1142/9789812701022_0005

Cruz A, Dias JC (2017) The Binomial CEV Model and the Greeks. *J Futures Mark* 37: 90–104. https://doi.org/10.1002/fut.21791

Cruz A, Dias JC (2020) Valuing American-Style Options under the CEV Model: An Integral Representation Based Method. *Rev Deri Res* 23: 63–83. https://doi.org/10.1007/s11147-019-09157-w

Deng G (2015) Pricing American Put Option on Zero-Coupon Bond in a Jump-Extended CIR Model. *Commun Nonlinear Sci* 22: 186–196. https://doi.org/10.1016/j.cnsns.2014.10.003

Dias JC, Nunes JPV, Cruz A (2020) A Note on Options and Bubbles under the CEV Model: Implications for Pricing and Hedging. *Rev Deriv Res* 23: 249–272. https://doi.org/10.1007/s11147-019-09164-x

Dias JC, Nunes JPV, Ruas JP (2015) Pricing and Static Hedging of European-Style Double Barrier Options under the Jump to Default Extended CEV Model. *Quant Financ* 15: 1995–2010. https://doi.org/10.1080/14697688.2014.971049

Feller W (1951) Two Singular Diffusion Problems. *Annal Math* 54: 173–182. https://doi.org/10.2307/1969318

Guo JH, Chang LF (2020) Repeated Richardson Extrapolation and Static Hedging of Barrier Options under the CEV Model. *J Futures Mark* 40: 974–988. https://doi.org/10.1002/fut.22100

Hull J, White A (1990) Valuing Derivative Securities Using the Explicit Finite Difference Method. *J Financ Quant Anal* 25: 87–100. https://doi.org/10.2307/2330889

Jamshidian F (1989) An Exact Bond Option Formula. *J Financ* 44: 205–209. https://doi.org/10.1111/j.1540-6261.1989.tb02413.x

Jamshidian F (1995) A Simple Class of Square-Root Interest-Rate Models. *Appl Math Financ* 2: 61–72. https://doi.org/10.1080/13504869500000004

Johnson NL, Kotz S, Balakrishnan N (1995) *Continuous Univariate Distributions*, Vol. 2, 2nd ed. (John Wiley & Sons, New York).

Larguinho M, Dias JC, Braumann CA (2013) On the Computation of Option Prices and Greeks under the CEV Model. *Quant Financ* 13: 907–917. https://doi.org/10.1080/14697688.2013.765958

Longstaff FA (1993) The Valuation of Options on Coupon Bonds. *J Bank Financ* 17: 27–42. https://doi.org/10.1016/0378-4266(93)90078-R

Longstaff FA, Schwartz ES (2001) Valuing American Options by Simulation: A Simple Least-Squares Approach. *Rev Financ Stud* 14: 113–147. https://doi.org/10.1093/rfs/14.1.113

Maghsoodi Y (1996) Solution of the Extended CIR Term Structure and Bond Option Valuation. *Math Financ* 6: 89–109. https://doi.org/10.1111/j.1467-9965.1996.tb00113.x

McKean Jr HP (1965) Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem of Mathematical Economics. *Ind Manage Rev* 6: 32–39.

Najafi AR, Mehrdoust F, Shirinpour S (2018) Pricing American Put Option On Zero-Coupon Bond Under Fractional CIR Model With Transaction Cost. *Commun Stat Simulation Comput* 47: 864–870. https://doi.org/10.1080/03610918.2017.1295153

Nawalkha SK, Beliaeva NA (2007) Efficient Trees for CIR and CEV Short Rate Models. *J Altern Invest* 10: 71–90. https://doi.org/10.3905/jai.2007.688995

Nelson DB, Ramaswamy K (1990) Simple Binomial Processes as Diffusion Approximations in Financial Models. *Rev Financ Stud* 3: 393–430. https://doi.org/10.1093/rfs/3.3.393

Nunes JPV, Ruas JP, Dias JC (2015) Pricing and Static Hedging of American-Style Knock-in Options on Defaultable Stocks. *J Bank Financ* 58: 343–360. https://doi.org/10.1016/j.jbankfin.2015.05.003

Nunes JPV, Ruas JP, Dias JC (2020) Early Exercise Boundaries for American-Style Knock-Out Options, *Eur J Oper Res* 285: 753–766. https://doi.org/10.1016/j.ejor.2020.02.006

Pelsser A, Vorst TC (1994) The Binomial Model and the Greeks. *J Deriv* 1: 45–49. https://doi.org/10.3905/jod.1994.407888

Peng Q, Henry S (2018) On the Distribution of Extended CIR Model. *Stat Pro Lett* 142: 23–29. https://doi.org/10.1016/j.spl.2018.06.011

Ruas JP, Dias JC, Nunes JPV (2013) Pricing and Static Hedging of American Options under the Jump to Default Extended CEV Model. *J Bank Financ* 37: 4059–4072. https://doi.org/10.1016/j.jbankfin.2013.07.019

Sankaran M (1963) Approximations to the Non-Central Chi-Square Distribution. *Biometrika* 50: 199–204. https://doi.org/10.2307/2333761

Shaw W (1998) *Modelling Financial Derivatives with Mathematica* (Cambridge University Press, Cambridge, UK).

ShuJi L, ShengHong L (2006) Pricing American Interest Rate Option on Zero-Coupon Bond Numerically. *Appl Math Comput* 175: 834–850. https://doi.org/10.1016/J.AMC.2005.08.008

Thakoor N, Tangman Y, Bhuruth M (2012) Numerical Pricing of Financial Derivatives Using Jain's High-Order Compact Scheme. *Math Sci* 6: 1–16. https://doi.org/10.1186/2251-7456-6-72

Tian Y (1992) A Simplified Binomial Approach to the Pricing of Interest-Rate Contingent Claims. *J Financ Eng* 1: 14–37.

Tian Y (1994) A Reexamination of Lattice Procedures for Interest Rate Contingent Claims. *Adv Futures Options Res* 7: 87–111. https://ssrn.com/abstract=5877

Vasicek O (1977) An Equilibrium Characterization of the Term Structure. *J Financ Econ* 5: 177–188. https://doi.org/10.1016/0304-405X(77)90016-2

Wei JZ (1997) A Simple Approach to Bond Option Pricing. *J Futures Mark* 17: 131–160.

Yang H (2004) American Put Options on Zero-Coupon Bonds and a Parabolic Free Boundary Problem. *Int J Numer Anal Model* 1: 203–215.

Zhou HJ, Yiu KFC, Li LK (2011) Evaluating American Put Options on Zero-Coupon Bonds by a Penalty Method. *J Comput Appl Math* 235: 3921–3931. https://doi.org/10.1016/j.cam.2011.01.038

A. Proof of Proposition 1

Let us first note that:

$$\frac{\partial Z(r,t,j)}{\partial r} = -B(t,j)Z(r,t,j), \quad j \in \{T,s\}, \tag{A.1}$$

$$\frac{\partial x_1(t,T,s,K)}{\partial r} = \frac{\partial x_2(t,T,s,K)}{\partial r} = 0, \tag{A.2}$$

$$\frac{\partial b_1(r,t,T,s)}{\partial r} = \frac{b_1(r,t,T,s)}{r}, \tag{A.3}$$

and

$$\frac{\partial b_2(r,t,T)}{\partial r} = \frac{b_2(r,t,T)}{r}. \tag{A.4}$$

The rho for a zero-coupon bond option is given by

$$\begin{aligned}\rho_v^{zc}(.) := \frac{\partial v^{zc}(.)}{\partial r} &= \alpha\frac{\partial Z(r,t,s)}{\partial r}Q[x_1(.);a,b_1(.);\alpha] + \alpha Z(r,t,s)\frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial r} \\ &\quad -\alpha K\left[\frac{\partial Z(r,t,T)}{\partial r}Q[x_2(.);a,b_2(.);\alpha] + Z(r,t,T)\frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial r}\right].\end{aligned} \tag{A.5}$$

Using expressions (17), (18), (A.2), (A.3), and (A.4) we are able to obtain the following partial derivatives:

$$\frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial r} = \frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial b_1(.)}\frac{\partial b_1(.)}{\partial r} = -\alpha\frac{b_1(.)}{r}p(x_1(.);a+2,b_1(.)), \tag{A.6}$$

and

$$\frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial r} = \frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial b_2(.)}\frac{\partial b_2(.)}{\partial r} = -\alpha\frac{b_2(.)}{r}p(x_2(.);a+2,b_2(.)). \tag{A.7}$$

Finally, substituting expressions (A.1), (A.6), and (A.7) into (A.5) yields expression (21).■

B. Proof of Proposition 2

We first recall that $\Gamma_{v,r}^{zc}(.) = \partial\rho_v^{zc}(.)/\partial r$. Hence, differentiating (21) w.r.t. r and using (20), (A.1), (A.2), (A.3), (A.4), (A.6), and (A.7), expression (23) is finally obtained after straightforward calculations.■

C. Proof of Proposition 3

Let us first note that:

$$\frac{\partial A(t,T)}{\partial t} = \frac{\kappa\theta}{\sigma^2}\frac{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)(2\gamma-(\kappa+\lambda+\gamma))}{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}A(t,T), \tag{C.1}$$

$$\frac{\partial B(t,T)}{\partial t} = -\frac{4\gamma^2 e^{\gamma(T-t)}}{\left[(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma\right]^2}, \tag{C.2}$$

and

$$\frac{\partial Z(r,t,j)}{\partial t} = Z(r,t,j)\left[\frac{1}{A(t,j)}\frac{\partial A(t,j)}{\partial t} - r\frac{\partial B(t,j)}{\partial t}\right] = Z(r,t,j)\,\zeta_j, \tag{C.3}$$

with

$$\zeta_j = \frac{1}{A(t,j)}\frac{\partial A(t,j)}{\partial t} - r\frac{\partial B(t,j)}{\partial t}, \tag{C.4}$$

for $j \in \{T, s\}$.

Let us also consider the following auxiliary functions:

$$\xi = \frac{\partial x_1\,(t,T,s,K)}{\partial t} = \frac{\partial x_2\,(t,T,s,K)}{\partial t} = \frac{4r^*\gamma^2 e^{\gamma(T-t)}}{\sigma^2(e^{\gamma(T-t)}-1)^2}, \tag{C.5}$$

$$\varrho_1 = \frac{\partial b_1\,(r,t,T,s)}{\partial t} = b_1\,(r,t,T,s)\,\gamma\frac{(\phi(t,T)+\psi+B(T,s)) + (\psi+B(T,s))e^{\gamma(T-t)}}{(e^{\gamma(T-t)}-1)(\phi(t,T)+\psi+B(T,s))}, \tag{C.6}$$

and

$$\varrho_2 = \frac{\partial b_2\,(r,t,T)}{\partial t} = b_2\,(r,t,T)\,\gamma\frac{(\phi(t,T)+\psi)+\psi e^{\gamma(T-t)}}{(e^{\gamma(T-t)}-1)(\phi(t,T)+\psi)}. \tag{C.7}$$

The theta for a zero-coupon bond option is given by

$$\begin{aligned}\theta_v^{zc}(.) := \frac{\partial v^{zc}(.)}{\partial t} \;&=\; \alpha\frac{\partial Z(r,t,s)}{\partial t}\,Q[x_1(.);a,b_1(.);\alpha] + \alpha Z(r,t,s)\,\frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial t} \\ &\quad -\alpha K\left[\frac{\partial Z(r,t,T)}{\partial t}\,Q[x_2(.);a,b_2(.);\alpha] + Z(r,t,T)\,\frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial t}\right].\end{aligned} \tag{C.8}$$

Using expressions (17), (18), (C.5), (C.6), and (C.7) we are able to compute the following partial derivatives:

$$\begin{aligned}\frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial t} &= \frac{\partial Q[x_1(.);a,b_1(.);\alpha)}{\partial x_1(.)}\,\frac{\partial x_1(.)}{\partial t} + \frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial b_1(.)}\,\frac{\partial b_1(.)}{\partial t} \\ &= \alpha p(x_1(.);a,b_1(.))\xi - \alpha p(x_1(.);a+2,b_1(.))\varrho_1,\end{aligned} \tag{C.9}$$

and

$$\begin{aligned}\frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial t} &= \frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial x_2(.)}\,\frac{\partial x_2(.)}{\partial t} + \frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial b_2(.)}\,\frac{\partial b_2(.)}{\partial t} \\ &= \alpha p(x_2(.);a,b_2(.))\xi - \alpha p(x_2(.);a+2,b_2(.))\varrho_2.\end{aligned} \tag{C.10}$$

Finally, substituting expressions (C.3), (C.9), and (C.10) into (C.8) yields expression (25).■

D. Proof of Proposition 4

Let us first note that:

$$\frac{\partial x_1\,(t,T,s,K)}{\partial K} = -\frac{2(\phi(t,T)+\psi+B(T,s))}{B(T,s)K}, \tag{D.1}$$

$$\frac{\partial x_2\,(t,T,s,K)}{\partial K} = -\frac{2(\phi(t,T)+\psi)}{B(T,s)K}, \tag{D.2}$$

and

$$\frac{\partial b_1\,(r,t,T,s)}{\partial K} = \frac{\partial b_2\,(r,t,T)}{\partial K} = 0. \tag{D.3}$$

The eta for a zero-coupon bond option is given by

$$\begin{aligned}\eta_v^{zc}(.) := \frac{\partial v^{zc}(.)}{\partial K} &= \alpha\, Z(r,t,s)\, \frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial K} \\ &\quad -\alpha Z(r,t,T)\left[Q[x_2(.);a,b_2(.);\alpha] + K\frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial K}\right].\end{aligned} \tag{D.4}$$

Using expressions (17), (18), (D.1), (D.2), and (D.3) we are able to compute the following partial derivatives:

$$\begin{aligned}&\frac{\partial Q[(x_1(.);a,b_1(.);\alpha]}{\partial K} \\ =\;& \frac{\partial Q[x_1(.);a,b_1(.);\alpha]}{\partial x_1(.)}\,\frac{\partial x_1(.)}{\partial K} = -2\alpha\, p(x_1(.);a,b_1(.))\frac{(\phi(t,T)+\psi+B(T,s))}{B(T,s)\,K},\end{aligned} \tag{D.5}$$

and

$$\frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial K} = \frac{\partial Q[x_2(.);a,b_2(.);\alpha]}{\partial x_2(.)}\,\frac{\partial x_2(.)}{\partial K} = -2\alpha\, p(x_2(.);a,b_2(.))\frac{(\phi(t,T)+\psi)}{B(T,s)\,K}. \tag{D.6}$$

Finally, substituting expressions (D.5) and (D.6) into (D.4) yields expression (31).■

E. Proof of Proposition 5

Let us first apply the decomposition technique of Jamshidian (1989) to obtain:

$$\eta_v^{cb}(.) := \frac{\partial v^{cb}(.)}{\partial K} = \sum_{i=1}^{N} a_i \frac{\partial v^{zc}(r,t,T,s_i,K_i;\alpha)}{\partial K}. \tag{E.1}$$

Let us also compute, for an arbitrary fixed i, the expression for $\frac{\partial v^{zc}(r,t,T,s_i,K_i;\alpha)}{\partial K} = \frac{\partial z_i}{\partial K}$, where $z_i = v^{zc}(r,t,T,s_i,K_i;\alpha)$. Note that the change of variable from K to r^{**} (keeping the remaining variables r, t, and T unchanged) is obtained as the implicit solution of $K = \sum_{j=1}^{N} a_j Z(r^{**},T,s_j)$ and so, applying the classical chain rule, one obtains

$$\begin{aligned}&\frac{\partial v^{zc}(r,t,T,s_i,K_i;\alpha)}{\partial K} \\ =\;& \frac{\partial z_i}{\partial K} = \frac{\partial z_i}{\partial r^{**}}\frac{\partial r^{**}}{\partial K} = \frac{\partial z_i}{\partial r^{**}}\frac{1}{\frac{\partial K}{\partial r^{**}}} = \frac{\partial z_i}{\partial r^{**}}\frac{1}{\sum_{j=1}^{N} a_j \frac{\partial Z(r^{**},T,s_j)}{\partial r^{**}}} = \frac{\partial z_i}{\partial r^{**}}\frac{1}{\sum_{j=1}^{N} a_j\left[-B(T,s_j)Z(r^{**},T,s_j)\right]}.\end{aligned} \tag{E.2}$$

Moreover, with a new change of variable from r^{**} to $K_i = Z(r^{**},T,s_i)$ (keeping the remaining variables r, t, and T unchanged) and the application of the chain rule, leads to

$$\frac{\partial z_i}{\partial r^{**}} = \frac{\partial z_i}{\partial K_i}\frac{\partial K_i}{\partial r^{**}}. \tag{E.3}$$

Since

$$\frac{\partial K_i}{\partial r^{**}} = \frac{\partial Z(r^{**}, T, s_i)}{\partial r^{**}} = -B(T, s_i)Z(r^{**}, T, s_i), \tag{E.4}$$

we obtain from (E.3) and (E.4),

$$\begin{aligned}\frac{\partial z_i}{\partial r^{**}} &= \frac{\partial z_i}{\partial K_i}\left[-B(T, s_i)Z(r^{**}, T, s_i)\right] = \frac{\partial v^{zc}(r, t, T, s_i, K_i; \alpha)}{\partial K_i}\left[-B(T, s_i)Z(r^{**}, T, s_i)\right] \\ &= \eta_v^{zc}(r, t, T, s_i, K_i; \alpha)\left[-B(T, s_i)Z(r^{**}, T, s_i)\right]. \end{aligned}\tag{E.5}$$

Therefore, using (E.2) and (E.5),

$$\begin{aligned}&\frac{\partial v^{zc}(r, t, T, s_i, K_i; \alpha)}{\partial K} \\ &= \eta_v^{zc}(r, t, T, s_i, K_i; \alpha)\left[-B(T, s_i)Z(r^{**}, T, s_i)\right] \frac{1}{\sum_{j=1}^{N} a_j \left[-B(T, s_j)Z(r^{**}, T, s_j)\right]}\end{aligned}\tag{E.6}$$

and so, using (E.1) and (E.6), we obtain expression (32).

F. Proof of Proposition 6

Let us first note that:

$$\Gamma_{v,Z}^{zc}(.) := \frac{\partial \Delta_v^{zc}(.)}{\partial Z(r, t, s)} = \frac{\partial \Delta_v^{zc}(.)}{\partial r}\,\frac{\partial r}{\partial Z(r, t, s)}. \tag{F.1}$$

Using Remark 5, we conclude that

$$\frac{\partial r}{\partial Z(r, t, s)} = -\frac{1}{B(t, s)Z(r, t, s)}. \tag{F.2}$$

Moreover,

$$\begin{aligned}\frac{\partial \Delta_v^{zc}(.)}{\partial r} &= -\frac{1}{B(t, s)}\,\frac{\partial}{\partial r}\left[\frac{\rho_v^{zc}(.)}{Z(r, t, s)}\right] \\ &= -\frac{1}{B(t, s)}\left[\frac{\frac{\partial \rho_v^{zc}(.)}{\partial r}\,Z(r, t, s) - \rho_v^{zc}(.)\,\frac{\partial Z(r,t,s)}{\partial r}}{[Z(r, t, s)]^2}\right].\end{aligned}\tag{F.3}$$

Noting that $\partial \rho_v^{zc}(.)/\partial r = \Gamma_{v,r}^{zc}(.)$, then expression (F.3) can be rewritten as

$$\begin{aligned}\frac{\partial \Delta_v^{zc}(.)}{\partial r} &= -\frac{1}{B(t, s)}\left[\frac{\Gamma_{v,r}^{zc}(.)\,Z(r, t, s) + \rho_v^{zc}(.)\,B(t, s)Z(r, t, s)}{[Z(r, t, s)]^2}\right] \\ &= -\frac{\Gamma_{v,r}^{zc}(.) + \rho_v^{zc}(.)\,B(t, s)}{B(t, s)Z(r, t, s)}.\end{aligned}\tag{F.4}$$

Substituting expressions (F.2) and (F.4) into expression (F.1), we obtain

$$\Gamma_{v,Z}^{zc}(.) = \frac{\Gamma_{v,r}^{zc}(.) + \rho_v^{zc}(.)\,B(t, s)}{[B(t, s)Z(r, t, s)]^2}. \tag{F.5}$$

Finally, using expression (33) in equation (F.5) yields equation (35).

G. Proof of Proposition 7

Let us first note that:

$$\Gamma_{v,Z}^{\text{cb}}(.) := \frac{\partial \Delta_v^{\text{cb}}(.)}{\partial P(r,t,s)} = \frac{\partial \Delta_v^{\text{cb}}(.)}{\partial r} \frac{\partial r}{\partial P(r,t,s)}. \tag{G.1}$$

Using Remark 5, we conclude that

$$\frac{\partial r}{\partial P(r,t,s)} = -\frac{1}{\sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i)}. \tag{G.2}$$

Moreover,

$$\begin{aligned} \frac{\partial \Delta_v^{\text{cb}}(.)}{\partial r} &= -\frac{\partial}{\partial r}\left[\frac{\rho_v^{\text{cb}}(.)}{\sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i)}\right] \\ &= -\left[\frac{\frac{\partial \rho_v^{\text{cb}}(.)}{\partial r} \sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i) - \rho_v^{\text{cb}}(.) \sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i)(-B(t,s_i))}{\left[\sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i)\right]^2}\right]. \end{aligned} \tag{G.3}$$

Noting that $\partial \rho_v^{\text{cb}}(.)/\partial r = \Gamma_{v,r}^{\text{cb}}(.)$, then expression (G.3) can be rewritten as

$$\frac{\partial \Delta_v^{\text{cb}}(.)}{\partial r} = -\left[\frac{\Gamma_{v,r}^{\text{cb}}(.) \sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i) + \rho_v^{\text{cb}}(.) \sum_{i=1}^N a_i \left[B(t,s_i)\right]^2 Z(r,t,s_i)}{\left[\sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i)\right]^2}\right]. \tag{G.4}$$

Substituting expressions (G.2) and (G.4) into expression (G.1), we obtain

$$\Gamma_{v,Z}^{\text{cb}}(.) = \frac{\Gamma_{v,r}^{\text{cb}}(.) \sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i) + \rho_v^{\text{cb}}(.) \sum_{i=1}^N a_i \left[B(t,s_i)\right]^2 Z(r,t,s_i)}{\left[\sum_{i=1}^N a_i B(t,s_i) Z(r,t,s_i)\right]^3}. \tag{G.5}$$

Finally, using expression (34) in equation (G.5) yields equation (36).

H. Proof of Proposition 8

Let us first note that:

$$\frac{\partial x_1(t,T,s,K)}{\partial Z} = \frac{\partial x_2(t,T,s,K)}{\partial Z} = 0, \tag{H.1}$$

$$\frac{\partial \bar{b}_1(Z,t,T,s)}{\partial Z} = -\frac{2\phi^2(t,T)\, e^{\gamma(T-t)}}{ZB(t,s)\left[\phi(t,T) + \psi + B(T,s)\right]}, \tag{H.2}$$

and

$$\frac{\partial \bar{b}_2(Z,t,T,s)}{\partial Z} = -\frac{2\phi^2(t,T)\, e^{\gamma(T-t)}}{ZB(t,s)\left[\phi(t,T) + \psi\right]}. \tag{H.3}$$

The delta for a zero-coupon bond option is computed as

$$\Delta_{\bar{v}}^{\text{zc}}(.) := \frac{\partial \bar{v}^{\text{zc}}(.)}{\partial Z} = \alpha Q[x_1(.); a, \bar{b}_1(.); \alpha] + \alpha Z \frac{\partial Q[x_1(.); a, \bar{b}_1(.); \alpha]}{\partial Z}$$

$$-\alpha K\frac{A(t,T)}{A(t,s)}\frac{B(t,T)}{B(t,s)}\left(\frac{Z}{A(t,s)}\right)^{\frac{B(t,T)}{B(t,s)}-1}Q[x_2(.);a,\bar{b}_2(.);\alpha]$$
$$-\alpha KA(t,T)\left(\frac{Z}{A(t,s)}\right)^{\frac{B(t,T)}{B(t,s)}}\frac{\partial Q[x_2(.);a,\bar{b}_2(.);\alpha]}{\partial Z}. \qquad \text{(H.4)}$$

Using expressions (17), (18), (H.1), (H.2), and (H.3) we are able to obtain the following partial derivatives:

$$\begin{aligned}\frac{\partial Q[x_1(.);a,\bar{b}_1(.);\alpha]}{\partial Z} &= \frac{\partial Q[x_1(.);a,\bar{b}_1(.);\alpha]}{\partial \bar{b}_1(.)}\frac{\partial \bar{b}_1(.)}{\partial Z} \\ &= 2\alpha p(x_1(.);a+2,\bar{b}_1(.))\frac{\phi^2(t,T)\,e^{\gamma(T-t)}}{ZB(t,s)\left[\phi(t,T)+\psi+B(T,s)\right]}, \qquad \text{(H.5)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Q[x_2(.);a,\bar{b}_2(.);\alpha]}{\partial Z} &= \frac{\partial Q[x_2(.);a,\bar{b}_2(.);\alpha]}{\partial \bar{b}_2(.)}\frac{\partial \bar{b}_2(.)}{\partial Z} \\ &= 2\alpha p(x_2(.);a+2,\bar{b}_2(.))\frac{\phi^2(t,T)\,e^{\gamma(T-t)}}{ZB(t,s)\left[\phi(t,T)+\psi\right]}. \qquad \text{(H.6)}\end{aligned}$$

Finally, substituting expressions (H.5) and (H.6) into (H.4) yields expression (41).■

I. Proof of Proposition 9

To verify the first inequality, we use expressions (48) and (55), along with the fact that $g^{-1}(x)$ is (positive and) increasing, to observe that this inequality becomes

$$\frac{1}{\gamma}\ln\left(1-\frac{2\gamma\rho_B^s}{(\kappa+\lambda+\gamma)\rho_B^s+2B^s(.)}\right)<\frac{1}{\gamma}\ln\left(1-\frac{2\gamma\rho_B^{sf}}{(\kappa+\lambda+\gamma)\rho_B^{sf}+2B^{sf}(.)}\right),$$

which is equivalent to $v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};-1)\rho_B^{sf}-B^s(r,t)\rho_p^{zc}<0$. To check the second inequality, we use now expression (49) and then follow the same reasoning to obtain $B^{cb}(r,t)\rho_c^{zc}-v^{zc}(r,t,t_1,t_2,(1+i_c)^{-1};1)\rho_B^{sf}<0$, which concludes the proof.■