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Research article

On radial extensions of the degenerate gamma and Laplace transforms

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Abstract: This paper develops radial extensions of the degenerate gamma and Laplace transforms on the Euclidean plane and, more generally, on $\mathbb{R}^n$, $n \geq 2$. Using spherical coordinates, the transforms are reduced to one-dimensional, degenerate transforms involving the spherical mean and the Jacobian r^{n-1}. This yields the convergence domain, beta-function representation, meromorphic continuation, recurrence relation, special values, and classical limit of the radial degenerate gamma integral. A logarithmic change of variable relates the radial transform to the ordinary Laplace transform; this gives injectivity, up to equality almost everywhere, on an admissible radial class, together with an explicit inversion formula. On unrestricted nonradial classes, the transform is not injective because it retains only the spherical mean. We also derive a general formula for transforms of radial-profile derivatives, showing how inverse-power terms and dimension-dependent boundary contributions arise from the radial Jacobian. Finally, in the logarithmic variable, the homogeneous radial transport equation becomes a pure translation, and the transform intertwines its divergence operator with multiplication by the transform parameter, giving an exact transform-domain formulation of the problem.

Keywords: degenerate Laplace transform; radial transform; degenerate gamma function; spherical mean; beta function; radial differentiation; radial transport equation

1. Introduction

We first recall the classical objects that are deformed below. The gamma function and the Laplace transform are

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, \mathrm{d}t \quad (\Re(z) > 0), \qquad \mathcal{L}h(s) = \int_0^\infty e^{-st} h(t)\, \mathrm{d}t,$$

the latter whenever the integral converges; see, for instance, [1, 2]. By $\Gamma(z+1) = z\Gamma(z)$, the gamma function extends meromorphically to $\mathbb{C}$, with simple poles at $0, -1, -2, \ldots$. The degenerate versions considered here replace the kernels e^{-t} and e^{-st} by $(1+\lambda t)^{-1/\lambda}$ and $(1+\lambda t)^{-s/\lambda}$, which reduce to them

as $\lambda \to 0^+$.

For $\lambda > 0$, the degenerate exponential function is defined by

$$e_\lambda^a(t) = (1 + \lambda t)^{a/\lambda} \qquad t \geq 0,$$

and satisfies $e_\lambda^a(t) \to e^{at}$ as $\lambda \to 0^+$. This deformation belongs to the degenerate framework initiated by Carlitz [3, 4]. Kim et al. [5] subsequently introduced the degenerate gamma function and degenerate Laplace transform

$$\Gamma_\lambda(z) = \int_0^\infty (1 + \lambda t)^{-1/\lambda} t^{z-1}\, \mathrm{d}t, \tag{1.1}$$

$$L_\lambda h(s) = \int_0^\infty (1 + \lambda t)^{-s/\lambda} h(t)\, \mathrm{d}t \tag{1.2}$$

whenever the integrals converge. Modified versions and further properties of these objects have been considered in [6, 7]; in particular, the meromorphic continuation of the one-dimensional, degenerate gamma function was studied in [7]. A degenerate Laplace-type transform unifying the degenerate Laplace, Sumudu, and Elzaki transforms was introduced in [8]. For general integral transform theory and tables of classical transforms, we refer to [1, 2].

This paper considers the rotation-invariant counterparts of Eqs (1.1) and (1.2) on Euclidean space. Here, a *radial extension* means that the one-dimensional kernel is evaluated at the Euclidean radius $r = |x|$ and integrated against Lebesgue measure on $\mathbb{R}^n$. The term "radial" refers to the kernel; the input function need not be radial. The resulting Laplace transform depends on a single complex parameter s rather than a vector of coordinatewise transform parameters. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, the radial kernel gives

$$\mathcal{L}_\lambda^{[n]} f(s) = \int_{\mathbb{R}^n} (1 + \lambda|x|)^{-s/\lambda} f(x)\, \mathrm{d}x.$$

The integral is understood in the sense of absolute convergence. In spherical coordinates, the Lebesgue measure contributes the Jacobian r^{n-1}, and angular integration retains only the spherical mean of the input. On their respective domains of absolute convergence, spherical coordinates reduce this transform and the corresponding radial gamma integral to

$$\mathcal{L}_\lambda^{[n]} f(s) = \omega_{n-1} L_\lambda(r^{n-1} \mathcal{M}_n f(r))(s), \qquad \Gamma_\lambda^{[n]}(z) = \omega_{n-1} \Gamma_\lambda(z + n - 1), \tag{1.3}$$

where $\mathcal{M}_n f$ is the spherical mean of f, and

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Consequently, the transform is not injective on unrestricted nonradial classes. For the classical theory of spherical means and radial functions on Euclidean spaces, we refer to [9].

The second identity in Eq (1.3) shows that the beta-function representation, recurrence relation, special values, and poles of the radial gamma function are dimensionally shifted consequences of the one-dimensional theory. We determine the exact convergence strip of its defining integral and distinguish it from the domain of its meromorphic continuation. For the radial Laplace transform, the change of variable

$$t = \frac{1}{\lambda} \log(1 + \lambda r)$$

gives a factorization through the ordinary Laplace transform. This factorization characterizes the null space on an admissible function class, establishes injectivity on radial profiles, and yields an explicit inversion formula for the spherical mean.

The Euclidean Jacobian r^{n-1} changes the operational formulas for profile derivatives. We derive a formula for $\mathcal{L}_\lambda^{[n]}[g^{(m)}]$ in which derivatives of the Jacobian produce transforms of the profiles $g(r)/r^q$. Boundary terms at the origin enter the formula when $m \geq n$. Their coefficients form a triangular combination of

$$g(0), g'(0), \ldots, g^{(m-n)}(0),$$

which gives a criterion for the absence of boundary contributions and allows the corresponding finite jet to be recovered from the transform identities. Its relation to the one-dimensional derivative rule is described in Remark 4.20.

The logarithmic radial variable also induces a forward semiflow with velocity $v_\lambda(r) = 1 + \lambda r$. The associated radial divergence operator is

$$\mathcal{B}_{\lambda,n} g(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \left[(1 + \lambda r) r^{n-1} g(r) \right].$$

Under explicit growth assumptions, the radial transform satisfies

$$\mathcal{L}_\lambda^{[n]}[\mathcal{B}_{\lambda,n} g](s) = s \mathcal{L}_\lambda^{[n]}[g](s).$$

This identity reduces the corresponding inhomogeneous radial transport problem to a first-order equation in the transform domain and gives an explicit representation in the homogeneous zero-inflow case. For the homogeneous problem, the same logarithmic radial variable turns the conservative form of the equation into a pure translation equation, which makes the roles of the radial Jacobian and of the degenerate kernel explicit.

Throughout, the identities inherited from the one-dimensional theory are kept separate from the dimension-dependent terms generated by the radial Jacobian.

Parameters and complex powers. Throughout, $\lambda > 0$, and $n \geq 2$ is an integer. The recurrence involving $\mu = \lambda/(1 - \lambda)$ requires $0 < \lambda < 1$. At $z = \ell \in \mathbb{N}_0$, the radial gamma integral requires $0 < \lambda < 1/(\ell + n - 1)$; outside this range, the finite-product formula is interpreted only through meromorphic continuation, away from its poles. For every positive real base a, we use $a^w = \exp(w \log a)$ with the real logarithm. In particular, this convention applies to r^w for $r > 0$, λ^w, and $(1-\lambda)^w$ in the recurrence range.

Notation. Throughout, Γ_λ and $\Gamma_\lambda^{[n]}$ denote functions defined by absolutely convergent integrals, whereas $\widetilde{\Gamma}_\lambda$ and $\widetilde{\Gamma}_\lambda^{[n]}$ denote their meromorphic continuations. We use L_λ for the one-dimensional Laplace transform and $\mathcal{L}_\lambda^{[n]}$ for the radial transform on $\mathbb{R}^n$. For a measurable profile $g : [0, \infty) \to \mathbb{C}$, let $\mathcal{R}_n g(x) := g(|x|)$. Whenever the integral converges absolutely, the profile notation means

$$\mathcal{L}_\lambda^{[n]}[g](s) := \mathcal{L}_\lambda^{[n]}(\mathcal{R}_n g)(s) = \omega_{n-1} \int_0^\infty (1 + \lambda r)^{-s/\lambda} g(r) r^{n-1} \, \mathrm{d}r.$$

Thus, $g^{(m)}$ in a profile transform denotes differentiation with respect to r, not a Cartesian derivative of $\mathcal{R}_n g$. Square brackets $\mathcal{L}_\lambda^{[n]}[g]$ are used exclusively when the argument is a profile on $[0, \infty)$ (so that

$\mathcal{L}_\lambda^{[n]}[g] = \mathcal{L}_\lambda^{[n]} f$ for $f = \mathcal{R}_n g$); for a function f on $\mathbb{R}^n$, we write $\mathcal{L}_\lambda^{[n]} f$. The real part of a complex number is denoted by $\mathfrak{R}$ throughout.

The paper is organized as follows. Section 2 recalls the one-dimensional preliminaries, and Section 3 treats the planar prototype. Section 4 develops the radial theory on $\mathbb{R}^n$: The radial gamma function, the radial Laplace transform and its factorization, and the operational formulas for radial-profile derivatives. Section 5 applies the transform to an associated radial transport problem, and Section 6 concludes the paper.

2. One-dimensional preliminaries

We recall the one-dimensional degenerate gamma and Laplace framework introduced by Kim et al. [5], together with the meromorphic continuation of the degenerate gamma function subsequently developed in [7]. The results are recorded below in the form needed for the radial constructions. Where appropriate, the parameter domains are extended directly from the defining integrals.

Proposition 2.1. *Let $\lambda > 0$. The following facts hold.*

(i) Convergence and meromorphic continuation. The integral defining the degenerate gamma function in Eq (1.1) *converges absolutely if and only if*

$$0 < \mathfrak{R}(w) < \frac{1}{\lambda}. \tag{2.1}$$

In this strip,

$$\Gamma_\lambda(w) = \lambda^{-w} B\left(w, \frac{1}{\lambda} - w\right). \tag{2.2}$$

Using the beta–gamma identity, the right-hand side of Eq (2.2) *extends meromorphically to $\mathbb{C}$ as*

$$\widetilde{\Gamma}_\lambda(w) := \lambda^{-w} \frac{\Gamma(w)\Gamma(1/\lambda - w)}{\Gamma(1/\lambda)}. \tag{2.3}$$

(ii) Recurrence relation. Let $0 < \lambda < 1$, and put

$$\mu = \frac{\lambda}{1-\lambda}.$$

If

$$0 < \mathfrak{R}(w) < \frac{1}{\lambda} - 1, \tag{2.4}$$

then both defining integrals converge, and

$$\Gamma_\lambda(w+1) = \frac{w}{(1-\lambda)^{w+1}} \Gamma_\mu(w). \tag{2.5}$$

The corresponding meromorphic identity is

$$\widetilde{\Gamma}_\lambda(w+1) = \frac{w}{(1-\lambda)^{w+1}} \widetilde{\Gamma}_\mu(w). \tag{2.6}$$

(iii) Degenerate Laplace transform of powers. Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > -1$. If

$$\Re(s) > \lambda(\Re(\alpha) + 1),$$

then

$$L_\lambda(t^\alpha)(s) = \lambda^{-(\alpha+1)} B\left(\alpha + 1, \frac{s}{\lambda} - \alpha - 1\right). \tag{2.7}$$

For real $\alpha > -1$ and real s, this is equivalent to the gamma form $L_\lambda(t^\alpha)(s) = s^{-\alpha-1}\Gamma_{\lambda/s}(\alpha + 1)$ in [5, Eq (3.19)]; for integer exponents, the beta form appears in [5, Eq (3.16)]. The stated complex-parameter form follows directly from the same defining integral.

Proof. (i) The gamma integrand has orders

$$t^{\Re(w)-1} \quad \text{as } t \to 0^+, \qquad t^{\Re(w)-1-1/\lambda} \quad \text{as } t \to \infty.$$

This proves Eq (2.1). The substitution $y = \lambda t$ gives Eq (2.2), and the beta–gamma identity gives the meromorphic extension in Eq (2.3); for $0 < \lambda < 1$, this is [7, Lemma 2.1], and the same argument applies for every $\lambda > 0$.

(ii) The integrals defining $\Gamma_\lambda(w + 1)$ and $\Gamma_\mu(w)$ converge simultaneously in Eq (2.4). Applying the ordinary gamma recurrence to Eq (2.2) gives Eq (2.5), which is the recurrence obtained by integration by parts in [5, Eq (2.4)]; see also [7, Eq (1.12)]. Because both sides extend meromorphically and agree on a nonempty open strip, Eq (2.6) follows by the identity theorem.

(iii) For t^α, integrability at zero requires $\Re(\alpha) > -1$, whereas integrability at infinity requires $\Re(s) > \lambda(\Re(\alpha) + 1)$. The substitution $y = \lambda t$ gives Eq (2.7).

□

For $j \in \mathbb{N}_0$, set

$$P_0(s;\lambda) = 1, \qquad P_j(s;\lambda) = \prod_{\nu=0}^{j-1}(s + \nu\lambda), \quad j \geq 1. \tag{2.8}$$

For

$$k_s(r) := (1 + \lambda r)^{-s/\lambda},$$

repeated differentiation gives

$$k_s^{(j)}(r) = (-1)^j P_j(s;\lambda)(1 + \lambda r)^{-(s+j\lambda)/\lambda}. \tag{2.9}$$

3. The planar prototype

To clarify how the Euclidean geometry interacts with the degenerate kernel, we first consider the planar case before passing to the general n-dimensional setting. Polar coordinates separate the angular and radial variables and thereby reduce the two-dimensional transform to a weighted one-dimensional radial integral involving the degenerate Laplace kernel. Let $x = (u, v) \in \mathbb{R}^2$, and $r = |x| = \sqrt{u^2 + v^2}$. For $\lambda > 0$, introduce

$$\Gamma_\lambda^{[2]}(z) := \int_{\mathbb{R}^2} (1 + \lambda|x|)^{-1/\lambda} |x|^{z-1}\, \mathrm{d}x, \tag{3.1}$$

$$\mathcal{L}_\lambda^{[2]} f(s) := \int_{\mathbb{R}^2} (1 + \lambda|x|)^{-s/\lambda} f(x)\,\mathrm{d}x \tag{3.2}$$

whenever the integrals converge absolutely. The circular mean of f is denoted by

$$\mathcal{M}_2 f(r) := \frac{1}{2\pi} \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta. \tag{3.3}$$

Proposition 3.1. *The integral defining $\Gamma_\lambda^{[2]}(z)$ converges absolutely if and only if*

$$-1 < \Re(z) < \frac{1}{\lambda} - 1.$$

In this strip,

$$\Gamma_\lambda^{[2]}(z) = 2\pi\lambda^{-(z+1)} B\left(z+1, \frac{1}{\lambda} - z - 1\right) = 2\pi\Gamma_\lambda(z+1). \tag{3.4}$$

If $\mathcal{L}_\lambda^{[2]} f(s)$ converges absolutely, then

$$\mathcal{L}_\lambda^{[2]} f(s) = 2\pi L_\lambda(r\mathcal{M}_2 f(r))(s). \tag{3.5}$$

For a radial function $f(x) = g(|x|)$, we use the profile notation $\mathcal{L}_\lambda^{[2]}[g](s)$; in this case,

$$\mathcal{L}_\lambda^{[2]}[g](s) = 2\pi L_\lambda(rg(r))(s). \tag{3.6}$$

Proof. Polar coordinates yield

$$\Gamma_\lambda^{[2]}(z) = 2\pi \int_0^\infty (1+\lambda r)^{-1/\lambda} r^z\,\mathrm{d}r.$$

The absolute integrand has orders $r^{\Re(z)}$ as $r \to 0^+$ and $r^{\Re(z)-1/\lambda}$ as $r \to \infty$. Hence, the integral converges if and only if

$$-1 < \Re(z) < \frac{1}{\lambda} - 1.$$

In this strip, the substitution $t = \lambda r$ gives Eq (3.4). Absolute convergence of Eq (3.2) permits the use of Fubini's theorem in polar coordinates, yielding

$$\mathcal{L}_\lambda^{[2]} f(s) = \int_0^\infty k_s(r) r \left[\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta\right] \mathrm{d}r,$$

which is Eq (3.5). The radial case follows from $\mathcal{M}_2 f(r) = g(r)$. □

Equation (3.5) shows that the transform depends only on the circular mean of its input. The factor r in the polar measure produces the shift in Eq (3.4) and modifies the transform formulas for profile derivatives.

Proposition 3.2. *Let $g \in C^2([0,\infty))$, and suppose that, for some $C > 0$ and $\beta \in \mathbb{R}$,*

$$|g^{(k)}(r)| \le C(1+r)^{\beta}, \qquad r \ge 0, \quad k = 0, 1, 2.$$

If

$$\Re(s) > \lambda(\beta + 2),$$

then

$$\mathcal{L}_\lambda^{[2]}[g'](s) = s\mathcal{L}_\lambda^{[2]}[g](s+\lambda) - \mathcal{L}_\lambda^{[2]}\left[\frac{g(r)}{r}\right](s), \tag{3.7}$$

$$\mathcal{L}_\lambda^{[2]}[g''](s) = s(s+\lambda)\mathcal{L}_\lambda^{[2]}[g](s+2\lambda) - 2s\mathcal{L}_\lambda^{[2]}\left[\frac{g(r)}{r}\right](s+\lambda) + 2\pi g(0). \tag{3.8}$$

The quotient $g(r)/r$ is used only for $r > 0$; its value at $r = 0$ does not affect the integrals.

Proof. Let $w_s(r) = rk_s(r)$. By Eq (2.9),

$$k_s'(r) = -sk_{s+\lambda}(r), \qquad k_s''(r) = s(s+\lambda)k_{s+2\lambda}(r).$$

Integration by parts gives

$$2\pi \int_0^\infty w_s(r)g'(r)\,\mathrm{d}r = -2\pi \int_0^\infty (k_s(r) + rk_s'(r))g(r)\,\mathrm{d}r,$$

which yields Eq (3.7). Similarly,

$$\begin{aligned} 2\pi \int_0^\infty w_s(r)g''(r)\,\mathrm{d}r &= 2\pi g(0) + 2\pi \int_0^\infty w_s''(r)g(r)\,\mathrm{d}r \\ &= 2\pi g(0) + 2\pi \int_0^\infty (2k_s'(r) + rk_s''(r))g(r)\,\mathrm{d}r. \end{aligned}$$

Substitution of the kernel derivatives gives Eq (3.8).

The growth assumption and the condition on $\Re(s)$ ensure absolute convergence and the vanishing of the boundary terms at infinity. At the origin,

$$w_s(0) = 0, \qquad w_s'(0) = 1,$$

which accounts for the term $2\pi g(0)$. □

Example 3.3. Consider the nonzero function

$$f(u,v) = (u^2 - v^2)e^{-(u^2+v^2)}.$$

Because

$$f(r\cos\theta, r\sin\theta) = r^2 e^{-r^2}\cos(2\theta),$$

its circular mean vanishes:

$$\mathcal{M}_2 f(r) = \frac{r^2 e^{-r^2}}{2\pi} \int_0^{2\pi} \cos(2\theta)\,\mathrm{d}\theta = 0.$$

The rapid decay of f guarantees absolute convergence for every $s \in \mathbb{C}$. Hence, Eq (3.5) gives

$$\mathcal{L}_\lambda^{[2]} f(s) = 0, \qquad s \in \mathbb{C}.$$

Thus, the planar transform is not injective on general, nonradial functions.

In higher dimensions, the Jacobian r^{n-1} produces additional terms under repeated integration by parts. The general theory below determines the resulting inverse-power profiles and origin contributions together with the corresponding n-dimensional reduction formulas.

4. Radial extension to $\mathbb{R}^n$

Having established the planar prototype, we now pass to the radial framework on $\mathbb{R}^n$, $n \geq 2$. The section has three main components. First, we define and analyze the radial degenerate gamma function: We determine its exact convergence strip, meromorphic continuation, and pole structure and relate these explicitly to the corresponding one-dimensional quantities. Second, we introduce the radial degenerate Laplace transform and derive its spherical-mean reduction and factorization through a one-dimensional transform. Third, we establish operational formulas for derivatives of radial profiles, showing how the Euclidean Jacobian r^{n-1} produces dimension-dependent inverse-power terms and boundary contributions at the origin.

4.1. The radial degenerate gamma function

Let $n \in \mathbb{N}$, $n \geq 2$, and let

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

be the surface measure of $\mathbb{S}^{n-1}$.

Definition 4.1. For $\lambda > 0$ and $z \in \mathbb{C}$, define

$$\Gamma_\lambda^{[n]}(z) := \int_{\mathbb{R}^n} (1 + \lambda|x|)^{-1/\lambda} |x|^{z-1} \, \mathrm{d}x \tag{4.1}$$

whenever the integral converges absolutely.

Theorem 4.2. *The integral in Eq* (4.1) *converges absolutely if and only if*

$$1 - n < \Re(z) < \frac{1}{\lambda} - n + 1. \tag{4.2}$$

In this strip,

$$\Gamma_\lambda^{[n]}(z) = \omega_{n-1} \lambda^{-(z+n-1)} B\left(z + n - 1, \frac{1}{\lambda} - z - n + 1\right) \tag{4.3}$$

$$= \omega_{n-1} \Gamma_\lambda(z + n - 1). \tag{4.4}$$

The function

$$\widetilde{\Gamma}_\lambda^{[n]}(z) := \omega_{n-1} \lambda^{-(z+n-1)} \frac{\Gamma(z + n - 1)\Gamma(1/\lambda - z - n + 1)}{\Gamma(1/\lambda)} \tag{4.5}$$

extends $\Gamma_\lambda^{[n]}$ *meromorphically to* $\mathbb{C}$*. Its poles are simple and are located at*

$$\{1 - n - k : k \in \mathbb{N}_0\} \cup \left\{\frac{1}{\lambda} - n + 1 + k : k \in \mathbb{N}_0\right\}.$$

For $k \in \mathbb{N}_0$, the corresponding residues are

$$\operatorname*{Res}_{z=1-n-k} \widetilde{\Gamma}_\lambda^{[n]}(z) = \omega_{n-1} \frac{(-1)^k \lambda^k}{k!} \frac{\Gamma(1/\lambda + k)}{\Gamma(1/\lambda)}, \tag{4.6}$$

$$\operatorname*{Res}_{z=1/\lambda-n+1+k} \widetilde{\Gamma}_\lambda^{[n]}(z) = \omega_{n-1} \frac{(-1)^{k+1} \lambda^{-1/\lambda-k}}{k!} \frac{\Gamma(1/\lambda + k)}{\Gamma(1/\lambda)}. \tag{4.7}$$

Proof. Spherical coordinates give

$$\Gamma_\lambda^{[n]}(z) = \omega_{n-1} \int_0^\infty (1 + \lambda r)^{-1/\lambda} r^{z+n-2} \, \mathrm{d}r.$$

The absolute integrand has orders

$$r^{\Re(z)+n-2} \quad \text{as } r \to 0^+, \qquad r^{\Re(z)+n-2-1/\lambda} \quad \text{as } r \to \infty.$$

Integrability at zero requires $\Re(z) + n - 2 > -1$, whereas integrability at infinity requires $\Re(z) + n - 2 - 1/\lambda < -1$. Equality at either endpoint gives a divergent integral of order r^{-1}. These conditions give Eq (4.2). Setting $t = \lambda r$, we obtain

$$\begin{aligned} \Gamma_\lambda^{[n]}(z) &= \omega_{n-1} \lambda^{-(z+n-1)} \int_0^\infty (1 + t)^{-1/\lambda} t^{z+n-2} \, \mathrm{d}t \\ &= \omega_{n-1} \lambda^{-(z+n-1)} B\left(z + n - 1, 1/\lambda - z - n + 1\right). \end{aligned}$$

Both beta parameters have positive real parts precisely in the stated strip. The reduction in Eq (4.4) follows directly from Eq (1.1).

The beta–gamma identity gives Eq (4.5). If the two pole families intersected, then for some $k, j \in \mathbb{N}_0$,

$$1 - n - k = 1/\lambda - n + 1 + j, \qquad 1/\lambda = -(k + j),$$

which is impossible for $\lambda > 0$. At each pole, the other gamma factor is finite and nonzero, and $\Gamma(1/\lambda) > 0$; hence, all poles are simple. Put $z_- = 1 - n - k$ and $z_+ = 1/\lambda - n + 1 + k$. At the first pole,

$$\lambda^{-(z_-+n-1)} = \lambda^k, \qquad \Gamma(1/\lambda - z_- - n + 1) = \Gamma(1/\lambda + k).$$

Because the derivative of $z + n - 1$ is 1, multiplying these factors by $(-1)^k/k!$ gives Eq (4.6). At the second pole,

$$\lambda^{-(z_++n-1)} = \lambda^{-1/\lambda-k}, \qquad \Gamma(z_+ + n - 1) = \Gamma(1/\lambda + k).$$

For $w = 1/\lambda - z - n + 1$, one has $w + k = -(z - z_+)$. Thus, the residue of $\Gamma(w)$, viewed as a function of z, is $(-1)^{k+1}/k!$, which proves Eq (4.7). For $0 < \lambda < 1$, in the variable $w = z + n - 1$, the poles and residues are those of the one-dimensional continuation in [7, Lemma 2.1], multiplied by ω_{n-1}; the computation above holds for every $\lambda > 0$. □

Remark 4.3. Equation (4.4) shows that the dependence on the dimension is given by the factor ω_{n-1} and the translation $z \mapsto z + n - 1$. Consequently, the poles of $\widetilde{\Gamma}_\lambda^{[n]}$ are the poles of the one-dimensional continuation [7] translated by $1 - n$.

Corollary 4.4. *Let* $0 < \lambda < 1$, *and*

$$\mu = \frac{\lambda}{1-\lambda}.$$

If

$$1 - n < \Re(z) < \frac{1}{\lambda} - n,$$

then both defining integrals converge, and

$$\Gamma_\lambda^{[n]}(z+1) = \frac{z+n-1}{(1-\lambda)^{z+n}}\Gamma_\mu^{[n]}(z). \tag{4.8}$$

The corresponding identity between the meromorphic continuations is

$$\widetilde{\Gamma}_\lambda^{[n]}(z+1) = \frac{z+n-1}{(1-\lambda)^{z+n}}\widetilde{\Gamma}_\mu^{[n]}(z), \qquad \textit{as an identity of meromorphic functions.} \tag{4.9}$$

Proof. The convergence strips of $\Gamma_\lambda^{[n]}(z+1)$ and $\Gamma_\mu^{[n]}(z)$ are, respectively,

$$-n < \Re(z) < \frac{1}{\lambda} - n$$

and

$$1 - n < \Re(z) < \frac{1}{\mu} - n + 1 = \frac{1}{\lambda} - n.$$

Their intersection is the strip stated above. Substituting $w = z+n-1$ into Eq (2.5) and using Eq (4.4) gives Eq (4.8). The meromorphic identity follows from analytic continuation. □

Corollary 4.5. *For* $\ell \in \mathbb{N}_0$ *and* $0 < \lambda < 1/(\ell+n-1)$,

$$\Gamma_\lambda^{[n]}(\ell) = \frac{\omega_{n-1}(\ell+n-2)!}{\prod_{j=1}^{\ell+n-1}(1-j\lambda)}. \tag{4.10}$$

Here, the restriction on λ *guarantees that Eq* (4.10) *is the value of the defining integral. If that restriction is dropped, but the denominator remains nonzero, the same finite product gives the value of the meromorphic continuation* $\widetilde{\Gamma}_\lambda^{[n]}(\ell)$, *not of a convergent integral. Moreover, for every fixed* z *with* $\Re(z) > 1-n$,

$$\lim_{\lambda\to 0^+}\Gamma_\lambda^{[n]}(z) = \omega_{n-1}\Gamma(z+n-1), \tag{4.11}$$

locally uniformly in the half-plane $\Re(z) > 1-n$.

Proof. Put $k = \ell+n-1 \geq 1$. By Eqs (4.4) and (2.2),

$$\Gamma_\lambda^{[n]}(\ell) = \omega_{n-1}\Gamma_\lambda(k) = \omega_{n-1}\lambda^{-k}\frac{(k-1)!\,\Gamma(1/\lambda-k)}{\Gamma(1/\lambda)},$$

and $\Gamma(1/\lambda)/\Gamma(1/\lambda-k) = \prod_{j=1}^{k}(1/\lambda-j) = \lambda^{-k}\prod_{j=1}^{k}(1-j\lambda)$. This gives Eq (4.10); see [5, Theorem 2.3]. For $0 < \lambda < 1$, the corresponding identity for the continuation $\widetilde{\Gamma}_\lambda$ at positive integers is [7, Theorem 2.3]. For the limit, put $w = z+n-1$ and $x = 1/\lambda$ in Eq (4.3). The locally uniform gamma-quotient asymptotic

$$x^w\frac{\Gamma(x-w)}{\Gamma(x)} \longrightarrow 1$$

proves Eq (4.11). □

4.2. The radial degenerate Laplace transform

Define the spherical mean by

$$\mathcal{M}_n f(r) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} f(r\theta)\, \mathrm{d}\sigma(\theta).$$

Here, $\mathrm{d}\sigma(\theta)$ is the standard surface measure on the unit sphere $\mathbb{S}^{n-1}$, and $\omega_{n-1} = \sigma(\mathbb{S}^{n-1})$.

Definition 4.6. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and $s \in \mathbb{C}$, define

$$\mathcal{L}_\lambda^{[n]} f(s) := \int_{\mathbb{R}^n} (1 + \lambda|x|)^{-s/\lambda} f(x)\, \mathrm{d}x \tag{4.12}$$

whenever the integral converges absolutely.

Polar coordinates and Fubini's theorem yield

$$\mathcal{L}_\lambda^{[n]} f(s) = \omega_{n-1} \int_0^\infty (1 + \lambda r)^{-s/\lambda} \mathcal{M}_n f(r) r^{n-1}\, \mathrm{d}r = \omega_{n-1} L_\lambda(r^{n-1} \mathcal{M}_n f(r))(s). \tag{4.13}$$

For a radial function $f(x) = g(|x|)$, we use the profile notation

$$\mathcal{L}_\lambda^{[n]}[g](s) = \omega_{n-1} \int_0^\infty (1 + \lambda r)^{-s/\lambda} g(r) r^{n-1}\, \mathrm{d}r. \tag{4.14}$$

The next factorization describes exactly what angular information is retained. Put

$$\phi_\lambda(t) = \frac{e^{\lambda t} - 1}{\lambda}, \qquad t \geq 0,$$

and whenever the spherical mean is defined, set

$$H_f(t) = \omega_{n-1} e^{\lambda t} \phi_\lambda(t)^{n-1} \mathcal{M}_n f(\phi_\lambda(t)). \tag{4.15}$$

For $a \in \mathbb{R}$, let $\mathcal{A}_a$ denote the *admissible class* of measurable $f : \mathbb{R}^n \to \mathbb{C}$ such that $\mathcal{M}_n|f|(r)$ is finite for almost every $r > 0$ and that the function $H_{|f|}$ obtained from Eq (4.15) by replacing f with $|f|$ satisfies $e^{-at} H_{|f|}(t) \in L^1(0, \infty)$. Because $\mathcal{M}_n|f_1 + f_2| \leq \mathcal{M}_n|f_1| + \mathcal{M}_n|f_2|$, each $\mathcal{A}_a$ is a linear space, and $\mathcal{A}_a \subset \mathcal{A}_{a'}$ for $a \leq a'$.

Theorem 4.7. *Let $a \in \mathbb{R}$ and $f \in \mathcal{A}_a$. Then,*

$$\mathcal{L}_\lambda^{[n]} f(s) = \int_0^\infty e^{-st} H_f(t)\, \mathrm{d}t, \qquad \Re(s) > a. \tag{4.16}$$

Consequently, for $f \in \mathcal{A}_a$,

$$\mathcal{L}_\lambda^{[n]} f(s) = 0 \quad (\Re(s) > a) \quad \Longleftrightarrow \quad \mathcal{M}_n f(r) = 0 \quad \text{for almost everywhere } r > 0. \tag{4.17}$$

Within the admissible class $\mathcal{A}_a$, the transform determines $\mathcal{M}_n f$ uniquely almost everywhere. In particular, on the radial subclass $\{f = \mathcal{R}_n g \in \mathcal{A}_a\}$ it is injective up to equality almost everywhere: If two radial functions in $\mathcal{A}_a$ have the same transform for $\Re(s) > a$, they coincide almost everywhere on $\mathbb{R}^n$. The transform is not injective on unrestricted nonradial classes. Moreover, if $c > a$, and $\xi \mapsto \mathcal{L}_\lambda^{[n]} f(c + i\xi)$ belongs to $L^1(\mathbb{R})$, then for almost every $r > 0$,

$$\mathcal{M}_n f(r) = \frac{(1 + \lambda r)^{c/\lambda - 1}}{2\pi\, \omega_{n-1}\, r^{n-1}} \int_{\mathbb{R}} e^{i\xi\psi_\lambda(r)}\, \mathcal{L}_\lambda^{[n]} f(c + i\xi)\, \mathrm{d}\xi, \qquad \psi_\lambda(r) = \tfrac{1}{\lambda} \log(1 + \lambda r). \tag{4.18}$$

Proof. In Eq (4.13), make the change of variables

$$t = \frac{1}{\lambda}\log(1+\lambda r), \qquad r = \phi_\lambda(t), \qquad \mathrm{d}r = e^{\lambda t}\,\mathrm{d}t.$$

This gives Eq (4.16). If the left-hand side vanishes throughout the half-plane, uniqueness of the ordinary Laplace transform gives $H_f = 0$ almost everywhere. Explicitly, for any $b > a$, extend $e^{-bt}H_f(t)$ by zero to the negative axis. This is an $L^1(\mathbb{R})$ function whose Fourier transform is zero, as the Laplace transform vanishes at $b + i\xi$ for every real ξ. Fourier uniqueness therefore gives the assertion. Because $e^{\lambda t}\phi_\lambda(t)^{n-1} > 0$ for $t > 0$, and ϕ_λ maps $(0,\infty)$ bijectively onto itself, this is equivalent to $\mathcal{M}_n f = 0$ almost everywhere. The converse follows directly from Eq (4.13). For radial $f(x) = g(|x|)$, one has $\mathcal{M}_n f = g$. For the radial injectivity statement, let $f_1 = \mathcal{R}_n g_1$ and $f_2 = \mathcal{R}_n g_2$ lie in $\mathcal{A}_{a_1}$ and $\mathcal{A}_{a_2}$ and have the same transform for $\Re(s) > \max\{a_1, a_2\}$. Then, both lie in $\mathcal{A}_a$ with $a = \max\{a_1, a_2\}$ and hence, so does $f_1 - f_2$ (linearity), and Eq (4.17) gives $g_1 - g_2 = \mathcal{M}_n(f_1 - f_2) = 0$ almost everywhere. This proves the uniqueness assertions.

For the inversion formula, let $c > a$, and put $G(t) = e^{-ct}H_f(t)$ for $t > 0$ and $G(t) = 0$ for $t \le 0$. Then, $G \in L^1(\mathbb{R})$, and by Eq (4.16), $\widehat{G}(\xi) = \int_{\mathbb{R}} e^{-i\xi t}G(t)\,\mathrm{d}t = \mathcal{L}_\lambda^{[n]} f(c + i\xi)$. If $\widehat{G} \in L^1(\mathbb{R})$, the Fourier inversion theorem gives $G(t) = 1/(2\pi)\int_{\mathbb{R}} e^{i\xi t}\widehat{G}(\xi)\,d\xi$ for almost every t. Now, substitute $t = \psi_\lambda(r)$, which maps $(0,\infty)$ diffeomorphically onto itself and therefore preserves null sets. Using $e^{ct} = (1+\lambda r)^{c/\lambda}$ and $e^{\lambda t}\phi_\lambda(t)^{n-1} = (1+\lambda r)r^{n-1}$, the definition Eq (4.15) gives $H_f(\psi_\lambda(r)) = \omega_{n-1}(1+\lambda r)r^{n-1}\mathcal{M}_n f(r)$, and solving for $\mathcal{M}_n f(r)$ yields Eq (4.18). □

Remark 4.8. Injectivity and null-space questions for transforms based on circular and spherical means have been studied extensively in integral geometry; see, for instance, [10]. In the present setting, a simple example already illustrates the loss of angular information. For every dimension $n \ge 2$,

$$f(x) = x_1 e^{-|x|^2}$$

is nonzero but has zero spherical mean at every positive radius. Hence, $\mathcal{L}_\lambda^{[n]} f$ vanishes identically on its half-plane of absolute convergence. Example 3.3 gives an even example of the same phenomenon in the plane. The angular factors in these examples are nonconstant spherical harmonics: x_1 is a homogeneous harmonic polynomial of degree one, and $u^2 - v^2$ is homogeneous harmonic of degree two in the plane. Their restrictions to the unit sphere have zero mean. The present transform retains only the degree-zero (constant angular) component.

Proposition 4.9. *Suppose that g is locally integrable with respect to $r^{n-1}\,\mathrm{d}r$ and that, for some $\beta \in \mathbb{R}$ and constants $C, R > 0$,*

$$|g(r)| \le C(1+r)^\beta, \qquad r \ge R.$$

Then, Eq (4.14) *converges absolutely and defines a holomorphic function in the half-plane*

$$\Re(s) > \lambda(\beta + n). \tag{4.19}$$

Proof. At infinity, the radial integrand is bounded in absolute value by a constant multiple of

$$r^{\beta+n-1-\Re(s)/\lambda},$$

which is integrable under Eq (4.19). Local integrability controls the origin. Let K be a compact subset of the stated half-plane, set $\sigma_K = \min_{s\in K} \Re(s)$, and put $\delta = \sigma_K/\lambda - \beta - n > 0$. For every integer $p \geq 0$ (see [5, Theorem 3.4]),

$$\partial_s^p k_s(r) = \left(-\lambda^{-1}\log(1+\lambda r)\right)^p k_s(r).$$

Choose $R_1 \geq \max\{1, R\}$. For $r \geq R_1$, uniformly in $s \in K$,

$$|\partial_s^p k_s(r) g(r)| r^{n-1} \leq C_{K,p} r^{-1-\delta}(1+\log r)^p \leq C'_{K,p} r^{-1-\delta/2}.$$

On $[0, R_1]$, the kernel derivatives are uniformly bounded for $s \in K$, so a constant multiple of $|g(r)|r^{n-1}$ is an integrable majorant. Together, these bounds justify differentiation under the integral sign of every order and establish holomorphy. □

Remark 4.10. The bound in Proposition 4.9 is only an upper growth estimate, so the displayed half-plane is sufficient rather than necessarily optimal. If $|g(r)|$ is comparable to r^β for all sufficiently large r, the same endpoint test shows that $\Re(s) > \lambda(\beta + n)$ is the exact half-plane of absolute convergence. More explicitly, if $cr^\beta \leq |g(r)| \leq Cr^\beta$ for $r \geq R_1$ with $c, C > 0$, the absolute radial integrand is comparable to $r^{\beta+n-1-\Re(s)/\lambda}$. It is integrable exactly when $\Re(s) > \lambda(\beta + n)$, and diverges at equality. Oscillatory cancellation can affect conditional improper integrals, but cannot alter this absolute-convergence test. For $n = 2$, the threshold becomes $\Re(s) > \lambda(\beta + 2)$.

Proposition 4.11. *Let g be measurable, locally integrable with respect to $r^{n-1}\,\mathrm{d}r$, and of at most polynomial growth at infinity. For every $s \in \mathbb{C}$ with $\Re(s) > 0$,*

$$\lim_{\lambda\to 0^+} \mathcal{L}_\lambda^{[n]}[g](s) = \omega_{n-1}\int_0^\infty e^{-sr} g(r) r^{n-1}\,\mathrm{d}r, \tag{4.20}$$

and the degenerate transform is absolutely convergent for all sufficiently small $\lambda > 0$.

Proof. Pointwise convergence follows from $(1+\lambda r)^{-s/\lambda} \to e^{-sr}$. If g has polynomial growth, choose $\lambda_0 > 0$ such that $\Re(s)/\lambda_0$ exceeds its growth order by more than n. For $0 < \lambda \leq \lambda_0$,

$$(1+\lambda r)^{-\Re(s)/\lambda} \leq (1+\lambda_0 r)^{-\Re(s)/\lambda_0}.$$

After multiplication by $|g(r)|r^{n-1}$, the right-hand side is an integrable majorant at infinity. Local integrability handles the origin, and dominated convergence proves Eq (4.20). □

Theorem 4.12. *Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > -n$. If*

$$\Re(s) > \lambda(\Re(\alpha) + n),$$

then

$$\mathcal{L}_\lambda^{[n]}[r^\alpha](s) = \omega_{n-1}\lambda^{-(\alpha+n)} B\left(\alpha + n, \frac{s}{\lambda} - \alpha - n\right). \tag{4.21}$$

When $s > 0$ is real, and the preceding convergence condition holds,

$$\mathcal{L}_\lambda^{[n]}[r^\alpha](s) = \frac{1}{s^{\alpha+n}}\Gamma_{\lambda/s}^{[n]}(\alpha+1). \tag{4.22}$$

For $\ell \in \mathbb{N}_0$ and $\Re(s) > (\ell + n)\lambda$,

$$\mathcal{L}_\lambda^{[n]}[r^\ell](s) = \frac{\omega_{n-1}(\ell + n - 1)!}{\prod_{j=1}^{\ell+n}(s - j\lambda)}. \tag{4.23}$$

Consequently,

$$\mathcal{L}_\lambda^{[n]}[1](s) = \frac{\omega_{n-1}(n-1)!}{\prod_{j=1}^{n}(s - j\lambda)}, \qquad \Re(s) > n\lambda. \tag{4.24}$$

Proof. The absolute radial integrand has order $r^{\Re(\alpha)+n-1}$ at zero and $r^{\Re(\alpha)+n-1-\Re(s)/\lambda}$ at infinity. Thus, the endpoint inequalities are

$$\Re(\alpha) + n - 1 > -1, \qquad \Re(\alpha) + n - 1 - \Re(s)/\lambda < -1.$$

If equality holds in either inequality, the absolute integrand has order r^{-1} at the corresponding endpoint, and the integral diverges. Under precisely these conditions, spherical coordinates and $t = \lambda r$ give Eq (4.21). Equivalently,

$$\mathcal{L}_\lambda^{[n]}[r^\alpha](s) = \omega_{n-1} L_\lambda(r^{\alpha+n-1})(s),$$

so Eq (4.21) is also the shifted form of Eq (2.7). The substitution $y = sr$ gives Eq (4.22) for real $s > 0$; this is the radial analog of [5, Eq (3.19)]. When $\alpha = \ell$, the gamma quotient in the beta function reduces to the finite product in Eq (4.23), in agreement with [5, Theorem 3.2] applied to the exponent $\ell + n - 1$. Taking $\ell = 0$ proves Eq (4.24). □

Remark 4.13. The factor $(n-1)!$ in Eq (4.24) is essential. It is equal to one in the planar case and can therefore be overlooked if one passes from $\mathbb{R}^2$ to $\mathbb{R}^n$ only formally.

4.3. Operational formulas for radial-profile derivatives

We now state the operational formula in which the effect of dimension is most pronounced. Recall from Eq (2.8) that

$$P_0(s;\lambda) = 1, \qquad P_j(s;\lambda) = \prod_{\nu=0}^{j-1}(s + \nu\lambda), \quad j \geq 1.$$

Definition 4.14. For $m \in \mathbb{N}_0$, let $C^m_{\mathrm{pol}}([0,\infty))$ be the class of functions $g \in C^m([0,\infty))$ for which there exist $C > 0$ and $\beta \in \mathbb{R}$, which may depend on g, such that

$$|g^{(k)}(r)| \leq C(1 + r)^\beta, \qquad r \geq 0, \quad k = 0, \ldots, m. \tag{4.25}$$

Write $w_{s,n}(r) = r^{n-1}k_s(r)$. In the Leibniz expansion, q counts the derivatives falling on the radial Jacobian:

$$\frac{d^m}{dr^m}(r^{n-1}k_s(r)) = \sum_{q=0}^{\min\{m,n-1\}} \binom{m}{q} \frac{(n-1)!}{(n-1-q)!} r^{n-1-q} k_s^{(m-q)}(r),$$

where

$$k_s^{(m-q)}(r) = (-1)^{m-q} P_{m-q}(s;\lambda) k_{s+(m-q)\lambda}(r).$$

After integration against g, the factor r^{n-1-q} produces profile transforms of $g(r)/r^q$. Moreover, $w_{s,n}$ has a zero of order $n-1$ at the origin: Its derivatives of order less than $n-1$ vanish there, while $w_{s,n}^{(n-1)}(0) = (n-1)!$. Because m integrations by parts use boundary derivatives only through order $m-1$, the first potentially nonzero origin term occurs at $m = n$.

Theorem 4.15. *Let $m \geq 1$, and let $g \in C^m_{\mathrm{pol}}([0,\infty))$ satisfy Eq* (4.25)*. If*

$$\Re(s) > \lambda(\beta + n), \tag{4.26}$$

then every transform in the following identity is absolutely convergent, all integrations by parts are justified, and

$$\begin{aligned}\mathcal{L}_\lambda^{[n]}[g^{(m)}](s) = &\sum_{q=0}^{\min\{m,n-1\}} (-1)^q \binom{m}{q} \frac{(n-1)!}{(n-1-q)!} P_{m-q}(s;\lambda) \\ &\times \mathcal{L}_\lambda^{[n]}\left[\frac{g(r)}{r^q}\right](s + (m-q)\lambda) \\ &+ \omega_{n-1}(-1)^n \sum_{j=n-1}^{m-1} \frac{j!}{(j-n+1)!} P_{j-n+1}(s;\lambda) g^{(m-1-j)}(0),\end{aligned} \tag{4.27}$$

where the last sum is empty when $m < n$. In the transformed terms, $g(r)/r^q$ is understood on $r > 0$; its value at the single point $r = 0$ is immaterial.

Proof. Write

$$k_s(r) = (1+\lambda r)^{-s/\lambda}, \qquad w_{s,n}(r) = r^{n-1} k_s(r).$$

We first verify all convergence and boundary requirements from the explicit hypotheses. Leibniz's rule and Eq (2.9) imply, for each fixed $j \geq 0$,

$$w_{s,n}^{(j)}(r) = O\left(r^{n-1-j-\Re(s)/\lambda}\right), \qquad r \to \infty. \tag{4.28}$$

Indeed, if q of the j derivatives fall on r^{n-1}, the corresponding term is a constant multiple of

$$r^{n-1-q}(1+\lambda r)^{-\Re(s)/\lambda-(j-q)},$$

and every such term has the order displayed in Eq (4.28). Consequently,

$$w_{s,n}^{(j)}(r) g^{(m-1-j)}(r) = O\left(r^{\beta+n-1-j-\Re(s)/\lambda}\right) \longrightarrow 0$$

for $j = 0, \ldots, m-1$, by Eq (4.26).

At infinity, the absolute radial integrand defining $\mathcal{L}_\lambda^{[n]}[g^{(m)}](s)$ is

$$O\left(r^{\beta+n-1-\Re(s)/\lambda}\right),$$

which is integrable under Eq (4.26). Similarly, $w_{s,n}^{(m)} g$ is absolutely integrable. For $0 \le q \le \min\{m, n-1\}$, the radial integrand of

$$\mathcal{L}_\lambda^{[n]}\left[\frac{g(r)}{r^q}\right](s + (m-q)\lambda)$$

has order

$$O\left(r^{\beta+n-1-m-\Re(s)/\lambda}\right)$$

at infinity and is therefore integrable. At the origin, g and all of its derivatives through order m are bounded, although these transformed integrands have the harmless factors r^{n-1-q}, with $n-1-q \ge 0$. Indeed,

$$r^{n-1}\frac{g(r)}{r^q} = r^{n-1-q} g(r),$$

which is locally integrable at zero. Thus, no value needs to be assigned to the quotient at that single point.

Repeated integration by parts now gives

$$\begin{aligned}\int_0^\infty w_{s,n}(r) g^{(m)}(r)\,\mathrm{d}r = &\sum_{j=0}^{m-1}(-1)^j\left[w_{s,n}^{(j)}(r) g^{(m-1-j)}(r)\right]_0^\infty \\ &+ (-1)^m \int_0^\infty w_{s,n}^{(m)}(r) g(r)\,\mathrm{d}r. \end{aligned} \tag{4.29}$$

For the integral term, Leibniz's rule and Eq (2.9) give

$$\begin{aligned}(-1)^m w_{s,n}^{(m)}(r) = &\sum_{q=0}^{\min\{m,n-1\}}(-1)^q\binom{m}{q}\frac{(n-1)!}{(n-1-q)!}r^{n-1-q} \\ &\times P_{m-q}(s;\lambda) k_{s+(m-q)\lambda}(r).\end{aligned}$$

Multiplying by $g(r)$, integrating, and then multiplying by ω_{n-1} produces the first sum in Eq (4.27).

It remains to evaluate the boundary terms at the origin. If $0 \le j < n-1$, then $w_{s,n}^{(j)}(0) = 0$. For $j \ge n-1$, exactly $n-1$ of the j derivatives must fall on r^{n-1} in order to obtain a nonzero value at zero. Hence,

$$w_{s,n}^{(j)}(0) = \frac{j!}{(j-n+1)!}(-1)^{j-n+1} P_{j-n+1}(s;\lambda). \tag{4.30}$$

The upper-endpoint terms vanish by the estimate already proved. The lower endpoint is subtracted, so its jth contribution has sign $(-1)^{j+1}$. Combining it with the sign in Eq (4.30) yields

$$(-1)^{j+1}(-1)^{j-n+1} = (-1)^{2j-n+2} = (-1)^n. \tag{4.31}$$

Thus, the boundary sign is independent of j. Substitution into Eq (4.29), followed by multiplication by ω_{n-1}, gives the second sum in Eq (4.27). □

For later use, denote the transformed part of Eq (4.27) by

$$\begin{aligned}\mathcal{A}_{m,n}(g;s) := &\sum_{q=0}^{\min\{m,n-1\}}(-1)^q\binom{m}{q}\frac{(n-1)!}{(n-1-q)!}P_{m-q}(s;\lambda) \\ &\times \mathcal{L}_\lambda^{[n]}\left[\frac{g(r)}{r^q}\right](s+(m-q)\lambda).\end{aligned} \tag{4.32}$$

Corollary 4.16. *Let $\ell \in \mathbb{N}_0$, and assume the hypotheses of Theorem 4.15 with $m = n + \ell$. Then,*

$$\begin{aligned}&\mathcal{L}_\lambda^{[n]}[g^{(n+\ell)}](s) - \mathcal{A}_{n+\ell,n}(g; s)\\ &\quad = \omega_{n-1}(-1)^n \sum_{k=0}^{\ell} \frac{(n+\ell-1-k)!}{(\ell-k)!} P_{\ell-k}(s;\lambda) g^{(k)}(0).\end{aligned} \tag{4.33}$$

In particular,

$$g(0) = \frac{(-1)^n}{\omega_{n-1}(n-1)!}\left[\mathcal{L}_\lambda^{[n]}[g^{(n)}](s) - \mathcal{A}_{n,n}(g; s)\right]. \tag{4.34}$$

More generally, Eq (4.33) *determines $g(0), g'(0), \dots, g^{(\ell)}(0)$ recursively. Moreover, the boundary-free identity*

$$\mathcal{L}_\lambda^{[n]}[g^{(n+\ell)}](s) = \mathcal{A}_{n+\ell,n}(g; s) \tag{4.35}$$

holds for every s in the half-plane Eq (4.26) *if and only if*

$$g^{(k)}(0) = 0, \qquad k = 0, 1, \dots, \ell. \tag{4.36}$$

Proof. In the boundary sum of Eq (4.27), put $m = n + \ell$, and set $k = m - 1 - j$. Then, $j = n + \ell - 1 - k$, which gives Eq (4.33). For $\ell = 0$, only $g(0)$ remains, and Eq (4.34) follows. For general ℓ, the coefficient of $g^{(\ell)}(0)$ is $\omega_{n-1}(-1)^n(n-1)!$, so the values of the jet are recovered successively from Eq (4.33). Finally, the polynomials $P_0(s;\lambda), \dots, P_\ell(s;\lambda)$ have distinct degrees and are therefore linearly independent. Hence, the boundary polynomial vanishes throughout a half-plane if and only if every coefficient $g^{(k)}(0)$ in Eq (4.33) vanishes. This proves the last assertion. □

Corollary 4.17. *Under the hypotheses of Theorem 4.15 with $m = 1$,*

$$\mathcal{L}_\lambda^{[n]}[g'](s) = s\mathcal{L}_\lambda^{[n]}[g](s+\lambda) - (n-1)\mathcal{L}_\lambda^{[n]}\left[\frac{g(r)}{r}\right](s). \tag{4.37}$$

Proof. In Eq (4.27), only $q = 0$ and $q = 1$ occur. Because $P_1(s;\lambda) = s$, the two terms are precisely those in Eq (4.37). The boundary sum is empty because $1 < n$. □

Remark 4.18. The first term on the right-hand side of Eq (4.37) is the direct analog of the one-dimensional derivative rule. The second term is geometric: Its coefficient $n - 1$ is the exponent of the polar Jacobian r^{n-1}.

Example 4.19. We illustrate the derivative formulas in dimensions two and three. For $n = 2$, the first two cases are exactly Eqs (3.7) and (3.8). For $n = 3$, let $g \in C^3_{\mathrm{pol}}([0,\infty))$ with growth exponent β, and suppose $\Re(s) > \lambda(\beta + 3)$. Because $\omega_2 = 4\pi$, the formulas become

$$\begin{aligned}\mathcal{L}_\lambda^{[3]}[g'](s) &= s\mathcal{L}_\lambda^{[3]}[g](s+\lambda) - 2\mathcal{L}_\lambda^{[3]}[g(r)/r](s),\\ \mathcal{L}_\lambda^{[3]}[g''](s) &= P_2(s;\lambda)\mathcal{L}_\lambda^{[3]}[g](s+2\lambda) - 4s\mathcal{L}_\lambda^{[3]}[g(r)/r](s+\lambda) + 2\mathcal{L}_\lambda^{[3]}[g(r)/r^2](s),\\ \mathcal{L}_\lambda^{[3]}[g^{(3)}](s) &= P_3(s;\lambda)\mathcal{L}_\lambda^{[3]}[g](s+3\lambda) - 6P_2(s;\lambda)\mathcal{L}_\lambda^{[3]}[g(r)/r](s+2\lambda)\\ &\quad + 6s\mathcal{L}_\lambda^{[3]}[g(r)/r^2](s+\lambda) - 8\pi g(0).\end{aligned}$$

The first origin contribution in dimension three therefore occurs at order three, with sign opposite to the planar order-two contribution. As a consistency check, take $g \equiv 1$ (so $g' = g'' = g^{(3)} = 0$ and

$g(0) = 1$): By Theorem 4.12 with $\alpha = 0, -1, -2$, each right-hand side above vanishes. For instance, in the second formula, the three terms equal $2\omega_2/(s-\lambda)$, $-4\omega_2/(s-\lambda)$, and $2\omega_2/(s-\lambda)$. The same check holds for the planar formulas (3.7) and (3.8), where the term $2\pi g(0)$ is needed to cancel the remaining contributions.

Remark 4.20. Let g, m, and β be as in Theorem 4.15. That theorem can be compared with the one-dimensional degenerate derivative rule [5, Theorem 3.3]. Because $(1+\lambda r)^{-m} k_\sigma = k_{\sigma+m\lambda}$, that rule reads, for $\Re(\sigma) > \lambda(\beta+1)$,

$$L_\lambda(g^{(m)})(\sigma) = P_m(\sigma;\lambda) L_\lambda(g)(\sigma+m\lambda) - \sum_{i=0}^{m-1} P_i(\sigma;\lambda)\, g^{(m-1-i)}(0); \tag{4.38}$$

see also [8, Theorem 14] for a related degenerate Laplace-type transform.

The two rules are linked by a λ-difference operator. Put $p = n-1$ and $D_p Q(s) = \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} Q(s - j\lambda)$. Because $r^p = \lambda^{-p}((1+\lambda r)-1)^p$ and $(1+\lambda r)^j k_s = k_{s-j\lambda}$,

$$\mathcal{L}_\lambda^{[n]}[h](s) = \omega_{n-1} \lambda^{-p}\, D_p[L_\lambda(h)](s),$$

which is the λ-analog of the classical rule $\mathcal{L}(t^p h)(s) = (-1)^p [\mathcal{L}(h)]^{(p)}(s)$. Taking $h = g^{(m)}$ and inserting Eq (4.38) gives, for $\Re(s) > \lambda(\beta+n)$, the equivalent representation

$$\mathcal{L}_\lambda^{[n]}[g^{(m)}](s) = \omega_{n-1}\lambda^{-p}\, D_p\Big[P_m(\cdot;\lambda) L_\lambda(g)(\cdot + m\lambda) - \sum_{i=0}^{m-1} P_i(\cdot;\lambda)\, g^{(m-1-i)}(0)\Big](s).$$

The origin terms of Eq (4.27) follow from this representation: A direct computation gives

$$D_p P_h(s;\lambda) = \begin{cases} 0, & h < p, \\ (-\lambda)^p \dfrac{h!}{(h-p)!} P_{h-p}(s;\lambda), & h \geq p, \end{cases}$$

so only the derivatives $g^{(k)}(0)$ with $k \leq m-n$ contribute, and the sign $-\lambda^{-p}(-\lambda)^p = (-1)^n$ is that of the boundary sum in Eq (4.27). In particular, no origin terms occur for $m < n$.

The two representations are complementary: Equation (4.27) uses radial transforms of the profiles $g(r)/r^q$ with coefficients containing no negative powers of λ and gives the origin coefficients explicitly (Corollary 4.16), whereas the representation above uses n shifted one-dimensional transforms with the prefactor $\lambda^{-(n-1)}$.

Finally, Eq (4.27) has a classical limit. Fix s with $\sigma = \Re(s) > 0$, choose $\lambda_0 > 0$ with $\sigma > \lambda_0(\beta+n)$, and let $0 < \lambda \leq \lambda_0$. Because $\lambda \mapsto \lambda^{-1}\log(1+\lambda r)$ is decreasing, for $0 \leq q \leq \min\{m, n-1\}$ and $\nu \geq 0$,

$$\left| r^{n-1-q} g(r) (1+\lambda r)^{-(s+\nu\lambda)/\lambda} \right| \leq C\, r^{n-1-q} (1+r)^\beta (1+\lambda_0 r)^{-\sigma/\lambda_0},$$

and the right-hand side is integrable on $(0,\infty)$; the same bound holds with g replaced by $g^{(m)}$ and $q = \nu = 0$. Hence, by dominated convergence, letting $\lambda \to 0^+$ termwise in Eq (4.27) gives the corresponding identity for the classical radial transform $\omega_{n-1}\int_0^\infty e^{-sr} h(r) r^{n-1}\,\mathrm{d}r$, with $P_j(s;\lambda)$ replaced by s^j.

5. Transform intertwining for a radial transport operator

In this section, the logarithmic radial variable associated with the degenerate Laplace kernel is used to define a forward semiflow with affine radial velocity. The corresponding radial divergence operator is introduced and shown to be intertwined with multiplication by the transform parameter through the radial degenerate Laplace transform. This property yields a transform-domain representation for the associated inhomogeneous transport problem and an explicit zero-inflow solution in the homogeneous case. For the general theory of transport equations and weak solutions, we refer to [11]. In the variable $t = \psi_\lambda(r)$ introduced below, the conservative form of the homogeneous problem is the translation equation $h_\tau + h_t = 0$. Let

$$\psi_\lambda(r) = \frac{1}{\lambda}\log(1+\lambda r), \qquad r \geq 0.$$

Then

$$k_s(r) = (1+\lambda r)^{-s/\lambda} = e^{-s\psi_\lambda(r)}.$$

Translation in the variable $\psi_\lambda(r)$ induces the forward semiflow

$$\Phi_\tau(r) = \frac{(1+\lambda r)e^{\lambda\tau} - 1}{\lambda}, \qquad \tau \geq 0. \tag{5.1}$$

It satisfies

$$\Phi_0(r) = r, \qquad \Phi_{\tau+\xi}(r) = \Phi_\tau(\Phi_\xi(r)), \qquad \frac{\partial}{\partial\tau}\Phi_\tau(r) = 1 + \lambda\Phi_\tau(r).$$

Moreover,

$$k_s(\Phi_\tau(r)) = e^{-s\tau}k_s(r), \qquad (1+\lambda r)k_s'(r) = -sk_s(r). \tag{5.2}$$

For a radial profile g, define

$$\mathcal{B}_{\lambda,n}g(r) := \frac{1}{r^{n-1}}\frac{d}{dr}\left[(1+\lambda r)r^{n-1}g(r)\right], \qquad r > 0. \tag{5.3}$$

Equivalently,

$$\mathcal{B}_{\lambda,n}g(r) = (1+\lambda r)g'(r) + \left[\lambda + \frac{(n-1)(1+\lambda r)}{r}\right]g(r).$$

If

$$V_\lambda(x) = (1+\lambda|x|)\frac{x}{|x|}, \qquad x \neq 0,$$

and $u(x) = g(|x|)$, then

$$\operatorname{div}(V_\lambda u)(x) = \mathcal{B}_{\lambda,n}g(|x|).$$

For bounded g and $n \geq 2$, the flux at the origin vanishes because

$$\lim_{r\to 0^+} r^{n-1}(1+\lambda r)g(r) = 0.$$

The following identity intertwines $\mathcal{B}_{\lambda,n}$ with multiplication by s on the transform side.

Theorem 5.1. *Let $n \geq 2$, and $g \in C^1([0,\infty))$. Suppose that, for some $C > 0$ and $\beta \in \mathbb{R}$,*

$$|g(r)| \leq C(1+r)^{\beta}, \qquad |g'(r)| \leq C(1+r)^{\beta-1}, \qquad r \geq 0. \tag{5.4}$$

If

$$\Re(s) > \lambda(\beta + n),$$

then the transforms of g and $\mathcal{B}_{\lambda,n}g$ converge absolutely, and

$$\mathcal{L}_{\lambda}^{[n]}[\mathcal{B}_{\lambda,n}g](s) = s\mathcal{L}_{\lambda}^{[n]}[g](s). \tag{5.5}$$

Proof. By Eq (5.4),

$$\mathcal{B}_{\lambda,n}g(r) = O(r^{\beta}), \qquad r \to \infty.$$

Hence, the radial integrands defining both transforms are

$$O\left(r^{\beta+n-1-\Re(s)/\lambda}\right)$$

at infinity. Near the origin, $\mathcal{B}_{\lambda,n}g(r) = O(r^{-1})$, so its radial integrand is $O(r^{n-2})$. The stated half-plane therefore guarantees absolute convergence.

Integration by parts gives

$$\begin{aligned}
&\mathcal{L}_{\lambda}^{[n]}[\mathcal{B}_{\lambda,n}g](s) \\
&\quad = \omega_{n-1}\left[k_s(r)(1+\lambda r)r^{n-1}g(r)\right]_0^{\infty} \\
&\qquad - \omega_{n-1}\int_0^{\infty} k_s'(r)(1+\lambda r)r^{n-1}g(r)\,\mathrm{d}r.
\end{aligned}$$

The boundary term vanishes at both endpoints. Using Eq (5.2) in the remaining integral yields

$$\mathcal{L}_{\lambda}^{[n]}[\mathcal{B}_{\lambda,n}g](s) = s\omega_{n-1}\int_0^{\infty} k_s(r)g(r)r^{n-1}\,\mathrm{d}r,$$

which proves Eq (5.5). □

As a consequence of Theorem 5.1, consider the radial initial-value problem

$$\frac{\partial u}{\partial \tau}(\tau, r) + \mathcal{B}_{\lambda,n}u(\tau, r) = F(\tau, r), \qquad u(0, r) = u_0(r), \tag{5.6}$$

for $0 < \tau \leq T$ and $r > 0$.

Corollary 5.2. *Let $u \in C^1([0,T]\times[0,\infty))$ solve Eq* (5.6)*, and let F be continuous. Suppose that, for some $C > 0$ and $\beta \in \mathbb{R}$,*

$$|u(\tau, r)| + |\partial_\tau u(\tau, r)| + |F(\tau, r)| \leq C(1+r)^{\beta}, \qquad |\partial_r u(\tau, r)| \leq C(1+r)^{\beta-1},$$

uniformly for $0 \leq \tau \leq T$ and $r \geq 0$. Define

$$U(\tau, s) := \mathcal{L}_{\lambda}^{[n]}[u(\tau,\cdot)](s), \qquad \widehat{F}(\tau, s) := \mathcal{L}_{\lambda}^{[n]}[F(\tau,\cdot)](s).$$

Then, for $\Re(s) > \lambda(\beta + n)$,

$$\frac{\partial U}{\partial \tau}(\tau, s) + sU(\tau, s) = \widehat{F}(\tau, s), \tag{5.7}$$

and

$$U(\tau, s) = e^{-s\tau}\mathcal{L}_\lambda^{[n]}[u_0](s) + \int_0^\tau e^{-s(\tau-\xi)}\widehat{F}(\xi, s)\,\mathrm{d}\xi. \tag{5.8}$$

Within the radial admissible class of Theorem 4.7, Eq (5.8) *determines at most one solution.*

Proof. The growth assumptions justify differentiation under the integral sign. Applying Theorem 5.1 to Eq (5.6) gives Eq (5.7). Formula (5.8) follows by solving this first-order equation. The final assertion follows from the injectivity of the transform on radial profiles. □

For the homogeneous problem, put

$$r_\tau := \Phi_\tau(0) = \frac{e^{\lambda\tau} - 1}{\lambda}, \qquad r_0(\tau, r) := \frac{(1 + \lambda r)e^{-\lambda\tau} - 1}{\lambda}.$$

Zero-inflow weak solutions. Put $v(r) = 1 + \lambda r$. With $t = \psi_\lambda(r)$, associate with a radial function $u(\tau, r)$ and an initial profile u_0 the functions

$$h(\tau, t) = v(\phi_\lambda(t))\phi_\lambda(t)^{n-1}u(\tau, \phi_\lambda(t)), \qquad h_0(t) = v(\phi_\lambda(t))\phi_\lambda(t)^{n-1}u_0(\phi_\lambda(t)). \tag{5.9}$$

A function $h \in L^1_{\mathrm{loc}}([0, \infty)^2)$ is called a *zero-inflow weak solution* with initial datum $h_0 \in L^1_{\mathrm{loc}}([0, \infty))$ if

$$\int_0^\infty \int_0^\infty h(\tau, t)(\eta_\tau + \eta_t)(\tau, t)\,\mathrm{d}t\,\mathrm{d}\tau + \int_0^\infty h_0(t)\eta(0, t)\,\mathrm{d}t = 0 \tag{5.10}$$

for every $\eta \in C^1_c([0, \infty) \times [0, \infty))$, where η is not required to vanish at $\tau = 0$ or at $t = 0$. The absence of a boundary term at $t = 0$ expresses the zero-inflow condition. A function u is a zero-inflow weak solution of Eq (5.6) with $F = 0$ if the associated h is one.

Corollary 5.3. *Let* $F = 0$*, and suppose that* u_0 *is continuous, bounded near the origin, and of polynomial growth at infinity. For* $\tau > 0$*, the zero-inflow weak solution in the sense defined above is*

$$u(\tau, r) = \begin{cases} e^{-\lambda\tau}\left(\dfrac{r_0(\tau, r)}{r}\right)^{n-1} u_0(r_0(\tau, r)), & r \geq r_\tau, \\ 0, & 0 \leq r < r_\tau. \end{cases} \tag{5.11}$$

It satisfies

$$\mathcal{L}_\lambda^{[n]}[u(\tau, \cdot)](s) = e^{-s\tau}\mathcal{L}_\lambda^{[n]}[u_0](s) \tag{5.12}$$

whenever the transforms converge absolutely. At $\tau = 0$*, set* $u(0, r) = u_0(r)$*, including at* $r = 0$*. If, additionally,* $u_0 \in C^1((0, \infty))$*, then for* $\tau > 0$ *the solution is classical for* $r > r_\tau$*, where the characteristic foot* $r_0(\tau, r)$ *is positive, and it vanishes identically for* $0 < r < r_\tau$*.*

Proof. Put $v(r) = 1 + \lambda r$ and $p(\tau, r) = r^{n-1}u(\tau, r)$. The conservative equation is $p_\tau + \partial_r(vp) = 0$. Characteristics satisfy

$$\frac{dr}{d\tau} = v(r), \qquad r = \Phi_\tau(r_0), \qquad \frac{\partial r}{\partial r_0} = e^{\lambda\tau}.$$

Consequently, conservation of $p\,\mathrm{d}r$ gives

$$u(\tau, r)r^{n-1}e^{\lambda\tau} = u_0(r_0)r_0^{n-1},$$

which yields the nonzero branch in Eq (5.11). For $r < r_\tau$, the backward characteristic reaches $r = 0$ at a positive time; the zero-inflow prescription gives the other branch.

Let h and h_0 be associated with u and u_0 as in Eq (5.9). Formally, $h_\tau + h_t = 0$. Because $n \geq 2$, and u_0 is bounded near the origin, h_0 is continuous with $h_0(0) = 0$; in particular, $h_0 \in L^1_{\mathrm{loc}}([0, \infty))$.

Existence. The function $h(\tau, t) = h_0(t - \tau)$ for $t \geq \tau$, $h(\tau, t) = 0$ for $t < \tau$ belongs to L^1_{loc} and satisfies Eq (5.10): With $a = t - \tau \geq 0$, the first integral equals $\int_0^\infty h_0(a) \int_0^\infty (d/d\tau)[\eta(\tau, a + \tau)]\,\mathrm{d}\tau\,\mathrm{d}a = -\int_0^\infty h_0(a)\eta(0, a)\,\mathrm{d}a$.

Uniqueness. Let h_1, h_2 be two zero-inflow weak solutions with the same datum h_0, and put $\varrho = h_1 - h_2$. Then, $\iint \varrho\,(\eta_\tau + \eta_t) = 0$ for all admissible η. Take $\eta(\tau, t) = \chi(\tau)\zeta(t - \tau)$ with $\chi \in C^1_c([0, \infty))$ and $\zeta \in C^1_c(\mathbb{R})$, extending ϱ by zero to $t < 0$ (the restriction of η to $t \geq 0$ is admissible and the extension adds nothing to the integral). Then, $\eta_\tau + \eta_t = \chi'(\tau)\zeta(t - \tau)$, and with $F_\zeta(\tau) = \int \varrho(\tau, a + \tau)\zeta(a)\,\mathrm{d}a \in L^1_{\mathrm{loc}}([0, \infty))$, $\int_0^\infty F_\zeta \chi'\,\mathrm{d}\tau = 0$ for all such χ. Taking χ supported in $(0, \infty)$ shows that F_ζ is almost everywhere equal to a constant c_ζ; then $\chi(0) = 1$ gives $-c_\zeta = 0$. Hence, $F_\zeta = 0$ almost everywhere for every ζ in a countable dense family, so $\varrho(\tau, \cdot) = 0$ for almost everywhere τ, that is, $\varrho = 0$.

Boundary trace. For the solution above, $h(\tau, t) = 0$ for $0 < t < \tau$, so $h(\tau, 0+) = 0$ for every $\tau > 0$; equivalently, $\lim_{r \downarrow 0} v(r)r^{n-1}u(\tau, r) = 0$. Hence, the solution has a vanishing trace at $t = 0$. Moreover, $u(\tau, r) = h(\tau, \psi_\lambda(r))/(v(r)r^{n-1})$ reproduces Eq (5.11), as $\phi_\lambda(\psi_\lambda(r) - \tau) = r_0(\tau, r)$, and $v(r_0)/v(r) = e^{-\lambda\tau}$. Finally, changing variables $r = \Phi_\tau(r_0)$ and using $k_s(\Phi_\tau(r_0)) = e^{-s\tau}k_s(r_0)$ proves Eq (5.12). Classical differentiability for $r > r_\tau$ follows under the additional C^1 assumption, as $r_0(\tau, r) > 0$ there, and $r \mapsto r_0(\tau, r)$ is smooth. □

Example 5.4. For $u_0(r) = 1$ and $\tau > 0$, the zero-inflow solution is

$$u(\tau, r) = \begin{cases} e^{-\lambda\tau}(r_0(\tau, r)/r)^{n-1}, & r \geq r_\tau, \\ 0, & 0 \leq r < r_\tau. \end{cases}$$

The change of variables along characteristics gives for $\Re(s) > n\lambda$,

$$\mathcal{L}^{[n]}_\lambda[u(\tau, \cdot)](s) = e^{-s\tau}\frac{\omega_{n-1}(n-1)!}{\prod_{j=1}^n (s - j\lambda)}.$$

For $\tau > 0$, this solution is continuous at $r = r_\tau$, because $r_0(\tau, r_\tau) = 0$, and $n \geq 2$. At $\tau = 0$, the initial profile is imposed separately; because $u(0, 0) = 1$, whereas $u(\tau, r) = 0$ for $0 \leq r < r_\tau$ and $\tau > 0$, this weak solution is discontinuous at $(\tau, r) = (0, 0)$ and therefore does not belong to the C^1 class of Corollary 5.2. Because $r_0(\tau, r) = e^{-\lambda\tau}(r - r_\tau)$, for fixed $\tau > 0$,

$$u(\tau, r) = e^{-n\lambda\tau}\left(\frac{r - r_\tau}{r}\right)^{n-1}, \qquad r > r_\tau.$$

Because $r_\tau > 0$, the radial derivatives of order $k \le n-2$ vanish from both sides at $r = r_\tau$, whereas the derivative of order $n-1$ jumps:

$$\partial_r^{n-1} u(\tau, r_\tau-) = 0, \qquad \partial_r^{n-1} u(\tau, r_\tau+) = \frac{(n-1)!\, e^{-n\lambda\tau}}{r_\tau^{n-1}} > 0.$$

Hence, at every positive time, $u(\tau, \cdot)$ is C^{n-2} but not C^{n-1} across the front. For $n = 2$, this is the jump $\partial_r u(\tau, r_\tau+) - \partial_r u(\tau, r_\tau-) = e^{-2\lambda\tau}/r_\tau$, so the solution is continuous but not differentiable there. For this constant initial profile, the order of vanishing at the front, and hence, the regularity there, is determined by the radial Jacobian r^{n-1}.

Remark 5.5. The C^1 hypothesis on u_0 cannot be dropped from the classical statement. Take $n = 2$, $\lambda = 1$, and $u_0(r) = |r-1|$. For $\tau > 0$, the solution (5.11) is for $r \ge r_\tau$, $u(\tau, r) = e^{-\tau}(r_0/r)\,|r_0 - 1|$ with $r_0 = (1+r)e^{-\tau} - 1$, and it has a corner along the characteristic $r = \Phi_\tau(1) = 2e^\tau - 1$: The one-sided slopes differ by $2e^{-2\tau}/(2e^\tau - 1) \ne 0$, so $u(\tau, \cdot)$ is not differentiable there. The formula (5.11) remains the unique zero-inflow weak solution in the sense above; only the classical-solution claim requires $u_0 \in C^1((0,\infty))$.

The change of variable $t = \psi_\lambda(r)$ converts the semiflow Φ_τ into translation and the kernel into e^{-st}. The factor r^{n-1} accounts for radial volume conservation and produces the dimension-dependent terms in $\mathcal{B}_{\lambda,n}$.

6. Conclusions

The radial extensions studied here reduce to the one-dimensional theory through the identities (1.3). For the radial gamma function, this reduction is essentially a dimensional shift. For the radial Laplace transform, the logarithmic variable $t = \psi_\lambda(r)$ turns it into an ordinary Laplace transform, which yields injectivity, up to equality almost everywhere, on an admissible radial class and an inversion formula for the spherical mean.

The dimension enters in an essential way through the Jacobian r^{n-1}. In the operational formula for radial-profile derivatives, it produces the inverse-power terms $g(r)/r^q$ and origin terms that first appear when the derivative order reaches the dimension. In the transport problem, it determines how the explicit solution vanishes at the front, as shown for a constant initial profile. In both cases, the one-dimensional theory supplies the mechanism, and the radial geometry determines the structure of the result.

The transform retains only the degree-zero angular component of its input. Natural extensions are transforms weighted by higher spherical harmonics, aimed at the remaining angular components, and kernels with a variable center, which lead to injectivity questions of the type studied in [10].

Use of AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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