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Research article

Unconditional stability and convergence of a discontinuous Galerkin method on graded meshes for the tempered fractional advection-diffusion equation with initial weak singularity

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Abstract: This work develops and examines a fully discrete local discontinuous Galerkin (LDG) approach on graded temporal partitions for the tempered fractional advection-diffusion problem. The tempered fractional derivative is approximated via an L1-type formula over a nonuniform time mesh, which efficiently resolves the weak singularity near the initial instant. In space, an LDG discretization with purely alternating fluxes is employed, and its stability is established through an energy estimate. It is demonstrated that the resulting scheme is unconditionally stable and converges with order $O(h^{k+1} + N^{-\min\{2-\alpha, r\alpha\}})$, where h denotes the spatial step, N is the number of time intervals, k stands for the polynomial degree, and $r \geq 1$ is the grading parameter. Several numerical experiments are performed to validate the theoretical findings and to illustrate the benefit of graded meshes over uniform ones.

Keywords: tempered fractional derivative; discontinuous Galerkin method; graded mesh; stability; error estimates

1. Introduction

Fractional partial differential equations (FPDEs) serve as essential mathematical tools for describing processes with memory and hereditary features, finding uses in physics, finance, biology, and hydrology [1, 2]. Anomalous diffusion, defined by the mean squared displacement $\langle x^2(t)\rangle \simeq t^\alpha$, appears widely across diverse complex media, including biological cell cytoplasm, amorphous semiconductors, and hydrological systems [3]. Classical fractional derivatives represent power-law memory, whereas the tempered fractional diffusion equation incorporates an exponential tempering factor into the power-law kernel. Tempered models can describe a cutoff in anomalous transport, with applications in geophysics and finance [4, 5]. Different tempered operators arise in these settings. In particular, the waiting-time-tail operator in [3] must be distinguished from the exponential-conjugation convention used below.

The construction of accurate and efficient numerical schemes for these models is thus highly important and remains an active research area. Nevertheless, a major obstacle in the numerical treatment of time-fractional equations, including the tempered version, is the weak singularity of their solutions at $t = 0$. Even when the given data are smooth, the solution often behaves as $u(x, t) - u_0(x) \sim t^\alpha$ for small t, leading to unbounded temporal derivatives [6, 7]. For related time-fractional diffusion problems [8, 9], typical solution regularity bounds take the form $|\partial_t^q u(x, t)| \le C(1 + t^{\alpha-q})$ for $q = 0, 1, 2$. Such singular behavior severely deteriorates the accuracy of conventional methods that assume high regularity and rely on uniform time steps.

To address this difficulty, Stynes et al. in their pioneering study [8] proved that the basic $L1$ scheme on properly graded temporal grids can recover an optimal convergence rate of $\min\{2 - \alpha, r\alpha\}$, with $r \ge 1$ denoting the grading parameter. This observation has inspired the creation of several high-order approximations on nonuniform meshes. For example, an $L1$-2 type formula was introduced by Qiao and Cheng [10], and Kumari et al. [11] constructed an even higher-order scheme based on cubic interpolation polynomials, attaining $\min\{4 - \alpha, r\alpha\}$ for the Caputo derivative on graded grids. Roul and Sundar [12] merged a compact finite difference discretization in space with an adaptive moving mesh in time for a weakly singular time-fractional advection-diffusion problem. In the context of tempered fractional equations, modified L1 schemes on graded meshes were proposed by Chen et al. [13] and Wang et al. [3] very recently derived a sharp α-robust L1 method on graded grids for the two-dimensional time tempered fractional Fokker-Planck equation. The notion of α-robustness, meaning that error constants remain bounded as $\alpha \to 1^-$, is essential for the reliability of a scheme over the whole interval of fractional orders [13, 14]. The error analysis in the present paper is for fixed α and does not assert α-uniform constants.

The discontinuous Galerkin (DG) framework, initially designed for hyperbolic conservation laws, has matured into a versatile and robust approach applicable to diverse partial differential equations, covering elliptic and parabolic classes [15]. Capitalizing on such merits, numerous DG discretizations have been extended to fractional systems, such as time-fractional ODEs [16], space-time fractional convection-diffusion problems [17], variable-order fractional equations [18, 19], and tempered fractional diffusion models [20, 21]. Recently, characteristic variants of DG were introduced for fractional diffusion involving local memory within time-dependent media [22]; additionally, superconvergence features of LDG schemes have been examined for related nonlinear wave equations [23]. Higher-order numerical treatments for multi-term fractional integro-differential equations featuring weakly singular kernels have also been put forward [24]. Building on these developments, we examine the graded-mesh LDG approximation for the specific tempered advection-diffusion-reaction model defined below. The present study focuses on the tempered fractional advection-diffusion problem.

$$\begin{cases} {}_0^C D_t^{\alpha,\gamma} u(x,t) + \rho u(x,t) + c\, u_x(x,t) - \dfrac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in (a,b) \times (0,T], \\ u(x,0) = u_0(x), \quad x \in [a,b], \end{cases} \tag{1.1}$$

where $\rho > 0$ is a reaction parameter, $c \ge 0$ denotes the advection speed, and f and u_0 are prescribed functions. The symbol ${}_0^C D_t^{\alpha,\gamma}$ stands for the tempered Caputo fractional derivative of order $0 < \alpha < 1$,

defined by

$$
\begin{aligned}
{}_0^C D_t^{\alpha,\gamma} u(x,t) &:= e^{-\gamma t}\, {}_0^C D_t^{\alpha}(e^{\gamma t} u(x,t)) \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\gamma(t-s)} (u_t(x,s) + \gamma u(x,s))\, ds.
\end{aligned} \tag{1.2}
$$

This is the exponential-conjugation convention, which differs from the tail-kernel operator in [3]. For example, for $\gamma > 0$ its action on 1 is $\gamma \int_0^t s^{-\alpha} e^{-\gamma s}\, ds/\Gamma(1-\alpha) > 0$, while its action on $e^{-\gamma t}$ is zero. Setting $V = e^{\gamma t} u$ transforms the model into the classical Caputo equation for V with source $e^{\gamma t} f$. The parameter $\gamma \geq 0$ controls the tempering effect. To simplify the presentation, we impose periodic boundary conditions or assume compactly supported solutions.

The principal contributions of this work are listed below.

- A graded temporal grid is constructed to discretize the tempered Caputo derivative by an $L1$-type formula on nonuniform meshes, thereby efficiently capturing the initial singularity.
- The spatial approximation relies on the LDG method with purely alternating numerical fluxes, and a suitable special projection is introduced to streamline the error analysis.
- We establish the unconditional stability of the resulting fully discrete scheme and prove an optimal convergence estimate of order $O(h^{k+1} + N^{-\min\{2-\alpha, r\alpha\}})$, where $r \geq 1$ is the mesh grading parameter.
- Several numerical experiments are carried out to confirm the sharpness of the theoretical convergence orders and to illustrate the improved performance of graded meshes over uniform ones.

The paper is organized as follows. In Section 2, we introduce some notations and auxiliary projection results that will be used in the analysis. Section 3 presents the fully discrete LDG scheme on graded temporal meshes, including the discretization of the tempered Caputo derivative and the spatial LDG formulation. Section 4 is devoted to the stability and error analysis of the fully discrete scheme. Numerical experiments are presented in Section 5. Finally, concluding remarks are given in Section 6.

2. Notations and auxiliary results

Let the computational interval be divided by the nodes

$$
a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = b,
$$

and define the cells

$$
I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \quad j = 1, \ldots, N_x,
$$

with lengths

$$
\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad 1 \leq j \leq N_x, \qquad h = \max_{1\leq j\leq N_x} \Delta x_j.
$$

For a function u, the symbols $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ stand for the values of u at the point $x_{j+\frac{1}{2}}$ approached from the right cell I_{j+1} and from the left cell I_j, respectively. The discontinuous finite element space is given by

$$
V_h^k = \{v : \, v|_{I_j} \in P^k(I_j), \ j = 1, 2, \ldots, N_x\}, \tag{2.1}
$$

where $P^k(I_j)$ denotes the set of polynomials of degree at most k on I_j.

To facilitate the subsequent error estimates, we introduce two projection operators on $[a, b]$. The first one is the standard L^2 projection $\mathcal{P}$ onto the local polynomial space, defined on each cell by

$$\int_{I_j} (\mathcal{P}\phi - \phi)v\,dx = 0, \qquad \forall\, v \in P^k(I_j). \tag{2.2}$$

The second type consists of two Gauss–Radau projections $\mathcal{P}^{\pm}$. For every j, they are specified by

$$\int_{I_j} (\mathcal{P}^{+}\phi - \phi)v\,dx = 0, \quad \forall\, v \in P^{k-1}(I_j), \qquad \mathcal{P}^{+}\phi(x^{+}_{j-\frac{1}{2}}) = \phi(x_{j-\frac{1}{2}}), \tag{2.3}$$

and

$$\int_{I_j} (\mathcal{P}^{-}\phi - \phi)v\,dx = 0, \quad \forall\, v \in P^{k-1}(I_j), \qquad \mathcal{P}^{-}\phi(x^{-}_{j+\frac{1}{2}}) = \phi(x_{j+\frac{1}{2}}). \tag{2.4}$$

These operators satisfy the approximation property [25]

$$\|\phi^e\| + h\|\phi^e\|_{\infty} + h^{\frac{1}{2}}\|\phi^e\|_{\tau_h} \le Ch^{k+1}, \tag{2.5}$$

where $\phi^e = \mathcal{P}\phi - \phi$ or $\phi^e = \mathcal{P}^{\pm}\phi - \phi$. Here $\|\cdot\|$ is the $L^2(\Omega)$ norm on $\Omega = (a, b)$ and $\|\cdot\|_{\tau_h}$ denotes the interface trace norm. We use quasi-uniform spatial meshes; the constant includes the required spatial Sobolev norm. For $k = 0$, the P^{k-1} moment conditions are empty. We write $[v] = v^{+} - v^{-}$ and include the periodic interface in all interface sums.

3. The fully discrete LDG scheme on graded meshes

For a given positive integer N, the temporal mesh is defined by

$$t_n = T\Big(\frac{n}{N}\Big)^r, \qquad n = 0, 1, \ldots, N, \tag{3.1}$$

where $r \ge 1$ denotes the mesh grading parameter. The corresponding step sizes are $\tau_n = t_n - t_{n-1}$ for $n = 1, \ldots, N$. One readily finds that $\tau_n = O(N^{-r}n^{r-1})$, which yields smaller steps close to $t = 0$ whenever $r > 1$.

To approximate the tempered Caputo derivative (1.2) at $t = t_n$, we begin with the expression

$$\begin{aligned} {}_0^C D_t^{\alpha,\gamma} u(x, t_n) &= \frac{e^{-\gamma t_n}}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha} \frac{\partial}{\partial s}\big(e^{\gamma s} u(x, s)\big)\, ds \\ &\simeq \frac{e^{-\gamma t_n}}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{e^{\gamma t_k} u(x, t_k) - e^{\gamma t_{k-1}} u(x, t_{k-1})}{\tau_k} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha}\, ds. \end{aligned} \tag{3.2}$$

Denoting by $U^n(x)$ the approximation to $u(x, t_n)$, the above formula leads to

$${}_0^C D_t^{\alpha,\gamma} u(x, t_n) \approx \sum_{k=0}^{n} \omega_{n,k} U^k(x), \tag{3.3}$$

with the coefficients $\omega_{n,k}$ given by

$$\begin{aligned}
\omega_{n,n} &= \frac{1}{\Gamma(2-\alpha)}\frac{(t_n - t_{n-1})^{1-\alpha}}{\tau_n},\\
\omega_{n,0} &= \frac{e^{-\gamma t_n}}{\Gamma(2-\alpha)}\Big[\frac{(t_n - t_1)^{1-\alpha} - (t_n - t_0)^{1-\alpha}}{\tau_1}\Big]e^{\gamma t_0},\\
\omega_{n,k} &= \frac{e^{-\gamma t_n}}{\Gamma(2-\alpha)}\Big[\frac{(t_n - t_{k-1})^{1-\alpha} - (t_n - t_k)^{1-\alpha}}{\tau_k}\\
&\quad - \frac{(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}}{\tau_{k+1}}\Big]e^{\gamma t_k}, \quad 1 \le k \le n-1.
\end{aligned} \tag{3.4}$$

For the analysis, let $V = e^{\gamma t}u$ and define the classical nonuniform L1 increment weights

$$a_{n,j} = \frac{(t_n - t_{j-1})^{1-\alpha} - (t_n - t_j)^{1-\alpha}}{\Gamma(2-\alpha)\tau_j}, \qquad \delta_\tau^\alpha z^n = \sum_{j=1}^{n} a_{n,j}(z^j - z^{j-1}).$$

Write $\delta_\tau^{\alpha,\gamma} z^n = \sum_{j=0}^{n} \omega_{n,j} z^j$. Then, $\delta_\tau^{\alpha,\gamma} z^n = e^{-\gamma t_n}\delta_\tau^\alpha (e^{\gamma t} z)^n$ exactly, and the truncation errors satisfy $R^n = \delta_\tau^{\alpha,\gamma} u(t_n) - {}_0^C D_t^{\alpha,\gamma} u(t_n) = e^{-\gamma t_n} R_V^n$. For V with $\|V'(t)\| \le C(1 + t^{\alpha-1})$ and $\|V''(t)\| \le C(1 + t^{\alpha-2})$, the relevant global consistency estimate is

$$\max_{1\le m\le N} \sum_{n=1}^{m} P_{m,n} e^{\gamma t_n} \|R^n\| \le C N^{-\min\{2-\alpha, r\alpha\}}. \tag{3.5}$$

Here $P_{m,n}$ are the nonnegative complementary weights defined in Appendix A. Lemmas A.1–A.2 give their properties and a complete proof of Eq (3.5). The constant may depend on r and α. A uniform local residual of this order is not assumed.

To apply the LDG framework, we first rewrite Eq (1.1) as a first-order system. Introducing the auxiliary variable $p = u_x$, we have

$$p = u_x, \qquad {}_0^C D_t^{\alpha,\gamma} u + \rho u + c\, u_x - p_x = f. \tag{3.6}$$

Combining the temporal approximation (3.3) with an LDG spatial discretization, the fully discrete scheme is stated as follows: For $n = 1, \dots, N$, find $u_h^n, p_h^n \in V_h^k$ such that for every $v, w \in V_h^k$,

$$\begin{aligned}
(\omega_{n,n} + \rho)(u_h^n, v) - c(u_h^n, v_x) &+ c\sum_{j=1}^{N_x}\Big(\widetilde{u_h^n} v^-\Big|_{j+\frac12} - \widetilde{u_h^n} v^+\Big|_{j-\frac12}\Big)\\
&+ (p_h^n, v_x) - \sum_{j=1}^{N_x}\Big(\widehat{p_h^n}\, v^-\Big|_{j+\frac12} - \widehat{p_h^n}\, v^+\Big|_{j-\frac12}\Big)\\
&= -\sum_{k=0}^{n-1} \omega_{n,k}(u_h^k, v) + (f^n, v),
\end{aligned} \tag{3.7a}$$

$$(p_h^n, w) + (u_h^n, w_x) - \sum_{j=1}^{N_x}\Big(\widehat{u_h^n}\, w^-\Big|_{j+\frac12} - \widehat{u_h^n}\, w^+\Big|_{j-\frac12}\Big) = 0. \tag{3.7b}$$

Here $(\cdot,\cdot)$ denotes the standard L^2 inner product over Ω. The initial value is $u_h^0 = \mathcal{P}_h^- u_0$, and we set $f^n = f(\cdot, t_n)$. For the numerical fluxes appearing in the boundary sums, we employ the alternating choice

$$\widehat{p_h^n} = (p_h^n)^+, \qquad \widehat{u_h^n} = (u_h^n)^-, \qquad \widetilde{u_h^n} = (u_h^n)^-. \tag{3.8}$$

4. Stability and convergence analysis

This section establishes the unconditional stability of the fully discrete scheme (3.7) and derives an optimal error bound. The projection operators introduced previously play an essential role in the subsequent analysis.

4.1. Unconditional stability

Theorem 4.1. *The fully discrete LDG scheme* (3.7) *on the graded time mesh is unconditionally stable. More precisely, for every $n = 1, \dots, N$, the numerical solution satisfies*

$$\|u_h^n\| \le \|u_h^0\| + C \max_{1\le m\le n} \|f^m\|,$$

where the constant C may depend on T, α, ρ, γ, and the coefficients appearing in the equation but is independent of h, N, and the grading parameter r.

Proof. Fix $n \ge 1$. Choose the test functions $v = u_h^n$ in Eq (3.7a) and $w = p_h^n$ in Eq (3.7b). Summing the two equations and applying the alternating flux definitions (3.8), we obtain the following cancellation identity:

$$\int_\Omega p_h^n (u_h^n)_x \, dx - \sum_{j=1}^{N_x} \Big(\widehat{p_h^n} (u_h^n)^- \Big|_{j+\frac{1}{2}} - \widehat{p_h^n} (u_h^n)^+ \Big|_{j-\frac{1}{2}} \Big)$$
$$+ \int_\Omega u_h^n (p_h^n)_x \, dx - \sum_{j=1}^{N_x} \Big(\widehat{u_h^n} (p_h^n)^- \Big|_{j+\frac{1}{2}} - \widehat{u_h^n} (p_h^n)^+ \Big|_{j-\frac{1}{2}} \Big) = 0.$$

This is a direct consequence of the fact that the two numerical fluxes are selected from opposite sides, causing the boundary terms to cancel exactly [15]. Hence, the diffusion coupling terms cancel after the addition.

For the advection term, integration by parts combined with the alternating fluxes gives

$$-(u_h^n, (u_h^n)_x) + \sum_{j=1}^{N_x} \Big(\widetilde{u_h^n} (u_h^n)^- \Big|_{j+\frac{1}{2}} - \widetilde{u_h^n} (u_h^n)^+ \Big|_{j-\frac{1}{2}} \Big) = \frac{1}{2} \sum_{j=1}^{N_x} [u_h^n]^2_{j-\frac{1}{2}},$$

which is nonnegative because it is a sum of squares [15].

Consequently, we arrive at the energy identity

$$\omega_{n,n} \|u_h^n\|^2 + \rho \|u_h^n\|^2 + \|p_h^n\|^2 + \frac{1}{2} c \sum_{j=1}^{N_x} [u_h^n]^2_{j-\frac{1}{2}} = - \sum_{k=0}^{n-1} \omega_{n,k} \int_\Omega u_h^k u_h^n \, dx + \int_\Omega f^n u_h^n \, dx. \tag{4.1}$$

Since $a_{n,j}$ are averages of the increasing kernel $(t_n - s)^{-\alpha}/\Gamma(1-\alpha)$, one has $0 < a_{n,1} < \cdots < a_{n,n}$. The coefficients in Eq (3.4) can be written as $\omega_{n,n} = a_{n,n}$, $\omega_{n,0} = -e^{-\gamma t_n} a_{n,1}$ and $\omega_{n,j} = e^{-\gamma (t_n - t_j)} (a_{n,j} - a_{n,j+1})$. Thus, $\omega_{n,n} > 0$ and all history coefficients are negative. Their absolute sum equals $a_{n,n}$ for $\gamma = 0$ by telescoping and can only decrease for $\gamma \ge 0$. Consequently $\sum_{j<n} |\omega_{n,j}| \le \omega_{n,n}$ and $\sum_{j=0}^{n} \omega_{n,j} \ge 0$. Using these properties in Eq (4.1) yields

$$(\omega_{n,n} + \rho) \|u_h^n\|^2 \le \sum_{k=0}^{n-1} |\omega_{n,k}| \, \|u_h^k\| \, \|u_h^n\| + \|f^n\| \, \|u_h^n\|.$$

If $\|u_h^n\| > 0$, divide by it. For fixed m, set $B = \max\{\|u_h^0\|, \rho^{-1} \max_{j\le m} \|f^j\|\}$. If earlier norms are at most B, the right-hand side after division is at most $(\omega_{n,n} + \rho)B$, so $\|u_h^n\| \le B$ by induction. The case of zero norm is immediate. This proves the stated bound with $C = \rho^{-1}$, without a step-size restriction or a Gronwall invocation. The same energy identity gives invertibility of the one-step system.

4.2. Error estimate

Let $u^n(x) = u(x, t_n)$ denote the exact solution and let u_h^n be its LDG approximation. The error is decomposed by means of the projections as

$$u^n - u_h^n = (u^n - \mathcal{P}_h^- u^n) + (\mathcal{P}_h^- u^n - u_h^n) =: \eta_u^n + \xi_u^n,$$

and similarly for p,

$$p^n - p_h^n = (p^n - \mathcal{P}_h^+ p^n) + (\mathcal{P}_h^+ p^n - p_h^n) =: \eta_p^n + \xi_p^n.$$

In view of the projection property (2.5), it is enough to bound the discrete error components $\xi_u^n,\ \xi_p^n \in\ V_h^k$.

Theorem 4.2. *Suppose that the exact solution u satisfies the regularity bounds*

$$\left|\frac{\partial^l u}{\partial t^l}(x,t)\right| \le C(1 + t^{\alpha-l}), \quad l = 0, 1, 2,$$

and has the spatial regularity

$$u \in C([0,T]; H^{k+2}_{\mathrm{per}}(\Omega)), \qquad u_t \in C((0,T]; H^{k+1}_{\mathrm{per}}(\Omega)),$$

$$\sup_{0\le t\le T} \|u(t)\|_{H^{k+2}} \le C_u, \qquad \|u_t(t)\|_{H^{k+1}} \le C_u(1 + t^{\alpha-1}).$$

Then, the error of the fully discrete LDG scheme (3.7) *on the graded mesh* (3.1) *obeys*

$$\|u(\cdot, t_n) - u_h^n\| \le C(h^{k+1} + N^{-\min\{2-\alpha, r\alpha\}}), \qquad n = 1, \dots, N,$$

where the constant C is independent of h and N but may depend on r, α, γ, T, and the stated spatial and temporal regularity bounds.

Proof. The exact pair (u^n, p^n) fulfills the variational form of Eq (3.6) up to the temporal truncation error R^n. Subtracting the discrete equations (3.7) from the exact ones and employing the above error splitting, the moment and selected-trace conditions in Eqs (2.3) and (2.4) give

$$\begin{aligned}(\omega_{n,n} + \rho)(\xi_u^n, v) - c(\xi_u^n, v_x) + c\sum_{j=1}^{N_x} \big((\xi_u^n)^- v^-\big|_{j+\frac12} - (\xi_u^n)^- v^+\big|_{j-\frac12}\big) \\ + (\xi_p^n, v_x) - \sum_{j=1}^{N_x} \big((\xi_p^n)^+ v^-\big|_{j+\frac12} - (\xi_p^n)^+ v^+\big|_{j-\frac12}\big) \\ = -\sum_{k=0}^{n-1} \omega_{n,k}(\xi_u^k, v) - \Big(\sum_{k=0}^{n} \omega_{n,k}\eta_u^k, v\Big) - \rho(\eta_u^n, v) + (R^n, v), \end{aligned} \tag{4.2}$$

$$(\xi_p^n, w) + \Big[(\xi_u^n, w_x) - \sum_{j=1}^{N_x} \big((\xi_u^n)^- w^-\big|_{j+\frac{1}{2}} - (\xi_u^n)^- w^+\big|_{j-\frac{1}{2}}\big)\Big] = -(\eta_p^n, w). \tag{4.3}$$

Only the differentiated and flux projection terms vanish. In particular, (η_p^n, w) is retained since w need not belong to P^{k-1}. To propagate the errors, use the exact exponential change of unknown and set

$$V = e^{\gamma t} u, \quad X^n = e^{\gamma t_n} \xi_u^n, \quad Z^n = e^{\gamma t_n} \xi_p^n, \quad E_u = V - \mathcal{P}_h^- V, \quad E_p = V_x - \mathcal{P}_h^+ V_x.$$

Multiply Eqs (4.2) and (4.3) by $e^{\gamma t_n}$ and test by X^n, Z^n. The same alternating-flux cancellation as in the stability proof yields

$$\begin{aligned}(\delta_\tau^\alpha X^n, X^n) + \rho\|X^n\|^2 + \|Z^n\|^2 + \frac{c}{2}\sum_e [X^n]_e^2 = (R_V^n - \delta_\tau^\alpha E_u^n - \rho E_u^n, X^n) \\ - (E_p^n, Z^n).\end{aligned} \tag{4.4}$$

Write $d_h = h^{k+1}$. The regularity assumptions and Eq (2.5) imply

$$\|E_u(t)\| + \|E_p(t)\| \le C d_h, \qquad \|E_u'(t)\| \le C d_h (1 + t^{\alpha-1}).$$

Young's inequality absorbs half of $\|Z^n\|^2$ and leaves a contribution $C d_h^2$. Define $y^n = (\|X^n\|^2 + d_h^2)^{1/2}$. Since δ_τ^α annihilates constants, the increasing L1 weights and Cauchy-Schwarz in $L^2(\Omega) \times \mathbb{R}$, applied to (X^n, d_h), give

$$(\delta_\tau^\alpha X^n, X^n) \ge y^n \delta_\tau^\alpha y^n.$$

As $y^n \ge d_h > 0$, division of Eq (4.4) after Young's inequality gives

$$\delta_\tau^\alpha y^n \le \|R_V^n\| + \|\delta_\tau^\alpha E_u^n\| + C d_h.$$

Here $X^0 = 0$ by the prescribed initial projection, so $y^0 = d_h$. Apply Lemma A.1 and Eq (3.5). The remaining history projection term is bounded without an absolute state-weight sum:

$$\begin{aligned}\sum_{j=1}^n P_{n,j} \|\delta_\tau^\alpha E_u^j\| &\le \sum_{j=1}^n P_{n,j} \sum_{k=1}^j a_{j,k} \int_{t_{k-1}}^{t_k} \|E_u'(s)\|\, ds \\ &= \int_0^{t_n} \|E_u'(s)\|\, ds \le C d_h (T + T^\alpha/\alpha).\end{aligned}$$

The bound on $\sum_j P_{n,j}$ in Lemma A.1 then yields $y^n \le C(h^{k+1} + N^{-\min\{2-\alpha, r\alpha\}})$. Finally, $\|\xi_u^n\| = e^{-\gamma t_n}\|X^n\| \le y^n$ and the triangle inequality give the stated estimate.

Remark 4.3. *If the solution is smooth in time, with* $\sup_{0\le t\le T} \|\partial_t^2 (e^{\gamma t} u)\| < \infty$*, one may set* $r = 1$*, corresponding to a uniform mesh, and the temporal error reduces to* $O(N^{-(2-\alpha)})$*, which matches the classical L1 approximation on uniform grids. This uses the separate smooth-data consistency estimate proved at the end of Appendix A. Under the weak-singularity assumptions alone,* $r = 1$ *gives the maximum-in-time rate* $N^{-\alpha}$*.*

5. Numerical experiments

In this section, we present a series of numerical experiments to verify the theoretical convergence rates established in Theorem 4.2. We consider the tempered fractional advection-diffusion Eq (1.1) on the domain $(x, t) \in (0, 1) \times (0, 1]$ with parameters $\rho = 1$, $c = 0.5$. The exact solution is chosen as

$$u(x, t) = e^{-\gamma t} t^{\alpha} \sin(2\pi x),$$

which exhibits the typical weak singularity $u(x, t) = O(t^{\alpha})$ near $t = 0$. The source term $f(x, t)$ is computed by substituting the exact solution into the governing equation. The initial condition is $u_0(x) = 0$, and periodic boundary conditions are imposed on the spatial domain.

We first examine the spatial accuracy of the LDG scheme. To check temporal contamination, we use $N = 4096$ time intervals, use the graded mesh parameter $r = \frac{2(2-\alpha)}{\alpha}$ in time, and double N to 8192 on the finest spatial grid. The relative changes in the final-time L^2 errors are at most 7.43×10^{-5} over the two fractional orders and both polynomial degrees. This is an empirical refinement check, not a rigorous bound on the remaining time error. The spatial domain is uniformly divided into N_x elements with mesh size $h = 1/N_x$. Both piecewise constant (P^0) and piecewise linear (P^1) polynomial spaces are considered.

Tables 1 and 2 display the final-time L^2 and sampled L^{∞} errors and the corresponding convergence orders for $\alpha = 0.7$ and $\alpha = 0.3$. For the P^0 case (Table 1), the observed convergence orders in both norms approach 1 for both values of α, which is consistent with the theoretical spatial accuracy $O(h^{k+1})$ with $k = 0$. For the P^1 case (Table 2), the convergence orders are close to 2 in both the L^2 and L^{∞} norms, again matching the expected rate $O(h^2)$ for $k = 1$. These results support the optimal convergence of the spatial discretization. The two fractional orders have the same exact profile at $T = 1$, so spatially dominated errors can be close; additional digits distinguish the computed entries. The sampled maximum uses 2001 points per cell.

Next, we investigate the temporal accuracy and highlight the advantage of using graded meshes over uniform ones. We fix the spatial discretization to be sufficiently fine by taking $N_x = 100$ elements with piecewise quadratic P^2 polynomials. The time interval $[0, 1]$ is divided into N subintervals, and we compare two choices of the mesh grading parameter: the uniform mesh ($r = 1$) and the graded mesh with $r = \frac{2(2-\alpha)}{\alpha}$.

Table 3 presents the results for $\alpha = 0.4$. Both the maximum nodal-in-time L^2 error and the final-time L^2 error are reported. These quantities need not have the same observed rate. The uniform-mesh rates on these grids are preasymptotic, while the graded mesh substantially reduces the maximum error. The theorem gives a maximum-in-time bound, not a sharp error law at each fixed positive time.

Additional tests address the fractional order, grading and tempering parameters. For these checks only, use $u = e^{-\gamma t}(b_0 + t^{\alpha})(1 + \sin(2\pi x))$, with the analytic source obtained by substitution, $N_x = 80$, and P^3 polynomials. The constant component makes the initial-layer error visible without relying only on the damped spatial harmonic. Table 4 tests $\alpha = 0.3, 0.7, 0.95, 0.99$; Table 5 tests $r = 1, 2, 4, 8$ at $\alpha = 0.4$ and $\gamma = 0, 0.5, 2, 5$ with nonzero initial data. In the latter cases $b_0 = 1$; otherwise $b_0 = 0$. Each source is derived from the continuous operator, not the discrete weights. The near-one cases are empirical checks and do not establish α-robustness. The direct tempered and transformed recurrences agree to 3.4×10^{-16} in a separate validation run, and an independently assembled full LDG matrix agrees with the periodic Fourier-block implementation to 9.4×10^{-15}. Source code and raw results are provided separately.

Table 1. Spatial accuracy test when $\gamma = 2.0$, $N = 4096$, $T = 1$ using piecewise P^0 polynomials. Results are shown for $r = 2(2-\alpha)/\alpha$. Errors are measured at T; $h = 1/N_x$. The maximum norm is sampled.

α	N_x	L^2-error	order	L^∞-error	order
0.7	5	3.46803262E-02	–	8.22304622E-02	–
	10	1.72660460E-02	1.01	4.20263542E-02	0.97
	20	8.67353779E-03	0.99	2.12054640E-02	0.99
	40	4.35417554E-03	0.99	1.06870617E-02	0.99
0.3	5	3.46813138E-02	–	8.22345614E-02	–
	10	1.72660697E-02	1.01	4.20267699E-02	0.97
	20	8.67354634E-03	0.99	2.12055577E-02	0.99
	40	4.35419103E-03	0.99	1.06871505E-02	0.99

Table 2. Spatial accuracy test when $\gamma = 2.0$, $N = 4096$, $T = 1$ using piecewise P^1 polynomials. Results are shown for $r = 2(2-\alpha)/\alpha$. Errors are measured at T; $h = 1/N_x$. The maximum norm is sampled.

α	N_x	L^2-error	order	L^∞-error	order
0.7	5	9.11388045E-03	–	3.35574442E-02	–
	10	2.29459003E-03	1.99	8.72350934E-03	1.94
	20	5.74522293E-04	2.00	2.20320443E-03	1.99
	40	1.43682299E-04	2.00	5.54808955E-04	1.99
0.3	5	9.11396931E-03	–	3.35579556E-02	–
	10	2.29459468E-03	1.99	8.72354733E-03	1.94
	20	5.74522538E-04	2.00	2.20322538E-03	1.99
	40	1.43682286E-04	2.00	5.54829649E-04	1.99

Table 3. Temporal errors and convergence rates when $N_x = 100$, $\gamma = 2$, $T = 1$, $\alpha = 0.4$, using piecewise P^2 polynomials. $E_{2,\max}$ is the maximum over the time nodes and $E_2(T)$ is the final-time error. Graded meshes use $r = 2(2-\alpha)/\alpha = 8$.

r	N	$E_{2,\max}$	order	$E_2(T)$	order
1	20	3.34949E-03	–	3.61112E-06	–
	40	3.42969E-03	-0.03	1.36213E-06	1.41
	80	3.39988E-03	0.01	5.31647E-07	1.36
	160	3.29827E-03	0.04	2.32682E-07	1.19
8	20	4.72569E-04	–	1.99463E-05	–
	40	1.76379E-04	1.42	6.96730E-06	1.52
	80	6.29155E-05	1.49	2.40195E-06	1.54
	160	2.19403E-05	1.52	8.25807E-07	1.54

Table 4. Additional fractional-order checks: $E_{2,\max}$ with $r = 2(2-\alpha)/\alpha$, $\gamma = 2$, $b_0 = 0$, $N_x = 80$, P^3. The last column is the predicted exponent.

α	r	$N = 512$	$N = 1024$	order	$2-\alpha$
0.30	11.333	2.13097E-05	6.77298E-06	1.65	1.70
0.70	3.714	5.45571E-05	2.22178E-05	1.30	1.30
0.95	2.211	3.75532E-05	1.81920E-05	1.05	1.05
0.99	2.040	9.58284E-06	4.77801E-06	1.00	1.01

Table 5. Parameter checks at $\alpha = 0.4$: $E_{2,\max}$, $N_x = 80$, P^3. The grading rows use $\gamma = 2$, $b_0 = 0$; the tempering rows use $r = 4$, $b_0 = 1$ and the direct tempered recurrence. The two error columns correspond to the time counts in the third column.

Check	Parameter	N_1/N_2	Error 1	Error 2	Order
Grading	$r = 1$	512/1024	1.61386E-02	1.25580E-02	0.36
	$r = 2$	512/1024	1.61299E-03	9.49801E-04	0.76
	$r = 4$	512/1024	2.95062E-05	1.01181E-05	1.54
	$r = 8$	512/1024	2.67851E-05	9.00009E-06	1.57
Tempering	$\gamma = 0.0$	128/256	2.41565E-04	8.51405E-05	1.50
	$\gamma = 0.5$	128/256	2.41564E-04	8.51404E-05	1.50
	$\gamma = 2.0$	128/256	2.41561E-04	8.51402E-05	1.50
	$\gamma = 5.0$	128/256	2.41555E-04	8.51398E-05	1.50

6. Conclusions

This paper has presented a fully discrete local discontinuous Galerkin method on graded temporal meshes for the tempered fractional advection-diffusion equation. The nonuniform $L1$ approximation effectively captures the initial weak singularity, while the LDG spatial discretization with purely alternating fluxes delivers high-order accuracy in space. By combining Gauss-Radau projections with an energy estimate and a complementary-kernel argument, unconditional stability and the optimal error bound $O(h^{k+1} + N^{-\min\{2-\alpha, r\alpha\}})$ were rigorously established. Numerical experiments validate the theoretical results and clearly illustrate the superior performance of graded meshes compared with uniform meshes. Extensions to two-dimensional problems and higher-order time discretizations will be addressed in future work.

A. Temporal estimates

For completeness, we prove the precise comparison and consistency estimates used in Sections 3 and 4. They are the zero-growth complementary-kernel estimates associated with the nonuniform L1 formula; see also [26, 27].

Lemma A.1. *Define* $P_{n,n} = a_{n,n}^{-1}$ *and, backward for* $1 \le j < n$,

$$P_{n,j} = \frac{1}{a_{j,j}} \sum_{\ell=j+1}^{n} P_{n,\ell}(a_{\ell,j+1} - a_{\ell,j}).$$

Then, $P_{n,j} \geq 0$ and

$$\sum_{j=k}^{n} P_{n,j} a_{j,k} = 1, \qquad z^n - z^0 = \sum_{j=1}^{n} P_{n,j} \delta_\tau^\alpha z^j. \tag{A.1}$$

In particular, $\delta_\tau^\alpha z^j \leq g^j$ implies $z^n \leq z^0 + \sum_{j=1}^n P_{n,j} g^j$, and

$$\sum_{j=1}^{n} P_{n,j} t_j^{-\alpha} \leq \Gamma(1-\alpha), \qquad \sum_{j=1}^{n} P_{n,j} \leq \Gamma(1-\alpha) t_n^\alpha. \tag{A.2}$$

Proof. The increasing increment weights give positivity in the backward recursion. The first identity holds at $k = n$. If it holds at $k+1$, the recursion gives $P_{n,k} a_{k,k} + \sum_{j=k+1}^n P_{n,j} a_{j,k} = \sum_{j=k+1}^n P_{n,j} a_{j,k+1} = 1$. Exchanging the finite increment sums proves the second identity and the comparison assertion. Since $a_{j,1} \geq t_j^{-\alpha}/\Gamma(1-\alpha)$, the identity at $k = 1$ gives the first bound in Eq (A.2); $t_j \leq t_n$ gives the second. Exponential conjugation replaces the inverse weights for zero initial data by $e^{-\gamma(t_n - t_j)} P_{n,j}$, preserving positivity.

Lemma A.2. *Under the temporal regularity in Section 3, $R_V^n = \delta_\tau^\alpha V(t_n) - {}_0^C D_t^\alpha V(t_n)$ satisfies*

$$\sum_{j=1}^{n} P_{n,j} \|R_V^j\| \leq C N^{-q}, \qquad q = \min\{2-\alpha, r\alpha\},$$

with C independent of N but possibly dependent on r, α, T, and the regularity constant. This proves Eq (3.5) *because $R_V^n = e^{\gamma t_n} R^n$.*

Proof. Let $I_\tau V$ be the piecewise linear interpolant and $e = V - I_\tau V$. On $[A, B] = [t_{k-1}, t_k]$, its interpolation Green function gives

$$\|e(s)\| \leq J_k, \qquad \|e(s)\| \leq \frac{B-s}{\tau_k} J_k, \qquad J_k = \int_A^B (s-A) \|V''(s)\| \, ds.$$

Indeed, split the Green-function integrals at s and use $(s-A)(B-\xi) \leq (B-s)(\xi - A)$ for $\xi \geq s$. At $A = 0$ the weighted integral is finite; continuity of V and the bound on V' justify the limiting formula. For fixed n put $K_n(s) = (t_n - s)^{-\alpha}/\Gamma(1-\alpha)$. Integration by parts bounds the residual by $\sum_k \int_{t_{k-1}}^{t_k} K_n'(s) \|e(s)\| \, ds$. The last interval contributes at most

$$\frac{J_n}{\tau_n} \int_{t_{n-1}}^{t_n} K_n'(s)(t_n - s) \, ds = J_n (a_{n,n} - K_n(t_{n-1})) \leq a_{n,n} J_n.$$

The endpoint term at t_n vanishes since $e(s) = O(t_n - s)$. On an earlier interval, convexity and monotonicity give

$$a_{n,k} \leq \frac{K_n(t_{k-1}) + K_n(t_k)}{2}, \quad a_{n,k+1} \geq K_n(t_k), \quad K_n(t_k) - K_n(t_{k-1}) \leq 2(a_{n,k+1} - a_{n,k}).$$

Therefore, with $G_k = 2J_k$,

$$\|R_V^n\| \leq a_{n,n} G_n + \sum_{k=1}^{n-1} (a_{n,k+1} - a_{n,k}) G_k. \tag{A.3}$$

Multiply by $P_{m,n}$, sum, and use its backward recursion to obtain

$$\sum_{n=1}^{m} P_{m,n}\|R_V^n\| \le 2\sum_{k=1}^{m} P_{m,k}a_{k,k}G_k.$$

Regularity gives $G_1 \le C\tau_1^\alpha$ and $G_k \le C\tau_k^2 t_{k-1}^{\alpha-2}$ for $k \ge 2$. On the graded mesh,

$$t_k^\alpha a_{k,k}G_k \le \begin{cases} CN^{-r\alpha}, & k = 1, \\ CN^{-r\alpha}k^{r\alpha-(2-\alpha)}, & k \ge 2, \end{cases} \le CN^{-q}.$$

Equation (A.2) proves the claimed global estimate. In contrast, for $V = t^\alpha$ the first local residual is $1/\Gamma(2-\alpha) - \Gamma(1+\alpha)$, independently of N.

For the smooth-data statement in Remark 4.3, the additional bound $\sup_t \|V''(t)\| < \infty$ gives $G_k \le C\tau^2$ on a uniform mesh. Equation (A.3) then implies

$$\|R_V^n\| \le C\tau^2\left(a_{n,n} + \sum_{k<n}(a_{n,k+1} - a_{n,k})\right) \le C\tau^{2-\alpha}.$$

Its complementary-kernel sum has the same order by Eq (A.2). This is the separate consistency estimate required for smooth solutions. The factor $\Gamma(1-\alpha)$ also shows why no α-uniform error constant is asserted here.

References

1. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006. https://doi.org/10.1016/S0304-0208(06)80001-0
3. C. Wang, W. Deng, X. Tang, A sharp α-robust L1 scheme on graded meshes for two-dimensional time tempered fractional Fokker-Planck equation, *Int. J. Numer. Anal. Model.*, **20** (2023), 739–771. https://doi.org/10.4208/ijnam2023-1033
4. B. Baeumer, M. M. Meerschaert, Tempered stable Lévy motion and transient superdiffusion, *J. Comput. Appl. Math.*, **233** (2010), 2438–2448. https://doi.org/10.1016/j.cam.2009.10.027
5. A. Cartea, D. del-Castillo-Negrete, Fractional diffusion models of option prices in markets with jumps, *Phys. A*, **374** (2007), 749–763. https://doi.org/10.1016/j.physa.2006.08.071
6. K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, Berlin, **2004** (2010). https://doi.org/10.1007/978-3-642-14574-2
7. L. Wei, Y. Chen, F. Qin, J. Huang, P. Li, The discontinuous Galerkin method for solving the two-dimensional transport equation with distributed-order memory kernel on triangular meshes, *Appl. Numer. Math.*, **228** (2026), 57–70. https://doi.org/10.1016/j.apnum.2026.05.009
8. M. Stynes, E. O'Riordan, J. L. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation, *SIAM J. Numer. Anal.*, **55** (2017), 1057–1079. https://doi.org/10.1137/16M1082329

9. Y. Luchko, Initial-boundary-value problems for the one-dimensional time fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **15** (2012), 141–160. https://doi.org/10.2478/s13540-012-0010-7

10. H. Qiao, A. Cheng, A fast high order method for time fractional diffusion equation with non-smooth data, *Discrete Contin. Dyn. Syst. B*, **27** (2022), 903–920. https://doi.org/10.3934/dcdsb.2021073

11. S. Kumari, A. K. Singh, V. Mehandiratta, M. Mehra, High-order approximation to Caputo derivative on graded mesh and time-fractional diffusion equation for nonsmooth solutions, *J. Comput. Nonlinear Dyn.*, **19** (2024), 101001. https://doi.org/10.1115/1.4066023

12. P. Roul, S. Sundar, Novel numerical methods based on graded, adaptive and uniform meshes for a time-fractional advection-diffusion equation subjected to weakly singular solution, *Numer. Algorithms*, **98** (2025), 531–561. https://doi.org/10.1007/s11075-024-01804-0

13. M. Chen, S. Jiang, W. Bu, Two L1 schemes on graded meshes for fractional Feynman-Kac equation, *J. Sci. Comput.*, **88** (2021), 58. https://doi.org/10.1007/s10915-021-01581-1

14. H. Chen, M. Stynes, Blow-up of error estimates in time-fractional initial-boundary value problems, *IMA J. Numer. Anal.*, **41** (2021), 974–997. https://doi.org/10.1093/imanum/draa015

15. B. Cockburn, C. W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, *SIAM J. Numer. Anal.*, **35** (1998), 2440–2463. https://doi.org/10.1137/S0036142997316712

16. W. Deng, J. S. Hesthaven, Local discontinuous Galerkin methods for fractional ordinary differential equations, *BIT Numer. Math.*, **55** (2015), 967–985. https://doi.org/10.1007/s10543-014-0531-z

17. M. Ahmadinia, Z. Safari, S. Fouladi, Analysis of local discontinuous Galerkin method for time-space fractional convection-diffusion equations, *Bit. Numer. Math.*, **58** (2018), 533–554. https://doi.org/10.1007/s10543-018-0697-x

18. M. Ahmadinia, Z. Safari, M. Abbasi, Local discontinuous Galerkin method for time variable order fractional differential equations with sub-diffusion and super-diffusion, *Appl. Numer. Math.*, **157** (2020), 602–618. https://doi.org/10.1016/j.apnum.2020.07.015

19. L. Wei, L. Liu, X. Zhang, An upwind interior penalty DG scheme for solute transport in 2D variable-order mobile-immobile model, *Entropy*, **28** (2026), 997. https://doi.org/10.3390/e28090997

20. L. Wei, L. Feng, I. Turner, Z. Mao, F. Liu, Numerical investigation of the 2D unsteady natural convection heat transfer equation with tempered fractional constitutive relationship, *Commun. Nonlinear Sci. Numer. Simul.*, **161** (2026), 110071. https://doi.org/10.1016/j.cnsns.2026.110071

21. Z. Safari, G. B. Loghmani, M. Ahmadinia, Convergence analysis of a LDG method for time-space tempered fractional diffusion equations with weakly singular solutions, *J. Sci. Comput.*, **91** (2022), 68. https://doi.org/10.1007/s10915-022-01835-6

22. D. Liu, W. Deng, Characteristic discontinuous Galerkin method for fractional diffusion equations with local memory in nonstatic media, *Phys. Fluids*, **37** (2025), 033611. https://doi.org/10.1063/5.0260459

23. M. Baccouch, Superconvergence of the local discontinuous Galerkin method for the sine-Gordon equation in one space dimension, *J. Comput. Appl. Math.*, **333** (2018), 292–313. https://doi.org/10.1016/j.cam.2017.11.007

24. S. Santra, Analysis of a higher-order scheme for multi-term time-fractional integro-partial differential equations with multi-term weakly singular kernels, *Numer. Algorithms*, **99** (2025), 1649–1695. https://doi.org/10.1007/s11075-024-01927-4

25. L. L. Wei, Y. N. He, Analysis of a fully discrete local discontinuous Galerkin method for time-fractional fourth-order problems, *Appl. Math. Model.*, **38** (2014), 1511–1522. https://doi.org/10.1016/j.apm.2013.07.040

26. H. L. Liao, D. Li, J. Zhang, Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations, *SIAM J. Numer. Anal.*, **56** (2018), 1112–1133. https://doi.org/10.1137/17M1131829

27. H. L. Liao, W. McLean, J. Zhang, A discrete Gronwall inequality with applications to numerical schemes for subdiffusion problems, *SIAM J. Numer. Anal.*, **57** (2019), 218–237. https://doi.org/10.1137/16M1175742



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