Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 21(6): 2329–2367.
DOI: 10.3934/nhm.2026087
Received: 22 July 2026
Revised: 14 August 2026
Accepted: 03 October 2026
Published: 02 November 2026

Research article

Algebraic and geometric properties of Fibonacci Cartan numbers

Bahar Kuloğlu[1], **Hasan Gökbaş**[2], **William Ramirez**[3,4,*] **and Clemente Cesarano**[4]

[1] Department of Engineering Basic Sciences, Sivas University of Science and Technology, Sivas, Türkiye

[2] Bitlis Eren University, Science and Arts Faculty, Bitlis, Türkiye

[3] Department of Natural and Exact Sciences, Universidad de la Costa, Barranquilla, Colombia

[4] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, Rome, Italy

* **Correspondence:** Email: wramirez4@cuc.edu.co.

Abstract: In this paper we developed an algebraic and geometric study of Fibonacci – Cartan numbers and Fibonacci Pauli–Cartan numbers by integrating Fibonacci sequences with the algebraic structure of Cartan numbers in pseudo–Euclidean spaces. Building on the concept of Fibonacci–Cartan numbers, we investigated their fundamental algebraic and geometric properties, including conjugation, character, norm, and causal classification, proving that they are spacelike elements and establishing their geometric interpretation within a Minkowski–type framework. Matrix, polar, and exponential representations were obtained, enabling the derivation of De Moivre-type identities, formulas for powers and roots, and characterizations in terms of the determinant and trace of real 2×2 matrices. In addition, the powers of Fibonacci–Cartan numbers were represented in terms of Chebyshev polynomials of the first and second kinds, yielding explicit polynomial expressions for their scalar and vector parts. Scalar and vector products were also analyzed, providing a consistent geometric interpretation in a three–dimensional Cartan space. Furthermore, Fibonacci Pauli–Cartan numbers were defined and studied through recurrence relations, Binet formulas, ordinary and exponential generating functions, summation formulas, and real matrix representations. Finally, rotational properties associated with lightlike directions were examined, showing that parabolic rotations degenerate into linear trajectories within the Cartan geometric framework.

Keywords: Fibonacci–Cartan numbers; Fibonacci Pauli–Cartan numbers; Cartan geometry; Pauli matrices; De Moivre's formula; parabolic rotations; Chebyshev polynomials

1. Introduction

Pseudo-Euclidean spaces equipped with indefinite or degenerate metrics play a fundamental role in differential geometry, relativity theory, and algebraic structures related to non–Euclidean kinematics [1, 2]. In particular, Cartan–type geometries provide a natural framework for studying algebraic numbers and geometric motions in spaces where the norm is not positive definite [3, 4]. Let $\mathbb{R}^3_1$ denote the three–dimensional real vector space endowed with a Cartan metric, defined via a symmetric bilinear form $\langle \cdot, \cdot \rangle$.

For a vector $v \in \mathbb{R}^3_1$, the causal character is determined by

$$\langle v, v \rangle \begin{cases} > 0, & \text{if } v \text{ is spacelike,} \\ < 0, & \text{if } v \text{ is timelike,} \\ = 0, & \text{if } v \text{ is lightlike (null),} \end{cases}$$

a classification that has no analogue in Euclidean geometry. In particular, lightlike vectors may be nonzero while having zero norm, which leads to degeneracies in both geometric constructions and algebraic representations [5, 6].

Cartan numbers arise as algebraic objects compatible with this metric structure. A Cartan number can be written in the form

$$P = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k},$$

where $a, b, c, d \in \mathbb{R}$ are real coefficients and i, j, k denote the basis elements of the Cartan algebra, whose multiplication rules are recalled below. Additionaly, the multiplication rules of the basis elements reflect the underlying Cartan metric. Similar to split–quaternions and hyperbolic numbers, the norm of a Cartan number is defined by a quadratic form that may vanish for nonzero elements, particularly in the presence of lightlike components [7, 8].

Cartan numbers provide an algebraic framework compatible with this pseudo–Euclidean setting. Their indefinite character underlies the spacelike, timelike, and lightlike cases considered in the sequel. A precise algebraic definition together with the corresponding multiplication rules is given below. Non-lightlike Cartan numbers admit hyperbolic polar and exponential representations. More precisely, a spacelike or timelike Cartan number P can be expressed as

$$P = \|P\| (\cosh\varphi + U \sinh\varphi) \quad \text{or} \quad P = \|P\| (\sinh\varphi + U \cosh\varphi),$$

where U is a unit Cartan vector satisfying $U^2 = 1$ and φ is a hyperbolic angle. Such representations generalize the classical polar form of complex numbers and have been widely used in the study of Lorentzian rotations and hyperbolic kinematics [9, 10].

From a geometric point of view, the causal character of vectors has a decisive influence on the theory of curves. While spacelike and timelike curves admit Frenet frames analogous to the Euclidean case, null and pseudo–null curves require modified Frenet constructions due to the degeneracy of the metric [11]. In particular, rotations about lightlike directions do not generate circular orbits; instead, the corresponding trajectories degenerate into straight lines, reflecting the parabolic nature of such rotations [12].

The algebraic framework of Cartan numbers provides an effective tool for describing these degenerate rotations. Exponential Cartan numbers associated with lightlike vectors naturally encode

parabolic rotations and yield explicit formulas for powers and roots. For a Cartan number written in exponential form

$$P = \|P\|e^{\varphi U},$$

generalized de Moivre–type identities can be derived, leading to distinct root structures for even and odd powers depending on the causal character of P [13].

Our aim of this paper is to establish a unified algebraic and geometric approach to Cartan numbers and their applications to rotations in $\mathbb{R}^3_1$. After introducing matrix, polar, and exponential representations of Cartan numbers, we investigate their root structures and rotational properties. Special attention is devoted to rotations about lightlike directions and their interpretation within the Frenet frame theory of null and pseudo–null curves.

Let $\mathbb{C}$ be the set of complex numbers z of the form $z = a + b\mathbf{i}$, where $a, b \in \mathbb{R}$ and $\mathbf{i}^2 = -1$. The sets of dual numbers $\mathbb{D}$ and hyperbolic numbers $\mathbb{H}$ are defined as $\mathbb{D} = \{a + c\varepsilon : a, c \in \mathbb{R}, \ \varepsilon^2 = 0\}$, $\mathbb{H} = \{a + d\mathbf{h} : a, d \in \mathbb{R}, \mathbf{h}^2 = 1\}$, respectively. Moreover, units of dual and hyperbolic numbers satisfy relations $\varepsilon \neq 0$, $\mathbf{h} \neq \pm 1$. There are many extensions and combinations of complex, dual, and hyperbolic numbers, e.g., bicomplex numbers, bihyperbolic numbers, dual-complex numbers, dual-hyperbolic numbers, etc.

Following [14], let S denote the set of Cartan numbers. A Cartan number is an element of the form

$$Y = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$, 1 is the multiplicative identity, and i, j, k are the basis elements of the Cartan algebra satisfying

$$i^2 = 1, \qquad j^2 = 0, \qquad k^2 = 0, \tag{1.1}$$

and

$$jk = 1 + i, \qquad kj = 1 - i. \tag{1.2}$$

These relations are sufficient to determine the multiplication in the Cartan algebra, provided that the algebra is associative and 1 is its multiplicative identity. Indeed, the remaining products follow from Eqs (1.1) and (1.2) as

$$ij = j, \qquad ji = -j, \qquad ik = -k, \qquad ki = k.$$

Thus, Table 1 does not introduce additional defining relations; it summarizes the complete multiplication table resulting from Eqs (1.1) and (1.2).

The complete multiplication rules for the basis elements i, j, k are summarized in Table 1.

Table 1. The Cartan number multiplication.

$\cdot$	i	j	k
i	1	j	$-k$
j	$-j$	0	$1 + i$
k	k	$1 - i$	0

In particular, besides Eqs (1.1) and (1.2),

$$ij = j, \qquad ji = -j, \qquad ik = -k, \qquad ki = k.$$

The conjugate of Y is defined by

$$\overline{Y} = a - bi - cj - dk,$$

and the real-valued character of Y is

$$C(Y) = Y\overline{Y} = \overline{Y}Y = a^2 - b^2 - 2cd.$$

The Cartan matrices that have been introduced by Öztürk form a set of 2×2 matrices as follows [14]:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

where the following relations hold:

$$\varphi_1^2 = 1, \quad \varphi_2^2 = \varphi_3^2 = 0, \quad \varphi_2\varphi_3 = 1 + \varphi_1, \quad \varphi_3\varphi_2 = 1 - \varphi_1. \tag{1.3}$$

These 2×2 real matrices provide a matrix representation of the Cartan basis elements and reproduce the corresponding Cartan multiplication rules [15].

Cartan numbers were defined in [14], providing the properties of the set of Cartan numbers and some geometric definitions, theorems, and applications. Fibonacci Cartan and Lucas Cartan numbers were introduced in [16]. The authors gave some algebraic properties of the Fibonacci Cartan and Lucas Cartan numbers, including generating functions, Binet's formula, summation formulas, and Catalan's and Cassini's identities. The purpose of the study [17] was to present a few geometric characteristics of dual Cartan numbers. It was demonstrated that the unit dual Cartan number with a lightlike dual vector also corresponds to a parabolic rotation and a translation transformation in the Cartan frame, as a unit timelike Cartan number with a lightlike vector portion corresponds to a parabolic rotation around a lightlike axis.

The Pauli matrices that have been introduced by Wolfgang Pauli form a set of 2×2 matrices as follows:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The multiplication rules are given by $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$, $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$, $\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1$, and $\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2$. These matrices are Hermitian and unitary. These 2×2 types of Hermitian matrices form a basis for the real vector space and the span of $\{I, i\sigma_1, i\sigma_2, i\sigma_3\}$ is isomorphic to the real algebra of Quaternions. Then, Kim [18] presented a representation of de Moivre's formula over Pauli quaternions. Aydın [19] investigated the Pauli–Fibonacci quaternions. İsbilir et al. [20] examined Pauli–Leonardo quaternions. Azak [21] obtained Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions. Yılmaz et al. [22] introduced generalized Pauli Fibonacci polynomial quaternions.

The Fibonacci sequence, which has the following recurrence relation, is obtained [23]:

$$F_n = F_{n-1} + F_{n-2}, \tag{1.4}$$

with $F_0 = 0$ and $F_1 = 1$. The characteristic equation of the Fibonacci sequence is $\delta^2 - \delta - 1 = 0$, and the roots of this equation can be given as: $\delta_1 = \frac{1+\sqrt{5}}{2}$ and $\delta_2 = \frac{1-\sqrt{5}}{2}$. Additionaly, the Fibonacci sequence can be extended to negative subscripts and has the following recurrence relation:

$$F_{(-n)} = F_{-(n-2)} - F_{-(n-1)}. \tag{1.5}$$

The Binet's formula of the Fibonacci sequence is written as [24]:

$$F_n = \frac{\delta_1^n - \delta_2^n}{\delta_1 - \delta_2}. \tag{1.6}$$

The following matrix is called the Fibonacci Q-matrix and is given as follows [25]:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \tag{1.7}$$

The Q-matrix provides a matrix representation of the Fibonacci recurrence. Indeed,

$$Q\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix},$$

and consequently, for $n \geq 1$,

$$Q^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Thus, successive powers of Q encode consecutive Fibonacci numbers and provide a compact matrix formulation of the recurrence relation.

In the literature, we can find generalizations of the Fibonacci numbers with respect to one or more parameters and applications of the Fibonacci numbers in different number systems. A list of many generalizations of Fibonacci numbers can be found in [26]. In [27], further generalizations of Fibonacci numbers and applications of these numbers in the theory of hypercomplex numbers were presented, such as, complex Fibonacci numbers (introduced in [8]), Fibonacci quaternions [28–30], and bicomplex Fibonacci numbers [31]. Fibonacci hybrid numbers were examined in [32]. Another generalization of Fibonacci numbers is Fibonacci polynomials. The polygon structure of Fibonacci polynomials was studied in [33]. Related analytical and computational techniques have also been successfully employed in the mathematical investigation of complex models in different applied settings [34–36].

Our motivation for this study arises from the interaction between recurrence-based number systems and the algebraic and geometric structure of Cartan numbers. Fibonacci numbers possess a simple recursive structure, whereas Cartan numbers provide a noncommutative algebraic framework equipped with an indefinite geometric character. Combining these two structures makes it possible to investigate how the classical properties of the Fibonacci sequence are reflected in Cartan algebra and, conversely, how the Cartan framework enriches Fibonacci-type constructions with geometric notions such as causal classification, polar and exponential representations, and rotations. This interaction motivates the study of Fibonacci-Cartan numbers from both algebraic and geometric viewpoints and provides a natural basis for their matrix and Pauli-type representations.

The three main ingredients of this study, Fibonacci numbers, Cartan numbers, and Pauli matrices, are connected through their complementary algebraic roles. The Fibonacci sequence provides the recursive scalar coefficients, while the Cartan basis embeds these coefficients into a noncommutative algebraic structure endowed with a pseudo-Euclidean geometric interpretation. In this setting, a Fibonacci-Cartan number combines consecutive Fibonacci numbers as its components, thereby transferring the recurrence structure of the Fibonacci sequence to the Cartan algebra. On the other hand, matrix representations provide a natural bridge between the abstract Cartan algebra and linear algebra. In particular, the

Pauli-type representation enables the corresponding Fibonacci components to be encoded in matrix form and leads naturally to the Fibonacci Pauli-Cartan construction. Thus, the Fibonacci sequence supplies the recursive structure, Cartan algebra supplies the geometric framework, and the Pauli-matrix approach supplies a matrix realization of the resulting algebraic objects.

The main contributions of this paper can be summarized as follows: First, building on the existing definition of Fibonacci–Cartan numbers, we develop their geometric structure by examining their character, norm, and causal classification and by interpreting the associated vectors in a pseudo–Euclidean framework. Second, we establish matrix-based characterizations of Fibonacci–Cartan numbers and relate their algebraic and geometric quantities to the determinant, trace, and discriminant of the corresponding real matrices. Third, polar and exponential representations are developed and used to obtain De Moivre-type relations and to investigate the corresponding power and root structures. Building on this power structure, we further derive a Chebyshev polynomial representation of Fibonacci–Cartan powers in terms of Chebyshev polynomials of the first and second kinds. This representation provides explicit polynomial expressions for the scalar and vector parts of P_n^m and establishes a direct connection between the De Moivre–type structure of Fibonacci–Cartan numbers and the recurrence framework of Chebyshev polynomials. Fourth, scalar and vector products are introduced to provide a geometric interpretation of Fibonacci–Cartan numbers in a three-dimensional Cartan space, and rotations associated with lightlike directions are analyzed, showing how the corresponding parabolic rotations lead to linear trajectories. Finally, Fibonacci Pauli–Cartan numbers are studied through their recurrence relations, Binet-type representation, ordinary and exponential generating functions, summation formulas, determinant representations, and associated real linear transformations. Thus, rather than redefining the previously introduced Fibonacci–Cartan numbers, we extend their study toward a unified algebraic, matrix-based, and geometric framework.

2. Fibonacci Cartan numbers

Definition 1. *For $n \geq 0$, the nth Fibonacci–Cartan number is defined by*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k, \tag{2.1}$$

where F_n denotes the nth Fibonacci number and $1, i, j, k$ are the basis elements of the associative unital Cartan algebra satisfying the defining relations (1.1) and (1.2), with the resulting complete multiplication rules summarized in Table 1.

On the other hand, Fibonacci–Cartan numbers can be expressed in terms of the Cartan basis units as follows:

$$\mathrm{P}_n = \begin{Bmatrix} F_n \leftrightarrow (F_n, 0, 0, 0), \\ F_{n+1}i \leftrightarrow (0, F_{n+1}, 0, 0), \\ F_{n+2}j \leftrightarrow (0, 0, F_{n+2}, 0), \\ F_{n+3}k \leftrightarrow (0, 0, 0, F_{n+3}) \end{Bmatrix}.$$

Equivalently, P_n can be written as

$$\mathrm{P}_n = F_n(1, 0, 0, 0) + F_{n+1}(0, 1, 0, 0) + F_{n+2}(0, 0, 1, 0) + F_{n+3}(0, 0, 0, 1),$$

where $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$ denote the Cartan units corresponding to 1, i, j, and k, respectively.

Furthermore, by defining

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

the term F_n is called the *scalar part* of P_n, while

$$F_{n+1}i + F_{n+2}j + F_{n+3}k$$

is referred to as the *vector part* of P_n.

2.1. Norm and classification of the Fibonacci Cartan numbers

Definition 2. *Let*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

be a Fibonacci–Cartan number. The conjugate of P_n is defined by

$$\overline{P_n} = F_n - F_{n+1}i - F_{n+2}j - F_{n+3}k.$$

Moreover, the product of a Fibonacci–Cartan number and its conjugate is commutative and given by

$$\begin{aligned} C(P_n) &= P_n\overline{P_n} = \overline{P_n}P_n \\ &= (F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k)(F_n - F_{n+1}i - F_{n+2}j - F_{n+3}k) \\ &= F_n^2 - F_{n+1}^2 - 2F_{n+2}F_{n+3}. \end{aligned}$$

The quantity $C(P_n)$ is called the *character* of the Fibonacci–Cartan number P_n.

Definition 3. *For the Fibonacci–Cartan number*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

the vector

$$\begin{aligned} V_{P_n} &= \left(F_n, \frac{F_{n+2} - F_{n+3}}{\sqrt{2}}, F_{n+1}, \frac{F_{n+2} + F_{n+3}}{\sqrt{2}}\right) \\ &= \left(F_n, -\frac{F_{n+1}}{\sqrt{2}}, F_{n+1}, \frac{F_{n+4}}{\sqrt{2}}\right) \end{aligned}$$

is called the Fibonacci–Cartan vector *associated with P_n.*

Furthermore, the character of P_n can be expressed as

$$\begin{aligned} C(P_n) &= -\langle V_{P_n}, V_{P_n}\rangle_{E_2^4} \\ &= F_n^2 - F_{n+1}^2 - 2F_{n+2}F_{n+3} \\ &= F_n^2 - F_{n+1}^2 + \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right) < 0. \end{aligned}$$

Indeed, by the Fibonacci recurrence,

$$F_{n+4} - F_{n+1} = 2F_{n+2}, \qquad F_{n+4} + F_{n+1} = 2F_{n+3},$$

and therefore

$$F_{n+4}^2 - F_{n+1}^2 = 4F_{n+2}F_{n+3}.$$

Hence,

$$-2F_{n+2}F_{n+3} = \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right),$$

which gives

$$C(P_n) = F_n^2 - F_{n+1}^2 + \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right).$$

For $n \geq 0$, the Fibonacci sequence satisfies $F_{n+1} \geq F_n$ and $F_{n+4} > F_{n+1}$. Consequently,

$$F_n^2 - F_{n+1}^2 \leq 0$$

and

$$\frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right) < 0.$$

Thus,

$$C(P_n) < 0 \qquad \text{for every } n \geq 0.$$

Therefore, according to the adopted causal classification, every Fibonacci–Cartan number P_n is spacelike.

The real number

$$\|P_n\| = \sqrt{|C(P_n)|} = \sqrt{\left|F_n^2 - F_{n+1}^2 + \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right)\right|}$$

is called the *norm* of the Fibonacci–Cartan number P_n.

Definition 4. *Let*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

be a Fibonacci–Cartan number. The vector

$$\begin{aligned} U_{P_n} &= \left(\frac{F_{n+2} - F_{n+3}}{\sqrt{2}},\ F_{n+1},\ \frac{F_{n+2} + F_{n+3}}{\sqrt{2}}\right) \\ &= \left(-\frac{F_{n+1}}{\sqrt{2}},\ F_{n+1},\ \frac{F_{n+4}}{\sqrt{2}}\right) \end{aligned}$$

is called the vector *of the Fibonacci–Cartan number P_n. This vector is regarded as an element of the Minkowski* 3*–space* E_1^3.

For the vector U_{P_n}, we define

$$\begin{aligned} C_U(P_n) &= -\frac{1}{2}F_{n+1}^2 + F_{n+1}^2 + \frac{1}{2}F_{n+4}^2 \\ &= \frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right). \end{aligned}$$

Since $C_U(P_n) > 0$, the Fibonacci–Cartan vector P_n is said to be *spacelike*. The real number

$$\|U_{P_n}\| = \sqrt{|C_U(P_n)|} = \sqrt{\frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right)}$$

is called the *norm* of the Fibonacci–Cartan vector.

Each Fibonacci–Cartan number admits a planar interpretation in a two–dimensional setting, in a manner reminiscent of, but not identical to, the classical Cartesian framework. Within this representation, one axis is associated with the scalar component of the Fibonacci–Cartan number and will be referred to as the *scalar axis*. The second axis, termed the *Fibonacci–Cartan axis*, is determined by the magnitude $\|U_{P_n}\|$ of the corresponding Cartan vector component. Accordingly, the scalar part of a Fibonacci–Cartan number determines its horizontal coordinate, while the value $\|U_{P_n}\|$ specifies its vertical coordinate in this plane.

Therefore, a Fibonacci–Cartan number can be expressed in the coordinate system as

$$P_n = (x, y) = (F_n,\ \|U_{P_n}\|) = \left(F_n,\ \sqrt{\left|\frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right)\right|}\right).$$

The Fibonacci–Cartan plane may be regarded as a subspace of the pseudo–Euclidean space E_2^4.

Example 5. *Let $n = 4$. Then*

$$P_4 = F_4 + F_5 i + F_6 j + F_7 k = 3 + 5i + 8j + 13k.$$

In Fibonacci–Cartan coordinates, we have

$$P_4 = \left(3,\ \sqrt{\left|\frac{1}{2}(25 + 441)\right|}\right) = (3,\ \sqrt{233}).$$

In this case, the Fibonacci–Cartan axis is determined by the unit Fibonacci–Cartan vector

$$U = \frac{5i + 8j + 13k}{\sqrt{233}}.$$

Definition 6. *Let*

$$P_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k, \qquad \|P_n\| \neq 0.$$

The inverse of the Fibonacci–Cartan number P_n is defined by

$$P_n^{-1} = \frac{\overline{P_n}}{C(P_n)}.$$

where $C(P_n) \neq 0$.

Corollary 7. *For every $n \geq 0$, the Fibonacci–Cartan number P_n is invertible.*

Proof. As established above,

$$C(P_n) < 0$$

for every $n \geq 0$. Hence, $C(P_n) \neq 0$ for all $n \geq 0$. Therefore, the inverse

$$P_n^{-1} = \frac{\overline{P_n}}{C(P_n)}$$

exists for every non-negative integer n. □

Example 8. *Consider the Fibonacci–Cartan number*

$$P_4 = F_4 + F_5 i + F_6 j + F_7 k = 3 + 5i + 8j + 13k.$$

Its inverse is given by

$$\begin{aligned} P_4^{-1} &= \frac{\overline{P_4}}{C(P_4)} = \frac{3 - 5i - 8j - 13k}{F_4^2 - F_5^2 + \frac{1}{2}\left(F_5^2 - F_8^2\right)} \\ &= \frac{3 - 5i - 8j - 13k}{9 - 25 + \frac{1}{2}(25 - 441)} = \frac{3 - 5i - 8j - 13k}{-224}. \end{aligned}$$

Remark 9. *The set*

$$S = \{P_n : n \geq 0\}$$

of Fibonacci–Cartan numbers is not, in general, closed under multiplication and therefore does not form a ring. Products such as $P_n P_m$ are taken in the ambient Cartan algebra.

2.2. Geometric interpretation of scalar and vector products in Fibonacci-Cartan numbers

The scalar product on the set of Fibonacci–Cartan numbers is defined by

$$\begin{gathered} \langle \cdot, \cdot \rangle_S : S \times S \to \mathbb{R}, \\ \begin{aligned} \langle P_n, P_m \rangle_S &= \frac{1}{2}\left(P_n \overline{P_m} + P_m \overline{P_n}\right) \\ &= F_n F_m - F_{n+1} F_{m+1} - F_{n+2} F_{m+3} - F_{n+3} F_{m+2}. \end{aligned} \end{gathered} \tag{2.2}$$

where

$$P_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k \quad \text{and} \quad P_m = F_m + F_{m+1} i + F_{m+2} j + F_{m+3} k.$$

This scalar product is a symmetric, bilinear, and non-degenerate indefinite form, and hence defines a pseudo-inner product structure. Moreover, the scalar product admits the following matrix representation:

$$\langle P_n, P_m \rangle_S = \begin{pmatrix} F_n \\ F_{n+1} \\ F_{n+2} \\ F_{n+3} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} F_m \\ F_{m+1} \\ F_{m+2} \\ F_{m+3} \end{pmatrix}. \tag{2.3}$$

Let

$$\mathcal{V} = \operatorname{span}_{\mathbb{R}}\{i, j, k\}$$

denote the pure vector subspace of the ambient Cartan algebra. For $P_n, P_m \in S$, the vector product is defined by

$$\begin{gathered} \times_S : S \times S \longrightarrow \mathcal{V}, \\ P_n \times_S P_m = \frac{1}{2}\left(P_n P_m - P_m P_n\right). \end{gathered}$$

Thus, $P_n \times_S P_m$ is a pure Cartan vector and need not itself be a Fibonacci–Cartan number belonging to S.

which yields

$$P_n \times_S P_m = \Big(0,\ F_{m+1}F_n - F_mF_{n+1} - F_{m+2}F_{n+3} + F_{m+3}F_{n+2},$$
$$F_{m+2}F_n - F_mF_{n+2} - F_{m+1}F_{n+3} + F_{m+3}F_{n+1},$$
$$F_nF_{m+3} - F_mF_{n+3} + F_{m+1}F_{n+2} - F_{m+2}F_{n+1}\Big).$$

Finally, the Cartan product of two Fibonacci–Cartan numbers decomposes as

$$P_n\overline{P_m} = \langle P_n, P_m\rangle_S + P_n \times_S P_m.$$

The scalar and vector products introduced above provide a natural geometric interpretation of Fibonacci–Cartan numbers within an indefinite metric framework. In particular, the scalar product $\langle\cdot,\cdot\rangle_S$ induces a pseudo–inner product structure compatible with Lorentzian geometry, while the vector product $\times_S$ encodes directional information analogous to the cross product in Minkowski spaces. This decomposition reveals a clear separation between magnitude and direction, enabling Fibonacci–Cartan numbers to be interpreted as geometric objects.

Consequently, the space of Fibonacci–Cartan numbers can be regarded as a geometric model closely related to Minkowski space, in which spacelike, timelike, and lightlike elements arise naturally. This viewpoint establishes a direct bridge between the algebraic structure developed here and the geometric analysis of curves and transformations in pseudo–Euclidean spaces, which will be explored in the following section through rotational transformations and Frenet frame formulations.

3. An algebraic framework for Fibonacci Cartan numbers

In what follows, Cartan numbers are examined through their matrix, polar, and exponential representations, each organized according to the underlying character and type. These formulations serve as the principal tools for establishing a de Moivre–type identity and for determining the roots of Cartan numbers. Furthermore, they form the geometric and algebraic basis required for the development of the rotation formula, which is treated in the subsequent section.

3.1. Matrix-based characterization of Fibonacci Cartan numbers

A faithful real 2×2 matrix representation of Fibonacci–Cartan numbers is established in this subsection. This representation provides a convenient matrix framework for studying their algebraic and geometric properties. This correspondence provides a rigorous framework for defining multiplication on Fibonacci–Cartan numbers and for deriving their fundamental algebraic properties. Moreover, Cartan numbers are classified via the determinants and discriminants of the characteristic equations associated with their corresponding matrix representations.

Theorem 10. *Let*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

be a Fibonacci–Cartan number. Then the mapping

$$\psi(P_n) = \begin{pmatrix} F_n + F_{n+2} + \dfrac{1}{2}F_{n+3} & F_{n+1} - F_{n+2} + \dfrac{1}{2}F_{n+3} \\ F_{n+1} + F_{n+2} - \dfrac{1}{2}F_{n+3} & F_n - F_{n+2} - \dfrac{1}{2}F_{n+3} \end{pmatrix} \tag{3.1}$$

gives a faithful real 2×2 *matrix representation of Fibonacci–Cartan numbers. Moreover, for any Fibonacci–Cartan numbers* P_n *and* P_m,

$$\psi(P_n + P_m) = \psi(P_n) + \psi(P_m)$$

and

$$\psi(P_n P_m) = \psi(P_n)\psi(P_m),$$

where the sum and product on the left-hand sides are taken in the ambient Cartan algebra.

Proof. The Cartan algebra admits the matrix representation

$$\psi(a + bi + cj + dk) = \begin{pmatrix} a + c + \dfrac{d}{2} & b - c + \dfrac{d}{2} \\ b + c - \dfrac{d}{2} & a - c - \dfrac{d}{2} \end{pmatrix}.$$

Taking

$$a = F_n, \qquad b = F_{n+1}, \qquad c = F_{n+2}, \qquad d = F_{n+3},$$

gives the stated matrix representation of P_n.

Since the matrices corresponding to the Cartan basis elements satisfy the same multiplication rules as $1, i, j, k$, the mapping preserves addition and multiplication in the ambient Cartan algebra. Hence, for Fibonacci–Cartan numbers P_n and P_m,

$$\psi(P_n + P_m) = \psi(P_n) + \psi(P_m)$$

and

$$\psi(P_n P_m) = \psi(P_n)\psi(P_m).$$

Finally, if

$$\psi(P_n) = \psi(P_m),$$

then the corresponding Cartan coefficients are equal, so that

$$F_n = F_m, \qquad F_{n+1} = F_{m+1}, \qquad F_{n+2} = F_{m+2}, \qquad F_{n+3} = F_{m+3},$$

and consequently $P_n = P_m$. Thus, the representation is faithful. □

The following result should be understood as the specialization of the general matrix identities for Cartan numbers to the Fibonacci–Cartan case. Indeed, for a general Cartan number $Y = a + bi + cj + dk$, its matrix representation satisfies

$$\det \psi(Y) = a^2 - b^2 - 2cd = C(Y).$$

Thus, the determinant and norm identities below follow by taking $a = F_n$, $b = F_{n+1}$, $c = F_{n+2}$, and $d = F_{n+3}$. The Fibonacci structure becomes relevant in expressing the vector character in terms of consecutive Fibonacci numbers.

Theorem 11. *Let $E \in M_{2\times 2}(\mathbb{R})$ be the matrix corresponding to a Fibonacci–Cartan number P_n. Then the following relations hold:*

$$C(P_n) = \det E, \qquad \|P_n\| = \sqrt{|\det E|},$$

and

$$C_U(P_n) = \frac{(\operatorname{tr} E)^2 - 4\det E}{4}.$$

Proof. Let

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

be a Fibonacci–Cartan number, and let $E = \psi(P_n)$. Using the matrix representation of the Cartan basis elements, we obtain Then

$$E = \begin{pmatrix} F_n + F_{n+2} + \frac{1}{2}F_{n+3} & F_{n+1} - F_{n+2} + \frac{1}{2}F_{n+3} \\ F_{n+1} + F_{n+2} - \frac{1}{2}F_{n+3} & F_n - F_{n+2} - \frac{1}{2}F_{n+3} \end{pmatrix}.$$

A direct computation gives

$$\begin{aligned} \det E &= \left(F_n + F_{n+2} + \frac{1}{2}F_{n+3}\right)\left(F_n - F_{n+2} - \frac{1}{2}F_{n+3}\right) \\ &\quad - \left(F_{n+1} - F_{n+2} + \frac{1}{2}F_{n+3}\right)\left(F_{n+1} + F_{n+2} - \frac{1}{2}F_{n+3}\right) \\ &= F_n^2 - F_{n+1}^2 - 2F_{n+2}F_{n+3} \\ &= C(P_n). \end{aligned}$$

Consequently,

$$\|P_n\| = \sqrt{|C(P_n)|} = \sqrt{|\det E|}.$$

Moreover,

$$\begin{aligned} \operatorname{tr} E &= F_n + F_{n+2} + \frac{1}{2}F_{n+3} + F_n - F_{n+2} - \frac{1}{2}F_{n+3} \\ &= 2F_n. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{(\operatorname{tr} E)^2 - 4\det E}{4} &= F_n^2 - \left(F_n^2 - F_{n+1}^2 - 2F_{n+2}F_{n+3}\right) \\ &= F_{n+1}^2 + 2F_{n+2}F_{n+3}. \end{aligned}$$

Since

$$F_{n+3} = F_{n+1} + F_{n+2}$$

and

$$F_{n+4} = F_{n+2} + F_{n+3} = F_{n+1} + 2F_{n+2},$$

we have

$$\begin{aligned} F_{n+1}^2 + 2F_{n+2}F_{n+3} &= F_{n+1}^2 + 2F_{n+2}(F_{n+1} + F_{n+2}) \\ &= \frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right) \\ &= C_U(P_n). \end{aligned}$$

Hence,

$$C_U(P_n) = \frac{(\operatorname{tr} E)^2 - 4\det E}{4}.$$

This completes the proof. □

Corollary 12. *Let*

$$\Delta_E = (\operatorname{tr} E)^2 - 4\det E$$

denote the discriminant of the characteristic polynomial of the real 2×2 *matrix* E*. Then*

$$C_U(P_n) = \frac{\Delta_E}{4}.$$

3.2. Polar analysis of Fibonacci Cartan numbers

In this section, the polar form of a Cartan number is developed in accordance with its character and classification. Because the polar representation is jointly determined by the causal nature of the corresponding Cartan vector and the inherent character of the Cartan number, the resulting expressions are derived through the theorems presented below.

Theorem 13. *Since Fibonacci–Cartan numbers are of spacelike type, every Fibonacci–Cartan number can be expressed in the polar form*

$$P_n = \|P_n\|\left(\sinh\varphi + U\cosh\varphi\right). \tag{3.2}$$

where

$$\sinh\varphi = \frac{F_n}{\|P_n\|}, \qquad \cosh\varphi = \frac{\sqrt{\frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right)}}{\|P_n\|},$$

and

$$U^2 = 1, \qquad U = \frac{F_{n+1}i + F_{n+2}j + F_{n+3}k}{\sqrt{\frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right)}}.$$

Proof. Let

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

be a spacelike Fibonacci–Cartan number. Then its character satisfies

$$C(P_n) = F_n^2 - F_{n+1}^2 + \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right) < 0,$$

and the norm of P_n is given by

$$\|P_n\| = \sqrt{-F_n^2 + F_{n+1}^2 - \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right)}.$$

Moreover, the associated Fibonacci–Cartan vector

$$U_{P_n} = (F_{n+1}, F_{n+2}, F_{n+3})$$

is spacelike, since

$$C_U(P_n) = \frac{1}{2}\left(F_{n+1}^2 + F_{n+4}^2\right) > 0.$$

Consequently,

$$0 < F_n^2 < F_{n+1}^2 - \frac{1}{2}\left(F_{n+1}^2 - F_{n+4}^2\right),$$

which enables P_n to be written in the form

$$P_n = \|P_n\|\left(\sinh\varphi + U\cosh\varphi\right),$$

where

$$\sinh\varphi = \frac{F_n}{\|P_n\|}, \qquad \cosh\varphi = \frac{\|U_{P_n}\|}{\|P_n\|}, \qquad U = \frac{F_{n+1}i + F_{n+2}j + F_{n+3}k}{\|U_{P_n}\|}.$$

This completes the proof. □

Example 14. *Let $n = 3$. Then*

$$P_3 = F_3 + F_4 i + F_5 j + F_6 k = 2 + 3i + 5j + 8k.$$

The character of P_3 is

$$C(P_3) = F_3^2 - F_4^2 + \frac{1}{2}\left(F_4^2 - F_7^2\right) = -85 < 0,$$

which shows that P_3 is spacelike. Furthermore,

$$C_U(P_3) = \frac{1}{2}\left(F_4^2 + F_7^2\right) = 89 > 0,$$

so that the corresponding Fibonacci–Cartan vector is also spacelike.

Hence, the polar form of P_3 is given by

$$P_3 = \sqrt{85}\left(\sinh\varphi + U\cosh\varphi\right),$$

where

$$\sinh\varphi = \frac{2}{\sqrt{85}}, \qquad \cosh\varphi = \frac{\sqrt{89}}{\sqrt{85}}, \qquad U = \frac{3i + 5j + 8k}{\sqrt{89}}, \qquad U^2 = 1.$$

Corollary 15. *The argument of a spacelike Fibonacci–Cartan number*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

is defined by

$$\varphi = \ln\left(\frac{|F_n| + \|U_{P_n}\|}{\|P_n\|}\right).$$

Example 16. *For $n = 3$, we have*

$$P_3 = 2 + 3i + 5j + 8k, \qquad \|P_3\| = \sqrt{85}, \qquad \|U_{P_3}\| = \sqrt{89}.$$

Thus, the argument of P_3 is

$$\varphi = \ln\left(\frac{2 + \sqrt{89}}{\sqrt{85}}\right).$$

3.3. Logarithmic analysis of Fibonacci Cartan numbers

Let P_n be a spacelike Fibonacci–Cartan number. Assume that $F_n > 0$ and set $s = \|P_n\|$. Let

$$U = \frac{F_{n+1}i + F_{n+2}j + F_{n+3}k}{\|U_{P_n}\|}$$

denote the corresponding unit Cartan vector. Then $U^2 = 1$, and the polar representation of P_n is

$$P_n = s\,(\sinh\varphi + U\cosh\varphi)\,.$$

Since U is fixed for a given Fibonacci–Cartan number, differentiating with respect to s and φ gives

$$\begin{aligned} dP_n &= (\sinh\varphi + U\cosh\varphi)\; ds + s\,(\cosh\varphi + U\sinh\varphi)\; d\varphi \\ &= P_n\frac{ds}{s} + P_n U\, d\varphi \\ &= P_n\left(\frac{ds}{s} + U\, d\varphi\right), \end{aligned}$$

where we have used $U^2 = 1$ and

$$P_n U = s\,(\cosh\varphi + U\sinh\varphi)\,.$$

On the other hand, since $U^2 = 1$, the exponential of $U\varphi$ satisfies

$$e^{U\varphi} = \cosh\varphi + U\sinh\varphi.$$

Multiplying this identity by U, we obtain

$$Ue^{U\varphi} = U\cosh\varphi + U^2\sinh\varphi = \sinh\varphi + U\cosh\varphi.$$

Therefore, the correct exponential representation of P_n is

$$P_n = sUe^{U\varphi}.$$

Equivalently,

$$UP_n = se^{U\varphi},$$

because $U^2 = 1$.

Since UP_n belongs to the commutative subalgebra generated by 1 and U, its logarithmic representation may be written as

$$\ln(UP_n) = \ln s + U\varphi.$$

Thus, the logarithmic and exponential representations associated with the spacelike polar form are

$$\ln(UP_n) = \ln \|P_n\| + U\varphi$$

and

$$P_n = \|P_n\| \, Ue^{U\varphi}.$$

Example 17. *Let*

$$P_3 = 2 + 3i + 5j + 8k.$$

From the previous computations,

$$\|P_3\| = \sqrt{85}, \qquad \|U_{P_3}\| = \sqrt{89},$$

and hence

$$U = \frac{3i + 5j + 8k}{\sqrt{89}}, \qquad U^2 = 1.$$

Moreover,

$$\varphi = \ln\left(\frac{2 + \sqrt{89}}{\sqrt{85}}\right).$$

Therefore, the exponential representation of P_3 is

$$P_3 = \sqrt{85}\,\frac{3i + 5j + 8k}{\sqrt{89}}\,\exp\left[\frac{3i + 5j + 8k}{\sqrt{89}}\,\ln\left(\frac{2 + \sqrt{89}}{\sqrt{85}}\right)\right].$$

Equivalently,

$$UP_3 = \sqrt{85}\,\exp\left[\frac{3i + 5j + 8k}{\sqrt{89}}\,\ln\left(\frac{2 + \sqrt{89}}{\sqrt{85}}\right)\right].$$

4. De Moivre–type relations and root structure of Fibonacci Cartan numbers

In this section, we investigate the power and root structures of Fibonacci–Cartan numbers by means of their polar representations. We first establish a De Moivre–type formula and then use it to derive the corresponding root representations.

4.1. De Moivre relations in the Fibonacci Cartan setting

The introduction of the Fibonacci–Cartan spacelike sphere is motivated by the pseudo-Euclidean geometry induced by the bilinear form considered above. In this setting, the level sets of the associated quadratic form provide the natural geometric loci on which the causal character of Fibonacci–Cartan vectors can be studied. In particular, the spacelike sphere represents a constant pseudo-norm surface for spacelike vectors and therefore provides a geometric setting for describing their directional behavior.

This construction is also closely related to the rotational properties considered below. Indeed, the rotations studied in the Fibonacci–Cartan space are transformations adapted to the underlying indefinite metric, and hence their geometric interpretation is naturally expressed in terms of the corresponding pseudo-Euclidean level surfaces. Thus, the Fibonacci–Cartan spacelike sphere is introduced not as an independent geometric object, but as a natural geometric framework connecting the metric structure of the space with the rotational transformations investigated in the subsequent results.

Theorem 18 (De Moivre-type formula for Fibonacci–Cartan numbers)**.** *Let P_n be a spacelike Fibonacci–Cartan number with polar representation*

$$P_n = \|P_n\|(\sinh\phi + U\cosh\phi), \qquad U^2 = 1.$$

Then, for every positive integer m,

$$P_n^m = \begin{cases} \|P_n\|^m(\sinh(m\phi) + U\cosh(m\phi)), & m \text{ is odd}, \\ \|P_n\|^m(\cosh(m\phi) + U\sinh(m\phi)), & m \text{ is even}. \end{cases} \tag{4.1}$$

Proof. We proceed by mathematical induction on m. For $m = 1$, the result follows immediately from the polar representation

$$P_n = \|P_n\|(\sinh\phi + U\cosh\phi).$$

For $m = 2$, using $U^2 = 1$, we obtain

$$\begin{aligned} P_n^2 &= \|P_n\|^2(\sinh\phi + U\cosh\phi)^2 \\ &= \|P_n\|^2\left(\sinh^2\phi + \cosh^2\phi + 2U\sinh\phi\cosh\phi\right) \\ &= \|P_n\|^2(\cosh(2\phi) + U\sinh(2\phi)), \end{aligned}$$

which is the required formula for the even case. Now suppose that the formula holds for some positive integer m. First, assume that m is even. Then

$$P_n^m = \|P_n\|^m(\cosh(m\phi) + U\sinh(m\phi)).$$

Multiplying by P_n, we obtain

$$\begin{aligned} P_n^{m+1} &= P_n^m P_n \\ &= \|P_n\|^{m+1}(\cosh(m\phi) + U\sinh(m\phi))(\sinh\phi + U\cosh\phi) \\ &= \|P_n\|^{m+1}\Big[\cosh(m\phi)\sinh\phi + \sinh(m\phi)\cosh\phi \\ &\qquad + U(\cosh(m\phi)\cosh\phi + \sinh(m\phi)\sinh\phi)\Big] \end{aligned}$$

$$= \|P_n\|^{m+1}(\sinh((m+1)\phi) + U\cosh((m+1)\phi)).$$

Since $m+1$ is odd, this is precisely the required odd-power formula. Next, assume that m is odd. Then

$$P_n^m = \|P_n\|^m(\sinh(m\phi) + U\cosh(m\phi)).$$

Therefore,

$$\begin{aligned} P_n^{m+1} &= P_n^m P_n \\ &= \|P_n\|^{m+1}(\sinh(m\phi) + U\cosh(m\phi))(\sinh\phi + U\cosh\phi) \\ &= \|P_n\|^{m+1}\Big[\sinh(m\phi)\sinh\phi + \cosh(m\phi)\cosh\phi \\ &\qquad + U(\sinh(m\phi)\cosh\phi + \cosh(m\phi)\sinh\phi)\Big] \\ &= \|P_n\|^{m+1}(\cosh((m+1)\phi) + U\sinh((m+1)\phi)). \end{aligned}$$

Since $m+1$ is even, this is the required even-power formula. Thus, both parity cases follow by induction, and the proof is complete. □

Example 19. *Let*

$$P_3 = 2 + 3i + 5j + 8k.$$

From the previous computations, we have

$$\|P_3\| = \sqrt{85}, \qquad U = \frac{3i + 5j + 8k}{\sqrt{89}},$$

and

$$\sinh\phi = \frac{2}{\sqrt{85}}, \qquad \cosh\phi = \frac{\sqrt{89}}{\sqrt{85}}.$$

Hence,

$$\phi = \ln\left(\frac{2 + \sqrt{89}}{\sqrt{85}}\right).$$

Since $m = 3$ is odd, Theorem 18 gives

$$P_3^3 = (\sqrt{85})^3\,(\sinh(3\phi) + U\cosh(3\phi)).$$

Using the triple-angle identities

$$\sinh(3\phi) = 3\sinh\phi + 4\sinh^3\phi$$

and

$$\cosh(3\phi) = 4\cosh^3\phi - 3\cosh\phi,$$

we obtain

$$\sinh(3\phi) = 3\left(\frac{2}{\sqrt{85}}\right) + 4\left(\frac{2}{\sqrt{85}}\right)^3 = \frac{542}{85\sqrt{85}},$$

and

$$\cosh(3\phi) = 4\left(\frac{\sqrt{89}}{\sqrt{85}}\right)^3 - 3\left(\frac{\sqrt{89}}{\sqrt{85}}\right) = \frac{101\sqrt{89}}{85\sqrt{85}}.$$

Therefore,

$$\begin{aligned} P_3^3 &= 85\sqrt{85}\left[\frac{542}{85\sqrt{85}} + \frac{3i + 5j + 8k}{\sqrt{89}}\frac{101\sqrt{89}}{85\sqrt{85}}\right] \\ &= 542 + 303i + 505j + 808k. \end{aligned}$$

4.2. Root structure of Fibonacci Cartan numbers

Theorem 20. *Let*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

be a Fibonacci–Cartan number. Then the mth roots of P_n are given as follows.

(i) If

$$P_n = \|P_n\|(\cosh\varphi + U\sinh\varphi),$$

then its mth roots are of the form

$$s = \|P_n\|^{1/m}\left(\cosh\frac{\varphi}{m} + U\sinh\frac{\varphi}{m}\right).$$

(ii) If

$$P_n = \|P_n\|(\sinh\varphi + U\cosh\varphi),$$

then for odd m, its mth roots are given by

$$s = \|P_n\|^{1/m}\left(\sinh\frac{\varphi}{m} + U\cosh\frac{\varphi}{m}\right).$$

(iii) If m is even, then the Fibonacci–Cartan number

$$P_n = \|P_n\|(\sinh\varphi + U\cosh\varphi)$$

has no mth roots.
On the other hand, if

$$P_n = F_n(1 + U),$$

then its roots are given by

$$s_1 = \frac{\sqrt[m]{2F_n}}{2}(1 + U), \qquad m \text{ odd},$$

and

$$s_2 = \pm\frac{\sqrt[m]{2F_n}}{2}(1 + U), \qquad m \text{ even}.$$

Proof. **(i)** Let

$$s = \|s\|(\cosh\beta + U\sinh\beta),$$

be an mth root of P_n, that is, $s^m = P_n$. By Theorem 18, we obtain

$$s^m = \|s\|^m(\cosh(m\beta) + U\sinh(m\beta)).$$

Comparing this expression with

$$P_n = \|P_n\|(\cosh\varphi + U\sinh\varphi),$$

we find

$$\|s\|^m = \|P_n\|, \qquad m\beta = \varphi.$$

Hence,

$$s = \|P_n\|^{1/m}\left(\cosh\frac{\varphi}{m} + U\sinh\frac{\varphi}{m}\right).$$

(ii) The proof follows analogously to part (i) by using the second form of Theorem 18 for odd m.

(iii) Let

$$s = s^*(1 + \varepsilon U)$$

be a root of the equation $s^m = P_n$, where m is odd and $s^* \in \mathbb{R}$ denotes the scalar coefficient to be determined. Using Theorem 18, we have

$$F_n(1 + U) = (s^*)^m 2^{m-1}(1 + \varepsilon U).$$

Solving for s^* yields for m is odd

$$s_1 = \frac{\sqrt[m]{2F_n}}{2}(1 + U),$$

If m is even, the roots occur in pairs, giving

$$s_2 = \pm\frac{\sqrt[m]{2F_n}}{2}(1 + U).$$

This completes the proof. □

5. The geometry of the Fibonacci Cartan number

Our purpose of this section is to provide a geometric framework for the Fibonacci–Cartan structures introduced in the preceding sections. To this end, we first describe the underlying three-dimensional space and its associated indefinite bilinear form, and then consider the Fibonacci vectors as a distinguished family in this space. The subsequent geometric constructions are introduced in order to examine the metric and rotational behavior of these vectors within the corresponding pseudo-Euclidean setting. In particular, the notions introduced below provide the geometric background for the spacelike sphere and the rotational transformations studied later in the section. Thus, the definitions in this section are intended to connect the algebraic structure of Fibonacci–Cartan numbers with their geometric interpretation.

In the final part of this work, we analyze the rotational properties of null and pseudo–null curves by means of the Frenet frame formalism. Owing to the fact that the Cartan number system is formulated on vectors in $\mathbb{R}^3$ equipped with a modified metric, the construction of a suitable metric space becomes essential. Following the introduction of the basic geometric concepts related to this space, we concentrate on the study of rotations generated along lightlike directions.

Throughout the remainder of the paper, a vector

$$\mathbf{m} = (F_m, F_{m+1}, F_{m+2}) \in \mathbb{R}^3,$$

will be identified with the corresponding pure Cartan number

$$F_m i + F_{m+1} j + F_{m+2} k.$$

Accordingly, the Cartan scalar product of the Cartan vectors

$$\mathbf{m} = (F_m, F_{m+1}, F_{m+2}) \quad \text{and} \quad \mathbf{n} = (F_n, F_{n+1}, F_{n+2})$$

is defined by

$$\langle \mathbf{m}, \mathbf{n} \rangle = F_m F_n + F_{n+1} F_{m+2} + F_{n+2} F_{m+1}.$$

For the geometric analysis in the three-dimensional Cartan space, we adopt the sign-reversed restriction of the pseudo-inner product introduced in Section 2.2 to the pure Cartan vector part. More precisely, if

$$V = bi + cj + dk, \qquad W = fi + gj + hk,$$

then

$$\langle V, W \rangle_3 = -\langle V, W \rangle_S = bf + ch + dg.$$

This sign convention is adopted throughout this section and determines the corresponding causal classification.

Definition 21 (Fibonacci–Cartan 3-space)**.** *Let*

$$\mathbb{R}^3_C = \{X = (x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}.$$

For arbitrary vectors

$$X = (x_1, x_2, x_3), \qquad Y = (y_1, y_2, y_3) \in \mathbb{R}^3_C,$$

we define the bilinear form

$$\langle X, Y \rangle_C = x_1 y_1 + x_2 y_3 + x_3 y_2.$$

Equivalently,

$$\langle X, Y \rangle_C = X^T G Y, \qquad G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since

$$\det G = -1 \neq 0,$$

this bilinear form is non-degenerate. Moreover, the eigenvalues of G are $1, 1, -1$*; hence, the form is indefinite and has signature* $(2, 1)$*. The vectors*

$$\mathbf{F}_n = (F_n, F_{n+1}, F_{n+2}), \qquad n \geq 0,$$

form a distinguished family of vectors in $\mathbb{R}^3_C$*. For two such vectors*

$$\mathbf{F}_m = (F_m, F_{m+1}, F_{m+2}) \quad \textit{and} \quad \mathbf{F}_n = (F_n, F_{n+1}, F_{n+2}),$$

the bilinear form becomes

$$\langle \mathbf{F}_m, \mathbf{F}_n \rangle_C = F_m F_n + F_{m+1} F_{n+2} + F_{m+2} F_{n+1}.$$

We refer to $\mathbb{R}^3_C$ *endowed with this bilinear form, together with the distinguished family* $\{\mathbf{F}_n\}_{n\geq 0}$*, as the Fibonacci–Cartan* 3*-space.*

For vectors

$$\mathbf{F} = (F_m, F_{m+1}, F_{m+2}) \quad \text{and} \quad \mathbf{F}^* = (F_n, F_{n+1}, F_{n+2}),$$

a Fibonacci–Cartan number will be regarded as a vector in the Cartan 3–space throughout the paper. Moreover, since

$$\Delta(U) = F_n^2 + 2F_{n+1}F_{n+2} > 0,$$

the vector U is spacelike. When negative Fibonacci indices are considered, including both Fibonacci–Cartan numbers and their negative counterparts, it may occur that

$$\langle \mathbf{F}, \mathbf{F}^* \rangle = 0.$$

In this case, the vectors $\mathbf{F}$ and $\mathbf{F}^*$ are said to be *pseudo–orthogonal*.

Definition 22. *The vector product in the Cartan* 3*–space is defined by*

$$\mathbf{F} \times \mathbf{F}^* = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_n & F_{n+1} & F_{n+2} \\ F_m & F_{m+1} & F_{m+2} \end{vmatrix}.$$

Hence, the Fibonacci–Cartan product of

$$P_n = S(P_n) + V(P_n) \quad \text{and} \quad P_m = S(P_m) + V(P_m)$$

can be written as

$$P_n P_m = S(P_n)S(P_m) + S(P_n)V(P_m) + S(P_m)V(P_n) + \langle V(P_n), V(P_m) \rangle + V(P_n) \times V(P_m).$$

Consequently, the vector triple product satisfies

$$\mathbf{F} \times (\mathbf{F}^* \times \mathbf{F}^{**}) = \langle \mathbf{F}, \mathbf{F}^* \rangle \mathbf{F}^{**} - \langle \mathbf{F}, \mathbf{F}^{**} \rangle \mathbf{F}^*,$$

where

$$\mathbf{F}^{**} = (F_k, F_{k+1}, F_{k+2}),$$

and F_k denotes the kth Fibonacci number and $F^{**} = (F_k, F_{k+1}, F_{k+2})$ denotes the third Fibonacci–Cartan vector in the vector triple product, and F_k denotes the kth Fibonacci number.

Definition 23. *The* Fibonacci–Cartan spacelike sphere *is defined as the set*

$$\mathcal{S} = \left\{(F_a, F_b, F_c) \in \mathbb{R}^3 \ : \ F_a^2 + 2F_bF_c > 0\right\} \cup \{(0,0,0)\},$$

where F_a, F_b, and F_c denote the ath, bth, and cth Fibonacci numbers, respectively.

Having established the algebraic and geometric structure of the Fibonacci–Cartan 3–space, we are now in a position to investigate rotational transformations within this framework. In particular, the classification of vectors into spacelike, lightlike, and pseudo–null types enables a precise formulation of rotations adapted to the underlying indefinite metric. In the following section, we focus on rotations generated about a lightlike direction, which play a fundamental role in the geometry of null and pseudo–null curves. These rotations are formulated in terms of Fibonacci–Cartan numbers and are analyzed through their action on Frenet frames, leading to explicit rotation formulas specific to the Cartan setting.

5.1. Rotation on the line

Throughout this study, the notations P_U and $P_U^{\perp}$ will be used to denote planes in the Fibonacci–Cartan 3–space. The planes

$$P_U = \text{span}\{0, U\} \quad \text{and} \quad P_U^{\perp}$$

are mutually pseudo–orthogonal. In this part, we demonstrate that if the position vector of a point undergoing rotation is pseudo–orthogonal to the corresponding rotation axis, then the usual rotation circle collapses into a linear trajectory. In the setting of Cartan 3–space, the subspaces span$\{i, j\}$ and span$\{i, k\}$ form a dual pair of planes. In this context, the vector j is pseudo–orthogonal to the plane span$\{i, j\}$, whereas vector k is pseudo–orthogonal to the plane span$\{i, k\}$.

Elements belonging to these dual planes can be described by ordered pairs (x, y), where x denotes the spacelike component and y corresponds to the lightlike component. For such a point (x, y), the argument in the sense of dual numbers is given by the ratio y/x. Exploiting these structural features of Cartan 3–space, we proceed to examine rotation transformations generated by the lightlike directions j and k.

Theorem 24. *Let*

$$P_n = 1 + \varphi j$$

and

$$F^* = F_m i + F_{m+1} j \in P_j^{\perp}, \qquad F_m \neq 0,$$

and define the mapping

$$L_{P_n} : P_j^{\perp} \longrightarrow P_j^{\perp}, \qquad F^* \longmapsto L_{P_n}(F^*) = P_n F^*.$$

Then $L_{P_n}(F^*)$ represents a parabolic rotation of F^* with angle $-\varphi$ in the plane $P_j^{\perp}$. The corresponding trajectory lies on the straight line

$$\ell(t) = F_m i + F_{m+1} j + t j,$$

and the radius of rotation is $|F_m|$.

Proof. Using the Cartan multiplication rules $ji = -j$ and $j^2 = 0$, we obtain

$$\begin{aligned} L_{P_n}(F^*) &= (1+\varphi j)(F_m i + F_{m+1} j) \\ &= F_m i + F_{m+1} j + \varphi F_m ji + \varphi F_{m+1} j^2 \\ &= F_m i + (F_{m+1} - F_m \varphi)\, j. \end{aligned}$$

Hence, $L_{P_n}(F^*) \in P_j^{\perp}$. Since $P_j^{\perp}$ is identified with the dual-number plane, the argument of F^* is

$$\arg(F^*) = \frac{F_{m+1}}{F_m},$$

whereas

$$\arg(L_{P_n}(F^*)) = \frac{F_{m+1} - F_m\varphi}{F_m} = \frac{F_{m+1}}{F_m} - \varphi.$$

Therefore, the argument changes by $-\varphi$, which gives the stated parabolic rotation.

Moreover, since $j^2 = 0$,

$$(1+\varphi j)^r = 1 + r\varphi j$$

for every positive integer r. Thus,

$$L_{P_n}^r(F^*) = F_m i + (F_{m+1} - rF_m\varphi)\, j,$$

so the iterates lie on

$$\ell(t) = F_m i + F_{m+1} j + tj.$$

Finally, the coefficient of the spacelike component i remains F_m; equivalently,

$$C(F^*) = C(L_{P_n}(F^*)) = -F_m^2.$$

Hence, the radius is preserved and is equal to $|F_m|$. □

Theorem 25. *Let*

$$P_n = 1 - \varphi j \in P_j \quad \text{and} \quad F^* = F_m i + F_{m+1} j \in P_j^{\perp},$$

and define the mappings

$$\begin{aligned} M_{P_n} &: P_j^{\perp} \to P_j^{\perp}, \qquad F^* \mapsto M_{P_n}(F^*) = P_n F^*, \\ N_{P_n} &: P_j^{\perp} \to P_j^{\perp}, \qquad F^* \mapsto N_{P_n}(F^*) = F^* P_n. \end{aligned}$$

Then:

(i) M_{P_n} *rotates* F^* *by the parabolic angle* φ *in the plane* $P_j^{\perp}$.
(ii) N_{P_n} *rotates* F^* *by the angle* $-\varphi$ *in the plane* $P_j^{\perp}$.

In both cases, the rotation circle degenerates into the straight line

$$\ell(t) = F_m i + F_{m+1} j + tj,$$

with radius $|F_m|$.

Proof. (i) Let $F^* = F_m i + F_{m+1} j \in P_j^{\perp}$. Then

$$M_{P_n}(F^*) = P_n F^* = F_m i + (F_m \varphi + F_{m+1}) j \in P_j^{\perp}.$$

The argument of F^* is F_{m+1}/F_m, while

$$\arg(M_{P_n}(F^*)) = \frac{F_{m+1} + F_m \varphi}{F_m}.$$

Thus, F^* is rotated by φ about the lightlike direction j. Iterating M_{P_n} gives

$$M_{P_n}^x(F^*) = F_m i + (F_{m+1} + x F_m \varphi) j,$$

which lies on the straight line $\ell(t)$.

(ii) Consider the right multiplication

$$N_{P_n}(F^*) = F^* P_n,$$

where

$$F^* = F_m i + F_{m+1} j, \qquad P_n = 1 - \phi j.$$

Using the Cartan multiplication rules

$$ij = j, \qquad j^2 = 0,$$

we obtain

$$\begin{aligned} N_{P_n}(F^*) &= (F_m i + F_{m+1} j)(1 - \phi j) \\ &= F_m i + F_{m+1} j - \phi F_m i j - \phi F_{m+1} j^2 \\ &= F_m i + (F_{m+1} - F_m \phi) j. \end{aligned}$$

Hence,

$$\arg(N_{P_n}(F^*)) = \frac{F_{m+1} - F_m \phi}{F_m} = \frac{F_{m+1}}{F_m} - \phi.$$

Therefore, right multiplication by P_n produces a parabolic rotation through the angle $-\phi$. Moreover, after x iterations,

$$N_{P_n}^x(F^*) = F_m i + (F_{m+1} - x F_m \phi) j,$$

which lies on the straight line

$$\ell(t) = F_m i + F_{m+1} j + t j.$$

Thus, in both the left- and right-multiplication cases, the parabolic rotation degenerates into a linear trajectory, while the direction of the rotation is determined by the sign of the angle. □

6. Fibonacci Pauli Cartan numbers

In this section, Fibonacci Pauli–Cartan numbers with non-negative and negative indices are investigated. We derive their recurrence relations, Binet-type formula, ordinary and exponential generating functions, summation identities, and matrix representations. A numerical example is also provided to illustrate the determinant representations, followed by real linear transformations associated with left and right multiplication.

Definition 26. *Let $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The nth Fibonacci Pauli–Cartan number is defined by*

$$\mathfrak{F}_n = F_n 1 + F_{n+1}\sigma_1 + F_{n+2}\sigma_2 + F_{n+3}\sigma_3. \tag{6.1}$$

Here, F_n denotes the nth Fibonacci number, and $\sigma_1, \sigma_2, \sigma_3$ are the basis elements introduced above, satisfy Pauli multiplication rules stated in Section 1.

Definition 27. *Let $n \in \mathbb{N}_0 \setminus \{0\}$. The Fibonacci Pauli–Cartan number with index $-n$ is defined by*

$$\mathfrak{F}_{-n} = F_{-n} 1 + F_{-n+1}\sigma_1 + F_{-n+2}\sigma_2 + F_{-n+3}\sigma_3. \tag{6.2}$$

Here, F_{-n} denotes the Fibonacci number with negative index, and $\sigma_1, \sigma_2, \sigma_3$ satisfy Pauli multiplication rules stated in Section 1, given in Eq. (1.3).

Corollary 28. *Let $\mathfrak{F}_n$ be the nth term of the Fibonacci Pauli Cartan and $\mathfrak{F}_{-n}$ be the $-$nth term of the Fibonacci Pauli Cartan. The following recurrence relations hold:*

$$\mathfrak{F}_{n+2} = \mathfrak{F}_{n+1} + \mathfrak{F}_n, \tag{6.3}$$

and

$$\mathfrak{F}_{-n} = -\mathfrak{F}_{-n+1} + \mathfrak{F}_{-n+2}. \tag{6.4}$$

Proof. Using Eqs (1.4) and (2.1), we obtain:

$$\begin{aligned}
\mathfrak{F}_{n+1} + \mathfrak{F}_n &= F_{n+1}\mathbf{1} + F_{n+2}\sigma_{\mathbf{1}} + F_{n+3}\sigma_{\mathbf{2}} + F_{n+4}\sigma_{\mathbf{3}} \\
&\quad + F_n\mathbf{1} + F_{n+1}\sigma_{\mathbf{1}} + F_{n+2}\sigma_{\mathbf{2}} + F_{n+3}\sigma_{\mathbf{3}} \\
&= (F_{n+1} + F_n)\mathbf{1} + (F_{n+2} + F_{n+1})\sigma_{\mathbf{1}} + (F_{n+3} \\
&\quad + F_{n+2})\sigma_{\mathbf{2}} + (F_{n+4} + F_{n+3})\sigma_{\mathbf{3}} \\
&= F_{n+2}\mathbf{1} + F_{n+3}\sigma_{\mathbf{1}} + F_{n+4}\sigma_{\mathbf{2}} + F_{n+5}\sigma_{\mathbf{3}} = \mathfrak{F}_{n+2}.
\end{aligned}$$

For the negative-index recurrence, using

$$F_{-n} = -F_{-n+1} + F_{-n+2},$$

we obtain

$$\begin{aligned}
\mathfrak{F}_{-n} &= F_{-n}\mathbf{1} + F_{-n+1}\sigma_1 + F_{-n+2}\sigma_2 + F_{-n+3}\sigma_3 \\
&= -\mathfrak{F}_{-n+1} + \mathfrak{F}_{-n+2}.
\end{aligned}$$

□

Theorem 29. *Let $\mathfrak{F}_n$ be the nth term of the Fibonacci Pauli Cartan. The following Binet's formula is satisfied for the Fibonacci Pauli Cartans:*

$$\mathfrak{F}_n = \frac{\delta_1^n \hat{\delta_1} - \delta_2^n \hat{\delta_2}}{\delta_1 - \delta_2} \tag{6.5}$$

where $\delta_1 = \frac{1+\sqrt{5}}{2}$, $\delta_2 = \frac{1-\sqrt{5}}{2}$, $\hat{\delta_1} = \mathbf{1} + \delta_1\sigma_{\mathbf{1}} + \delta_1^2\sigma_{\mathbf{2}} + \delta_1^3\sigma_{\mathbf{3}}$, $\hat{\delta_2} = \mathbf{1} + \delta_2\sigma_{\mathbf{1}} + \delta_2^2\sigma_{\mathbf{2}} + \delta_2^3\sigma_{\mathbf{3}}$, *and* $\sigma_{\mathbf{1}}$, $\sigma_{\mathbf{2}}$, $\sigma_{\mathbf{3}}$ *satisfy the Pauli multiplication rules stated in Section 1.*

Proof. Using the Binet formula

$$F_n = \frac{\delta_1^n - \delta_2^n}{\delta_1 - \delta_2},$$

we obtain

$$\begin{aligned}\mathfrak{F}_n &= F_n\mathbf{1} + F_{n+1}\sigma_1 + F_{n+2}\sigma_2 + F_{n+3}\sigma_3 \\ &= \frac{\delta_1^n\left(\mathbf{1} + \delta_1\sigma_1 + \delta_1^2\sigma_2 + \delta_1^3\sigma_3\right) - \delta_2^n\left(\mathbf{1} + \delta_2\sigma_1 + \delta_2^2\sigma_2 + \delta_2^3\sigma_3\right)}{\delta_1 - \delta_2}.\end{aligned}$$

Therefore,

$$\mathfrak{F}_n = \frac{\delta_1^n\widehat{\delta_1} - \delta_2^n\widehat{\delta_2}}{\delta_1 - \delta_2},$$

which proves the result. □

Theorem 30. *Let $\mathfrak{F}_n$ be the nth term of the Fibonacci Pauli Cartan. The following generating function is satisfied for the Fibonacci Pauli Cartans:*

$$\sum_{n=0}^{\infty} \mathfrak{F}_n a^n = \frac{a(\mathfrak{F}_1 - \mathfrak{F}_0) + \mathfrak{F}_0}{1 - a - a^2}. \tag{6.6}$$

Proof. Let $m(a)$ be the generating function for the Fibonacci Pauli Cartan numbers as $\sum_{n=0}^{\infty} \mathfrak{F}_n a^n$. We get the following equations, $am(a) = \sum_{n=0}^{\infty} \mathfrak{F}_n a^{n+1}$ and $a^2 m(a) = \sum_{n=0}^{\infty} \mathfrak{F}_n a^{n+2}$. After the needed calculations, the generating function for the Fibonacci Pauli Cartan numbers is obtained as

$$\sum_{n=0}^{\infty} \mathfrak{F}_n a^n = \frac{a(\mathfrak{F}_1 - \mathfrak{F}_0) + \mathfrak{F}_0}{1 - a - a^2}.$$

□

Theorem 31. *Let $\mathfrak{F}_n$ be the nth term of the Fibonacci Pauli Cartan. The exponential generating function is satisfied for the Fibonacci Pauli Cartans:*

$$\sum_{n=0}^{\infty} \mathfrak{F}_n \frac{t^n}{n!} = \left(\frac{\hat{\delta_1}}{\delta_1 - \delta_2}\right) e^{\delta_1 t} - \left(\frac{\hat{\delta_2}}{\delta_1 - \delta_2}\right) e^{\delta_2 t}. \tag{6.7}$$

Proof. Using Eq (6.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{F}_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{\delta_1^n \hat{\delta_1} - \delta_2^n \hat{\delta_2}}{\delta_1 - \delta_2} \right) \frac{t^n}{n!} \\ &= \left(\frac{\hat{\delta_1}}{\delta_1 - \delta_2} \right) \sum_{n=0}^{\infty} \frac{\delta_1^n t^n}{n!} - \left(\frac{\hat{\delta_2}}{\delta_1 - \delta_2} \right) \sum_{n=0}^{\infty} \frac{\delta_2^n t^n}{n!} \\ &= \left(\frac{\hat{\delta_1}}{\delta_1 - \delta_2} \right) e^{\delta_1 t} - \left(\frac{\hat{\delta_2}}{\delta_1 - \delta_2} \right) e^{\delta_2 t}.\end{aligned}$$

□

Theorem 32. *Let $\mathfrak{F}_n$ be the nth term of the Fibonacci Pauli Cartan. The following summation formulas are satisfied:*

$$\sum_{m=0}^{n} \mathfrak{F}_m = \mathfrak{F}_{n+2} - \mathfrak{F}_1,$$

$$\sum_{m=0}^{n} \mathfrak{F}_{2m} = \mathfrak{F}_{2n+1} - \mathfrak{F}_1 + \mathfrak{F}_0,$$

$$\sum_{m=0}^{n} \mathfrak{F}_{2m+1} = \mathfrak{F}_{2n+2} - \mathfrak{F}_0.$$

Proof. According to the recurrence relation in Eq (6.3), we can get the following:

$$\mathfrak{F}_2 = \mathfrak{F}_1 + \mathfrak{F}_0,$$

$$\mathfrak{F}_3 = \mathfrak{F}_2 + \mathfrak{F}_1,$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathfrak{F}_{n+2} = \mathfrak{F}_{n+1} + \mathfrak{F}_n.$$

If we add both of sides of the identities above, we obtain,

$$\sum_{m=0}^{n} \mathfrak{F}_m = \mathfrak{F}_{n+2} - \mathfrak{F}_1.$$

The other equalities can be proved in a similar way. □

Theorem 33. *Let $\mathfrak{F}_n$ be the nth term of the Fibonacci Pauli Cartan and $\mathfrak{F}_{-n}$ be the $-$nth term of the Fibonacci Pauli Cartan. These determinant matrices are constructed directly from the recurrence relations (6.3) and (6.4), namely*

$$\mathfrak{F}_{n+2} = \mathfrak{F}_{n+1} + \mathfrak{F}_n$$

and

$$\mathfrak{F}_{-n} = -\mathfrak{F}_{-n+1} + \mathfrak{F}_{-n+2},$$

respectively.

The following equations are satisfied:

$$\mathfrak{F}_n = \begin{vmatrix} \mathfrak{F}_0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \mathfrak{F}_1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)} = F_n\mathfrak{F}_1 + F_{n-1}\mathfrak{F}_0,$$

$$\mathfrak{F}_{-(n+1)} = \begin{vmatrix} \mathfrak{F}_{-1} & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \mathfrak{F}_{-2} & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{vmatrix}_{(n+1)\times(n+1)} = (-1)^n F_{n-1}\mathfrak{F}_{-1} - (-1)^n F_n\mathfrak{F}_{-2}.$$

where F_n is the nth term of Fibonacci number.

Proof. The proof follows by mathematical induction, using recurrence relation (6.3) for the first equality and recurrence relation (6.4) for the second equality.

□

Example 34. *Consider $n = 5$. We illustrate the determinant representations given in the preceding theorem.*

- *First, let us compute the fifth Fibonacci Pauli–Cartan number. We have*

$$\mathfrak{F}_5 = \begin{vmatrix} \mathfrak{F}_0 & -1 & 0 & 0 & 0 & 0 \\ \mathfrak{F}_1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix}_{6\times 6}.$$

Expanding the determinant along the first column, only the first two entries contribute. The corresponding cofactors are $F_4 = 3$ and $F_5 = 5$, respectively. Hence,

$$\mathfrak{F}_5 = F_4\mathfrak{F}_0 + F_5\mathfrak{F}_1.$$

Using

$$\mathfrak{F}_0 = \sigma_1 + \sigma_2 + 2\sigma_3$$

and

$$\mathfrak{F}_1 = \mathbf{1} + \sigma_1 + 2\sigma_2 + 3\sigma_3,$$

we obtain

$$\begin{aligned}\mathfrak{F}_5 &= 3(\sigma_1+\sigma_2+2\sigma_3)+5(\mathbf{1}+\sigma_1+2\sigma_2+3\sigma_3)\\ &= 5+8\sigma_1+13\sigma_2+21\sigma_3.\end{aligned}$$

- *Next, let us compute the* (-6)*th Fibonacci Pauli–Cartan number. We have*

$$\mathfrak{F}_{-6}=\begin{vmatrix} \mathfrak{F}_{-1} & -1 & 0 & 0 & 0 & 0\\ \mathfrak{F}_{-2} & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & -1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{vmatrix}_{6\times 6}.$$

Again, expanding along the first column, only the first two entries contribute. The corresponding cofactors give

$$\mathfrak{F}_{-6}=(-1)^5F_4\mathfrak{F}_{-1}-(-1)^5F_5\mathfrak{F}_{-2},$$

and therefore

$$\mathfrak{F}_{-6}=-3\mathfrak{F}_{-1}+5\mathfrak{F}_{-2}.$$

Since

$$\mathfrak{F}_{-1}=\mathbf{1}+\sigma_2+\sigma_3$$

and

$$\mathfrak{F}_{-2}=-\mathbf{1}+\sigma_1+\sigma_3,$$

it follows that

$$\begin{aligned}\mathfrak{F}_{-6} &= -3(\mathbf{1}+\sigma_2+\sigma_3)+5(-\mathbf{1}+\sigma_1+\sigma_3)\\ &= -8+5\sigma_1-3\sigma_2+2\sigma_3.\end{aligned}$$

Let $\mathfrak{F}_a, \mathfrak{F}_b \in \mathfrak{F}$ to construct the matrix representation of the $\mathbb{R}$-linear transformations representing the left and right product in $\mathfrak{F}$ using de Moivre's formula.

$$\zeta_{L_{\mathfrak{F}_a}} : \mathfrak{F} \longrightarrow \mathfrak{F}$$

where $\zeta_{L_{\mathfrak{F}_a}}$ is written as:

$$A\zeta_{L_{\mathfrak{F}_a}}=\begin{bmatrix} \mathfrak{F}_a & \mathfrak{F}_{a+1} & \mathfrak{F}_{a+3} & \mathfrak{F}_{a+2}\\ \mathfrak{F}_{a+1} & \mathfrak{F}_a & -\mathfrak{F}_{a+3} & \mathfrak{F}_{a+2}\\ \mathfrak{F}_{a+2} & -\mathfrak{F}_{a+2} & \mathfrak{F}_a+\mathfrak{F}_{a+1} & 0\\ \mathfrak{F}_{a+3} & \mathfrak{F}_{a+3} & 0 & \mathfrak{F}_a-\mathfrak{F}_{a+1} \end{bmatrix},$$

and

$$\zeta_{R_{\mathfrak{F}_a}} : \mathfrak{F} \longrightarrow \mathfrak{F}$$

where $\zeta_{R_{\mathfrak{F}_a}}$ is written as:

$$A\zeta_{R_{\mathfrak{F}_a}} = \begin{bmatrix} \mathfrak{F}_a & \mathfrak{F}_{a+1} & \mathfrak{F}_{a+3} & \mathfrak{F}_{a+2} \\ \mathfrak{F}_{a+1} & \mathfrak{F}_a & \mathfrak{F}_{a+3} & -\mathfrak{F}_{a+2} \\ \mathfrak{F}_{a+2} & \mathfrak{F}_{a+2} & \mathfrak{F}_a - \mathfrak{F}_{a+1} & 0 \\ \mathfrak{F}_{a+3} & -\mathfrak{F}_{a+3} & 0 & \mathfrak{F}_a + \mathfrak{F}_{a+1} \end{bmatrix}.$$

Let $\mathcal{A}$ denote the ambient algebra generated by $1, \sigma_1, \sigma_2, \sigma_3$, and let $X \in \mathcal{A}$. For a fixed Fibonacci Pauli–Cartan number $\mathfrak{F}_a$, define the left- and right-multiplication operators by

$$\zeta^L_{\mathfrak{F}_a}(X) = \mathfrak{F}_a X, \qquad \zeta^R_{\mathfrak{F}_a}(X) = X\mathfrak{F}_a.$$

The matrices displayed above are the matrix representations of these operators with respect to the chosen basis.

For all $X, Y \in \mathcal{A}$ and $\lambda \in \mathbb{R}$, the left- and right-multiplication operators satisfy

$$\zeta^L_{\mathfrak{F}_a}(X + Y) = \zeta^L_{\mathfrak{F}_a}(X) + \zeta^L_{\mathfrak{F}_a}(Y),$$
$$\zeta^R_{\mathfrak{F}_a}(X + Y) = \zeta^R_{\mathfrak{F}_a}(X) + \zeta^R_{\mathfrak{F}_a}(Y),$$

and

$$\zeta^L_{\mathfrak{F}_a}(\lambda X) = \lambda\zeta^L_{\mathfrak{F}_a}(X), \qquad \zeta^R_{\mathfrak{F}_a}(\lambda X) = \lambda\zeta^R_{\mathfrak{F}_a}(X).$$

Hence, both operators are $\mathbb{R}$-linear. Moreover, associativity of the ambient algebra gives

$$\zeta^L_{\mathfrak{F}_a} \circ \zeta^R_{\mathfrak{F}_b} = \zeta^R_{\mathfrak{F}_b} \circ \zeta^L_{\mathfrak{F}_a}.$$

Indeed, for every $X \in \mathcal{A}$,

$$\mathfrak{F}_a(X\mathfrak{F}_b) = (\mathfrak{F}_a X)\mathfrak{F}_b.$$

The left-multiplication operators satisfy

$$\zeta^L_{\mathfrak{F}_a\mathfrak{F}_b} = \zeta^L_{\mathfrak{F}_a} \circ \zeta^L_{\mathfrak{F}_b},$$

whereas the right-multiplication operators satisfy

$$\zeta^R_{\mathfrak{F}_a\mathfrak{F}_b} = \zeta^R_{\mathfrak{F}_b} \circ \zeta^R_{\mathfrak{F}_a}.$$

Consequently, at the matrix level,

$$A^L_{\mathfrak{F}_a\mathfrak{F}_b} = A^L_{\mathfrak{F}_a} A^L_{\mathfrak{F}_b},$$

while

$$A^R_{\mathfrak{F}_a\mathfrak{F}_b} = A^R_{\mathfrak{F}_b} A^R_{\mathfrak{F}_a}.$$

Thus, the left multiplication gives a matrix representation, whereas the right multiplication has the corresponding anti-representation property. In particular, no claim is made that these representations are surjective onto the full matrix algebra $M_{4\times4}(\mathbb{R})$.

Remark 35. *The formulas developed in this paper may provide useful tools for further investigations involving recurrence-based algebraic systems and their matrix and geometric representations. In particular, the matrix representations enable Fibonacci–Cartan numbers to be studied by means of standard linear algebraic techniques, while the polar and exponential forms, together with the De Moivre-type formulas, facilitate the computation of powers and roots. Moreover, the parabolic rotation formulas may be useful in the geometric analysis of transformations associated with lightlike directions. These results may also serve as a basis for extending this framework to other recursive sequences and related Cartan-type number systems.*

7. An application of Chebyshev polynomials to Fibonacci Cartan powers

Chebyshev polynomials provide a natural polynomial framework for multiple-angle identities and therefore fit naturally with the De Moivre–type structure developed for Fibonacci–Cartan numbers. Direct connections between Fibonacci and Chebyshev polynomials have been established through connection formulas and generating-function methods [37, 38]. In particular, finite products involving Chebyshev polynomials of the second kind and Fibonacci polynomials were investigated in [39], while generalized Chebyshev polynomials and their connections with classical polynomial sequences were studied in [40]. Related recurrence- and matrix-based approaches include augmented Pascal-type matrix constructions with connections to Fibonacci matrices [41] and Euler–Seidel matrix methods for recurrence-based sequences [42].

Motivated by these connections, we express the powers of Fibonacci–Cartan numbers in terms of Chebyshev polynomials. This provides an alternative polynomial formulation of the hyperbolic multiple-angle functions occurring in the De Moivre–type formula established above. Recall that the Chebyshev polynomials of the first kind $T_m(x)$ and the second kind $U_m(x)$ are defined by

$$T_0(x) = 1, \qquad T_1(x) = x,$$

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), \qquad m \geq 1,$$

and

$$U_0(x) = 1, \qquad U_1(x) = 2x,$$

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x), \qquad m \geq 1.$$

For $x = \cosh\varphi$, the corresponding hyperbolic identities are

$$T_m(\cosh\varphi) = \cosh(m\varphi)$$

and

$$\sinh(m\varphi) = \sinh\varphi\, U_{m-1}(\cosh\varphi), \qquad m \geq 1.$$

To avoid confusion between the Chebyshev polynomial U_m and the unit Cartan vector used in the preceding sections, we denote the latter by

$$\widehat{U}_n = \frac{F_{n+1}i + F_{n+2}j + F_{n+3}k}{\|U_{P_n}\|}.$$

Then

$$\widehat{U}_n^2 = 1.$$

Theorem 36. *Let*

$$P_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k, \qquad n \geq 0,$$

be a spacelike Fibonacci–Cartan number with polar representation

$$P_n = \|P_n\|\left(\sinh\varphi + \widehat{U}_n\cosh\varphi\right).$$

Define

$$\eta_n = \frac{F_n}{\|P_n\|} = \sinh\varphi$$

and

$$\xi_n = \frac{\|U_{P_n}\|}{\|P_n\|} = \cosh\varphi.$$

Then, for every positive integer m,

$$P_n^m = \begin{cases} \|P_n\|^m \left[\eta_n U_{m-1}(\xi_n) + \widehat{U}_n T_m(\xi_n)\right], & m \text{ is odd}, \\ \|P_n\|^m \left[T_m(\xi_n) + \widehat{U}_n \eta_n U_{m-1}(\xi_n)\right], & m \text{ is even}. \end{cases}$$

Proof. From the De Moivre–type formula established above, if m is odd, then

$$P_n^m = \|P_n\|^m \left(\sinh(m\varphi) + \widehat{U}_n \cosh(m\varphi)\right).$$

Using

$$T_m(\cosh\varphi) = \cosh(m\varphi)$$

and

$$\sinh(m\varphi) = \sinh\varphi\, U_{m-1}(\cosh\varphi),$$

together with

$$\eta_n = \sinh\varphi, \qquad \xi_n = \cosh\varphi,$$

we obtain

$$\cosh(m\varphi) = T_m(\xi_n)$$

and

$$\sinh(m\varphi) = \eta_n U_{m-1}(\xi_n).$$

Therefore,

$$P_n^m = \|P_n\|^m \left[\eta_n U_{m-1}(\xi_n) + \widehat{U}_n T_m(\xi_n)\right].$$

If m is even, the De Moivre–type formula gives

$$P_n^m = \|P_n\|^m \left(\cosh(m\varphi) + \widehat{U}_n \sinh(m\varphi)\right).$$

Applying the same Chebyshev identities yields

$$P_n^m = \|P_n\|^m \left[T_m(\xi_n) + \widehat{U}_n \eta_n U_{m-1}(\xi_n)\right].$$

Hence, the stated representation holds for both parity cases. □

Corollary 37. *The scalar and vector parts of P_n^m admit explicit representations in terms of Chebyshev polynomials. If m is odd, then*

$$S(P_n^m) = \|P_n\|^m \eta_n U_{m-1}(\xi_n)$$

and

$$V(P_n^m) = \|P_n\|^m \widehat{U}_n T_m(\xi_n).$$

If m is even, then

$$S(P_n^m) = \|P_n\|^m T_m(\xi_n)$$

and

$$V(P_n^m) = \|P_n\|^m \widehat{U}_n \eta_n U_{m-1}(\xi_n).$$

Example 38. *Consider*

$$P_3 = 2 + 3i + 5j + 8k.$$

From the previous computations,

$$\|P_3\| = \sqrt{85}, \qquad \|U_{P_3}\| = \sqrt{89}.$$

Therefore,

$$\eta_3 = \frac{2}{\sqrt{85}}, \qquad \xi_3 = \frac{\sqrt{89}}{\sqrt{85}},$$

and

$$\widehat{U}_3 = \frac{3i + 5j + 8k}{\sqrt{89}}.$$

For $m = 2$, the even case of the preceding theorem gives

$$P_3^2 = 85\left[T_2(\xi_3) + \widehat{U}_3\eta_3 U_1(\xi_3)\right].$$

Since

$$T_2(x) = 2x^2 - 1, \qquad U_1(x) = 2x,$$

we have

$$T_2(\xi_3) = 2\left(\frac{89}{85}\right) - 1 = \frac{93}{85}$$

and

$$\eta_3 U_1(\xi_3) = \frac{2}{\sqrt{85}}\left(\frac{2\sqrt{89}}{\sqrt{85}}\right) = \frac{4\sqrt{89}}{85}.$$

Consequently,

$$\begin{aligned} P_3^2 &= 85\left[\frac{93}{85} + \frac{3i + 5j + 8k}{\sqrt{89}}\frac{4\sqrt{89}}{85}\right] \\ &= 93 + 12i + 20j + 32k. \end{aligned}$$

Thus, the Chebyshev representation reproduces the power of the Fibonacci–Cartan number directly from its polar data.

The preceding results show that the Chebyshev representation is not an independent structure imposed on Fibonacci–Cartan numbers. Rather, it provides a polynomial reformulation of their De Moivre–type power structure. In particular, the hyperbolic multiple-angle terms appearing in powers of Fibonacci–Cartan numbers can be evaluated through the recurrence relations of the Chebyshev polynomials. This establishes a direct computational connection between Fibonacci–Cartan numbers and the classical theory of Chebyshev polynomials.

8. Conclusions

In this work, we established a comprehensive algebraic and geometric framework for Fibonacci–Cartan and Fibonacci Pauli–Cartan numbers. Their structural properties were characterized through conjugation, norm, causal classification, and matrix representations, revealing a natural interpretation in pseudo–Euclidean geometry. The derivation of polar and exponential forms enabled the extension of classical identities such as De Moivre's formula to this generalized setting. Building on this De Moivre–type power structure, we obtained a Chebyshev polynomial representation of Fibonacci–Cartan powers in terms of the first- and second-kind Chebyshev polynomials. This representation yielded explicit polynomial expressions for the scalar and vector parts of P_n^m and provides a direct connection between Fibonacci–Cartan numbers and the recurrence structure of Chebyshev polynomials. Moreover, the introduction of generating functions, Binet–type formulas, and summation identities provided a coherent analytic foundation for these sequences. The geometric analysis of rotations in lightlike directions highlighted the distinctive behavior of parabolic transformations, which reduced to linear motions in the Cartan space.

This study also has some limitations that indicate possible directions for further research. The analysis is restricted to Fibonacci-based Cartan and Pauli–Cartan structures, and therefore the extent to which the proposed framework can be transferred to other recurrence sequences remains to be investigated. Moreover, this work mainly focuses on algebraic, matrix, and geometric properties, whereas analytical and computational aspects have not been considered in detail.

Finally, although the connection with Chebyshev polynomials provides a concrete theoretical application of the developed framework, its potential use in specific applied or computational models remains to be investigated. These limitations suggest several natural directions for extending this study. These results contribute to the interaction between Fibonacci theory and non–Euclidean algebraic structures, while the Chebyshev representation further connects the proposed framework with the theory of special polynomials. This opens potential directions for further research in geometric algebra, recurrence-based polynomial systems, mathematical physics, and generalized number systems.

Use of AI tools declaration

AI tools were used to assist with the preparation of this manuscript. The authors reviewed the content and take full responsibility for its accuracy.

Acknowledgments

The authors thank the Editor and the anonymous reviewers for their constructive comments and suggestions, which helped improve the quality of this manuscript.

Conflict of interest

Clemente Cesarano is an editorial board member for Networks and Heterogeneous Media and was not involved in the editorial review or the decision to publish this article. All authors declare that they have no competing interests.

Author contributions

Bahar Kuloğlu, Hasan Gökbaş, William Ramírez, and Clemente Cesarano contributed equally to the conceptualization, methodology, formal analysis, investigation, validation, and writing of this manuscript. All authors have read and approved the final version of the manuscript.

References

1. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983. Available from: https://shop.elsevier.com/books/semi-riemannian-geometry-with-applications-to-relativity/oneill/978-0-12-526740-3.
2. H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1955. Available from: https://ci.nii.ac.jp/ncid/BA0358873X.
3. E. Cartan, *Lectures on the Geometry of Riemannian Spaces [Leçons sur la geometrie des espaces de Riemann]*, Gauthier-Villars, Paris, 1928. Available from: https://rcin.org.pl/dlibra/publication/11909/edition/45404/lecons-sur-la-geometrie-des-espaces-de-riemann-cartan-elie-joseph-1869-1951?language=en.
4. R. Lopez, Differential geometry of curves and surfaces in Lorentz–Minkowski space, *Int. Electron. J. Geom.* **7** (2014), 44–107. https://doi.org/10.36890/iejg.594497
5. K. L. Duggal, A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, Dordrecht, 1996. https://doi.org/10.1007/978-94-017-2089-2
6. K. L. Duggal, B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Birkhäuser, Basel, 2010. https://doi.org/10.1007/978-3-0346-0251-8
7. J. Cockle, On systems of algebra involving more than one imaginary; and on equations of the fifth degree, *Philos. Mag.*, **35** (1849), 434–437. https://doi.org/10.1080/14786444908646384
8. A. F. Horadam, Complex Fibonacci numbers and Fibonacci quaternions, *Amer. Math. Monthly*, **70** (1963), 289–291. https://doi.org/10.2307/2313129
9. I. M. Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis*, Springer, New York, 1979. https://doi.org/10.1007/978-1-4612-6135-3
10. G. Sobczyk, The hyperbolic number plane, *College Math. J.*, **26** (1995), 268–280. https://doi.org/10.1080/07468342.1995.11973712
11. A. Ferrandez, A. Gimenez, P. Lucas, Null helices in Lorentzian space forms, *Int. J. Mod. Phys. A*, **16** (2001), 4845–4863. https://doi.org/10.1142/S0217751X01005821
12. K. L. Duggal, D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, Singapore, 2007. https://doi.org/10.1142/6449
13. J. P. Ward, *Quaternions and Cayley Numbers*, Springer, Dordrecht, 1997. https://doi.org/10.1007/978-94-011-5768-1
14. I. Ozturk, Introduction to Cartan numbers and some geometric applications, *Math. Methods Appl. Sci.*, **46** (2023), 7514–7542. https://doi.org/10.1002/mma.8982

15. R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 2012. https://doi.org/10.1017/CBO9781139020411

16. I. Ozturk, H. Cakir, Fibonacci Cartan and Lucas Cartan numbers, *Open Math.*, **23** (2025), 20250204. https://doi.org/10.1515/math-2025-0204

17. I. Ozturk, Dual Cartan numbers and pseudo-null lines, *Math. Methods Appl. Sci.*, **48** (2025), 207–217. https://doi.org/10.1002/mma.10323

18. J. E. Kim, A representation of de Moivre's formula over Pauli quaternions, *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, **9** (2017), 145–151. Available from: https://aos.ro/wp-content/anale/MVol9Nr2Art.3.pdf.

19. F. T. Aydin, Pauli-Fibonacci quaternions, *Notes Number Theory Discret. Math.*, **27** (2021), 184–193. https://doi.org/10.7546/nntdm.2021.27.3.184-193

20. Z. Işbilir, M. Akyigit, M. Tosun, Pauli-Leonardo quaternions, *Notes Number Theory Discret. Math.*, **29** (2023), 1–16. https://doi.org/10.7546/nntdm.2023.29.1.1-16

21. A. Z. Azak, Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions, *Mathematics*, **10** (2022), 4655. https://doi.org/10.3390/math10244655

22. B. Yılmaz, N. G. Bilgin, Y. Soykan, Generalized Pauli Fibonacci polynomial quaternions, *Axioms*, **14** (2025), 449. https://doi.org/10.3390/axioms14060449

23. T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001. https://doi.org/10.1002/9781118033067

24. J. P. M. Binet, Memoire sur lintegration des equations lineaires aux differences finies, dun ordre quelconque, a coefficients variables, *C. R. Acad. Sci. Paris*, **17** (1843), 559–567. Available from: https://mathshistory.st-andrews.ac.uk/Extras/Binet_Papers/.

25. T. Koshy, *Fibonacci and Lucas Numbers With Applications*, John Wiley & Sons, 2018. https://doi.org/10.1002/9781118742297

26. U. Bednarz, I. Wloch, M. Wolowiec-Musial, Total graph interpretation of the numbers of the Fibonacci type, *J. Appl. Math.*, **2015** (2015), 837917. https://doi.org/10.1155/2015/837917

27. A. Szynal-Liana, I. Wloch, *Hypercomplex Numbers of the Fibonacci Type*, Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów, 2019.

28. C. Kızılateş, W. Ramirez, Y. Quintana, D. Bedoya, M. F. Heredia-Moyano, Mixed-type Fibonacci-Mittag-Leffler and Lucas-Mittag-Leffler polynomials: Some properties, *Comput. Appl. Math.*, **45** (2026), 32. https://doi.org/10.1007/s40314-025-03379-5

29. S. Halici, On Fibonacci quaternions, *Adv. Appl. Clifford Algebras*, **22** (2012), 321–327. https://doi.org/10.1007/s00006-011-0317-1

30. M. R. Iyer, A note on Fibonacci quaternions, *Fibonacci Quart.*, **7** (1969), 225–229. https://doi.org/10.1080/00150517.1969.12431146

31. S. K. Nurkan, I. A. Güven, A note on bicomplex Fibonacci and Lucas numbers, *Int. J. Pure Appl. Math.*, **120** (2018), 365–377. https://doi.org/10.12732/ijpam.v120i3.7

32. A. Szynal-Liana, I. Wloch, The Fibonacci hybrid numbers, *Util. Math.*, **110** (2019), 3–10. Available from: https://utilitasmathematica.com/index.php/Index/issue/view/59.

33. B. Kuloglu, E. Ozkan, M. Marin, Fibonacci and Lucas polynomials in n-gon, *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.*, **31** (2023), 127–140. https://doi.org/10.2478/auom-2023-0023

34. A. Shahid, H. L. Huang, M. M. Bhatti, M. Marin, Numerical computation of magnetized bioconvection nanofluid flow with temperature-dependent viscosity and Arrhenius kinetic, *Math. Comput. Simul.*, **200** (2022), 377–392. https://doi.org/10.1016/j.matcom.2022.04.032

35. S. Askar, A. E. Abouelregal, M. Marin, A. Foul, Photo-thermoelasticity heat transfer modeling with fractional differential actuators for stimulated nano-semiconductor media, *Symmetry*, **15** (2023), 656. https://doi.org/10.3390/sym15030656

36. A. E. Abouelregal, M. Marin, A. Öchsner, A modified spatiotemporal nonlocal thermoelasticity theory with higher-order phase delays for a viscoelastic micropolar medium exposed to short-pulse laser excitation, *Contin. Mech. Thermodyn.*, **37** (2025), 15. https://doi.org/10.1007/s00161-024-01342-z

37. W. M. Abd-Elhameed, Y. H. Youssri, N. El-Sissi, M. Sadek, New hypergeometric connection formulae between Fibonacci and Chebyshev polynomials, *Ramanujan J.*, **42** (2017), 347–361. https://doi.org/10.1007/s11139-015-9712-x

38. R. Frontczak, T. Goy, Chebyshev–Fibonacci polynomial relations using generating functions, *Integers*, **21** (2021), A100. https://doi.org/10.5281/zenodo.10839350

39. T. Kim, D. S. Kim, D. V. Dolgy, J. W. Park, Sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials, *J. Inequal. Appl.*, **2018** (2018), 148. https://doi.org/10.1186/s13660-018-1744-5

40. T. Kim, D. S. Kim, On generalized Chebyshev polynomials, preprint, arXiv:2607.29661, 2026. https://doi.org/10.48550/arXiv.2607.29661

41. G. Kizilaslan, T. Kim, The k-augmented Pascal matrix and its properties, *J. Math.*, **2025** (2025), 6280240. https://doi.org/10.1155/jom/6280240

42. T. Kim, D. S. Kim, Degenerate Euler-Seidel matrix method and their applications, *Math. Methods Appl. Sci.*, **49** (2026), 8209–8223. https://doi.org/10.1002/mma.70468



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)