



Research article

Influence of fractional order on the dynamic behavior of a cross-reaction diffusion model

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Abstract: This paper investigates influence of fractional order on the dynamic behavior of a cross-reaction diffusion model. Stability analysis determines the existence conditions of equilibrium points, and bifurcation analyses are performed. A numerical method combining high-order time-precision generating functions and a nine-point difference spatial discretization scheme is proposed, incorporating the short memory principle to improve computational efficiency. Numerical simulations demonstrate the spatial pattern formation of the system at different parameters, validating the effectiveness of the theoretical results. Some novel patterns are shown.

Keywords: a numerical method; stability analysis; bifurcation analysis; novel pattern; cross-reaction diffusion model

1. Introduction

The study of fractional differential equations (FDEs) has evolved into a mature discipline, encompassing rigorous theory, sophisticated numerical methods, stability and qualitative analysis, and diverse applications across science and engineering. Foundational texts by [1–6] have established the core principles of fractional calculus and FDEs.

In recent years, fractional-order reaction diffusion systems have garnered significant attention for modeling spatiotemporal dynamics in ecology, particularly in predator prey and competition systems. The introduction of temporal or spatial fractional derivatives has enabled the study of complex pattern formation, chaotic behaviors, and system stability. For instance, Ma et al. [7] investigated the effects of time delay and cross-fractional diffusion on Turing patterns in a predator prey model; Owolabi

and Karaagac [8] studied chaotic behaviors and spatiotemporal oscillations in fractional reaction–diffusion systems; and Manimaran et al. [9] analyzed weak solutions and performed simulations for a time-fractional competition model with cross-diffusion. Further, Owolabi [10] examined cross reaction–diffusion systems with Caputo–Fabrizio and Riesz operators, while Owolabi [11] explored chaotic pattern formation in a diffusive predator prey system with Caputo derivatives. Additionally, Zhi et al. [12] derived conditions for Turing instability and pattern formation in a model with Smith-type growth and additive predation, revealing a rich variety of stationary patterns driven by cross-diffusion. These works demonstrate that fractional derivatives and cross-diffusion profoundly influence dynamical behaviors.

Some studies have further expanded the scope of fractional-order ecological dynamics, encompassing diverse perspectives such as fear effects, Allee effects, stochastic disturbances, memory effects, discrete models, and higher-order food webs. For example, Mani et al. [13] investigated the dynamic behavior of a fractional-order eco-epidemiological model with a fear effect; Ali [14] analyzed a stochastic fractional-order predator prey system with nonlinear functional response and Allee effect; Madani et al. [15] employed a hybrid genetic algorithm–interior point algorithm (GA-IPA) and neural network approach to solve fractional-order nonlinear predator prey dynamical systems; Berkal et al. [16] discussed the qualitative behavior of a discretized conformable fractional-order predator prey model; Zeng et al. [17] conducted bifurcation analysis of a fractional-order eco-epidemiological system with two delays; Chakraborty et al. [18] employed a multiscale approach to study diffusive instability, patterns, and limit cycles in a generalized Samardzija–Greller model; Mondal et al. [19] explored the influence of hunting cooperation, fear, and incommensurate memory effects on a generalist predator prey model; Kumar et al. [20] systematically analyzed the complex dynamic behavior of fractional predator prey models; Li and Sun [21] investigated a delayed diffusive model with hunting cooperation and anti-predator behavior. Lv [22] established global boundedness and analyzed spatially inhomogeneous Hopf bifurcation in a delayed predator prey model with dual taxis. Mondal et al. [23] investigated the role of memory in the transition from chaos to stability in tri-trophic food chain dynamics; and Xie et al. [24] performed dynamic analysis of a fractional-order three-dimensional predator prey system with competitive and fear effects.

These achievements not only highlight the advantages of fractional derivatives in characterizing ecological memory and nonlocal effects, but also underscore the significant impacts of cross-diffusion, time delays, stochastic factors, and other elements on system stability and pattern formation, and numerical methods are essential for analyzing such complex models. Building upon the aforementioned research, this paper systematically investigates the stability, bifurcation, and numerical simulation of a fractional-order predator prey cross-reaction diffusion model, aiming to further reveal the dynamical mechanisms under the coupling effects of fractional-order and cross-diffusion.

The classical predator prey model with habitat complexity proposed by [7] is described by the following system of ordinary differential equations:

$$\begin{cases} \frac{du}{dt} = ru \left(1 - \frac{u}{K} \right) - \frac{(1-\beta)cu^\gamma v}{(1-\beta)c\zeta u^\gamma + 1}, \\ \frac{dv}{dt} = v \left(\frac{(1-\beta)ceu^\gamma}{(1-\beta)c\zeta u^\gamma + 1} - s \right), \end{cases}$$

where u and v denote the prey and predator densities. The parameter $\beta \in (0, 1)$ measures the strength or degree of habitat complexity, e is the conversion efficiency, c and ζ represent the attack coefficient and

handling time, r is intrinsic per capita growth rate of prey population, K is the environmental carrying capacity, γ quantifies a kind of aggregation efficiency, and s represents the death rate of predator population.

To incorporate spatial movement and interactions, reference [25] extends the model by introducing diffusion and cross-diffusion terms, leading to the following cross-reaction diffusion system:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ru \left(1 - \frac{u}{K}\right) - \frac{(1-\beta)eu^\gamma v}{(1-\beta)c\zeta u^\gamma + 1}, \\ \frac{\partial v}{\partial t} = d\Delta u + d_2 \Delta v + v \left(\frac{(1-\beta)ceu^\gamma}{(1-\beta)c\zeta u^\gamma + 1} - s \right), \end{cases}$$

where d_1 and d_2 are self-diffusion coefficients, d is the cross-diffusion coefficient, and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

In this paper, we investigate the following fractional-order cross-reaction diffusion model:

$$\begin{cases} \mathcal{D}_t^{\alpha_1} u = d_1 \Delta u + ru \left(1 - \frac{u}{K}\right) - \frac{(1-\beta)eu^\gamma v}{(1-\beta)c\zeta u^\gamma + 1}, \\ \mathcal{D}_t^{\alpha_2} v = d\Delta u + d_2 \Delta v + v \left(\frac{(1-\beta)ceu^\gamma}{(1-\beta)c\zeta u^\gamma + 1} - s \right), \end{cases} \quad (1.1)$$

where $\mathcal{D}_t^{\alpha_i}$ denotes the Caputo fractional derivative of order $\alpha_i \in (0, 2)$, and α_i are the rational numbers.

Numerical solutions of FDEs poses significant challenges due to the nonlocal nature and inherent singularities of fractional operators. Pioneering contributions have been made from various perspectives: fundamental numerical analysis and algorithms [26–28], comprehensive treatments of numerical methods [5, 6]; fractional-order control and approximations [29, 30], recent applications in pattern formation [31–35], matrix-based discrete fractional calculus [36, 37], finite element methods for time-fractional PDEs [38], and finite difference methods for fractional PDEs [39, 40].

This paper makes the following main contributions:

- A high-order numerical scheme for fractional-order system (1.1) is proposed. This scheme integrates polynomial generating functions for achieving p th order temporal accuracy in fractional derivatives with a fourth-order spatial discretization. It provides a reliable tool for long-term, large-scale numerical simulations of fractional-order reaction diffusion systems.
- The crucial role of cross-diffusion in fractional-order dynamics is elucidated. Through theoretical analysis and numerical experiments, it is demonstrated that the cross-diffusion term—where predator dispersal is influenced by prey density gradients—serves as a core mechanism for inducing Turing instability and generating complex spatial patterns. This deepens the understanding of spatial self-organization in ecological systems.
- A systematic linkage between fractional-order parameters and system dynamics is established. The study systematically analyzes how the fractional orders α_1 and α_2 affect stability, bifurcation, and pattern evolution, providing a theoretical foundation for using fractional-order models to capture memory effects and nonlocal interactions in ecological processes.

This paper is organized as follows. In Section 2, we conduct the stability analysis of the fractional-order predator prey cross-reaction diffusion model. In Section 3, we investigate the bifurcation analysis,

including both Turing and Hopf bifurcations. In Section 4, we develop a high-precision numerical scheme with polynomial generating functions and a fourth-order spatial discretization. In Section 5, we perform numerical simulations to demonstrate pattern formation and cross-diffusion effects. Finally, in Section 6, we present the conclusion and summarize the main findings.

2. Stability analysis

The equilibrium point is found by solving the system:

$$ru\left(1 - \frac{u}{K}\right) - \frac{(1-\beta)eu^\gamma v}{(1-\beta)c\zeta u^\gamma + 1} = 0, \quad \frac{(1-\beta)ceu^\gamma}{(1-\beta)c\zeta u^\gamma + 1} - s = 0.$$

From the second equation, we derive an explicit expression for u^* :

$$u^* = \left[\frac{s}{(1-\beta)c(e - s\zeta)} \right]^{1/\gamma}. \quad (2.1)$$

This equilibrium exists only when $e > s\zeta$ (the energy gain from predation exceeds the maintenance cost).

Then, v^* is calculated from the first equation:

$$v^* = \frac{r(1 - \frac{u^*}{K})[(1-\beta)c\zeta u^{*\gamma} + 1]}{(1-\beta)e}. \quad (2.2)$$

The Jacobian matrix at the equilibrium point $E_* = (u^*, v^*)$ is

$$J(E_*) = \begin{pmatrix} a_{11} - d_1 k^2 & a_{12} \\ a_{21} - dk^2 & a_{22} - d_2 k^2 \end{pmatrix},$$

where k is the spatial wave-number, and

$$\begin{aligned} a_{11} &= r\left(1 - \frac{2u^*}{K}\right) - \frac{(1-\beta)e\gamma(u^*)^{\gamma-1}v^*}{[(1-\beta)c\zeta(u^*)^\gamma + 1]^2}, & a_{12} &= -\frac{(1-\beta)e(u^*)^\gamma}{(1-\beta)c\zeta(u^*)^\gamma + 1}, \\ a_{21} &= \frac{(1-\beta)ce\gamma(u^*)^{\gamma-1}v^*}{[(1-\beta)c\zeta(u^*)^\gamma + 1]^2}, & a_{22} &= \frac{(1-\beta)ce(u^*)^\gamma}{(1-\beta)c\zeta(u^*)^\gamma + 1} - s. \end{aligned}$$

The corresponding characteristic equation is

$$\begin{aligned} \lambda^{m(\alpha_1+\alpha_2)} &- \left[\frac{(1-\beta)ce(u^*)^\gamma}{(1-\beta)c\zeta(u^*)^\gamma + 1} - s - d_2 k^2 \right] \lambda^{m\alpha_1} - \left[r\left(1 - \frac{2u^*}{K}\right) - \frac{(1-\beta)e\gamma(u^*)^{\gamma-1}v^*}{[(1-\beta)c\zeta(u^*)^\gamma + 1]^2} - d_1 k^2 \right] \lambda^{m\alpha_2} \\ &+ \left[r\left(1 - \frac{2u^*}{K}\right) - \frac{(1-\beta)e\gamma(u^*)^{\gamma-1}v^*}{[(1-\beta)c\zeta(u^*)^\gamma + 1]^2} - d_1 k^2 \right] \times \left[\frac{(1-\beta)ce(u^*)^\gamma}{(1-\beta)c\zeta(u^*)^\gamma + 1} - s - d_2 k^2 \right] \\ &- \frac{(1-\beta)e(u^*)^\gamma}{(1-\beta)c\zeta(u^*)^\gamma + 1} \left[\frac{(1-\beta)ce\gamma(u^*)^{\gamma-1}v^*}{[(1-\beta)c\zeta(u^*)^\gamma + 1]^2} - dk^2 \right] = 0, \end{aligned} \quad (2.3)$$

where $m = \text{lcm}(\text{denom}(\alpha_1), \text{denom}(\alpha_2))$. Here, $\text{lcm}(\text{denom}(\alpha_1), \text{denom}(\alpha_2))$ represents the least common multiple of $\text{denom}(\alpha_1)$ and $\text{denom}(\alpha_2)$, and $\text{denom}(\alpha_i)$ denotes the denominator (a positive integer) of the reduced fraction of the rational number α_i .

Define

$$f(\alpha_1, \alpha_2, k^2) = \min_i |\arg(\lambda_i)| - \frac{\pi}{2m}, \quad (2.4)$$

where λ_i are the eigenvalues of the characteristic equation (2.3).

If all eigenvalues λ_i satisfy $f(\alpha_1, \alpha_2, k^2) > 0$, then System (1.1) at the equilibrium point (u^*, v^*) is locally asymptotically stable. In particular, if all eigenvalues λ_i satisfy $f(\alpha_1, \alpha_2, 0) > 0$, then the System (1.1) without diffusion at the equilibrium point (u^*, v^*) is locally asymptotically stable [1].

We set the parameters as $r = 1.04$, $K = 8.0$, $\beta = 0.8$, $e = 0.4$, $c = 1.6$, $\zeta = 0.8$, $s = 0.39$, $d = 0$, $d_1 = 0$, $d_2 = 0$, $\gamma = 2.1$. The stability and the eigenvalue distribution of System (1.1) at the equilibrium $E_*(3.4957, 8.3977)$ are shown in Figure 1.

Phase portraits of System (1.1) at different fractional orders, with $d_1 = 0$, $d_2 = 0$, $d = 0$, $r = 0.8$, $K = 10.0$, $\beta = 0.8$, $e = 0.4$, $c = 1.6$, $\zeta = 0.8$, $s = 0.4$, $\gamma = 2.1$, are shown in Figure 2.

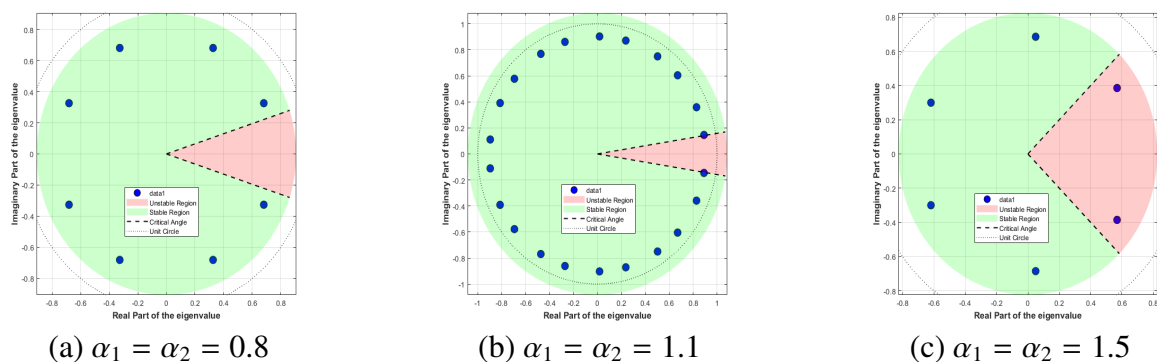


Figure 1. Stability and eigenvalue distribution of System (1.1) at the equilibrium $E_*(3.4957, 8.3977)$ for $r = 1.04$, $K = 8.0$, $\beta = 0.8$, $e = 0.4$, $c = 1.6$, $\zeta = 0.8$, $s = 0.39$, $d = 0$, $d_1 = 0$, $d_2 = 0$, $\gamma = 2.1$. The dots denote the eigenvalues λ_i of the characteristic equation, the dashed lines mark the critical angles $\pm\pi/(2m)$, the shaded sectors indicate the stable region $|\arg(\lambda)| > \pi/(2m)$ and the unstable region $|\arg(\lambda)| < \pi/(2m)$. The unit circle is included only as a visual reference and does not enter the stability criterion.

Figure 1 illustrates the stability and eigenvalue distribution of the fractional-order predator prey system at the equilibrium point $E_*(3.4957, 8.3977)$ under the parameter set given above, in the absence of diffusion. For all three fractional orders shown, the eigenvalues lie in the stable sector, which is consistent with the local asymptotic stability of the equilibrium predicted by the theoretical analysis of Section 2.

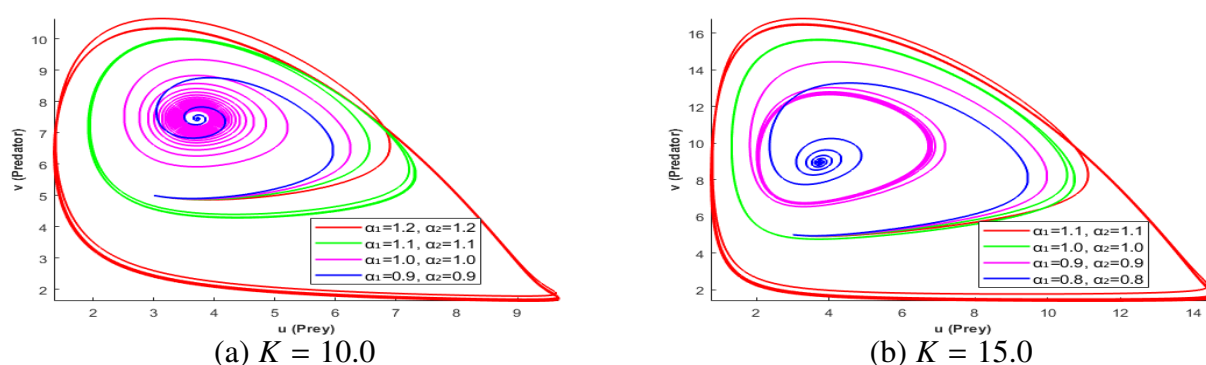


Figure 2. Phase portraits of System (1.1) at different fractional orders, with $d_1 = 0$, $d_2 = 0$, $d = 0$, $r = 0.8$, $K = 10.0$, $\beta = 0.8$, $e = 0.4$, $c = 1.6$, $\zeta = 0.8$, $s = 0.4$, $\gamma = 2.1$. Panel (a) corresponds to $K = 10.0$ and panel (b) to $K = 15.0$.

Figure 2 presents phase portraits of the system under different fractional orders α_1 and α_2 . A comparison of the phase trajectories for different orders shows how the fractional orders affect the dynamical behavior of the system. These numerical observations are consistent with the way in which α_1 and α_2 enter the characteristic equation and shift the argument of the eigenvalues, as discussed in Section 2.

3. Bifurcation analysis

The necessary condition for the Turing bifurcation of System (1.1) is

$$f(\alpha_1, \alpha_2, 0) > 0, \quad f(\alpha_1, \alpha_2, k^2) < 0, k \neq 0. \quad (3.1)$$

Suppose there exists k_c^2 such that $f(\alpha_1, \alpha_2, k_c^2) = 0$. k_c^2 is called the critical wave-number. Suppose there exists (α_1^*, α_2^*) such that $f(\alpha_1^*, \alpha_2^*, k^2) = 0$. Then, a Hopf bifurcation occurs at (α_1^*, α_2^*) .

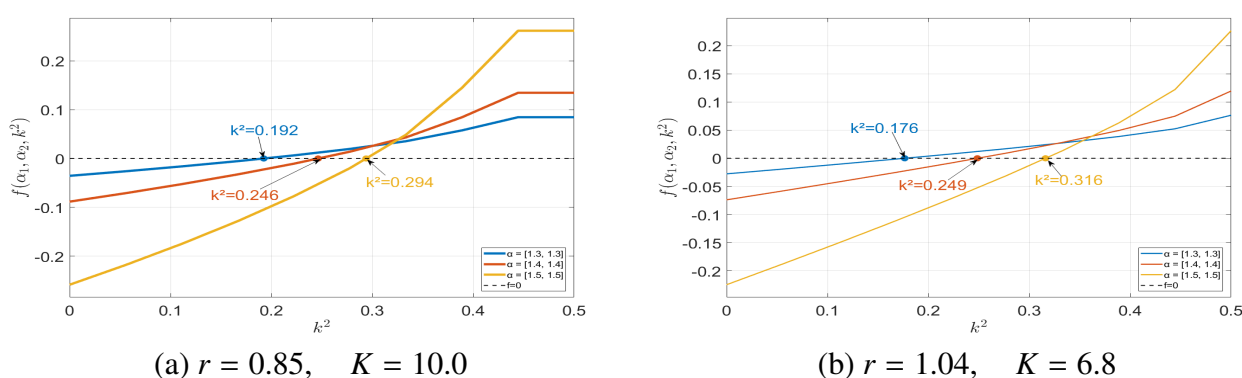


Figure 3. The critical wave-number at $r = 0.85$, $K = 10.0$, $\beta = 0.4$, $e = 0.5$, $c = 1.6$, $\zeta = 0.4$, $ds = 0.8$, $d = 0.9$, $d_1 = 0.2$, $d_2 = 2.5$, $\gamma = 2.1$.

We set parameters $r = 0.85$, $K = 10.0$, $\beta = 0.4$, $e = 0.5$, $c = 1.6$, $\zeta = 0.4$, $ds = 0.8$, $d = 0.9$, $d_1 = 0.2$, $d_2 = 2.5$, $\gamma = 2.1$. We call the equilibrium $(u^*, v^*) = (2.0746, 2.7951)$, and we compute k^2 for given

(α_1, α_2) to solve $f(\alpha_1, \alpha_2, k^2) = 0$ for the critical wave-number k^2 . When $\alpha_1 = 1.30, \alpha_2 = 1.30$, we can get $k_1^2 = 0.192381$. When $\alpha_1 = 1.40, \alpha_2 = 1.40$, we can get $k_1^2 = 0.246309$. When $\alpha_1 = 1.50, \alpha_2 = 1.50$, we can get $k_1^2 = 0.293853$. Figure 3 shows the results of bifurcation analysis. This provides a theoretical basis for the formation of Turing patterns.

Next, we compute the pairs (α_1, α_2) for a given k^2 satisfying $f(\alpha_1, \alpha_2, k^2) = 0$. When $k^2 = 0.00$, we obtain

$$\alpha_1 = 1.031, \alpha_2 = 1.000, \quad \text{and} \quad \alpha_1 = 1.000, \alpha_2 = 1.031.$$

When $k^2 = 0.01$, we obtain

$$\alpha_1 = 1.036, \alpha_2 = 1.000; \quad \alpha_1 = 1.033, \alpha_2 = 1.006; \quad \alpha_1 = 1.006, \alpha_2 = 1.033.$$

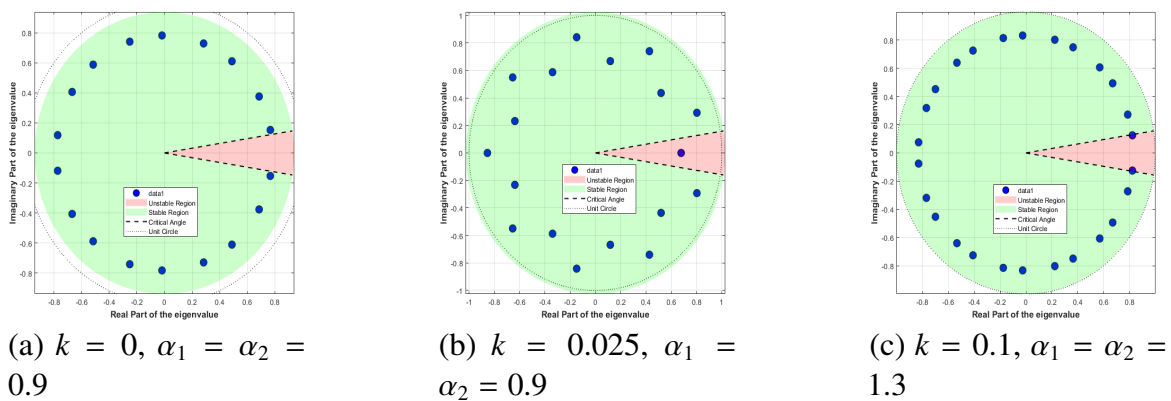


Figure 4. Stability and eigenvalue distribution of System (1.1) at the equilibrium $E_*(3.4957, 8.3977)$ for $r = 0.85, K = 8.0, \beta = 0.8, e = 0.4, c = 1.6, \zeta = 0.4, s = 0.8, d = 0.9, d_1 = 0.2, d_2 = 2.5, \gamma = 2.1$. The three panels correspond to different wave-numbers k and different fractional orders $\alpha_1 = \alpha_2$, as indicated in the sub captions. The dots denote the eigenvalues λ_i of the characteristic equation, the dashed lines mark the critical angles $\pm\pi/(2m)$, the shaded sectors indicate the stable region $|\arg(\lambda)| > \pi/(2m)$ and the unstable region $|\arg(\lambda)| < \pi/(2m)$. The unit circle is included only as a visual reference and does not enter the stability criterion.

Figure 4 compares the stability characteristics of the system for different wave-numbers k , with the diffusion coefficients $d = 0.9, d_1 = 0.2, d_2 = 2.5$. Panel (a) corresponds to $k = 0$, panel (b) to $k = 0.025$, and panel (c) to $k = 0.1$. As k increases, the eigenvalues move towards the boundary of the stable sector, which illustrates how the diffusion terms affect the linear stability of the equilibrium and can lead to spatial heterogeneity.

4. Numerical scheme

This section constructs a high-order numerical framework for the time-fractional cross-reaction–diffusion system (1.1). The scheme combines a p th order generating-function approximation of the fractional temporal derivative with a fourth-order nine-point spatial discretization, and incorporates the short-memory principle to reduce computational and storage costs. Special attention is paid to the treatment of nonzero initial values, which is essential for the Caputo-type system (1.1).

4.1. Discretization of the fractional time derivative

We adopt the Grünwald–Letnikov (GL) definition as the point of departure [4, 41].

Definition 4.1. For a sufficiently smooth function $f(t)$, the α -th order Grünwald–Letnikov fractional derivative over $[t_0, t]$ is defined as

$${}^{GL}_{t_0}\mathcal{D}_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor (t-t_0)/h \rfloor} (-1)^j \binom{\alpha}{j} f(t-jh), \quad (4.1)$$

where, $\lfloor \cdot \rfloor$ denotes the floor function, and the generalized binomial coefficient is

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}. \quad (4.2)$$

On a uniform temporal grid $t_k = t_0 + k\tau$, the GL derivative admits the discrete approximation

$${}^{GL}_{t_0}\mathcal{D}_t^\alpha f(t_k) \approx \tau^{-\alpha} \sum_{j=0}^k \omega_j f(t_{k-j}), \quad \omega_j = (-1)^j \binom{\alpha}{j}. \quad (4.3)$$

To avoid direct evaluation of gamma functions at large indices, the coefficients ω_j are computed recursively [41]:

$$\omega_0 = 1, \quad \omega_j = \left(1 - \frac{\alpha+1}{j}\right) \omega_{j-1}, \quad j = 1, 2, \dots \quad (4.4)$$

The approximations (4.3) and (4.4) are first-order accurate in time, that is $O(\tau)$.

To achieve p th order temporal accuracy, we employ the generating-function approach of Lubich [42].

Definition 4.2. The p th order generating function for the first-order derivative operator is

$$\mathcal{G}_p(z) = \sum_{\ell=1}^p \frac{1}{\ell} (1-z)^\ell = \sum_{\ell=0}^p g_\ell z^\ell, \quad (4.5)$$

where, the coefficients $\{g_\ell\}_{\ell=0}^p$ are uniquely determined by requiring $\mathcal{G}_p(z)$ to approximate $\ln \frac{1-z}{z}$ to order p near $z = 1$.

Theorem 4.3. The coefficients $\{g_\ell\}_{\ell=0}^p$ satisfy the linear system

$$\sum_{\ell=0}^p \ell^r g_\ell = \begin{cases} 0, & r = 0, \\ -r, & r = 1, 2, \dots, p. \end{cases} \quad (4.6)$$

Equivalently, in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & p+1 \\ 1 & 2^2 & 3^2 & \cdots & (p+1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^p & 3^p & \cdots & (p+1)^p \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_p \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ p \end{bmatrix}. \quad (4.7)$$

Exponentiating the generating function extends the construction to fractional orders.

Definition 4.4. The fractional p -th order generating function of order α is

$$\mathcal{G}_p^{(\alpha)}(z) = (\mathcal{G}_p(z))^\alpha = \left(\sum_{\ell=0}^p g_\ell z^\ell \right)^\alpha = \sum_{j=0}^{\infty} w_j^{(\alpha)} z^j. \quad (4.8)$$

Theorem 4.5. The expansion coefficients $w_j^{(\alpha)}$ are computed recursively. For $k \geq p$,

$$w_k^{(\alpha)} = -\frac{1}{g_0} \sum_{i=1}^p g_i \left(1 - i \frac{1+\alpha}{k} \right) w_{k-i}^{(\alpha)}, \quad (4.9)$$

whereas for $m = 1, 2, \dots, p-1$,

$$w_m^{(\alpha)} = -\frac{1}{g_0} \sum_{i=1}^m g_i \left(1 - i \frac{1+\alpha}{m} \right) w_{m-i}^{(\alpha)}, \quad (4.10)$$

with $w_0^{(\alpha)} = g_0^\alpha$ and $w_k^{(\alpha)} = 0$ for $k < 0$.

Proof. The result follows by differentiating both sides of Eq (4.8) with respect to z , multiplying by $\mathcal{G}_p(z)$, and equating the coefficients of equal powers of z ; see [41] for details.

Remark 4.6. For $k < p$, Eq (4.9) cannot be used directly because $w_{k-i}^{(\alpha)}$ with $k-i < 0$ would be involved. The initial coefficients must be obtained from Eq (4.10). This distinction is essential for a correct numerical implementation.

The resulting high-order approximation of the fractional derivative is

$${}^{GL}\mathcal{D}_t^\alpha f(t_k) \approx \tau^{-\alpha} \sum_{j=0}^k w_j^{(\alpha)} f(t_{k-j}), \quad (4.11)$$

with local truncation error [41]

$${}^{RL}\mathcal{D}_t^\alpha f(t_k) = \tau^{-\alpha} \sum_{j=0}^k w_j^{(\alpha)} f(t_{k-j}) + O(h^v) + O(h^p), \quad (4.12)$$

where v depends on the behavior of f near t_0 .

The generating-function approximation (4.11) is rigorous only for functions with zero initial values. For a function $y(t)$ with nonzero initial values, the term $O(h^v)$ in Eq (4.12) may dominate the $O(h^p)$ term and lead to severe oscillations near $t = t_0$ [41]. To remedy this, we introduce an auxiliary function

$$u(t) = \sum_{k=0}^p c_k (t - t_0)^k, \quad (4.13)$$

where the coefficients $\{c_k\}_{k=0}^p$ are chosen so that $u(t)$ and $y(t)$ coincide at the first $p + 1$ grid points, that is,

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & h & h^2 & \cdots & h^p \\ 1 & 2h & (2h)^2 & \cdots & (2h)^p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & ph & (ph)^2 & \cdots & (ph)^p \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} y(t_0) \\ y(t_0 + h) \\ y(t_0 + 2h) \\ \vdots \\ y(t_0 + ph) \end{bmatrix}. \quad (4.14)$$

The fractional-order derivative of $u(t)$ admits the closed-form expression

$${}^{RL}_{t_0} \mathcal{D}_t^\alpha (t - t_0)^k = \frac{\Gamma(k + 1)}{\Gamma(k + 1 - \alpha)} (t - t_0)^{k - \alpha}, \quad k = 0, 1, \dots, p. \quad (4.15)$$

Decomposing $y(t) = u(t) + v(t)$, the residual $v(t)$ has zero initial values up to order p , so that the high-order approximation (4.11) can be applied to $v(t)$. The final approximation reads

$${}^{RL}_{t_0} \mathcal{D}_t^\alpha y(t) = {}^{RL}_{t_0} \mathcal{D}_t^\alpha u(t) + {}^{RL}_{t_0} \mathcal{D}_t^\alpha v(t). \quad (4.16)$$

Remark 4.7. Without the compensation (4.16), the numerical derivative exhibits spurious oscillations near $t = t_0$, and the error may even increase as p grows [41]. This is particularly important for the Caputo system (1.1), where nonzero initial values are unavoidable.

4.2. Spatial discretization

For the two-dimensional Laplacian in System (1.1), we use the fourth-order nine-point compact stencil. At a grid point (x_i, y_j) with uniform mesh spacing h , the discrete Laplacian is

$$\Delta_h \phi_{i,j} = \frac{1}{6h^2} [\phi_{i+1,j+1} + \phi_{i-1,j-1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + 4(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}) - 20\phi_{i,j}], \quad (4.17)$$

which is $O(h^4)$ -accurate. For periodic boundary conditions on an $n \times n$ interior grid, the wraparound indexing

$$\phi_{i+n,j} = \phi_{i,j}, \quad \phi_{i,j+n} = \phi_{i,j}, \quad (4.18)$$

is applied to all eight neighbors in Eq (4.17). At the four corner points, Eq (4.18) is used twice along both directions, yielding the stencil

$$\Delta_h \phi_{1,1} = \frac{1}{6h^2} [\phi_{2,2} + \phi_{n,n} + \phi_{2,n} + \phi_{n,2} + 4(\phi_{2,1} + \phi_{n,1} + \phi_{1,2} + \phi_{1,n}) - 20\phi_{1,1}], \quad (4.19)$$

with analogous expressions at $(1, n)$, $(n, 1)$ and (n, n) .

4.3. Fully discrete formulation

Let $u_{i,j}^k$ and $v_{i,j}^k$ denote the numerical solutions at grid node (i, j) and time level t_k . Applying Eqs (4.16)–(4.11) to the fractional derivatives in Eq (1.1) and using Eq (4.17) for the diffusion terms, we obtain

$$\begin{aligned} \tau^{-\alpha_1} \sum_{m=0}^k w_m^{(\alpha_1)} u_{i,j}^{k-m} &= d_1 \Delta_h u_{i,j}^k + r u_{i,j}^k \left(1 - \frac{u_{i,j}^k}{K} \right) - \frac{(1 - \beta)e (u_{i,j}^k)^\gamma v_{i,j}^k}{(1 - \beta)c \zeta (u_{i,j}^k)^\gamma + 1}, \\ \tau^{-\alpha_2} \sum_{m=0}^k w_m^{(\alpha_2)} v_{i,j}^{k-m} &= d \Delta_h u_{i,j}^k + d_2 \Delta_h v_{i,j}^k + v_{i,j}^k \left(\frac{(1 - \beta)ce (u_{i,j}^k)^\gamma}{(1 - \beta)c \zeta (u_{i,j}^k)^\gamma + 1} - s \right). \end{aligned} \quad (4.20)$$

Isolating the $m = 0$ term and shifting the summation index $m \rightarrow m + 1$, we solve for the solutions at the next time level:

$$\begin{aligned} u_{i,j}^{k+1} &= \tau^{\alpha_1} \left[d_1 \Delta_h u_{i,j}^k + r u_{i,j}^k \left(1 - \frac{u_{i,j}^k}{K} \right) - \frac{(1-\beta)e(u_{i,j}^k)^\gamma v_{i,j}^k}{(1-\beta)c\zeta(u_{i,j}^k)^\gamma + 1} \right] - \sum_{m=1}^{\min(k,M)} w_m^{(\alpha_1)} u_{i,j}^{k+1-m}, \\ v_{i,j}^{k+1} &= \tau^{\alpha_2} \left[d \Delta_h u_{i,j}^k + d_2 \Delta_h v_{i,j}^k + v_{i,j}^k \left(\frac{(1-\beta)ce(u_{i,j}^k)^\gamma}{(1-\beta)c\zeta(u_{i,j}^k)^\gamma + 1} - s \right) \right] - \sum_{m=1}^{\min(k,M)} w_m^{(\alpha_2)} v_{i,j}^{k+1-m}, \end{aligned} \quad (4.21)$$

where the discrete Laplacian Δ_h is given by Eq (4.17) for interior and boundary nodes, and by Eq (4.19) for the four corner nodes. For $k = 0$, the memory sums are interpreted as zero.

The upper summation limit M in Eq (4.21) implements the short-memory principle [41]: The coefficients $w_m^{(\alpha)}$ with $m > M$ are neglected. This truncation reduces both the computational complexity per time step from $O(k)$ to $O(M)$ and the storage requirement from $O(k)$ to $O(M)$.

Following [41], the memory length L (with $M = L/\tau$) should satisfy

$$L \geq \left(\frac{M_f}{\epsilon |\Gamma(1-\alpha)|} \right)^{1/\alpha}, \quad (4.22)$$

where M_f is an upper bound of $|f(t)|$ on the interval of interest and ϵ is the prescribed error tolerance. It should be noted that Eq (4.22) is a conservative estimate; in some cases the short-memory approximation may fail even when Eq (4.22) is satisfied, and the full-memory version should then be used.

5. Numerical simulation

The spatial domain is discretized into an $n \times n$ grid. Next, we construct the initial conditions. Define four patch regions centered around the domain center as follows:

$$\begin{aligned} R_1 &= \{(i, j) : n/2 - 80 \leq i \leq n/2 - 48, n/2 - 16 \leq j \leq n/2 + 16\}, \\ R_2 &= \{(i, j) : n/2 + 48 \leq i \leq n/2 + 80, n/2 - 16 \leq j \leq n/2 + 16\}, \\ R_3 &= \{(i, j) : n/2 - 16 \leq i \leq n/2 + 16, n/2 - 80 \leq j \leq n/2 - 48\}, \\ R_4 &= \{(i, j) : n/2 - 16 \leq i \leq n/2 + 16, n/2 + 48 \leq j \leq n/2 + 80\}. \end{aligned} \quad (5.1)$$

Then, define the patch indicator function:

$$\mathcal{P}(i, j) = \begin{cases} 1, & \text{if } (i, j) \in \bigcup_{k=1}^4 R_k, \\ 0, & \text{otherwise.} \end{cases} \quad (5.2)$$

The initial prey population matrix is $\mathbf{U}_0(i, j) = u^* [1 + 0.1 \cdot \mathcal{P}(i, j)]$, $\forall i, j = 1, \dots, n$. The initial predator population matrix is $\mathbf{V}_0(i, j) = v^* [1 + 0.1 \cdot \mathcal{P}(i, j)]$, $\forall i, j = 1, \dots, n$.

We set $n = 200$, $\tau = 0.1$, $h = 1$, $p = 2$, $L = 50$, the parameters are shown in Tables 1 and 2. The periodic boundary condition is used. The numerical simulations of patterns are shown in Figures 5 and 6.

Figure 5 displays the spatial patterns at $\alpha_1 = \alpha_2 = 1.2$, $t = 20,000$, and different parameters.

Table 1. Parameters of Figure 5. Each row corresponds, from left to right and top to bottom, to one panel of the figure.

d_1	d_2	d	r	K	e	c	ζ	s	β	γ
0.2	1.0	0.81	0.85	8.1	0.9	1.2	0.5	0.9	0.5	2.1
0.2	1.2	0.82	0.85	8.2	0.7	1.3	0.4	0.8	0.4	2.2
0.2	1.3	0.85	0.68	8.0	0.8	1.7	0.5	0.9	0.4	2.6
0.2	1.4	0.83	0.85	8.3	0.8	1.4	0.4	0.8	0.3	2.3
0.2	1.2	0.84	0.85	8.3	0.6	1.5	0.3	0.8	0.3	2.4
0.2	1.0	0.85	0.82	8.2	0.7	1.6	0.4	0.9	0.4	2.5

Table 2. Parameters of Figure 6. Each row corresponds, from left to right and top to bottom, to one panel of the figure.

d_1	d_2	d	r	K	e	c	ζ	s	β	γ
0.2	1.9	0.9	1.04	6.8	0.8	1.6	0.4	0.8	0.39	2.1
0.2	2.5	0.9	0.85	6.8	0.8	1.6	0.4	0.8	0.39	2.1
0.2	1.9	0.9	0.95	6.8	0.8	1.6	0.4	0.8	0.39	2.1
0.2	2.4	0.9	0.85	6.8	0.8	1.6	0.4	0.8	0.39	2.1
0.2	2.0	0.9	0.85	6.8	0.8	1.6	0.4	0.8	0.39	2.1
0.2	1.9	0.9	1.04	8.0	0.8	1.6	0.4	0.8	0.39	2.1

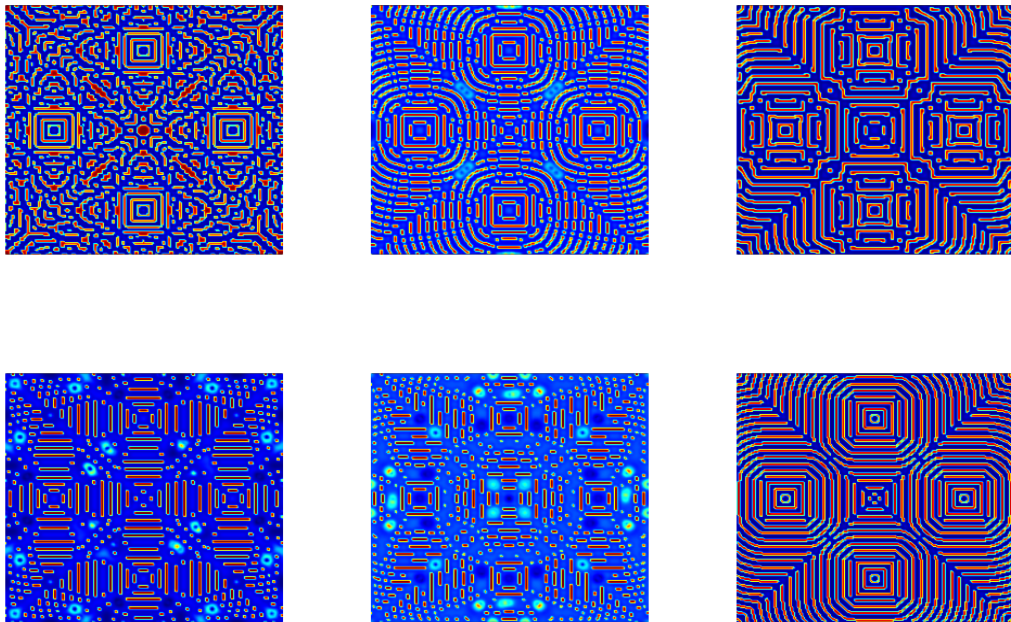


Figure 5. Patterns at $\alpha_1 = \alpha_2 = 1.2$, $t = 20,000$. Parameters (from left to right and top to bottom) are shown in Table 1.

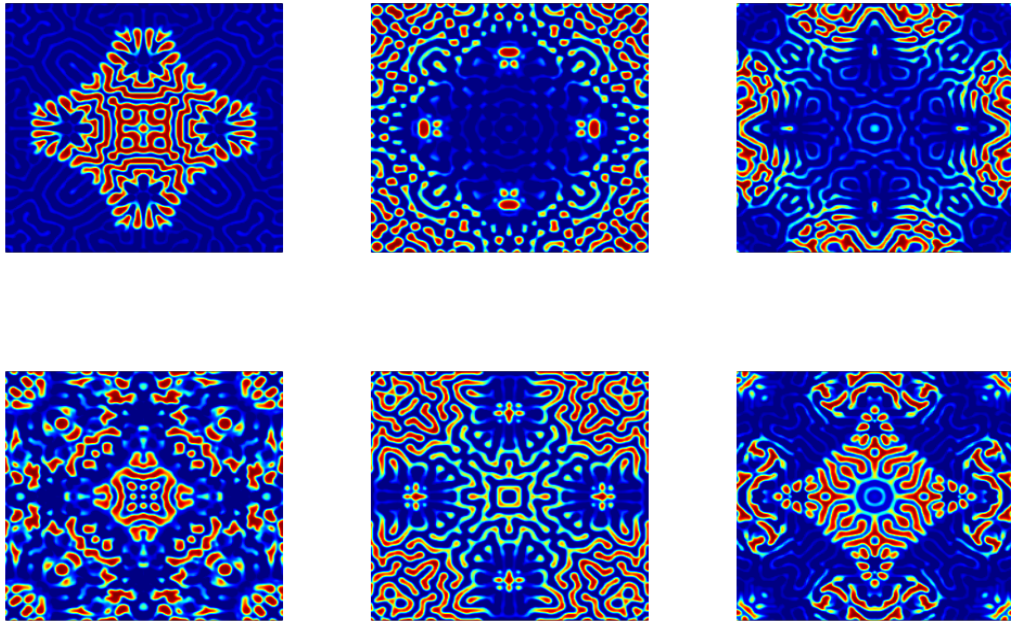
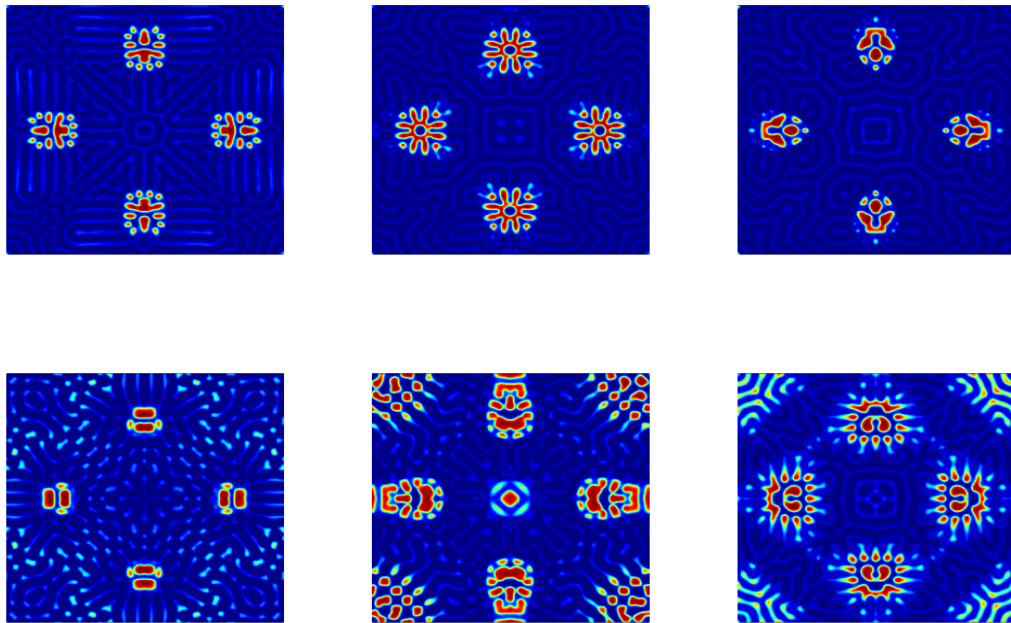
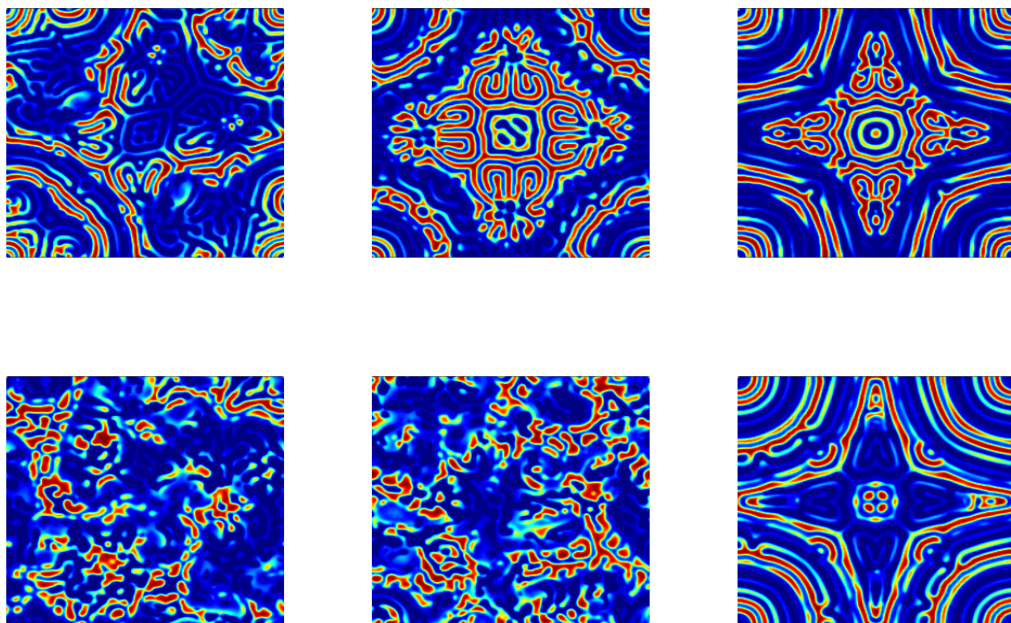


Figure 6. Patterns at $\alpha_1 = \alpha_2 = 1.3$, $t = 5000$. Parameters (from left to right and top to bottom) are shown in Table 2.

Table 3. Parameters of Figure 7. Each row corresponds, from left to right, to one panel of the corresponding row of the figure.

d_1	d_2	d	r	K	e	c	ζ	s	β	γ
0.2	2.0	0.9	0.85	7.0	0.8	1.6	0.4	0.8	0.4	2.1
0.2	2.2	0.9	0.85	8.0	0.8	1.6	0.4	0.8	0.4	2.1
0.2	2.4	0.9	0.85	9.0	0.8	1.6	0.4	0.8	0.4	2.1
0.2	2.5	0.9	0.85	6.8	0.8	1.6	0.4	0.8	0.4	2.1
0.2	2.5	0.9	0.82	6.8	0.8	1.6	0.4	0.8	0.4	2.1
0.2	2.3	0.9	0.68	7.0	0.8	1.6	0.4	0.8	0.4	2.1

Figure 6 presents patterns at $\alpha_1 = \alpha_2 = 1.3$ and $t = 5000$, with different parameters. These results further verify the significant impact of cross-diffusion on pattern complexity and dynamic evolution, demonstrating the system's capability for spatial self-organization in various ecological scenarios. Figure 7 explores the effect of different fractional orders (α_1, α_2) on pattern formation. The results show that variations in the orders significantly alter the scale, texture, and stability of the patterns, reflecting the regulatory role of fractional orders in spatial structure formation.

(a) $\alpha_1 = \alpha_2 = 1.3$ (b) $\alpha_1 = \alpha_2 = 1.25$ **Figure 7.** Patterns at different $\alpha_1 = \alpha_2$. Parameters (from left to right) are shown in Table 3.

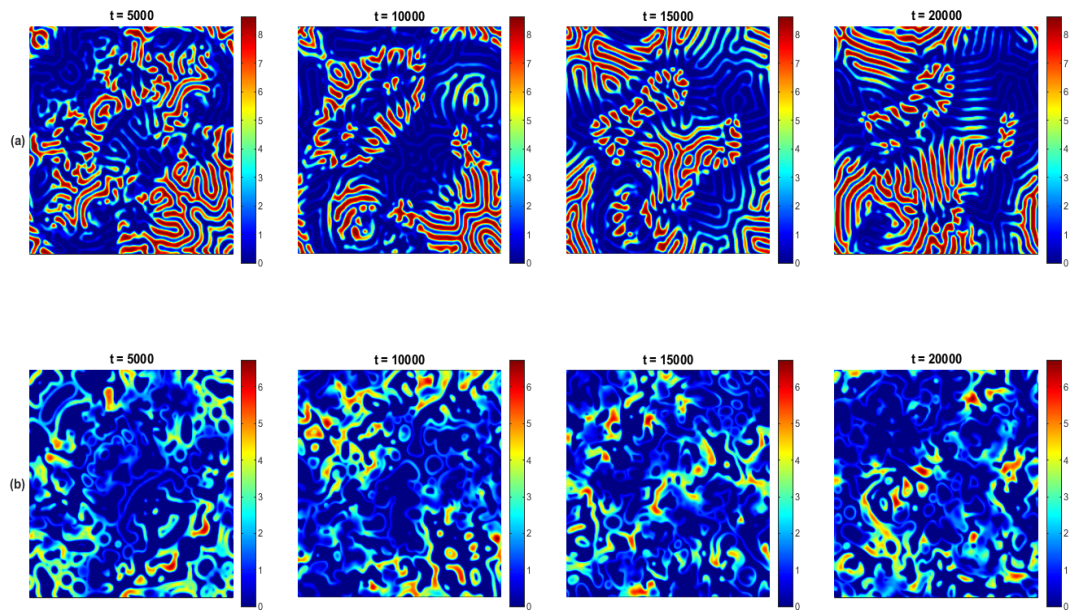


Figure 8. Effect of the sign of the cross-diffusion coefficient d on pattern formation in System (1.1) at $\alpha_1 = \alpha_2 = 1.3$, $d_1 = 0.2$, $d_2 = 2.0$, $r = 0.8$, $K = 8.1$, $e = 0.9$, $c = 1.8$, $\zeta = 0.5$, $s = 0.8$, $\beta = 0.5$, $\gamma = 2.5$: (a) $d = +0.8$ (positive cross-diffusion), producing large-scale stripe–patch patterns; (b) $d = -0.8$ (negative cross-diffusion), producing small-scale spot patterns. From left to right, $t = 5000, 10,000, 15,000, 20,000$.

Figure 8 illustrates the decisive influence of the sign of the cross-diffusion coefficient d on the spatiotemporal pattern formation of System (1.1). The parameters are fixed at $\alpha_1 = \alpha_2 = 1.3$, $d_1 = 0.2$, $d_2 = 2.0$, $r = 0.8$, $K = 8.1$, $e = 0.9$, $c = 1.8$, $\zeta = 0.5$, $s = 0.8$, $\beta = 0.5$ and $\gamma = 2.5$, while only the sign of d is varied. The evolution is recorded at four time instants, $t = 5000, 10,000, 15,000$ and $20,000$.

When $d = +0.8$ (positive cross-diffusion), the predator flux is directed opposite to the prey density gradient, that is, the predator tends to move away from regions of high prey density. At $t = 5000$, a mixed structure of stripes and spots emerges; at $t = 10,000$, the stripes become coarser and large-scale connected structures are formed; at $t = 15,000$, the pattern evolves into a labyrinthine morphology; and at $t = 20,000$, a stable coexistence of wide stripes and large patches is established.

When $d = -0.8$ (negative cross-diffusion), the predator flux is directed along the prey density gradient, that is, the predator tends to aggregate toward regions of high prey density. At $t = 5000$, fine and densely distributed spots appear; at $t = 10,000$, the spots grow slightly but remain fine-grained, at $t = 15,000$, local connectivity begins to emerge; and at $t = 20,000$, a mixture of fine patches and short stripes is formed, with a characteristic scale significantly smaller than that of the positive case.

These results confirm the theoretical prediction that d modulates the critical wave-number k_c^2 through the term $a_{21} - dk^2$ in the characteristic equation: $d > 0$ corresponds to a smaller critical wave-number and thus large-scale patterns, whereas $d < 0$ corresponds to a larger critical wave-number and thus small-scale patterns. Therefore, the sign of the cross-diffusion coefficient is a key factor in determining the scale, morphology, and temporal evolution pathway of Turing patterns.

6. Conclusions

This paper systematically investigates the stability, bifurcation, and pattern formation of a fractional-order predator prey cross-reaction diffusion model. Based on the characteristic equation of the fractional-order system, stability criteria are established. It is shown that the system without diffusion can remain stable, and the introduction of diffusion terms can induce Turing instability and lead to spatial pattern formation. The fractional orders α_1 and α_2 are found to play a crucial role in determining the stability and Hopf bifurcation of the system.

A high-order numerical method is proposed, which combines polynomial generating functions to achieve p th order temporal accuracy for fractional derivatives with a fourth-order spatial discretization. The method incorporates a short-memory principle to enhance computational efficiency, making it suitable for long-term simulations of fractional-order reaction diffusion systems.

Numerical simulations reveal a variety of spatial patterns. The numerical results clearly demonstrate the cross-diffusion process, where the predator's diffusion is influenced by the prey density gradient via the term $d\Delta u$. This cross-diffusion mechanism significantly enhances the complexity of spatial structures and is identified as a key factor in driving the formation and evolution of diverse patterns, highlighting the important role of cross-diffusion in the spatiotemporal dynamics of predator prey systems.

Author contributions

Conceptualization, methodology, software, data, formal analysis, funding acquisition, writing-original draft, and writing-review and editing: Hao Lu Zhang, Xiao Yu Li, Zhi Yuan Li. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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