



Research article

Parameter-uniform superconvergence of multiscale simulation on layer-adapted meshes for singularly perturbed problems

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Abstract: A robust multiscale finite element method is proposed for singularly perturbed convection–reaction–diffusion problems, featuring high accuracy, computational efficiency, and numerically observed parameter-uniform superconvergence. Within the multiscale finite element framework, multiscale basis functions are enriched with adequate local fine-scale information to enable global model reduction. The nontrivial relation between standard finite element basis functions and multiscale basis functions is analyzed and encoded in a global mapping matrix for practical implementation. By utilizing parameter-independent layer-adapted graded meshes, large errors induced by boundary layers are effectively suppressed. Numerical comparisons against existing approaches confirm the efficiency, reliability, and applicability of the presented scheme. Further numerical tests demonstrate this parameter-uniform superconvergence on graded meshes, as well as its promising capability for simulations in complex media and high-dimensional singularly perturbed systems.

Keywords: singular perturbation; multiscale scheme; layer-adapted mesh; uniform superconvergence

1. Introduction

Singularly perturbed problems pervade a wide spectrum of scientific and engineering disciplines, including turbulent wave propagation with chaotic and irregular dynamics, fluid dynamics requiring elaborate modeling of flow behavior, magnetohydrodynamic flows involving the coupling between magnetic fields and fluid motion, and biochemical reactors supporting various biological and chemical processes.

These problems are governed by differential equations with small perturbation parameters, which typically act as coefficients of the highest-order terms. The presence of such small parameters gives rise

to the well-known boundary layer phenomenon. A key characteristic is that while the global solution remains bounded, the solution exhibits extremely large derivatives within narrow layers adjacent to the domain boundary. Understanding this phenomenon is vital for clarifying the complex interaction between convection, defined as the heat or mass transfer driven by bulk fluid movement, and diffusion, referring to the substance transport from high-concentration to low-concentration regions. Therefore, investigating singularly perturbed problems is pivotal to unraveling the inherent mechanisms of these convection–reaction–diffusion dynamics.

The behaviors of singular perturbations in convection–diffusion and reaction–diffusion systems have been extensively investigated through a diverse array of numerical approaches, as documented in seminal monographs such as [1, 2]. These foundational works lay a solid theoretical groundwork and contribute to a comprehensive understanding of how singular perturbations manifest in practical engineering problems and can be effectively analyzed through rigorous numerical means. The spectrum of applicable methods is broad and continues to expand; it includes, but is not limited to, the finite difference method (FDM) [3–6], the finite volume method (FVM) [7, 8], the finite element method (FEM) [9–14], the streamline–diffusion FEM (SDFEM) [15, 16], the interior penalty Galerkin method [17], the local discontinuous Galerkin (LDG) [18–20], equidistributed moving-mesh methods [21, 22], and the multiscale FEM (MsFEM) [23–29].

The FDM approximates derivatives by finite differences. It divides the domain of interest into a grid of discrete points and replaces the derivatives in the governing equations with algebraic expressions based on the values of the function at neighboring grid points. This method is relatively straightforward to implement and is especially effective for problems with simple geometries.

The FEM, on the other hand, divides the domain into smaller elements. It constructs approximate solutions over these elements using basis functions. The FEM is highly versatile as it can handle complex geometries and a wide variety of boundary conditions. For instance, reference [11] investigated the use of piecewise polynomials of degree p for the approximation of singularly perturbed reaction–diffusion problems. By adopting an exponentially graded mesh, the authors established robust optimal convergence rates in a number of norms and verified the theoretical findings through computational demonstrations. Liu et al. [13] studied a high-order FEM on a Vulcanovi–Bakhvalov mesh for a singularly perturbed convection–diffusion problem. Building upon a novel Lagrange interpolation technique, they proved an optimal order of uniform convergence with respect to the perturbation parameter. In [14], Zhang et al. studied the uniform convergence properties of the FEM applied to two-dimensional convection–diffusion problems featuring exponential boundary layers on a Bakhvalov–type mesh, and derived the optimal-order uniform convergence result of order k in the energy norm. In [20], a detailed investigation was carried out on the reaction–diffusion equation with exponential boundary layers. For the LDG numerical scheme, the authors successfully deduced an optimal-order convergence in the balanced norm, where the computation relied on alternating fluxes and was validated across three distinct types of layer-adapted meshes. Apart from a priori layer-adapted meshes, a posteriori equidistributed moving-mesh (r -refinement) methods have been devised for singularly perturbed problems. Das et al. [21] constructed a cubic-spline hybrid finite difference scheme on curvature-monitor equidistributed meshes for coupled reaction–diffusion systems, attaining second-order parameter-uniform convergence without predefining boundary layer positions. In contrast, for k -star-graph reaction–diffusion problems with non-smooth sources under Kirchhoff conditions, they adopted

Shishkin meshes with three-point junction approximations to obtain nearly second order ε -uniform convergence [22].

Simultaneously, the MsFEM has been introduced, bringing with its substantial benefits. This method adeptly captures the fine-scale nuances within heterogeneous media without overly refined meshes. By minimizing the degrees of freedom, it not only cuts down on computational costs but also facilitates parallelization. Moreover, its high versatility allows for applications across a wide variety of scientific problems, and it can be integrated with other numerical techniques. Franca et al. [23] proposed a Petrov–Galerkin MsFEM, and they conducted a rigorous convergence analysis for the method across different asymptotic regimes. In [24], we put forward an adapted Petrov–Galerkin MsFEM to address singularly perturbed reaction–diffusion problems, constructed Petrov–Galerkin multiscale basis functions and graded meshes, and through numerical experiments, demonstrated the effectiveness in capturing boundary layers, reducing errors, and achieving stable results. Chen et al. [27] contributed to the exponentially convergent multiscale method (ExpMsFEM). Their work introduced a general and elegant multiscale finite element framework, extending their previous findings and providing a robust theoretical foundation. Li et al. [28] extended prior work by presenting a convergent iterative generalized multiscale finite element method (GMsFEM) approach that leverages a reduced multiscale space and parameter-independent reformulation to efficiently handle stochastic dual continuum problems, substantially lowering computational costs. Jiang et al. [29] developed a superconvergent multiscale finite element scheme for Helmholtz problems in rapidly oscillating heterogeneous media, relevant to wave propagation applications.

Now the characteristics of this work are briefly stated:

1. Within the proposed multiscale framework, multiscale basis functions are enriched with local fine-scale information to facilitate global model reduction. The inherent relation between standard finite element basis functions and multiscale basis functions is formulated and encoded in a global mapping matrix for practical implementation.
2. A graded mesh is constructed recursively using a mesh characterization function. This mesh design is independent of the perturbation parameter, enabling automatic detection of boundary layers and effectively mitigating the large errors arising in boundary layers.
3. Numerical tests and comparative studies demonstrate the efficiency, reliability, and applicability of our graded-mesh MsFEM for singularly perturbed problems. Numerical observations show parameter-uniform superconvergence on graded meshes without additional post-processing; this numerically demonstrated behavior constitutes the main novelty of the present work.

The paper is organized as follows: Section 2 introduces the model problem and its solution decomposition. Two numerical strategies are then considered, and the detailed multiscale computational principles are elaborated in Section 3. In Section 4, several non-uniform meshes, including an updated graded mesh, are constructed for adaptivity. This updated graded mesh is independent of the perturbation parameter value and can automatically resolve subtle boundary-layer regions. In Section 5, numerical results are presented and compared to demonstrate the advantages of the proposed method. Finally, concluding remarks are provided in Section 6.

2. Model problem and solution decomposition

In the paper, we investigate a two-dimensional convection-reaction-diffusion problem

$$\begin{cases} \mathcal{L}u := -\varepsilon \Delta u + \vec{b} \cdot \nabla u + cu = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where ε is a small positive parameter, $\vec{b} = (b_1, b_2)^T$ is a velocity vector, and c , f , and g are smooth functions defined on the square domain $\Omega = (0, 1)^2$ and its boundary $\partial\Omega$, with u denoting the solution to the problem. Specifically, the term $-\varepsilon \Delta u$ corresponds to the diffusion part (describing the spatial spread of the quantity u), $\vec{b} \cdot \nabla u$ represents the convection part (capturing the transport of u by the flow field $\vec{b}$), and cu stands for the reaction part (typically modeling processes like the generation or consumption of u in physical/chemical systems). These three components together constitute the convection–reaction–diffusion system targeted in this work.

This equation is commonly regarded as a linearized form of the Navier–Stokes equation, and it is widely applied in the field of fluid dynamics. For the model problem above, an extremely small value of ε (relative to the convection and reaction effects) gives rise to boundary and/or interior layer phenomena, which refer to regions where the solution u changes sharply over a narrow spatial range. In such scenarios, standard numerical methods may produce significant errors in the form of large oscillations, which are typical challenges of singularly perturbed problems.

Since the solution exhibits a multiscale nature, it can be analyzed via scale decomposition. Suppose u can be decomposed into several components:

$$u = u^s + \sum_{k=1}^4 u_k^e + \sum_{k=1}^4 u_k^c. \quad (2.2)$$

Here u^s denotes the smooth component, u_k^e represent edge boundary-layer components, and u_k^c denote corner boundary-layer components of the solution.

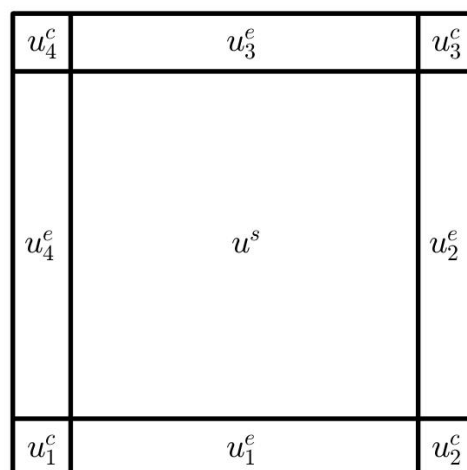


Figure 1. Domain decomposition and the associated solutions.

Lemma 2.1. [1, 8] For some constant $\delta > 0$, the derivatives of decomposition (2.2) satisfy the following bounds:

$$\begin{aligned} \left| \frac{\partial^{i+j} u^s}{\partial x^i \partial y^j} \right| &\leq C, \\ \left| \frac{\partial^{i+j} u_1^e}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-j} \exp^{-\frac{\delta y}{\varepsilon}}, \\ \left| \frac{\partial^{i+j} u_2^e}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-i} \exp^{-\frac{\delta(1-x)}{\varepsilon}}, \\ \left| \frac{\partial^{i+j} u_3^e}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-j} \exp^{-\frac{\delta(1-y)}{\varepsilon}}, \\ \left| \frac{\partial^{i+j} u_4^e}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-i} \exp^{-\frac{\delta x}{\varepsilon}}, \\ \left| \frac{\partial^{i+j} u_1^c}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-(i+j)} \left(\exp^{-\frac{\delta x}{\varepsilon}} \cdot \exp^{-\frac{\delta y}{\varepsilon}} \right), \\ \left| \frac{\partial^{i+j} u_2^c}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-(i+j)} \left(\exp^{-\frac{\delta(1-x)}{\varepsilon}} \cdot \exp^{-\frac{\delta y}{\varepsilon}} \right), \\ \left| \frac{\partial^{i+j} u_3^c}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-(i+j)} \left(\exp^{-\frac{\delta(1-x)}{\varepsilon}} \cdot \exp^{-\frac{\delta(1-y)}{\varepsilon}} \right), \\ \left| \frac{\partial^{i+j} u_4^c}{\partial x^i \partial y^j} \right| &\leq C \varepsilon^{-(i+j)} \left(\exp^{-\frac{\delta x}{\varepsilon}} \cdot \exp^{-\frac{\delta(1-y)}{\varepsilon}} \right), \end{aligned}$$

for $(x, y) \in \Omega$. See Figure 1.

For the well-posed problem, the weak form of Eq (2.1) is to find $u \in H^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (2.3)$$

where

$$a(u, v) = \int_{\Omega} (\varepsilon \nabla u \cdot \nabla v + (\vec{b} \cdot \nabla u) v + cuv) \, d\mathbf{x}, \quad (2.4)$$

$$(f, v) = \int_{\Omega} f v \, d\mathbf{x}, \quad (2.5)$$

where $\mathbf{x}$ represents a two-dimensional spatial variable which consists of x, y . Suppose that

$$\vec{b} \geq (\beta_1, \beta_2)^T \geq (0, 0)^T, \quad c - \frac{1}{2} \nabla \cdot \vec{b} \geq \gamma > 0, \quad \text{for } (x, y) \in \Omega.$$

Given that the bilinear form $a(u, v)$ is both V -elliptic and continuous and the linear functional (f, v) is continuous on the function space, the problem (2.1) admits a unique weak solution by the Lax–Milgram theorem. This theoretical guarantee lays a solid foundation for subsequent numerical discretization, ensuring the well-posedness essential for reliable computational results. Such uniqueness also validates the rationality of our multiscale scheme's design, as it targets a mathematically rigorous solution framework.

3. Numerical strategies

3.1. Finite element method

The weak formulation given by Eq (2.3) corresponds to an infinite-dimensional problem, so appropriate numerical strategies must be employed to derive a finite-dimensional approximation. Within the context of the standard FEM, we let $\mathcal{K}^h$ represent a fine mesh partition of the domain Ω that may be non-uniform, consisting of individual elements. For each element $K \in \mathcal{K}^h$, the element size h_K satisfies $0 < h_K \ll 1$, with $h = \max_K h_K$ denoting the global mesh size. Over each fine element K , a set of nodal basis functions is constructed, typically associated with the four vertices of the rectangular elements used in the discretization.

Over each fine-scale element K , a set of nodal basis functions is constructed, typically associated with the four vertices of the rectangular elements used in the discretization. The finite element space is constructed using piecewise bilinear polynomials $\mathbb{Q}^1(K)$, with its definition given by

$$V^h = \{v_h \in H^1(\Omega) : v_{h|_K} \in \mathbb{Q}^1(K), \forall K \in \mathcal{K}^h\}. \quad (3.1)$$

The Galerkin formula seeks a solution $u_g \in V^h$ such that

$$a(u_g, v) = (f, v), \quad \forall v \in V_0^h, \quad (3.2)$$

where u_g denotes the Galerkin FEM solution. This solution can be represented via local interpolation as $u_g^I = \sum_i u_g(\mathbf{x}_i) \psi_i(\mathbf{x})$, with ψ_i denoting the finite element basis functions.

Within this framework, let NM denote the number of fine-scale elements along the x - and y -directions, respectively, where N, M stand for partition numbers along each spatial direction. For a two-dimensional problem, the total number of elements at the fine-mesh level amounts to $N^2 M^2$. It is well known that the computational cost of the FEM scales as $O(N^2 M^2)$.

However, the standard FEM has inherent limitations: To resolve steep boundary or interior layers induced by small ε , a hallmark of singularly perturbed problems, it requires extremely dense mesh refinement. This leads to a sharp increase in computational cost and may still fail to suppress numerical oscillations. The nodal basis functions lack the ability to capture the solution's local multiscale features adaptively, which further restricts the method's performance for such problems.

3.2. Multiscale finite element method

In contrast, the weak formulation of the multiscale strategy aims to restrict the solution of the variational problem (2.3) to a multiscale finite-dimensional subspace $U^H \subset H^1(\Omega)$. Specifically, it seeks a multiscale solution $u_H \in U^H$ such that

$$a(u_H, v) = (f, v), \quad \forall v \in U^H, \quad (3.3)$$

where the multiscale function space U^H will be formally defined in Eq (3.5) later in this section. Analogous to the standard finite element interpolation, the multiscale solution u_H can be expressed in terms of its nodal values and multiscale basis functions as $u_H^I = \sum_i u_H(\mathbf{x}_i) \varphi_i(\mathbf{x})$, with φ_i denoting the multiscale basis functions tailored to capture the problem's intrinsic small-scale features.

We now proceed to construct the multiscale basis functions by solving a series of local boundary value problems defined on coarse elements. For each coarse-scale element K that partitions the computational domain Ω , consider the following local auxiliary problem:

$$\begin{cases} \mathcal{L}\varphi_i := -\varepsilon\Delta\varphi_i + \vec{b} \cdot \nabla\varphi_i + c\varphi_i = 0, & \text{in } K, \\ \varphi_i = \theta_i, & \text{on } \partial K, \end{cases} \quad (3.4)$$

where the differential operator $\mathcal{L}$ is consistent with that of the original model problem (2.1). A key constraint imposed on the multiscale basis functions is the Kronecker delta property: $\varphi_i(\mathbf{x}_j) = \delta_{ij}$, which ensures the inter-element continuity of the multiscale space and simplifies the assembly of the global stiffness matrix. Here, $\mathbf{x}_i \in \bar{K}$ ($i = 1, \dots, 4$) denote the four vertices of the rectangular coarse element K , labeled in a counter-clockwise order starting from the lower-left corner as (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , and (x_4, y_4) . In this work, we adopt linear boundary conditions θ_i for each coarse element K to enforce the delta property and enhance the basis functions' ability to resolve boundary layer effects. For illustrative purposes, the boundary condition θ_1 corresponding to the vertex $\mathbf{x}_1 = (x_1, y_1)$ is defined as follows:

$$\theta_1 = \begin{cases} \frac{x_2 - x}{x_2 - x_1}, & \text{on } [x_1, x_2], \\ \frac{y_4 - y}{y_4 - y_1}, & \text{on } [y_1, y_4]. \end{cases}$$

This definition directly enforces $\theta_1(\mathbf{x}_1) = 1$ and $\theta_1(\mathbf{x}_j) = 0$ for $j = 2, 3, 4$, satisfying the delta constraint. The boundary conditions θ_2 , θ_3 , and θ_4 corresponding to the other three vertices are defined analogously, with each θ_i taking the value 1 at its associated vertex and 0 at all other vertices of the coarse element K .

Given the above-specified boundary conditions, the local problem (3.4) on each coarse element K can be efficiently solved using the standard FEM on a fine subelement mesh. Specifically, we discretize each coarse element K into smaller subelements with a uniform subelement size $h = H_K/M$, where H_K denotes the characteristic size of the coarse element K and M represents the number of uniform subpartitions in each spatial direction (i.e., x - and y -directions) within K . This fine-scale discretization enables the local problem to capture the small-scale features induced by singular perturbations, which are otherwise difficult to resolve using only coarse-scale meshes.

Upon obtaining the multiscale basis functions φ_i for each coarse element throughout the domain, the global multiscale finite element space U^H is constructed as the span of these functions, whose formal definition is given by

$$U^H = \text{span}\{\varphi_i^K : i = 1, \dots, 4, \forall K \in \mathcal{K}^H\}, \quad (3.5)$$

where $\mathcal{K}^H$ denotes the set of all coarse elements partitioning Ω and φ_i^K represents the i -th multiscale basis function associated with coarse element K . A critical advantage of this construction is that the multiscale basis functions inherently encode the microscopic details of the original macroscopic problem (2.1), such as the steep gradients and boundary layer structures caused by singular perturbations. As a consequence, these small-scale characteristics are seamlessly incorporated into the macroscopic multiscale solution u_H without requiring a globally fine mesh for the entire computational domain. It is also worth emphasizing that the multiscale basis functions φ_i are adaptive to different values of the perturbation parameter ε in Eq (3.4): As ε varies, the basis functions automatically adjust to capture local perturbations of different scales across the domain. This unique

property endows the multiscale basis functions with the flexibility to adapt to a wide range of singular perturbation scenarios, making them highly effective in accurately representing the multiscale behavior of the solution.

Based on this fine subelement discretization, each multiscale basis function φ_i defined on the coarse element $K_{i,j}$ can be expanded as a linear combination of the standard finite element basis functions ψ_l , which are piecewise bilinear on the fine subelement mesh. Specifically, we have

$$\varphi_i = \sum_{l=1}^{(M+1)^2} r_{il} \psi_l, \quad (3.6)$$

where r_{il} are the corresponding combination coefficients between the multiscale basis functions and the standard finite element basis functions; these coefficients are computed by solving the local problem (3.4) and are subsequently stored as entries in a global mapping matrix. This mapping matrix plays a pivotal role in the efficient assembly of the multiscale stiffness matrix, as it allows the multiscale method to leverage precomputed fine-scale information while maintaining the computational efficiency of a coarse-scale global discretization.

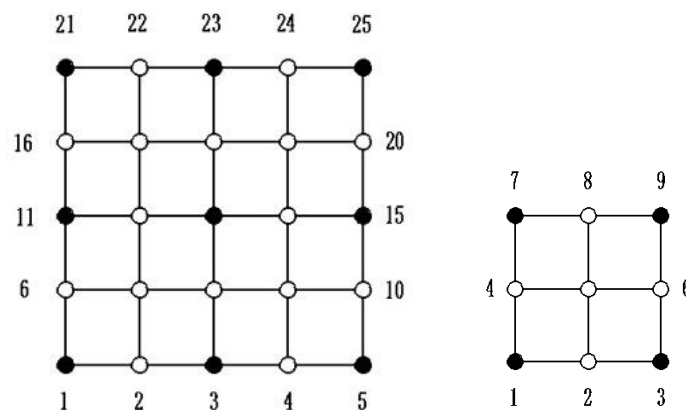


Figure 2. Global nodes and local nodes, with $N = 2$ and $M = 2$.

To clarify, we refer to Figure 2 with coarse mesh refinement level $N = 2$ and local fine mesh refinement level $M = 2$. In this schematic, solid points represent the nodes of the coarse mesh, while hollow points together with solid points collectively form all nodes of the underlying fine mesh. For this configuration, the coarse mesh contains nine solid nodes, which define the degrees of freedom for the global multiscale basis functions φ_i . By contrast, the fine mesh contains 25 nodes in total, and these nodes correspond to the degrees of freedom for the standard finite element basis functions ψ_i . The coarse mesh is constructed by partitioning the computational domain into $N \times N$ coarse elements. Within each coarse element, a local fine mesh with $M \times M$ subdivisions is generated to resolve local

Conversely, the standard FEM requires extremely fine meshes to resolve the sharp boundary layers arising in singularly perturbed problems, leading to a very large discrete linear system:

$$A^f u_g = F^f. \quad (3.11)$$

This contrast highlights the strength of MsFEM: It delivers comparable accuracy while substantially cutting computational expense. The present work develops and validates this customized multiscale method, with the main goal of capturing the boundary-layer phenomena for the two-dimensional singularly perturbed convection–reaction–diffusion equation.

3.3. Computational criteria

It is a well-established fact that for the singularly perturbed problem described by Eq (2.1) when $h_K^2 \ll \varepsilon$, the FEM can yield accurate results. However, when ε becomes extremely small, specifically when $\varepsilon \ll h_K^2$, the FEM solution u_g obtained via the FEM may exhibit spurious oscillations and substantial errors within the boundary layers. Significantly, the MsFEM capitalizes on the use of multiscale basis functions within coarse elements, with the submesh size fulfilling the condition $h^2 = (H_K/M)^2 \leq \varepsilon$. In the subsequent sections, numerical examples illustrate that the proposed multiscale scheme effectively reduces boundary layer errors and yields numerically observed superconvergent results on graded meshes.

In the following error tests and numerical experiments, we will evaluate the approximation effect by employing L^∞ , L^2 , H^1 , and ε -weighted energy norms in the following forms:

$$\|u - u_{\text{approx}}\|_{L^\infty} = \max_{\Omega} |u - u_{\text{approx}}|, \quad (3.12)$$

$$\|u - u_{\text{approx}}\|_{L^2}^2 = \int_{\Omega} (u - u_{\text{approx}})^2 d\mathbf{x}, \quad (3.13)$$

$$\|u - u_{\text{approx}}\|_{H^1}^2 = \int_{\Omega} \left((u - u_{\text{approx}})^2 + (u - u_{\text{approx}})_x^2 + (u - u_{\text{approx}})_y^2 \right) d\mathbf{x}, \quad (3.14)$$

$$\|u - u_{\text{approx}}\|_{en}^2 = \|u - u_{\text{approx}}\|_{L^2}^2 + \varepsilon \|u - u_{\text{approx}}\|_{H^1}^2, \quad (3.15)$$

where u_{approx} represents the numerical approximations u_g in Eq (3.2) or u_H in Eq (3.3), respectively.

3.4. Stability result

For the continuous singularly perturbed convection–reaction–diffusion problem, well-posedness and continuous stability follow from the Lax–Milgram theorem, since the associated bilinear form is continuous and V -elliptic.

For fitted-mesh finite difference schemes of similar singularly perturbed problems, rigorous discrete stability estimates can be established via discrete maximum principle and barrier function techniques under suitable mesh assumptions [5]. However, such rigorous discrete stability analysis is non-trivial for the present MsFEM, due to the construction of multiscale coarse basis functions. We therefore study

the numerical stability of MsFEM(G) by numerical experiments in later sections, and full rigorous discrete stability estimates are reserved for future work.

Lemma 3.1. Let $u(x, y)$ be the solution of problem (2.1). Then u satisfies the L^∞ stability estimate

$$\|u\|_{\bar{\Omega}} \leq \frac{1}{c_0} \|f\|_{\bar{\Omega}} + \max_{\partial\Omega} |u|, \quad (3.16)$$

where $c_0 > 0$ is the positive lower bound of the reaction coefficient $c(x, y)$.

Proof. We construct the piecewise linear barrier function

$$\phi^\pm(x, y) = C + \frac{\|f\|_{\bar{\Omega}}}{c_0} (1 + x + y) \pm u(x, y), \quad (3.17)$$

where $C = \max_{\partial\Omega} |u|$. Since Ω is a bounded square domain, $1 + x + y$ remains bounded on $\bar{\Omega}$. On the boundary $\partial\Omega$, we have $\phi^\pm \geq 0$. Under our assumption $c(x, y) \geq c_0 > 0$, direct computation yields $\mathcal{L}\phi^\pm \geq 0$ for all $(x, y) \in \Omega$. By the maximum principle, $\phi^\pm(x, y) \geq 0$ over $\bar{\Omega}$, which implies the desired bound (3.16). $\square$

4. Construction of non-uniform meshes and adaptation

4.1. Shishkin mesh

In this section, we introduce domain partitions denoted as $\mathcal{K}^h$. Consider the model problem (2.1) first: If the parameter ε assumes a relatively large value, no boundary layer phenomenon arises. It is widely acknowledged that traditional numerical methods yield satisfactory results when applied on a uniform mesh with mesh size $h = \frac{1}{N}$.

However, when ε in Eq (2.1) becomes extremely small, pronounced boundary layers will emerge in the solution domain. Even with a sufficiently large partition number N , the standard FEM may still generate non-physical spurious oscillations. To address this issue, we adopt the Shishkin mesh technique [2, 4], which partitions the two-dimensional computational domain into nine distinct subdomains that cover both smooth solution regions and sharp boundary layer regions, as depicted in Figure 1. Based on a priori error estimates, we define a transition point as $\tau = \min\{\frac{1}{4}, \frac{\varepsilon \log N}{\beta_i}\}$, which serves to pinpoint the precise location of boundary layers. For the Shishkin mesh, the total partition number N is pre-specified. In each spatial direction, the mesh is composed of three parts with the partition ratio $\frac{N}{4} : \frac{N}{2} : \frac{N}{4}$. Correspondingly, each of these three segments is discretized into piecewise equidistant subintervals, as shown in

$$x_i = \begin{cases} \frac{4\tau}{N} \cdot (i-1), & i = 1, \dots, \frac{N}{4}, \\ \tau + \frac{2(1-2\tau)}{N} \cdot (i - \frac{N}{4} - 1), & i = \frac{N}{4} + 1, \dots, \frac{3N}{4}, \\ 1 + \frac{4\tau}{N} \cdot (i - N - 1), & i = \frac{3N}{4} + 1, \dots, N + 1. \end{cases} \quad (4.1)$$

For the two-dimensional case, the nodes y_j are likewise arranged in a piecewise equidistant pattern across their corresponding mesh segments.

4.2. Graded mesh

Although the Shishkin mesh has proven to be a valuable tool for singularly perturbed problems and significantly simplifies the theoretical error analysis of numerical schemes, it does exhibit inherent limitations rooted in its piecewise equidistant discretization structure, as explicitly depicted in Eq (4.1). When addressing the two-dimensional singularly perturbed convection–reaction–diffusion problem with pronounced boundary layer effects, our primary focus is on devising a more efficient and robust layer-adapted mesh that can precisely resolve the steep solution gradients within thin boundary layers without over-refining the entire computational domain. In one of our previous works [24], we developed a graded mesh where the mesh parameter λ , an integer greater than one, was introduced to implement exponential mesh refinement near layer regions. Now we are set to introduce an updated and enhanced version of this graded mesh framework. The advanced graded mesh presented in this paper is meticulously crafted to be fully layer-adapted and highly anisotropic, with its refinement intensity directly linked to the magnitude of the perturbation parameter ε . This tailored design enables it to tackle more effectively the numerical challenges posed by the multiscale features of the problem under consideration.

The graded mesh is generated through a tensor product approach in both the x and y directions of the two-dimensional domain. It relies on a well-defined mesh-characterizing function ω , which can be rigorously expounded in terms of a smooth mesh-generating function g that encodes the layer-adaptive refinement rules. The explicit analytical form of this function is given by

$$\omega = C_1 \exp^{-\frac{\delta x_i}{\varepsilon}} + C_2 \exp^{-\frac{\delta(1-x_i)}{\varepsilon}} \quad (4.2)$$

for some positive constant $\delta > 0$. This function serves as a fundamental foundation for recursive iteration, enabling the systematic and iterative construction of the graded mesh, starting from the initial parameters $\sigma = 1$, $h = \frac{1}{2^k}$, $k \in \mathbb{N}$ as follows:

$$x_i = \begin{cases} 0, & i = 1, \\ \sigma \varepsilon h, & i = 2, \\ (1 + \sigma h)x_{i-1}, & 3 \leq i \leq N_0 - 1, \\ \frac{1}{2}, & i = N_0, \\ 1 - (1 + \sigma h)(1 - x_{i+1}), & N_0 + 1 \leq i \leq N - 1, \\ 1 - \sigma \varepsilon h, & i = N, \\ 1, & i = N + 1. \end{cases} \quad (4.3)$$

It should be noted that the partition number N of the proposed graded mesh differs distinctly from that of the Shishkin mesh. In this context, a critical mid-point N_0 (an integer index of mesh nodes) is required to satisfy the specific conditions $x_{N_0-1} < \frac{1}{2}$ and $(1 + h)x_{N_0-1} > \frac{1}{2}$. As mentioned previously, the nodes y_j in the transverse direction can be generated in an entirely analogous manner by following the same mesh construction logic.

5. Numerical results

The capabilities of the proposed multiscale method for solving the singularly perturbed problem will be demonstrated in this section. All the codes are compiled using Matlab 2020b and run on a desktop computer featuring an Intel Core i9-10900K CPU @ 3.70GHz with 32GB of RAM.

Two numerical strategies introduced in Section 3, namely the FEM and the MsFEM, will be employed respectively. Regarding the mesh discretization, their numerical outcomes obtained on the Shishkin-type mesh and on the graded-type mesh detailed in Section 4 will be compared. The L^∞ , L^2 , H^1 , and energy norms defined in Eqs (3.12)–(3.15) will be utilized to assess the corresponding accuracy and efficiency. For the sake of simplicity, we adopt the abbreviated notations FEM(S) and MsFEM(S) to denote the results derived by applying FEM and MsFEM on the Shishkin mesh, respectively. Similarly, FEM(G) and MsFEM(G) will represent the results obtained on the graded mesh.

Throughout the numerical experiments, we specify the subgrid number M used for solving the local problem (3.4) in each coarse element. Small values such as $M = 2$ lack sufficient spatial resolution to capture fine-scale oscillations and degrade the quality of multiscale basis functions. In contrast, larger choices like $M = 10$ would enlarge the dimension of local boundary value problems and increase memory consumption and runtime cost. So, we adopt $M = 4$ for a balanced setup in what follows. This choice delivers adequate local resolution to resolve microscale features while avoiding excessive computational burden.

Example 1. In the problem (2.1), an exact solution is given by

$$u = 2 + (1 - \exp^{-\frac{x}{\varepsilon}})(\exp^{\frac{x-1}{\varepsilon}} - 1) + (1 - \exp^{-\frac{y}{\varepsilon}})(\exp^{\frac{y-1}{\varepsilon}} - 1), \quad (5.1)$$

and $\vec{b} = (0, 0)^T$, $c = 2$. Subsequently, we can verify the right-hand side expression

$$f = -\frac{1}{\varepsilon}(\exp^{-\frac{x}{\varepsilon}} + \exp^{\frac{x-1}{\varepsilon}} + \exp^{-\frac{y}{\varepsilon}} + \exp^{\frac{y-1}{\varepsilon}}) + cu.$$

This benchmark problem is also investigated in [24]. It is known that for relatively large ε , the solution given by Eq (5.1) is smooth over the entire domain, and traditional methods (e.g., the FEM) on a uniform mesh perform satisfactorily. However, for extremely small ε , the solution exhibits sharp transitions near the domain boundaries, which yields substantial and unacceptably large errors. In the square computational domain, thin boundary layers emerge along all four edges, where the solution undergoes rapid variation.

At present, we implement the FEM and MsFEM on both the Shishkin mesh and the graded mesh. Since the function u exhibits sharp transitions along all four edges of the square domain in two-dimensional space, the two-sided Shishkin mesh (cf. Eq (4.1)) and graded mesh (cf. Eq (4.3)) are employed in the x -direction, and the same discretization strategy is adopted in the y -direction. As illustrated in Figure 3, the numerical solution obtained by FEM(S) fails to approximate the exact solution satisfactorily. In contrast, the FEM(G) solution presented in Figure 4(a) appears quite reasonable. It should be noted that the FEM operates on a fine mesh discretization with a computational complexity of $O(N^2M^2)$. Conversely, the MsFEM is implemented on a coarse mesh discretization with a complexity of $O(N^2)$. Moreover, as observed in Figure 4(b), the MsFEM(G) solution on the coarse mesh demonstrates excellent performance in terms of both accuracy and efficiency.

To better visualize the corresponding errors, Figures 5 and 6 show that the errors arising from the Shishkin-type mesh are rather unsatisfactory (with maximum errors reaching 3 and 0.6, respectively). However, the maximum errors of FEM(G) and MsFEM(G) are of the same order of magnitude, a relatively small value of $4e-3$. It is worth emphasizing that the MsFEM(G) error is derived from a

coarse mesh level with $N^2 = 48^2$, while the FEM(G) error is obtained from a fine mesh level with $N^2 M^2 = 192^2$.

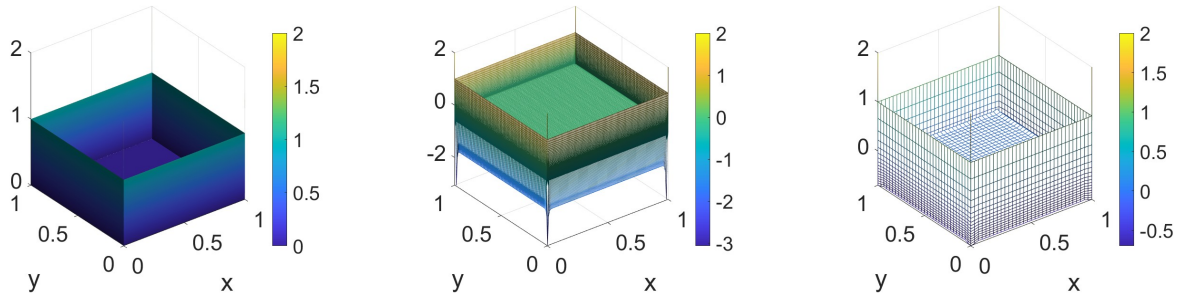


Figure 3. Exact solution, FEM(S) solution on the fine mesh ($N^2 M^2 = 256^2$), and MsFEM(S) solution on the coarse mesh ($N^2 = 64^2$) (Example 1, $\varepsilon = 10^{-4}$).

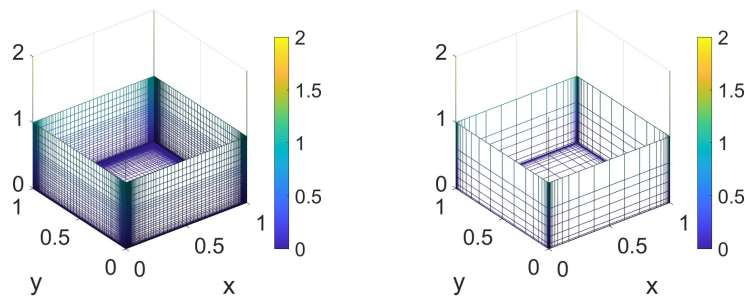


Figure 4. FEM(G) solution on the fine mesh ($N^2 M^2 = 192^2$) and MsFEM(G) solution on the coarse mesh ($N^2 = 48^2$) (Example 1, $\varepsilon = 10^{-4}$).

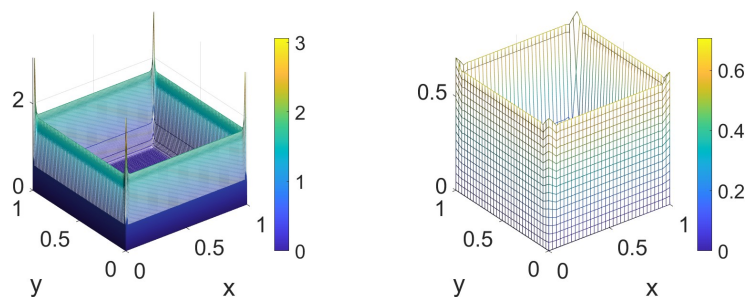


Figure 5. Pointwise errors of FEM(S) solution on the fine mesh ($N^2 M^2 = 256^2$) and MsFEM(S) solution on the coarse mesh ($N^2 = 64^2$) (Example 1, $\varepsilon = 10^{-4}$).

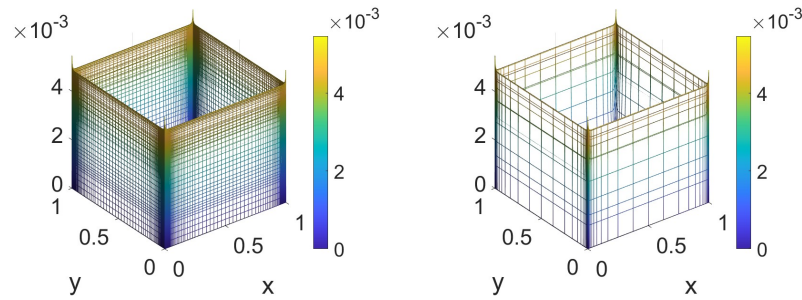


Figure 6. Pointwise errors of FEM(G) solution on the fine mesh ($N^2 M^2 = 192^2$) and MsFEM(G) solution on the coarse mesh ($N^2 = 48^2$) (Example 1, $\varepsilon = 10^{-4}$).

Table 1. Convergence history and convergence order of L^∞ and H^1 norm errors for FEM and MsFEM on meshes (Example 1, $\varepsilon = 10^{-4}$).

	NM	FEM(S)	order	N	MsFEM(S)	order	NM	FEM(G)	order	N	MsFEM(G)	order
$\ u - u_h\ _{L^\infty}$	128	1.261e+1	-	32	2.408e+0	-	112	1.792e-2	-	28	1.810e-2	-
	256	3.062e+0	2.04	64	7.061e-1	1.77	192	5.456e-3	1.72	48	5.467e-3	1.73
	512	6.096e-1	2.33	128	2.028e-1	1.80	368	1.584e-3	1.78	92	1.589e-3	1.78
	1024	1.149e-1	2.41	256	6.701e-2	1.60	728	4.358e-4	1.86	182	4.370e-4	1.86
$\ u - u_h\ _{H^1}$	128	1.128e+2	-	32	1.383e+1	-	112	8.253e-2	-	28	2.122e-1	-
	256	3.297e+1	1.77	64	3.238e+0	2.10	192	2.078e-2	1.99	48	4.972e-2	2.09
	512	5.313e+0	2.63	128	6.068e-1	2.42	368	5.430e-3	1.94	92	1.154e-2	2.11
	1024	4.020e-1	3.72	256	1.125e-1	2.43	728	1.399e-3	1.96	182	2.793e-3	2.05

Table 1 displays the respective accuracy and the corresponding convergence of norm errors in relation to mesh refinement for Example 1. It should be noted that the Shishkin mesh is double-refined based on Eq (4.1) by selecting an initial integer $N = 32$. This value is nearly equal to the initially generated integer $N = 28$ of the graded mesh (4.3). Subsequently, for each coarse element K in the local problem (3.4) (irrespective of whether it is of the Shishkin or graded type), we divide it by using a subpartition number $M = 4$ throughout the entire numerical experiment as mentioned above. We clarify that the L^∞ error measures the maximum pointwise error of the solution itself, while the H^1 error includes both the error of the solution and the error of its gradient. For singularly perturbed problems with boundary layers, the gradient changes sharply within boundary layers, which contributes substantially to the H^1 norm error. Consequently, it is observed that the magnitude of $\|\cdot\|_{L^\infty}$ is smaller than $\|\cdot\|_{H^1}$ in our graded-mesh computations, as seen in Table 1. In this process, the microscopic perturbation information is conveyed via these multiscale basis functions. The multiscale scheme is then implemented on the coarse meshes to conserve computational resources. Conversely, the traditional finite element scheme is carried out at the fine-mesh level of $N^2 M^2$.

Upon comparison, it becomes clear that the results obtained on the graded mesh are substantially better than those on the Shishkin mesh. This superiority stems from the fact that the graded-type mesh utilizes its mesh-characterizing function (4.2) to more accurately approximate the location of its boundary layers. The numerical solutions obtained from both the FEM(S) and the MsFEM(S) are

relatively poor, despite the more manageable computational costs of the MsFEM(S). However, both the FEM(G) and the MsFEM(G) produce accurate and convergent results. Due to the advantageous effect of the adaptively graded mesh, the convergence order in the H^1 norm is improved to the second order. Moreover, an important benefit is that the MsFEM(G) implemented on the relatively coarse partition proves to be as effective as the FEM(G) implemented on the relatively fine partition. To illustrate, the outcomes obtained from the MsFEM with $N^2 = 92^2$ and 182^2 outperform those derived from the FEM with $N^2 M^2 = 112^2$ and 192^2 , respectively. This convincingly demonstrates the numerical potential of the multiscale scheme. The MsFEM(G) demonstrates remarkable robustness and adaptability in this two-dimensional problem. As the coarse meshes are successively refined to $N^2 = 48^2, 92^2, 182^2$, the superiority of MsFEM(G) becomes increasingly evident.

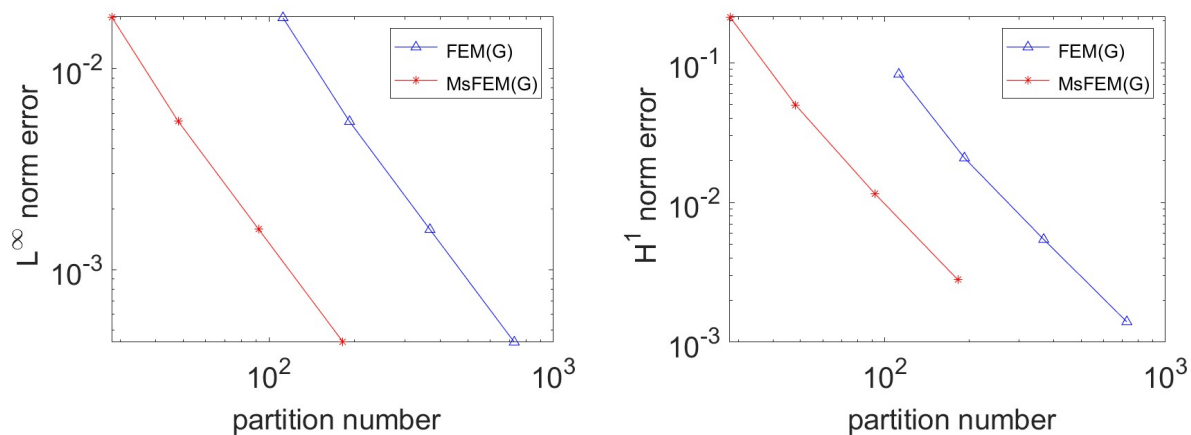


Figure 7. L^∞ and H^1 norms of FEM(G) error and MsFEM(G) error on a log-log scale (Example 1, $\varepsilon = 10^{-4}$).

Figure 7 offers the convergence history of the L^∞ and H^1 norm errors for FEM(G) and MsFEM(G). It is readily apparent that both the L^∞ and H^1 norms strictly follow second-order convergence, regardless of whether the calculations are performed on fine or coarse meshes. These numerical results indicate parameter-uniform superconvergence can be observed for our scheme on graded meshes. Clearly, the multiscale strategy shows greater cost-effectiveness in terms of computational resources.

Example 2. Another exact solution reads

$$u = \left(x - \frac{1 - \exp^{-\frac{x}{\varepsilon}}}{1 - \exp^{-\frac{1}{\varepsilon}}} \right) \left(y - \frac{1 - \exp^{-\frac{y}{\varepsilon}}}{1 - \exp^{-\frac{1}{\varepsilon}}} \right) \exp^{x+y}. \quad (5.2)$$

In Eq (2.1), it reads $\vec{b} = (2\varepsilon - 1, 2\varepsilon - 1)^T$, $c = 2(1 - \varepsilon)$, so

$$f = \left(-x + \frac{1 - \exp^{-\frac{x}{\varepsilon}}}{1 - \exp^{-\frac{1}{\varepsilon}}} - y + \frac{1 - \exp^{-\frac{y}{\varepsilon}}}{1 - \exp^{-\frac{1}{\varepsilon}}} \right) \exp^{x+y}.$$

This is a benchmark example also used in [9, 10].

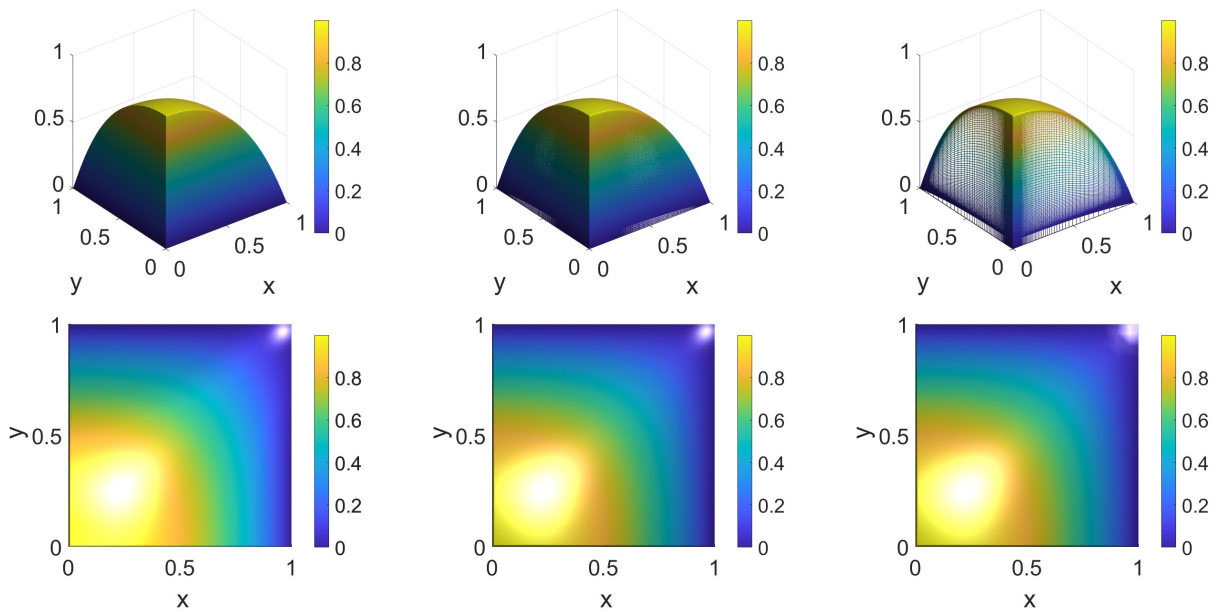


Figure 8. Exact solution, FEM(G) solution on the fine mesh ($N^2 M^2 = 644^2$), and MsFEM(G) solution on the coarse mesh ($N^2 = 161^2$). (Example 2, $\varepsilon = 10^{-3}$, top: 3D mesh figure, bottom: 2D surf contour).

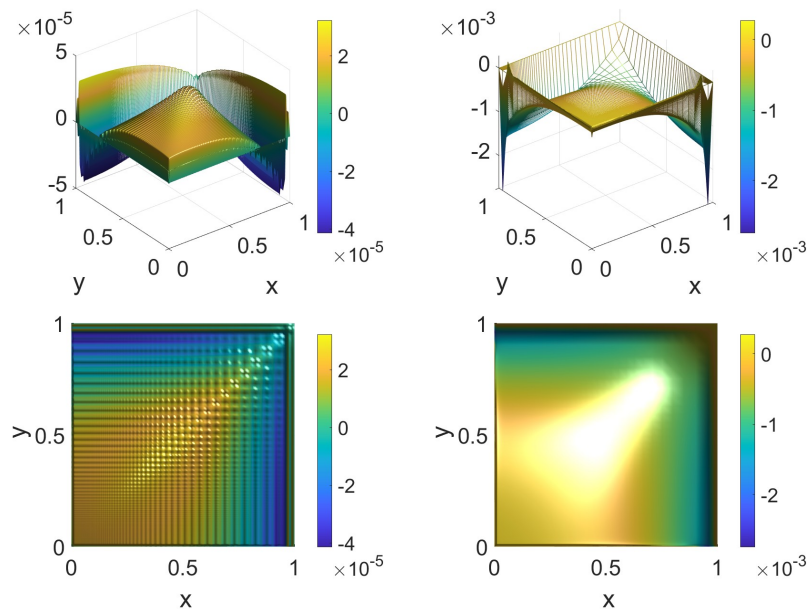


Figure 9. Pointwise errors of FEM(G) solution on the fine mesh ($N^2 M^2 = 644^2$) and MsFEM(G) solution on the coarse mesh ($N^2 = 161^2$). (Example 2, $\varepsilon = 10^{-3}$, top: 3D mesh figure, bottom: 2D surf contour).

Figure 8 presents the three-dimensional mesh visualizations and two-dimensional surface contours

of the exact solution (cf. Eq (5.2)), the solution obtained by the FEM on the graded fine mesh, and the solution derived from the MsFEM on the graded coarse mesh. Notably, in the vicinity of the boundaries at $x = 0$ and $y = 0$, substantial sharp transitions are observed, which are attributed to the small value of ε . In this case, the one-sided recursive formula (half of the two-sided formula (cf. Eq (4.3))) is employed to generate the graded mesh. Evidently, both the FEM(G) and MsFEM(G) solutions closely approximate the exact solution. Subsequently, Figure 9 illustrates the errors of FEM(G) (on the fine mesh) and MsFEM(G) (on the coarse mesh), respectively. Significantly, the errors of MsFEM(G) are relatively smoother on the coarse partition with $N = 161$ (as shown in Figures 9(b),(d)), even though this does not represent the optimal scenario, and boundary layer phenomena cannot be completely eliminated.

Table 2. Convergence history and convergence order of L^2 and energy norm errors for FEM and MsFEM on meshes (Example 2, $\varepsilon = 10^{-3}$).

	NM	FEM(S)	order	N	MsFEM(S)	order	NM	FEM(G)	order	N	MsFEM(G)	order
$\ u - u_h\ _{L^2}$	32	1.106e-3	-	8	5.597e-3	-	44	8.617e-4	-	11	1.877e-2	-
	64	2.067e-4	2.42	16	1.237e-3	2.18	80	2.410e-4	1.84	20	4.512e-3	2.06
	128	5.736e-5	1.85	32	3.456e-4	1.84	156	7.830e-5	1.62	39	1.124e-3	2.01
	256	1.659e-5	1.79	64	1.240e-4	1.48	312	2.469e-5	1.67	78	2.255e-4	2.32
	512	4.773e-6	1.80	128	4.485e-5	1.47	644	9.048e-6	1.45	161	3.473e-5	2.70
	1024	1.362e-6	1.81	256	2.059e-5	1.12	1356	1.508e-6	2.58	339	3.029e-6	3.52
$\ u - u_h\ _{en}$	32	1.630e-3	-	8	5.645e-3	-	44	8.305e-4	-	11	1.996e-2	-
	64	3.991e-4	2.03	16	1.240e-3	2.19	80	2.356e-4	1.82	20	4.685e-3	2.09
	128	1.091e-4	1.87	32	3.436e-4	1.85	156	8.447e-5	1.48	39	5.048e-4	3.21
	256	2.834e-5	1.94	64	1.266e-4	1.44	312	2.675e-5	1.66	78	6.552e-5	2.95
	512	6.937e-6	2.03	128	4.770e-5	1.41	644	9.186e-6	1.54	161	3.928e-5	0.74
	1024	1.672e-6	2.05	256	2.227e-5	1.10	1356	2.153e-6	2.09	339	1.816e-5	1.11

Table 3. Convergence history and convergence order of L^2 and energy norm errors for FEM and MsFEM on meshes (Example 2, $\varepsilon = 10^{-6}$).

	NM	FEM(S)	order	N	MsFEM(S)	order	NM	FEM(G)	order	N	MsFEM(G)	order
$\ u - u_h\ _{L^2}$	64	7.995e-4	-	16	1.039e-3	-	84	1.264e-3	-	21	1.348e-2	-
	128	3.486e-4	1.20	32	7.291e-5	3.78	148	4.029e-4	1.65	37	3.960e-3	1.77
	256	1.538e-4	1.18	64	9.807e-5	-0.43	280	1.693e-4	1.25	70	8.728e-4	2.18
	512	6.097e-5	1.33	128	9.458e-5	0.05	544	5.142e-5	1.72	136	4.145e-4	1.07
	1024	2.655e-5	1.20	256	9.958e-5	-0.07	1100	1.244e-5	2.05	275	1.279e-5	5.02
$\ u - u_h\ _{en}$	64	8.005e-4	-	16	1.039e-3	-	84	1.264e-3	-	21	1.348e-2	-
	128	3.496e-4	1.20	32	7.288e-5	3.83	148	4.029e-4	1.65	37	3.960e-3	1.77
	256	1.548e-4	1.18	64	9.810e-5	-0.43	280	1.692e-4	1.25	70	8.715e-4	2.18
	512	6.179e-5	1.33	128	9.461e-5	0.05	544	5.142e-5	1.72	136	4.146e-4	1.07
	1024	2.677e-5	1.21	256	9.972e-5	-0.08	1100	1.243e-5	2.05	275	1.280e-5	5.02

To comprehensively investigate the numerical behaviors of Example 2, we present detailed results in Table 2. Generally, the graded mesh outperforms the Shishkin mesh in terms of handling singularly perturbed problems. Furthermore, the multiscale scheme has convincingly proven to be more computationally efficient than the traditional finite element scheme, demonstrating great potential for computational applications. Regarding specific details: Although the results obtained by MsFEM(G) on the coarse mesh with $N = 339$ do not outperform those of FEM(G) on the extremely fine mesh where $NM = 1356$, the accuracy difference between them is negligible. At the coarse-grained level, MsFEM(G) with $NM = 339$ also outperforms FEM(G) with $NM = 312$ in a similar way. In essence, the numerical performance of MsFEM(G) offers more grounds for confidence. In the L^2 norm, it exhibits a superconvergent order (exceeding second-order convergence). Additionally, it is worth highlighting that our FEM(G) outperforms the results presented in Table 1 of [10] in terms of accuracy in both the L^2 norm and the energy norm. Notably, the effectiveness of MsFEM(G) arises from its use of a relatively coarse mesh, which holds great potential for significantly reducing computational costs.

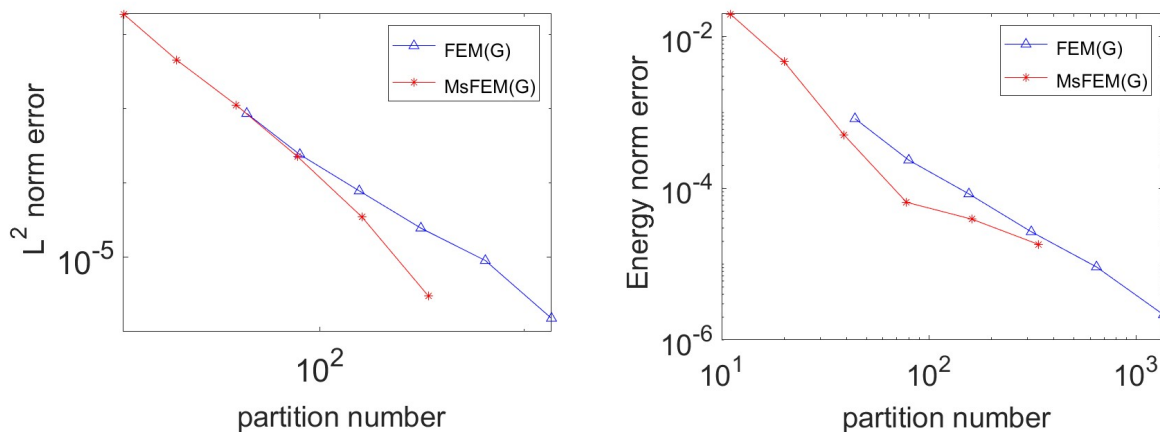


Figure 10. L^2 and energy norms of FEM(G) error and MsFEM(G) error on a log-log scale (Example 2, $\varepsilon = 10^{-3}$).

In Figure 10, when considering their respective computational performance, the errors of FEM(G) and MsFEM(G) in the L^2 norm and energy norm are of a similar magnitude. Evidently, MsFEM(G) generally exhibits a superconvergent property that exceeds second-order convergence in the L^2 norm. Regarding the superconvergence of MsFEM(G) in the energy norm, it initially takes a leading position; nevertheless, as the mesh is refined, the convergence rate of the energy norm gradually slows down.

We further test a smaller value $\varepsilon = 10^{-6}$ in Table 3, together with the graphical comparisons shown in Figures 11 and 12. For instance, similarly the results obtained by MsFEM(G) on the coarse mesh with $N = 275$ are nearly indistinguishable from those of FEM(G) on the extremely fine mesh where $NM = 1100$. For sufficiently small ε , the numerical results in Table 3 show that the convergence orders of the proposed MsFEM(G) remain stable and do not degrade with decreasing ε . Such numerical observations suggest that the superconvergence behavior appears to be insensitive to the perturbation parameter ε . Our numerical experiments show that MsFEM(G) can deliver favorable superconvergence-like rates in the energy norm even for very small values of ε . Moreover, this

multiscale strategy exhibits strong robustness and superior efficiency when applied to high-dimensional problems.

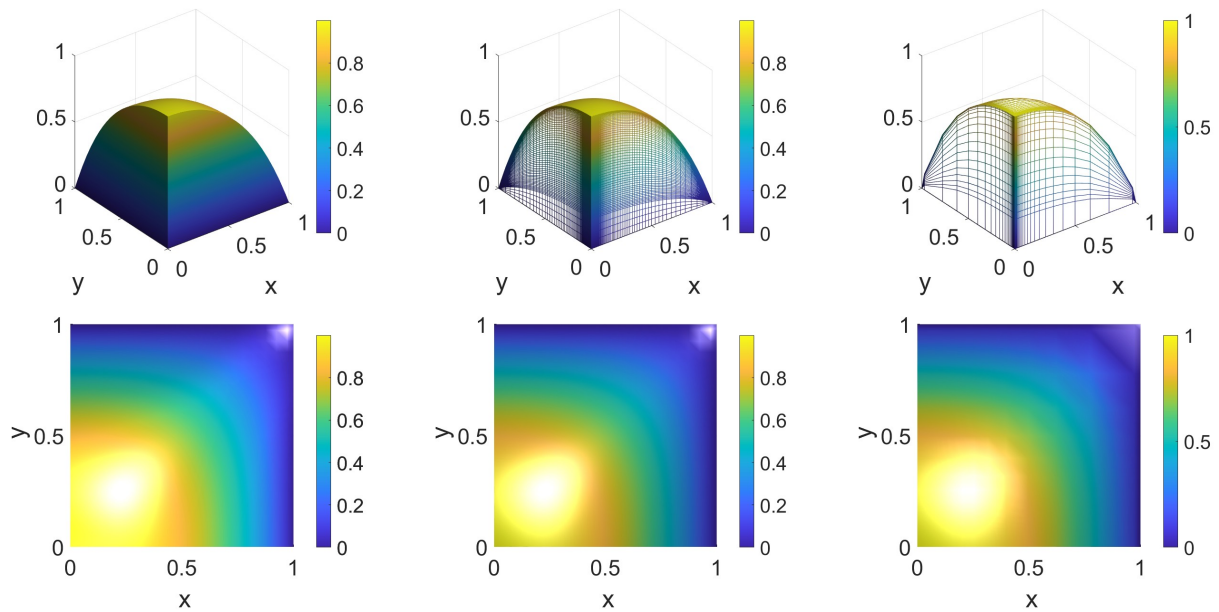


Figure 11. Exact solution, FEM(G) solution on the fine mesh ($N^2 M^2 = 280^2$), and MsFEM(G) solution on the coarse mesh ($N^2 = 70^2$). (Example 2, $\varepsilon = 10^{-6}$, top: 3D mesh figure, bottom: 2D surf contour).

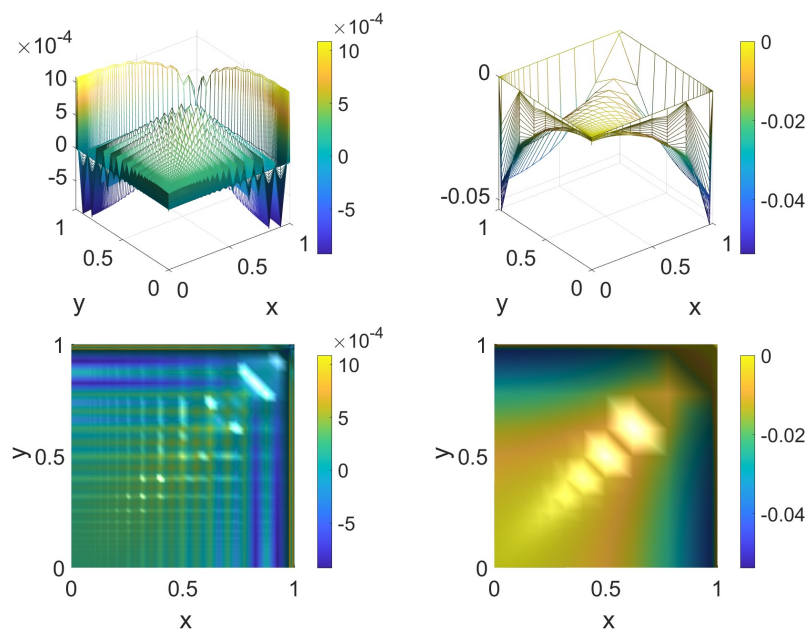


Figure 12. Pointwise errors of FEM(G) solution on the fine mesh ($N^2 M^2 = 280^2$) and MsFEM(G) solution on the coarse mesh ($N^2 = 70^2$). (Example 2, $\varepsilon = 10^{-6}$, top: 3D mesh figure, bottom: 2D surf contour).

Besides, Table 4 presents the CPU times of FEM(G) and MsFEM(G) to illustrate their computational efficiency. MsFEM(G) consists of offline preprocessing and online solving stages; although offline computation dominates the total overhead, the online solving time constitutes only a tiny fraction of the total cost. As the partition density of the graded mesh is gradually increased to resolve fine-scale structures, the CPU time of FEM(G) increases rapidly, eventually becoming nearly 10 times that of MsFEM(G). This difference arises from the fact that FEM(G) relies on dense fine-mesh discretization, whereas MsFEM(G) employs multiscale basis functions to maintain accuracy on coarse grids. This clearly shows that MsFEM(G) has significant potential advantages in dealing with large-scale and high-dimensional problems. To better illustrate this efficiency gap, the CPU times from Table 4 are visualized in Figure 13, which shows the mild linear growth of the computational cost for MsFEM(G), in contrast to the much steeper growth exhibited by FEM(G) as the mesh is refined.

Table 4. CPU time (seconds) for Example 2, $\varepsilon = 10^{-3}$.

NM	FEM(G)	N	MsFEM(G)	
	all time		offline	online
44	0.13	11	0.254	0.004
80	0.49	20	0.831	0.006
156	8.79	39	3.588	0.021
312	155	78	23.17	0.13
644	2860	161	314	1.45
1356	58709	339	6178	5.28

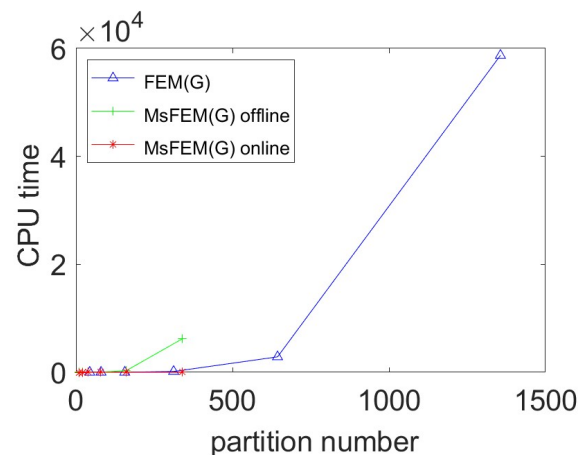


Figure 13. CPU time of FEM(G) and MsFEM(G) for Example 2, $\varepsilon = 10^{-3}$.

6. Conclusions

In this work, an MsFEM on graded meshes is developed for two-dimensional singularly perturbed convection–reaction–diffusion problems. Through the iterative construction of graded meshes, boundary layer regions can be accurately resolved for arbitrary values of the perturbation parameter.

Local fine-scale information is efficiently transferred into the global discrete system by means of multiscale basis functions, which allows the scheme to solve a reduced coarse-scale problem and substantially lower computational expenses. Numerical experiments demonstrate the efficiency of the scheme. They also yield numerical evidence of robust parameter-uniform behavior, where accuracy and numerical stability appear to be insensitive to ε , together with numerically observed superconvergence on graded meshes. Accordingly, it offers a competitive numerical tool for treating singularly perturbed problems involving small perturbation parameters. Rigorous theoretical error analysis for the numerically observed superconvergence phenomenon, including the derivation of error bounds in the H^1 or energy norms, is reserved for future research.

Author contributions

Meiling Sun: methodology, writing original draft; Jin Zhang: reorganization, validation; Guoyun Zhang: conceptualization, data curation; Shan Jiang: methodology, reviewing original draft.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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