



Research article

Dynamic analysis of Gnedenko system with multiple working vacations

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Abstract: We present a dynamic analysis of the Gnedenko repairable system with multiple working vacations, which is comprised of N operating units and one warm standby unit. The system dynamics are governed by partial differential equations with integral boundary conditions. By applying the C_0 -semigroup theory, we establish the well-posedness of the system and the strong convergence of the time-dependent solution to its steady-state solution. Numerical examples illustrate the effect of key parameters on time-dependnet reliability indices, providing practical insights into the system performance.

Keywords: Gnedenko system; multiple working vacations; C_0 -semigroup; dynamic solution; asymptotic behavior

Abbreviations: T-DS: Time-dependent solution; S-SS: Steady-state solution; WV: Working vacation; MWV: Multiple working vacation; SVT: Supplementary variable technique; ACP: Abstract Cauchy problem; T-DRI: Time-dependnet reliability indices.

1. Introduction

The working vacation (WV) policy has become a widely adopted operational strategy in queueing and reliability modeling over recent decades. Servi and Finn [1] first introduced this concept in the classical $M/M/1$ queue, where the server continues to provide service at a reduced rate during vacation periods rather than completely halting operations. This policy finds numerous practical applications. In the energy sector, power stations can operate some generating units at reduced capacity during low-demand periods, thereby conserving energy and reducing equipment wear while enabling rapid responses to peak loads. Similarly, escalators in shopping malls and subway stations can run at lower speeds during off-peak hours, saving energy and extending service life, and immediately resume normal speed when passengers arrive. These advantages have motivated extensive research on WV queueing and reliability systems. For steady-state performance, $M/G/1$ queues with multiple or single WV and vacation interruption were analyzed in [2–4], while WV were incorporated into

cold standby repairable systems [5] and machine repair problems with two failure modes [6]. From a dynamic-analysis perspective, M/G/1 single WV models were studied in [7, 8], and k/n(G) retrial systems with multiple working vacations (MWV) were examined in [9].

Standby repairable systems constitute a fundamental class of reliability models in which redundant units are maintained to enhance the system availability and operational continuity. In recent years, numerous studies have continued to address the reliability modeling and analysis of such systems from various perspectives [10–13]. Within this broad class, the Gnedenko system occupies holds particular significance as one of the most fundamental configurations. In its classical form, it comprises of N operating units connected in series, one warm standby unit, and a single repairman. The system only remains operational if all N operating units are functioning. When an operating unit fails, the warm standby unit immediately replaces it, and the failed unit is sent for repair. If another unit fails before the repair is completed, then the system is inoperative. Gnedenko [14] first studied this system and derived the reliability function and several limit theorems.

Subramanian [15] extended the analysis by assuming that, upon system failure, the remaining operating units have failure rates equal to those of the standby units, and obtained the system availability. Cao [16] further investigated the stochastic behavior of the Gnedenko system using Takács' method [17] and Markov renewal processes, deriving explicit expressions for steady-state availability and failure frequency. Classical Gnedenko systems assume the repairman is always available. In practice, repairmen may take vacations or attend to other duties, causing repair delays [18]. To model this, Liu and Kasim [19] developed a Gnedenko system with MWV, where the repairman continues servicing failed units at a reduced rate during vacations. Using the supplementary variable technique (SVT) and the generalized Markov process theory, the authors established this model and derived explicit steady-state formulas for important reliability indices. Nevertheless, the dynamic properties of this system—existence, uniqueness, and asymptotic stability of the time-dependent solution (T-DS)—were left unresolved.

Dynamic analyses are important both theoretically and practically. The existence and uniqueness of the T-DS ensure that the system is well-posed and that all time-dependent reliability indices (T-DRI) are uniquely defined for all $t \geq 0$. The Steady-state solution (S-SS) depends on the T-DS, which clearly reflects the system's operational trend. Asymptotic convergence to the S-SS further guarantees that long-term reliability and availability are well defined, meaning the S-SS can reliably approximate the system's behavior over sufficiently long periods. These properties are particularly significant for the Gnedenko system with MWV because the vacation and standby mechanisms lead to complex dynamic evolution that cannot be inferred from the steady-state analysis alone.

The system under investigation consists of N operating units, one warm standby unit, and a single repairman who operates under the MWV discipline. At time zero, all N units are new and functional, and the repairman commences a WV. If an operating unit fails, then the standby unit immediately replaces it, and the failed unit becomes a standby after repair. A system breakdown occurs when two units are simultaneously in the failed state; at that instant, all remaining $N - 1$ operating units cease to be subject to failure. During a WV, any failed unit is repaired at a reduced rate. Upon completion of a vacation, if any failed unit exists, then the repairman switches to the normal busy period and repairs at the full rate; if no failed units are present, then he immediately begins the next WV. Should a failure occur after the vacation has ended, the repairman enters the normal busy period without delay. This cycle repeats indefinitely. The system evolution is governed by a system of partial differential

equations with integral boundary conditions, which we analyze in detail in the subsequent sections.

As in [19], the new Gnedenko system Based on the SVT takes the following form:

$$\frac{dH_1^0(t)}{dt} = -(N\lambda_1 + \lambda_2)H_1^0(t) + \int_0^\infty H_1^1(t, x)\beta_1(x)dx + \int_0^\infty H_2^1(t, x)\beta_2(x)dx, \quad (1.1a)$$

$$\frac{\partial H_1^1(t, x)}{\partial t} + \frac{\partial H_1^1(t, x)}{\partial x} = -(N\lambda_1 + \beta_1(x) + \theta)H_1^1(t, x), \quad (1.1b)$$

$$\frac{\partial H_1^2(t, x)}{\partial t} + \frac{\partial H_1^2(t, x)}{\partial x} = -(\beta_1(x) + \theta)H_1^2(t, x) + N\lambda_1 H_1^1(t, x), \quad (1.1c)$$

$$\frac{\partial H_2^1(t, x)}{\partial t} + \frac{\partial H_2^1(t, x)}{\partial x} = -(N\lambda_1 + \beta_2(x))H_2^1(t, x) + \theta H_1^1(t, x), \quad (1.1d)$$

$$\frac{\partial H_2^2(t, x)}{\partial t} + \frac{\partial H_2^2(t, x)}{\partial x} = -\beta_2(x)H_2^2(t, x) + \theta H_1^2(t, x) + N\lambda_1 H_2^1(t, x), \quad (1.1e)$$

$$H_1^1(t, 0) = (N\lambda_1 + \lambda_2)H_1^0(t) + \int_0^\infty H_1^2(t, x)\beta_1(x)dx, \quad (1.1f)$$

$$H_1^2(t, 0) = 0, \quad (1.1g)$$

$$H_2^1(t, 0) = \int_0^\infty H_2^2(t, x)\beta_2(x)dx, \quad (1.1h)$$

$$H_2^2(t, 0) = 0, \quad (1.1i)$$

$$H_1^0(0) = 1, \quad H_1^1(0, x) = 0, \quad H_1^2(0, x) = 0, \quad H_2^1(0, x) = 0, \quad H_2^2(0, x) = 0, \quad x \in (0, \infty), \quad (1.1j)$$

where $(t, x) \in [0, \infty) \times [0, \infty)$. The notations used in the model are summarized in Table 1.

Table 1. Notations used in the mathematical model.

Symbol	Description
$H_1^0(t)$	The probability that the system is functioning in regular efficiency at time t , and the repairman is on the WV.
$H_1^j(t, x) dx$	The probability that there are j failed units in the system, the repairman is on the WV, and the consumed repair time exists in $(x, x + \Delta x)$, $j = 1, 2$.
$H_2^j(t, x) dx$	The probability that there are j failed units in the system, the repairman is in a normal busy period, and the consumed repair time exists in $(x, x + \Delta x)$, $j = 1, 2$.
λ_1	The failure rate of the active unit.
λ_2	The failure rate of the standby unit.
θ	The parameter of the exponential distribution of the repairman's WV period.
$\beta_1(x)$	The repair rate of a unit on the WV, satisfying $\beta_1(x) \geq 0$ and $\int_0^\infty \beta_1(x) dx = \infty$.
$\beta_2(x)$	The repair rate of a unit during the normal busy period, satisfying $\beta_2(x) \geq 0$ and $\int_0^\infty \beta_2(x) dx = \infty$.

The steady-state reliability of the above Gnedenko system with MWVs has been analyzed, whereas its dynamic behavior remains unexplored. Motivated by this gap, this paper conducts a rigorous dynamic analysis of the system. For the dynamic analysis, we employ the theory of C_0 -semigroups of linear operators, a powerful functional-analytic tool. Gupur's monograph [20] was among the first to apply this theory to queueing and reliability models and provides a detailed introduction to the relevant

ideas and methods. Inspired by this line of research, many authors have contributed to the dynamic analysis of various queueing models [21–23]. Similar methods have also been applied to reliability models [24–27].

The overall solution procedure and the main contributions of this paper are summarized as follows:

- We formulate the system as an abstract Cauchy problem (ACP) and establish its well-posedness by means of the C_0 -semigroup theory, which guarantees the existence and uniqueness of a non-negative T-DS;
- Through spectral analysis, we demonstrate that the T-DS strongly converges to its S-SS, thereby providing a rigorous theoretical foundation for the long-run behavior of the system;
- We verify this convergence numerically and examine the sensitivity of T-DRI to various system parameters, offering practical insights for system design and maintenance.

The remainder of this paper is organized as follows: Section 2 transforms the original system into an ACP and establishes its well-posedness; Section 3 demonstrates that the T-DS strongly converges to its S-SS; Section 4 presents numerical examples that illustrate the influence of various parameters on the system performance and numerically validate that the T-DRI tend to fixed values as time tends to infinity; a summary of the results and the recommended future research directions are provided in Section 5.

2. Well-posedness of the system

In this section, we demonstrate the well-posedness of the system based on the C_0 -semigroup theory by first converting the model to an ACP. For completeness, we briefly recall this notion. Let X be a Banach space and A be a linear operator on X . An ACP consists of the evolution equation

$$\frac{du(t)}{dt} = Au(t), \quad t > 0,$$

together with the initial condition $u(0) = u_0$. This framework recasts the original system of partial differential equations with integral boundary conditions into a single operator equation on a suitably chosen state space. It allows the well-posedness, existence, uniqueness, and asymptotic behavior of the solution to be analyzed within the theory of C_0 -semigroups. In our setting, the state space, the operator, and the initial value are chosen as follows.

Take the state space as below:

$$X = \left\{ H \left| \begin{array}{l} H \in \mathbb{R} \times L^1[0, \infty) \times L^1[0, \infty) \times L^1[0, \infty) \times L^1[0, \infty), \\ \|H\| = |H_1^0| + \sum_{i=1}^2 \sum_{j=1}^2 \|H_i^j\|_{L^1[0, \infty)} < \infty \end{array} \right. \right\}.$$

Obviously, X is a Banach space. The operators and their domains are defined as follows:

$$D(\mathfrak{A}) = \left\{ H \in X \left| \begin{array}{l} \frac{dH_i^j(x)}{dx} \in L^1[0, \infty), i, j = 1, 2 \\ H_i^j(x)(i, j = 1, 2) \text{ are absolutely continuous} \\ \text{and fulfil } H(0) = \int_0^\infty \Gamma H(x) dx \end{array} \right. \right\},$$

where

$$\Gamma = \begin{pmatrix} e^{-x} & 0 & & & \\ (N\lambda_1 + \lambda_2)e^{-x} & 0 & \beta_1(x) & & \\ & 0 & 0 & 0 & \\ & & 0 & 0 & \beta_2(x) \\ & & & 0 & 0 \end{pmatrix}.$$

Define, for $H \in D(\mathfrak{A})$,

$$\mathfrak{A} \begin{pmatrix} H_1^0 \\ H_1^1 \\ H_1^2 \\ H_2^1 \\ H_2^2 \end{pmatrix}(x) = \begin{pmatrix} -(N\lambda_1 + \lambda_2) & & & & \\ & a_{1,1} & & & \\ & & a_{1,2} & & \\ & & & a_{2,1} & \\ & & & & a_{2,2} \end{pmatrix} \begin{pmatrix} H_1^0 \\ H_1^1(x) \\ H_1^2(x) \\ H_2^1(x) \\ H_2^2(x) \end{pmatrix},$$

where

$$\begin{aligned} a_{1,1} &= -\frac{d}{dx} - (N\lambda_1 + \beta_1(x) + \theta), & a_{1,2} &= -\frac{d}{dx} - (\beta_1(x) + \theta), \\ a_{2,1} &= -\frac{d}{dx} - (N\lambda_1 + \beta_2(x)), & a_{2,2} &= -\frac{d}{dx} - \beta_2(x), \end{aligned}$$

and define, for $H \in X$,

$$\begin{aligned} \mathfrak{G} \begin{pmatrix} H_1^0 \\ H_1^1 \\ H_1^2 \\ H_2^1 \\ H_2^2 \end{pmatrix}(x) &= \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & N\lambda_1 & 0 & & \\ & \theta & 0 & 0 & \\ & & \theta & N\lambda_1 & 0 \end{pmatrix} \begin{pmatrix} H_1^0 \\ H_1^1(x) \\ H_1^2(x) \\ H_2^1(x) \\ H_2^2(x) \end{pmatrix}, \\ \mathfrak{R} \begin{pmatrix} H_1^0 \\ H_1^1 \\ H_1^2 \\ H_2^1 \\ H_2^2 \end{pmatrix}(x) &= \begin{pmatrix} \int_0^\infty H_1^1(t, x)\beta_1(x)dx + \int_0^\infty H_2^1(t, x)\beta_2(x)dx \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad D(\mathfrak{G}) = D(\mathfrak{R}) = X. \end{aligned}$$

Then, the original Eq (1.1) can be reformulated as an ACP on X as follows:

$$\begin{cases} \frac{dH(t)}{dt} = (\mathfrak{A} + \mathfrak{G} + \mathfrak{R})H(t), & \forall t \in (0, \infty), \\ H(0) = (1, 0, 0, 0, 0), \end{cases} \quad (2.1)$$

where we first get the following main result.

Theorem 2.1. If $\mathbb{M}_i = \sup_{x \in [0, \infty)} \beta_i(x) < \infty, i = 1, 2$, then $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ generates a positive contraction C_0 -semigroup $\mathcal{T}(t)$.

Proof. First, we estimate $\|(\ell I - \mathfrak{A})^{-1}\|$. With $(\ell I - \mathfrak{A})H = Z, \forall Z \in X$, we get the following:

$$(\ell + N\lambda_1 + \lambda_2)H_1^0 = Z_1^0, \quad (2.2a)$$

$$\frac{dH_1^1(x)}{dx} = -(\ell + N\lambda_1 + \beta_1(x) + \theta)H_1^1(x) + Z_1^1(x), \quad (2.2b)$$

$$\frac{dH_1^2(x)}{dx} = -(\ell + \beta_1(x) + \theta)H_1^2(x) + Z_1^2(x), \quad (2.2c)$$

$$\frac{dH_2^1(x)}{dx} = -(\ell + N\lambda_1 + \beta_2(x))H_2^1(x) + Z_2^1(x), \quad (2.2d)$$

$$\frac{dH_2^2(x)}{dx} = -(\ell + \beta_2(x))H_2^2(x) + Z_2^2(x), \quad (2.2e)$$

$$H_1^1(0) = (N\lambda_1 + \lambda_2)H_1^0 + \int_0^\infty H_1^2(x)\beta_1(x)dx, \quad (2.2f)$$

$$H_1^2(0) = 0, \quad (2.2g)$$

$$H_2^1(0) = \int_0^\infty H_2^2(x)\beta_2(x)dx, \quad (2.2h)$$

$$H_2^2(0) = 0. \quad (2.2i)$$

By solving Eqs (2.2a)–(2.2e), we have the following:

$$H_1^0 = \frac{1}{\ell + N\lambda_1 + \lambda_2} Z_1^0, \quad (2.3a)$$

$$H_1^1(x) = e^{-\int_0^x (\ell + N\lambda_1 + \theta + \beta_1(v))dv} (H_1^1(0) + \int_0^x Z_1^1(\varsigma) e^{\int_0^\varsigma (\ell + N\lambda_1 + \theta + \beta_1(v))dv} d\varsigma), \quad (2.3b)$$

$$H_1^2(x) = e^{-\int_0^x (\ell + \theta + \beta_1(v))dv} \int_0^x Z_1^2(\varsigma) e^{\int_0^\varsigma (\ell + \theta + \beta_1(v))dv} d\varsigma, \quad (2.3c)$$

$$H_2^1(x) = e^{-\int_0^x (\ell + N\lambda_1 + \beta_2(v))dv} (H_2^1(0) + \int_0^x Z_2^1(\varsigma) e^{\int_0^\varsigma (\ell + N\lambda_1 + \beta_2(v))dv} d\varsigma), \quad (2.3d)$$

$$H_2^2(x) = e^{-\int_0^x (\ell + \beta_2(v))dv} \int_0^x Z_2^2(\varsigma) e^{\int_0^\varsigma (\ell + \beta_2(v))dv} d\varsigma. \quad (2.3e)$$

By combining Eqs (2.2f) and (2.2h) with Eq (2.3a), and using the conditions of the Theorem, we deduce the following:

$$\begin{aligned} |H_1^1(0)| &\leq (N\lambda_1 + \lambda_2)|H_1^0| + \int_0^\infty \beta_1(x)|H_1^2(x)|dx \\ &\leq \frac{N\lambda_1 + \lambda_2}{|\ell + N\lambda_1 + \lambda_2|} |Z_1^0| + \mathbb{M}_1 \|H_1^2\|_{L^1[0,\infty)}, \end{aligned} \quad (2.4a)$$

$$|H_2^1(0)| \leq \int_0^\infty \beta_2(x)|H_2^2(x)|dx \leq \mathbb{M}_2 \|H_2^2\|_{L^1[0,\infty)}. \quad (2.4b)$$

Using

$$e^{-\int_\varsigma^x \beta_i(v)dv} \leq 1, \quad 0 < \varsigma < x, \quad i = 1, 2,$$

and the Fubini theorem, we estimate (assume $\ell > 0$) the following:

$$\|H_1^2\|_{L^1[0,\infty)} \leq \int_0^\infty e^{-(\ell+\theta)x} \int_0^x |Z_1^2(\varsigma)| e^{(\ell+\theta)\varsigma} d\varsigma dx = \frac{1}{\ell + \theta} \|Z_1^2\|_{L^1[0,\infty)}, \quad (2.5a)$$

$$\begin{aligned}
\|H_2^2\|_{L^1[0,\infty)} &= \int_0^\infty e^{-\ell x} \int_0^x |Z_2^2(\varsigma)| e^{\ell \varsigma} e^{-\int_\varsigma^x \beta_2(v) dv} d\varsigma dx \leq \int_0^\infty e^{-\ell x} \int_0^x |Z_2^2(\varsigma)| e^{\ell \varsigma} d\varsigma dx \\
&= \int_0^\infty |Z_2^2(\varsigma)| e^{\ell \varsigma} \int_\varsigma^\infty e^{-\ell x} dx d\varsigma = \frac{1}{\ell} \|Z_2^2\|_{L^1[0,\infty)}.
\end{aligned} \tag{2.5b}$$

Similarly,

$$\begin{aligned}
\|H_1^1\|_{L^1[0,\infty)} &\leq |H_1^1(0)| \int_0^\infty e^{-(\ell+N\lambda_1+\theta)x} dx \\
&\quad + \int_0^\infty e^{-(\ell+N\lambda_1+\theta)x} \int_0^x |Z_1^1(\varsigma)| e^{(\ell+N\lambda_1+\theta)\varsigma} e^{-\int_\varsigma^x \beta_1(v) dv} d\varsigma dx \\
&\leq \frac{|H_1^1(0)|}{\ell+N\lambda_1+\theta} + \frac{1}{\ell+N\lambda_1+\theta} \|Z_1^1\|_{L^1[0,\infty)} \\
&\leq \frac{N\lambda_1+\lambda_2}{(\ell+N\lambda_1+\theta)(\ell+N\lambda_1+\lambda_2)} |Z_1^0| + \frac{\mathbb{M}_1}{(\ell+\theta)(\ell+N\lambda_1+\theta)} \|Z_1^2\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\ell+N\lambda_1+\theta} \|Z_1^1\|_{L^1[0,\infty)},
\end{aligned} \tag{2.6a}$$

$$\begin{aligned}
\|H_2^1\|_{L^1[0,\infty)} &\leq \frac{|H_2^1(0)|}{\ell+N\lambda_1} + \frac{1}{\ell+N\lambda_1} \|Z_2^1\|_{L^1[0,\infty)} \\
&\leq \frac{\mathbb{M}_2}{\ell(\ell+N\lambda_1)} \|Z_2^2\|_{L^1[0,\infty)} + \frac{1}{\ell+N\lambda_1} \|Z_2^1\|_{L^1[0,\infty)}.
\end{aligned} \tag{2.6b}$$

By combining Eqs (2.5) and (2.6) with Eq (2.3a) and using the following inequalities:

$$\frac{\ell+2N\lambda_1+\lambda_2}{(\ell+N\lambda_1)(\ell+N\lambda_1+\lambda_2)} < \frac{1}{\ell-\mathbb{M}}, \quad \frac{1}{\ell+N\lambda_1} < \frac{1}{\ell-\mathbb{M}}, \quad \frac{\mathbb{M}+N\lambda_1+\ell}{\ell(\ell+N\lambda_1)} < \frac{1}{\ell-\mathbb{M}},$$

where $\mathbb{M} = \max\{\mathbb{M}_1, \mathbb{M}_2\}$, we deduce, for $\ell > \mathbb{M}$,

$$\begin{aligned}
\|H\| &= |H_1^0| + \sum_{j=1}^2 \sum_{k=1}^2 \|H_j^k\|_{L^1[0,\infty)} \\
&\leq \frac{\ell+2N\lambda_1+\lambda_2}{(\ell+N\lambda_1)(\ell+N\lambda_1+\lambda_2)} |Z_1^0| + \frac{1}{\ell+N\lambda_1+\theta} \|Z_1^1\|_{L^1[0,\infty)} + \frac{\mathbb{M}_1+\ell+N\lambda_1+\theta}{(\ell+\theta)(\ell+N\lambda_1+\theta)} \\
&\quad \times \|Z_1^2\|_{L^1[0,\infty)} + \frac{1}{\ell+N\lambda_1} \|Z_2^1\|_{L^1[0,\infty)} + \frac{\mathbb{M}_2+N\lambda_1+\ell}{\ell(\ell+N\lambda_1)} \|Z_2^2\|_{L^1[0,\infty)} \\
&\leq \frac{\ell+2N\lambda_1+\lambda_2}{(\ell+N\lambda_1)(\ell+N\lambda_1+\lambda_2)} |Z_1^0| + \frac{1}{\ell+N\lambda_1} \|Z_1^1\|_{L^1[0,\infty)} + \frac{\mathbb{M}+N\lambda_1+\ell}{\ell(\ell+N\lambda_1)} \|Z_1^2\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\ell+N\lambda_1} \|Z_2^1\|_{L^1[0,\infty)} + \frac{\mathbb{M}+N\lambda_1+\ell}{\ell(\ell+N\lambda_1)} \|Z_2^2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\ell-\mathbb{M}} \|Z\|_{L^1[0,\infty)}.
\end{aligned} \tag{2.7}$$

Equation (2.7) shows that when $\ell > \mathbb{M}$, we have the following:

$$(\ell I - \mathfrak{A})^{-1} : X \rightarrow D(\mathfrak{A}), \quad \|(\ell I - \mathfrak{A})^{-1}\| \leq \frac{1}{\ell - \mathbb{M}}.$$

Next, show that $D(\mathfrak{A})$ is dense in X . Let

$$F = \left\{ H(x) = (H_1^0, H_1^1(x), H_1^2(x), H_2^1(x), H_2^2(x)) \left| \begin{array}{l} H_i^j \in C_0^1[0, \infty), \text{ and there exist } c_{i,j} > 0, \\ \text{such that } H_i^j(x) = 0, \ i, j = 1, 2, \ \forall x \in [0, c_{i,j}] \end{array} \right. \right\}.$$

Then, F is dense in X (i.e., $\overline{F} = X$). If we assume $F \subset \overline{D(\mathfrak{A})}$, then

$$\begin{aligned} D(\mathfrak{A}) \subset X &\Rightarrow F \subset \overline{D(\mathfrak{A})} \subset X \Rightarrow X = \overline{F} \subset \overline{\overline{D(\mathfrak{A})}} \subset \overline{X} = X \\ &\Rightarrow \overline{D(\mathfrak{A})} = \overline{\overline{D(\mathfrak{A})}} = X, \end{aligned}$$

that is, $D(\mathfrak{A})$ is dense in X . The following proves that $F \subset \overline{D(\mathfrak{A})}$ holds.

Take any $H = (H_1^0, H_1^1, H_1^2, H_2^1, H_2^2) \in F$; then, there exists $c_{i,j} > 0$ such that $H_i^j(x) = 0, i, j = 1, 2$, for $\forall x \in [0, c_{i,j}]$. Therefore, when $x \in [0, q]$, there is $H_i^j(x) = 0$, where $0 < q < \min_{i,j=1,2} \{c_{i,j}\}$. Define

$$\begin{aligned} f^q(0) &= (H_1^0, f_{1,1}^q(0), f_{1,2}^q(0), f_{2,1}^q(0), f_{2,2}^q(0)) \\ &= (H_1^0, (N\lambda_1 + \lambda_2)H_1^0 + \int_q^\infty H_1^2(x)\beta_1(x)dx, 0, \int_q^\infty H_2^2(x)\beta_2(x)dx, 0), \\ f^q(x) &= (H_1^0, f_{1,1}^q(x), f_{1,2}^q(x), f_{2,1}^q(x), f_{2,2}^q(x)), \end{aligned}$$

where

$$f_{i,1}^q(x) = \begin{cases} f_{i,1}^q(0)(1 - \frac{x}{q})^2, & x \in [0, q] \\ H_i^1(x), & x \in [q, \infty) \end{cases}, \ i = 1, 2,$$

$$f_{i,2}^q(x) = H_i^2(x), \quad \forall x \in [0, \infty), \quad i = 1, 2.$$

It is not hard to demonstrate that $f^q \in D(\mathfrak{A})$. Furthermore,

$$\begin{aligned} \|H - f^q\| &= \int_0^\infty |H_1^1(x) - f_{1,1}^q(x)|dx + \int_0^\infty |H_2^1(x) - f_{2,1}^q(x)|dx \\ &= \int_0^q |f_{1,1}^q(0)|(1 - \frac{x}{q})^2 dx + \int_0^q |f_{2,1}^q(0)|(1 - \frac{x}{q})^2 dx \\ &= \frac{q}{3} (|f_{1,1}^q(0)| + |f_{2,1}^q(0)|) \rightarrow 0, \quad q \rightarrow 0. \end{aligned}$$

This indicates that $F \subset \overline{D(\mathfrak{A})}$ (i.e., $D(\mathfrak{A})$ is dense in X). Thus, by the previous two steps and the Hille-Yosida theorem (see [20, Theorem 1.68]), it follows that $\mathfrak{A}$ generates a C_0 -semigroup.

We can easily check that $\mathfrak{G}$ and $\mathfrak{R}$ are bounded linear operators. Thus, by the perturbation theorem of the C_0 -semigroup (see [20, Theorem 1.80]), it follows that $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ generates a C_0 -semigroup $\mathcal{T}(t)$. Clearly, $\mathfrak{A} + \mathfrak{G}$ also generates a C_0 -semigroup $\mathcal{E}(t)$, and we will use this in the next section.

Lastly, we verify that $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ is a dispersive operator. With $\forall H \in D(\mathfrak{A})$, let's take

$$\varpi(x) = \left(\frac{[H_1^0]^+}{H_1^0}, \frac{[H_1^1(x)]^+}{H_1^1(x)}, \frac{[H_1^2(x)]^+}{H_1^2(x)}, \frac{[H_2^1(x)]^+}{H_2^1(x)}, \frac{[H_2^2(x)]^+}{H_2^2(x)} \right),$$

where

$$[H_1^0]^+ = \begin{cases} H_1^0, & H_1^0 > 0 \\ 0, & H_1^0 \leq 0 \end{cases},$$

$$[H_i^j(x)]^+ = \begin{cases} H_i^j(x), & H_i^j(x) > 0 \\ 0, & H_i^j(x) \leq 0 \end{cases}, i, j = 1, 2.$$

Let $W_{i,j} = \{x \in [0, \infty) | H_i^j(x) > 0\}$ and $R_{i,j} = \{x \in [0, \infty) | H_i^j(x) \leq 0\}$, $i, j = 1, 2$. Then,

$$\begin{aligned} \int_0^\infty \frac{dH_i^j(x)}{dx} \frac{[H_i^j(x)]^+}{H_i^j(x)} dx &= \int_{W_{i,j}} \frac{dH_i^j(x)}{dx} \frac{[H_i^j(x)]^+}{H_i^j(x)} dx + \int_{R_{i,j}} \frac{dH_i^j(x)}{dx} \frac{[H_i^j(x)]^+}{H_i^j(x)} dx \\ &= \int_{W_{i,j}} \frac{dH_i^j(x)}{dx} dx = \int_0^\infty \frac{d[H_i^j(x)]^+}{dx} dx = -[H_i^j(0)]^+, i, j = 1, 2, \end{aligned} \quad (2.8)$$

which is used here as $H_i^j \in L^1[0, \infty) \Rightarrow H_i^j(\infty) = 0$, $i, j = 1, 2$.

For $H \in D(\mathfrak{A})$ and ϖ , by using Eqs (2.2f)–(2.2i), (2.8), and

$$\begin{aligned} \int_0^\infty \beta_i(x) H_i^2(x) dx &\leq \int_0^\infty \beta_i(x) [H_i^2(x)]^+ dx, \int_0^\infty H_1^i(x) \frac{[H_2^i(x)]^+}{H_2^i(x)} dx \leq \int_0^\infty [H_1^i(x)]^+ dx, \\ \int_0^\infty H_i^1(x) \frac{[H_i^2(x)]^+}{H_i^2(x)} dx &\leq \int_0^\infty [H_i^1(x)]^+ dx, i = 1, 2, \end{aligned}$$

we estimate

$$\begin{aligned} &\langle (\mathfrak{A} + \mathfrak{G} + \mathfrak{R})H, \varpi \rangle \\ &\leq -(N\lambda_1 + \lambda_2)[H_1^0]^+ + \frac{[H_1^0]^+}{H_1^0} \int_0^\infty \beta_1(x) [H_1^1(x)]^+ dx + \frac{[H_1^0]^+}{H_1^0} \int_0^\infty \beta_2(x) [H_2^1(x)]^+ dx \\ &\quad + [H_1^1(0)]^+ - (N\lambda_1 + \theta) \int_0^\infty [H_1^1(x)]^+ dx - \int_0^\infty \beta_1(x) [H_1^1(x)]^+ dx + [H_1^2(0)]^+ \\ &\quad - \theta \int_0^\infty [H_1^2(x)]^+ dx - \int_0^\infty \beta_1(x) [H_1^2(x)]^+ dx + N\lambda_1 \int_0^\infty [H_1^1(x)]^+ dx + [H_2^1(0)]^+ \\ &\quad - N\lambda_1 \int_0^\infty [H_2^1(x)]^+ dx - \int_0^\infty \beta_2(x) [H_2^1(x)]^+ dx + \theta \int_0^\infty [H_1^1(x)]^+ dx + [H_2^2(0)]^+ \\ &\quad - \int_0^\infty \beta_2(x) [H_2^2(x)]^+ dx + \theta \int_0^\infty [H_1^2(x)]^+ dx + N\lambda_1 \int_0^\infty [H_2^1(x)]^+ dx \\ &\leq \left(\frac{[H_1^0]^+}{H_1^0} - 1 \right) \left(\int_0^\infty \beta_1(x) [H_1^1(x)]^+ dx + \int_0^\infty \beta_2(x) [H_2^1(x)]^+ dx \right) \\ &\leq 0. \end{aligned} \quad (2.9)$$

Equation (2.9) shows that $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ is a dispersive operator. By employing the results from Steps 1 and 2, along with Eq (2.9) and [20, Theorem 1.79], we deduce that the operator $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ generates a positive contraction C_0 -semigroup. Subsequently, appealing to the uniqueness property of the C_0 -semigroup, we confirm that this semigroup is indeed $\mathcal{T}(t)$. $\square$

Remark 2.1. Similarly to the estimate for Eq (2.9), we get

$$\langle (\mathfrak{A} + \mathfrak{G})H, \varpi \rangle \leq 0.$$

Thus, by *Phillips theorem* (see [20, Theorem 1.79]), $\mathcal{E}(t)$ is a positive contraction C_0 -semigroup. Obviously,

$$X^* = \left\{ G^* \left| \begin{array}{l} G^* \in \mathbb{R} \times L^\infty[0, \infty) \times L^\infty[0, \infty) \times L^\infty[0, \infty) \times L^\infty[0, \infty), \\ |||G^*||| = \sup \left\{ |G_1^{0*}|, \|G_1^{1*}\|_{L^\infty[0, \infty)}, \|G_1^{2*}\|_{L^\infty[0, \infty)}, \right. \right. \\ \left. \left. \|G_2^{1*}\|_{L^\infty[0, \infty)}, \|G_2^{2*}\|_{L^\infty[0, \infty)} \right\} \right. \end{array} \right\},$$

and the adjoint space of X is a Banach space. Introduce the following cone in X :

$$\mathfrak{Y} = \left\{ H \in X \left| \begin{array}{l} H(x) = (H_1^0, H_1^1(x), H_1^2(x), H_2^1(x), H_2^2(x)) \\ H_1^0 \geq 0, H_i^j(x) \geq 0, i, j = 1, 2, \forall x \in [0, \infty) \end{array} \right. \right\}.$$

Then, by Theorem 2.1, we have $\mathcal{T}(t)\mathfrak{Y} \subset \mathfrak{Y}$.

For $H \in D(\mathfrak{A}) \cap \mathfrak{Y}$, take $G^*(x) = \|H\|(1, 1, 1, 1, 1)$, hence $G^* \in X^*$. Furthermore, by using $H_i^j \in L^1[0, \infty) \Rightarrow H_i^j(\infty) = 0$, $i, j = 1, 2$, we have the following:

$$\begin{aligned} & \langle (\mathfrak{A} + \mathfrak{G} + \mathfrak{R})H, G^* \rangle \\ &= \left\{ -(N\lambda_1 + \lambda_2)H_1^0 + \int_0^\infty H_1^1(x)\beta_1(x)dx + \int_0^\infty H_2^1(x)\beta_2(x)dx \right\} \|H\| \\ &+ \int_0^\infty \left\{ -\frac{dH_1^1(x)}{dx} - (N\lambda_1 + \beta_1(x) + \theta)H_1^1(x) \right\} \|H\| dx \\ &+ \int_0^\infty \left\{ -\frac{dH_1^2(x)}{dx} - (\beta_1(x) + \theta)H_1^2(x) + N\lambda_1 H_1^1(x) \right\} \|H\| dx \\ &+ \int_0^\infty \left\{ -\frac{dH_2^1(x)}{dx} - (N\lambda_1 + \beta_2(x))H_2^1(x) + \theta H_1^1(x) \right\} \|H\| dx \\ &+ \int_0^\infty \left\{ -\frac{dH_2^2(x)}{dx} - \beta_2(x)H_2^2(x) + \theta H_1^2(x) + N\lambda_1 H_2^1(x) \right\} \|H\| dx \\ &= -(N\lambda_1 + \lambda_2)\|H\|H_1^0 + (N\lambda_1 + \lambda_2)\|H\|H_1^0 + \|H\| \int_0^\infty \beta_1(x)H_1^1(x)dx \\ &- \|H\| \int_0^\infty \beta_1(x)H_1^2(x)dx + \|H\| \int_0^\infty \beta_2(x)H_2^2(x)dx - \|H\| \int_0^\infty \beta_2(x)H_2^1(x)dx \\ &= 0. \end{aligned}$$

Based on this equation, $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ is a conservative operator (see [28]) for

$$\Theta(H) = \left\{ G^* \in X^* \mid \langle H, G^* \rangle = \|H\|^2 = |||G^*|||^2 \right\}.$$

Owing to $H(0) \in D(\mathfrak{A}^2) \cap \mathfrak{Y}$, it follows from the Fattorini theorem (see [28]) that $\mathcal{T}(t)$ is an isometric operator for $D(\mathfrak{A}^2)$. This leads to the following result.

Theorem 2.2. The operator $\mathcal{T}(t)$ is isometric for the initial value $H(0)$ of System (2.1), i.e.,

$$\|\mathcal{T}(t)H(0)\| = \|H(0)\|, \quad \forall t \in [0, \infty).$$

Finally, we get the well-posedness of System (2.1) as follows.

Theorem 2.3. If $\mathbb{M}_i = \sup_{x \in [0, \infty)} \beta_i(x) < \infty$, $i = 1, 2$, then there exists a unique positive T-DS $H(t, x)$ for the System (2.1) such that

$$\|H(t, \cdot)\| = 1, \quad \forall t \in [0, \infty).$$

Proof. Due to $H(0) \in D(\mathfrak{A}^2) \cap \mathfrak{Y}$, we know that $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ generates a unique C_0 -semigroup by Theorem 2.1. Then, from the well-posedness of the ACP (see [20, Theorem 1.81]), it follows that System (2.1) has a unique positive T-DS $H(t, x)$ and could be represented as follows:

$$H(t, x) = \mathcal{T}(t)H(0), \quad \forall t \in [0, \infty).$$

As can be seen from this equation and Theorem 2.2,

$$\|H(t, \cdot)\| = \|\mathcal{T}(t)H(0)\| = \|H(0)\| = 1, \quad \forall t \in [0, \infty).$$

□

3. The asymptotic behavior of the T-DS

This section examines the asymptotic behavior of the system. We begin with the following proposition.

proposition 3.1. Let $H(t, x) = (\mathcal{E}(t)\varpi)(x)$ be the solution to the following

$$\begin{cases} \frac{dH(t)}{dt} = (\mathfrak{A} + \mathfrak{G})H(t), & \forall t \in (0, \infty), \\ H(0) = \varpi(x). \end{cases} \quad (3.1)$$

Then,

$$H(t, x) = \begin{cases} \begin{pmatrix} \varpi_{1,0}e^{-(N\lambda_1+\lambda_2)t} \\ H_1^1(t-x, 0)e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} \\ H_1^1(t-x, 0)e^{-\theta x-\int_0^x \beta_1(v)dv}(1-e^{-N\lambda_1 x}) \\ e^{-N\lambda_1 x-\int_0^x \beta_2(v)dv}[H_2^1(t-x, 0) + \theta H_1^1(t-x, 0) \int_0^x e^{-\theta z-\int_0^z (\beta_1(v)-\beta_2(v))dv} dz] \\ \Upsilon_1 \end{pmatrix}, & x < t, \\ \begin{pmatrix} \varpi_{1,0}e^{-(N\lambda_1+\lambda_2)t} \\ \varpi_{1,1}(x-t)e^{-(N\lambda_1+\theta)t-\int_{x-t}^x \beta_1(v)dv} \\ e^{-\theta t-\int_{x-t}^x \beta_1(v)dv}[\varpi_{1,2}(x-t) + \varpi_{1,1}(x-t)(1-e^{-N\lambda_1 t})] \\ e^{-N\lambda_1 t-\int_{x-t}^x \beta_2(v)dv}[\varpi_{2,1}(x-t) + \theta \varpi_{1,1}(x-t) \int_{x-t}^x e^{-\theta(l-x+t)-\int_{x-t}^l (\beta_1(v)-\beta_2(v))dv} dl] \\ \Upsilon_2 \end{pmatrix}, & x > t, \end{cases}$$

where

$$\begin{aligned}\Upsilon_1 &= (1 - e^{-N\lambda_1 x})[\theta H_1^1(t-x, 0)e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v))dv} dz \\ &\quad + N\lambda_1 H_2^1(t-x, 0)e^{-\int_0^x \beta_2(v)dv}], \\ \Upsilon_2 &= \varpi_{2,2}(x-t)e^{-\int_{x-t}^x \beta_2(v)dv} + \theta e^{-\theta(t-x) - \int_{x-t}^x \beta_2(v)dv} \int_{x-t}^x e^{-\theta l - \int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} \\ &\quad \times [\varpi_{1,2}(x-t) + \varpi_{1,1}(x-t)(1 - e^{-N\lambda_1(l-x+t)})]dl + N\lambda_1 \varpi_{2,1}(x-t)e^{-\int_{x-t}^x \beta_2(v)dv} \\ &\quad \times (1 - e^{-N\lambda_1 t}) + \theta \varpi_{1,1}(x-t)e^{-(N\lambda_1 + \theta)(t-x) - \int_{x-t}^x \beta_2(v)dv} \int_{x-t}^x e^{-\theta l - \int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} \\ &\quad \times (e^{-N\lambda_1 l} - e^{-N\lambda_1 x})dl,\end{aligned}$$

$H_1^1(t-x, 0)$ and $H_2^1(t-x, 0)$ are determined by Eqs (1.1f) and (1.1h), respectively.

Proof. Since $H(t, x)$ is a solution of System (3.1), we have the following:

$$\frac{dH_1^0(t)}{dt} = -(N\lambda_1 + \lambda_2)H_1^0(t), \quad (3.2a)$$

$$\frac{\partial H_1^1(t, x)}{\partial t} + \frac{\partial H_1^1(t, x)}{\partial x} = -(N\lambda_1 + \beta_1(x) + \theta)H_1^1(t, x), \quad (3.2b)$$

$$\frac{\partial H_1^2(t, x)}{\partial t} + \frac{\partial H_1^2(t, x)}{\partial x} = -(\beta_1(x) + \theta)H_1^2(t, x) + N\lambda_1 H_1^1(t, x), \quad (3.2c)$$

$$\frac{\partial H_2^1(t, x)}{\partial t} + \frac{\partial H_2^1(t, x)}{\partial x} = -(N\lambda_1 + \beta_2(x))H_2^1(t, x) + \theta H_1^1(t, x), \quad (3.2d)$$

$$\frac{\partial H_2^2(t, x)}{\partial t} + \frac{\partial H_2^2(t, x)}{\partial x} = -\beta_2(x)H_2^2(t, x) + \theta H_1^2(t, x) + N\lambda_1 H_2^1(t, x), \quad (3.2e)$$

$$H_1^1(t, 0) = (N\lambda_1 + \lambda_2)H_1^0(t) + \int_0^\infty H_1^1(t, x)\beta_1(x)dx, \quad (3.2f)$$

$$H_1^2(t, 0) = 0, \quad (3.2g)$$

$$H_2^1(t, 0) = \int_0^\infty H_2^1(t, x)\beta_2(x)dx, \quad (3.2h)$$

$$H_2^2(t, 0) = 0, \quad (3.2i)$$

$$H_1^0(0) = \varpi_{1,0}, \quad H_i^j(0, x) = \varpi_{i,j}(x), \quad i, j = 1, 2. \quad (3.2j)$$

If we define $\varsigma = x - t$ and $G_i^j(t) = H_i^j(t, t + \varsigma)$, $i, j = 1, 2$, then Eqs (3.2b)–(3.2e) will become the following:

$$\begin{aligned}\frac{dG_1^1(t)}{dt} &= -(N\lambda_1 + \beta_1(\varsigma + t) + \theta)G_1^1(t), \\ \frac{dG_1^2(t)}{dt} &= -(\beta_1(\varsigma + t) + \theta)G_1^2(t) + N\lambda_1 G_1^1(t), \\ \frac{dG_2^1(t)}{dt} &= -(N\lambda_1 + \beta_2(\varsigma + t))G_2^1(t) + \theta G_1^1(t), \\ \frac{dG_2^2(t)}{dt} &= -\beta_2(\varsigma + t)G_2^2(t) + \theta G_1^2(t) + N\lambda_1 G_2^1(t).\end{aligned} \quad (3.3)$$

In the case where $\varsigma < 0$ (i.e., $x < t$), integrating Eq (3.3) from $-\varsigma$ to t and applying $G_i^j(-\varsigma) = H_i^j(-\varsigma, \varsigma - \varsigma) = H_i^j(-\varsigma, 0) = H_i^j(t - x, 0)$, $i, j = 1, 2$, $G_1^j(z - \varsigma) = H_1^j(z - \varsigma, z - \varsigma + \varsigma) = H_1^j(z - \varsigma, z)$, $j = 1, 2$, $G_2^1(z - \varsigma) = H_2^1(z - \varsigma, z - \varsigma + \varsigma) = H_2^1(z - \varsigma, z)$, Eqs (3.2g) and (3.2i) give the following:

$$H_1^1(t, x) = H_1^1(t - x, 0)e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v)dv}, \quad (3.4a)$$

$$\begin{aligned} H_1^2(t, x) &= H_1^2(t - x, 0)e^{-\theta x - \int_0^x \beta_1(y)dy} + N\lambda_1 e^{-\theta x - \int_0^x \beta_1(y)dy} \int_{-\varsigma}^t G_1^1(\eta) e^{\theta(\eta + \varsigma) + \int_0^{\varsigma + \eta} \beta_1(y)dy} d\eta \\ &= N\lambda_1 e^{-\theta x - \int_0^x \beta_1(v)dv} \int_0^x H_1^1(z - \varsigma, z) e^{\theta z + \int_0^z \beta_1(v)dv} dz \\ &= N\lambda_1 e^{-\theta x - \int_0^x \beta_1(v)dv} \int_0^x H_1^1(-\varsigma, 0) e^{-(N\lambda_1 + \theta)z - \int_0^z \beta_1(v)dv + \theta z + \int_0^z \beta_1(v)dv} dz \\ &= H_1^1(t - x, 0) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}), \end{aligned} \quad (3.4b)$$

$$\begin{aligned} H_2^1(t, x) &= H_2^1(t - x, 0) e^{-N\lambda_1 x - \int_0^x \beta_2(y)dy} + \theta e^{-N\lambda_1 x - \int_0^x \beta_2(y)dy} \\ &\quad \times \int_{-\varsigma}^t G_1^1(\eta) e^{N\lambda_1(\eta + \varsigma) + \int_0^{\varsigma + \eta} \beta_2(y)dy} d\eta \\ &= H_2^1(t - x, 0) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} + \theta e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} \\ &\quad \times \int_0^x H_1^1(z - \varsigma, z) e^{N\lambda_1 z + \int_0^z \beta_2(v)dv} dz \\ &= H_2^1(t - x, 0) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} + \theta e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} \\ &\quad \times \int_0^x H_1^1(-\varsigma, 0) e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v))dv} dz \\ &= e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} [H_2^1(t - x, 0) + \theta H_1^1(t - x, 0) \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v))dv} dz], \end{aligned} \quad (3.4c)$$

$$\begin{aligned} H_2^2(t, x) &= H_2^2(t - x, 0) e^{-\int_0^x \beta_2(y)dy} + e^{-\int_0^x \beta_2(y)dy} \int_{-\varsigma}^t [\theta G_1^2(\eta) + N\lambda_1 G_2^1(\eta)] e^{\int_0^{\varsigma + \eta} \beta_2(y)dy} d\eta \\ &= H_2^2(t - x, 0) e^{-\int_0^x \beta_2(v)dv} + e^{-\int_0^x \beta_2(v)dv} \int_0^x [\theta G_1^2(z - \varsigma) + N\lambda_1 G_2^1(z - \varsigma)] e^{\int_0^z \beta_2(v)dv} dz \\ &= e^{-\int_0^x \beta_2(v)dv} \int_0^x [\theta H_1^2(z - \varsigma, z) + N\lambda_1 H_2^1(z - \varsigma, z)] e^{\int_0^z \beta_2(v)dv} dz \\ &= e^{-\int_0^x \beta_2(v)dv} \int_0^x \left\{ \theta H_1^1(-\varsigma, 0) e^{-\theta z - \int_0^z \beta_1(v)dv} (1 - e^{-N\lambda_1 z}) + N\lambda_1 e^{-N\lambda_1 z - \int_0^z \beta_2(v)dv} \right. \\ &\quad \times \left[H_2^1(-\varsigma, 0) + \theta H_1^1(-\varsigma, 0) \int_0^z e^{-\theta w - \int_0^w (\beta_1(v) - \beta_2(v))dv} dw \right] \left. \right\} e^{\int_0^z \beta_2(v)dv} dz \\ &= (1 - e^{-N\lambda_1 x}) \left[\theta H_1^1(t - x, 0) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v))dv} dz \right. \\ &\quad \left. + N\lambda_1 H_2^1(t - x, 0) e^{-\int_0^x \beta_2(v)dv} \right]. \end{aligned} \quad (3.4d)$$

By solving Eq (3.2b), we have the following:

$$H_1^0(t) = H_1^0(0) e^{-(N\lambda_1 + \lambda_2)t} = \varpi_{1,0} e^{-(N\lambda_1 + \lambda_2)t}. \quad (3.5)$$

In the case where $\varsigma > 0$ (namely $x > t$), integrating Eq (3.3) from 0 to t and applying the relation $G_i^j(0) = H_i^j(0, \varsigma) = \varpi_{i,j}(\varsigma) = \varpi_{i,j}(x - t)$, $i, j = 1, 2$, we have the following:

$$H_1^1(t, x) = \varpi_{1,1}(x - t)e^{-(N\lambda_1 + \theta)t - \int_{x-t}^x \beta_1(v)dv}, \quad (3.6a)$$

$$\begin{aligned} H_1^2(t, x) &= G_1^2(0)e^{-\theta t - \int_0^t \beta_1(\varsigma+v)dv} + N\lambda_1 e^{-\theta t - \int_0^t \beta_1(\varsigma+v)dv} \int_0^t G_1^1(y)e^{\theta y + \int_0^y \beta_1(\varsigma+v)dv} dy \\ &\stackrel{z=\varsigma+v}{=} \varpi_{1,2}(x - t)e^{-\theta t - \int_{\varsigma}^{\varsigma+t} \beta_1(z)dz} + N\lambda_1 e^{-\theta t - \int_{\varsigma}^{\varsigma+t} \beta_1(z)dz} \int_0^t G_1^1(y)e^{\theta y + \int_{\varsigma}^{\varsigma+y} \beta_1(z)dz} dy \\ &\stackrel{l=\varsigma+y}{=} \varpi_{1,2}(x - t)e^{-\theta t - \int_{x-t}^x \beta_1(z)dz} + N\lambda_1 e^{-\theta t - \int_{x-t}^x \beta_1(z)dz} \int_{x-t}^x H_1^1(l - \varsigma, l)e^{\theta(l-\varsigma)} dl \\ &\quad \times e^{\int_{x-t}^l \beta_1(z)dz} \\ &= \varpi_{1,2}(x - t)e^{-\theta t - \int_{x-t}^x \beta_1(z)dz} + N\lambda_1 \varpi_{1,1}(x - t)e^{-\theta t - \int_{x-t}^x \beta_1(z)dz} \int_{x-t}^x e^{-N\lambda_1(l-x+t)} dl \\ &= e^{-\theta t - \int_{x-t}^x \beta_1(v)dv} [\varpi_{1,2}(x - t) + \varpi_{1,1}(x - t)(1 - e^{-N\lambda_1 t})], \end{aligned} \quad (3.6b)$$

$$\begin{aligned} H_2^1(t, x) &= G_2^1(0)e^{-N\lambda_1 t - \int_0^t \beta_2(\varsigma+v)dv} + \theta e^{-N\lambda_1 t - \int_0^t \beta_2(\varsigma+v)dv} \int_0^t G_1^1(y)e^{N\lambda_1 y + \int_0^y \beta_2(\varsigma+v)dv} dy \\ &\stackrel{z=\varsigma+v}{=} \varpi_{2,1}(x - t)e^{-N\lambda_1 t - \int_{\varsigma}^{\varsigma+t} \beta_2(z)dz} + \theta e^{-N\lambda_1 t - \int_{\varsigma}^{\varsigma+t} \beta_2(z)dz} \int_0^t G_1^1(y)e^{N\lambda_1 y + \int_{\varsigma}^{\varsigma+y} \beta_2(z)dz} dy \\ &\stackrel{l=\varsigma+y}{=} \varpi_{2,1}(x - t)e^{-N\lambda_1 t - \int_{x-t}^x \beta_2(z)dz} + \theta e^{-N\lambda_1 t - \int_{x-t}^x \beta_2(z)dz} \int_{x-t}^x H_1^1(l - \varsigma, l)e^{N\lambda_1(l-\varsigma)} dl \\ &\quad \times e^{\int_{x-t}^l \beta_2(z)dz} \\ &= e^{-N\lambda_1 t - \int_{x-t}^x \beta_2(v)dv} [\varpi_{2,1}(x - t) + \theta \varpi_{1,1}(x - t) \int_{x-t}^x e^{-\theta(l-x+t) - \int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} dl], \end{aligned} \quad (3.6c)$$

$$\begin{aligned} H_2^2(t, x) &= G_2^2(0)e^{-\int_0^t \beta_2(\varsigma+v)dv} + e^{-\int_0^t \beta_2(\varsigma+v)dv} \int_0^t [\theta G_1^2(y) + N\lambda_1 G_2^1(y)]e^{\int_0^y \beta_2(\varsigma+v)dv} dy \\ &\stackrel{z=\varsigma+v}{=} \varpi_{2,2}(x - t)e^{-\int_{\varsigma}^{\varsigma+t} \beta_2(z)dz} + e^{-\int_{\varsigma}^{\varsigma+t} \beta_2(z)dz} \int_0^t [\theta G_1^2(y) + N\lambda_1 G_2^1(y)]e^{\int_{\varsigma}^{\varsigma+y} \beta_2(z)dz} dy \\ &\stackrel{l=\varsigma+y}{=} \varpi_{2,2}(x - t)e^{-\int_{x-t}^x \beta_2(z)dz} + e^{-\int_{x-t}^x \beta_2(z)dz} \int_{x-t}^x [\theta H_1^2(l - \varsigma, l) + N\lambda_1 H_2^1(l - \varsigma, l)] \\ &\quad \times e^{\int_{x-t}^l \beta_2(z)dz} dl \\ &= \varpi_{2,2}(x - t)e^{-\int_{x-t}^x \beta_2(v)dv} + \theta e^{-\int_{x-t}^x \beta_2(v)dv - \theta(t-x)} \int_{x-t}^x e^{-\int_{x-t}^l (\beta_1(v) - \beta_2(v))dv - \theta l} \\ &\quad \times [\varpi_{1,2}(x - t) + \varpi_{1,1}(x - t)(1 - e^{-N\lambda_1(l-x+t)})] dl + N\lambda_1 \varpi_{2,1}(x - t)e^{-\int_{x-t}^x \beta_2(v)dv} \\ &\quad \times (1 - e^{-N\lambda_1 t}) + \theta \varpi_{1,1}(x - t)e^{-\int_{x-t}^x \beta_2(v)dv - (N\lambda_1 + \theta)(t-x)} \int_{x-t}^x e^{-\int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} \\ &\quad \times e^{-\theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl. \end{aligned} \quad (3.6d)$$

Equations (3.4)–(3.6) show that the conclusion of this proposition is justified. $\square$

For any $\beta \in X$, define two operators as follows:

$$(Q(t)\vartheta)(x) = \begin{cases} 0, & x \in [0, t) \\ (\mathcal{E}(t)\vartheta)(x), & x \in [t, \infty) \end{cases}, \quad (3.7a)$$

$$(O(t)\vartheta)(x) = \begin{cases} (\mathcal{E}(t)\vartheta)(x), & x \in [0, t) \\ 0, & x \in [t, \infty) \end{cases}. \quad (3.7b)$$

Evidently, $\mathcal{E}(t)\vartheta = Q(t)\vartheta + O(t)\vartheta$, $\forall \vartheta \in X$.

Theorem 3.1. If $\beta_i(x)$ ($i = 1, 2$) is Lipschitz continuous and there are $\overline{\beta_i}$ and $\underline{\beta_i}$ to make $0 < \underline{\beta_i} \leq \beta_i(x) \leq \overline{\beta_i} < \infty$ ($i = 1, 2$), then $O(t)$ is a compact operator in X .

Proof. Similar results to those obtained using Corollary 1.37 in [20] can be obtained with ease. Similarly, one only needs to verify conditions analogous to those in Corollary 1.37 of [20], specifically Condition (2). In the case of $\varpi \in D(\mathfrak{U})$, by Proposition 3.1 $H(t, x) = (\mathcal{E}(t)\varpi)(x)$, $x \in [0, t)$ is a solution of System (3.1). Thus, for $x \in [0, t)$, $g \in [0, t)$, $x + g \in [0, t)$, we have the following:

$$\begin{aligned} & \sum_{i=1}^2 \sum_{j=1}^2 \int_0^t |H_i^j(t, x+g) - H_i^j(t, x)| dx \\ &= \int_0^t \left| H_1^1(t-x-g, 0) e^{-(N\lambda_1+\theta)(x+g)-\int_0^{x+g} \beta_1(v)dv} - H_1^1(t-x, 0) e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} \right| dx \\ &+ \int_0^t \left| H_1^1(t-x-g, 0) e^{-\theta(x+g)-\int_0^{x+g} \beta_1(v)dv} (1 - e^{-N\lambda_1(x+g)}) - H_1^1(t-x, 0) e^{-\theta x - \int_0^x \beta_1(v)dv} \right. \\ &\times (1 - e^{-N\lambda_1 x}) \Big| dx + \int_0^t \left| e^{-N\lambda_1(x+g)-\int_0^{x+g} \beta_2(v)dv} [H_2^1(t-x-g, 0) + \theta H_1^1(t-x-g, 0) \right. \\ &\times \int_0^{x+g} e^{-\theta z - \int_0^z (\beta_1(v)-\beta_2(v))dv} dz] \\ &- e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} [H_2^1(t-x, 0) + \theta H_1^1(t-x, 0) \int_0^x e^{-\theta z - \int_0^z (\beta_1(v)-\beta_2(v))dv} dz] \Big| dx \\ &+ \int_0^t \left| \theta H_1^1(t-x-g, 0) e^{-\int_0^{x+g} \beta_2(v)dv} (1 - e^{-N\lambda_1(x+g)}) \int_0^{x+g} e^{-\theta z - \int_0^z (\beta_1(v)-\beta_2(v))dv} dz \right. \\ &+ N\lambda_1 H_2^1(t-x-g, 0) e^{-\int_0^{x+g} \beta_2(v)dv} (1 - e^{-N\lambda_1(x+g)}) \\ &- \theta H_1^1(t-x, 0) e^{-\int_0^x \beta_2(v)dv} (1 - e^{-N\lambda_1 x}) \int_0^x e^{-\theta z - \int_0^z (\beta_1(v)-\beta_2(v))dv} dz \\ &- N\lambda_1 H_2^1(t-x, 0) e^{-\int_0^x \beta_2(v)dv} (1 - e^{-N\lambda_1 x}) \Big| dx \\ &\leq \int_0^t \left| H_1^1(t-x-g, 0) \right| \left| e^{-(N\lambda_1+\theta)(x+g)-\int_0^{x+g} \beta_1(v)dv} - e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} \right| dx \\ &+ \int_0^t \left| H_1^1(t-x-g, 0) - H_1^1(t-x, 0) \right| e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} dx \\ &+ \int_0^t \left| H_1^1(t-x-g, 0) \right| \left| e^{-\theta(x+g)-\int_0^{x+g} \beta_1(v)dv} (1 - e^{-N\lambda_1(x+g)}) - e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) \right| dx \end{aligned}$$

$$\begin{aligned}
& + \int_0^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-\theta x - \int_0^x \beta_1(v) dv} (1 - e^{-N\lambda_1 x}) dx \\
& + \int_0^t |H_2^1(t-x-g, 0)| \left| e^{-N\lambda_1(x+g) - \int_0^{x+g} \beta_2(v) dv} - e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \right| dx \\
& + \int_0^t |H_2^1(t-x-g, 0) - H_2^1(t-x, 0)| e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} dx \\
& + \theta \int_0^t |H_1^1(t-x-g, 0)| \left| e^{-N\lambda_1(x+g) - \int_0^{x+g} \beta_2(v) dv} \int_0^{x+g} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz \right. \\
& \quad \left. - e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz \right| dx \\
& + \theta \int_0^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz dx. \tag{3.8}
\end{aligned}$$

Now, separately estimate each term of the above equation. From Eq (3.2f) and *Theorem 2.1*, it follows that

$$\begin{aligned}
|H_1^1(t-x-g, 0)| & \leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \left[|H_1^0(t-x-g)| + \int_0^\infty |H_1^2(t-x-g, q)| dq \right] \\
& \leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\mathcal{E}(t-x-g)\varpi(\cdot)\|_X \leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X.
\end{aligned} \tag{3.9}$$

Estimate the first, third, and seventh terms of Eq (3.8) using Eq (3.9) as follow:

$$\begin{aligned}
& \int_0^t |H_1^1(t-x-g, 0)| \left| e^{-(N\lambda_1+\theta)(x+g) - \int_0^{x+g} \beta_1(v) dv} - e^{-(N\lambda_1+\theta)x - \int_0^x \beta_1(v) dv} \right| dx \\
& \leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^t \left| e^{-(N\lambda_1+\theta)(x+g) - \int_0^{x+g} \beta_1(v) dv} - e^{-(N\lambda_1+\theta)x - \int_0^x \beta_1(v) dv} \right| dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \\
& \int_0^t |H_1^1(t-x-g, 0)| \left| e^{-\theta(x+g) - \int_0^{x+g} \beta_1(v) dv} (1 - e^{-N\lambda_1(x+g)}) - e^{-\theta x - \int_0^x \beta_1(v) dv} (1 - e^{-N\lambda_1 x}) \right| dx \\
& \leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^t \left| e^{-\theta(x+g) - \int_0^{x+g} \beta_1(v) dv} (1 - e^{-N\lambda_1(x+g)}) \right. \\
& \quad \left. - e^{-\theta x - \int_0^x \beta_1(v) dv} (1 - e^{-N\lambda_1 x}) \right| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \tag{3.10} \\
& \theta \int_0^t |H_1^1(t-x-g, 0)| \left| e^{-N\lambda_1(x+g) - \int_0^{x+g} \beta_2(v) dv} \int_0^{x+g} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz \right. \\
& \quad \left. - e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz \right| dx \\
& \leq \theta \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^t \left| e^{-N\lambda_1(x+g) - \int_0^{x+g} \beta_2(v) dv} \int_0^{x+g} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz \right. \\
& \quad \left. - e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz \right| dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned}$$

From Eq (3.2f) and *Proposition 3.1*, and applying the new integral variables $t-x-g-q=y$ and

$t - x - q = y$, we know the following:

$$\begin{aligned}
& |H_1^1(t - x - g, 0) - H_1^1(t - x, 0)| \\
& \leq (N\lambda_1 + \lambda_2) |H_1^0(t - x - g) - H_1^0(t - x)| \\
& \quad + \left| \int_0^\infty H_1^2(t - x - g, q) \beta_1(q) dq - \int_0^\infty H_1^2(t - x, q) \beta_1(q) dq \right| \\
& \leq (N\lambda_1 + \lambda_2) |\varpi_{1,0}| |e^{-(N\lambda_1 + \lambda_2)(t-x-g)} - e^{-(N\lambda_1 + \lambda_2)(t-x)}| \\
& \quad + \left| \int_0^{t-x-g} H_1^2(t - x - g, t - x - g - y) \beta_1(t - x - g - y) dy \right. \\
& \quad \left. - \int_0^{t-x} H_1^2(t - x, t - x - y) \beta_1(t - x - y) dy \right| \\
& \quad + \left| \int_{t-x-g}^\infty \beta_1(q) [\varpi_{1,2}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_1(v) dv - \theta(t-x-g)} \right. \\
& \quad + \varpi_{1,1}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_1(v) dv - \theta(t-x-g)} (1 - e^{-N\lambda_1(t-x-g)})] dq \\
& \quad - \int_{t-x}^\infty \beta_1(q) [\varpi_{1,2}(q - t + x) e^{-\int_{q-t+x}^q \beta_1(v) dv - \theta(t-x)} \\
& \quad + \varpi_{1,1}(q - t + x) e^{-\int_{q-t+x}^q \beta_1(v) dv - \theta(t-x)} (1 - e^{-N\lambda_1(t-x)})] dq \Big|. \tag{3.11}
\end{aligned}$$

In the following, we will meticulously evaluate each term in Eq (3.11). By employing the relationships $q - t + x + g = y$, $q - t + x = y$, along with the Lipschitz continuity of $\beta_1(x)$ (without loss of generality, and assuming the Lipschitz constant to be 1), we can estimate the third term in Eq (3.11).

$$\begin{aligned}
& \left| \int_{t-x-g}^\infty \beta_1(q) [\varpi_{1,2}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_1(v) dv - \theta(t-x-g)} \right. \\
& \quad + \varpi_{1,1}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_1(v) dv - \theta(t-x-g)} (1 - e^{-N\lambda_1(t-x-g)})] dq \\
& \quad - \int_{t-x}^\infty \beta_1(q) [\varpi_{1,2}(q - t + x) e^{-\int_{q-t+x}^q \beta_1(v) dv - \theta(t-x)} \\
& \quad + \varpi_{1,1}(q - t + x) e^{-\int_{q-t+x}^q \beta_1(v) dv - \theta(t-x)} (1 - e^{-N\lambda_1(t-x)})] dq \Big| \\
& \leq \|\varpi_{1,2}\|_{L^1[0,\infty)} \left\{ |g| \sup_{y \in [0,\infty)} e^{-\int_y^{y+t-x-g} \beta_1(v) dv - \theta(t-x-g)} \right. \\
& \quad + \overline{\beta_1} \sup_{y \in [0,\infty)} \left| e^{-\int_y^{y+t-x-g} \beta_1(v) dv - \theta(t-x-g)} - e^{-\int_y^{y+t-x} \beta_1(v) dv - \theta(t-x)} \right| \Big\} \\
& \quad + \|\varpi_{1,1}\|_{L^1[0,\infty)} \left\{ |g| \sup_{y \in [0,\infty)} e^{-\int_y^{y+t-x-g} \beta_1(v) dv - \theta(t-x-g)} (1 - e^{-N\lambda_1(t-x-g)}) \right. \\
& \quad + \overline{\beta_1} \sup_{y \in [0,\infty)} \left| e^{-\int_y^{y+t-x-g} \beta_1(v) dv - \theta(t-x-g)} (1 - e^{-N\lambda_1(t-x-g)}) \right. \\
& \quad \left. - e^{-\int_y^{y+t-x} \beta_1(v) dv - \theta(t-x)} (1 - e^{-N\lambda_1(t-x)}) \right| \Big\} \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \tag{3.12}
\end{aligned}$$

By combining Eq (3.9), Proposition 3.1, and the Lipschitz continuity of $\beta_1(x)$, our estimation of the second term in Eq (3.11) is as follows:

$$\begin{aligned}
& \left| \int_0^{t-x-g} \beta_1(t-x-g-y) H_1^2(t-x-g, t-x-g-y) dy \right. \\
& \quad \left. - \int_0^{t-x} \beta_1(t-x-y) H_1^2(t-x, t-x-y) dy \right| \\
& \leq \bar{\beta}_1 \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_{t-x-g}^{t-x} e^{-\theta(t-x-g-y) - \int_0^{t-x-g-y} \beta_1(v) dv} (1 - e^{-N\lambda_1(t-x-g-y)}) dy \\
& \quad + |g| \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^{t-x} e^{-\theta(t-x-g-y) - \int_0^{t-x-g-y} \beta_1(v) dv} (1 - e^{-N\lambda_1(t-x-g-y)}) dy \\
& \quad + \bar{\beta}_1 \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^{t-x} \left| e^{-\theta(t-x-g-y) - \int_0^{t-x-g-y} \beta_1(v) dv} (1 - e^{-N\lambda_1(t-x-g-y)}) \right. \\
& \quad \left. - e^{-\theta(t-x-y) - \int_0^{t-x-y} \beta_1(v) dv} (1 - e^{-N\lambda_1(t-x-y)}) \right| dy \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.13}$$

The combination of Eqs (3.11)–(3.13) gives the following:

$$|H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \tag{3.14}$$

The above means that the second, fourth, and eighth terms in Eq (3.8) satisfy the following:

$$\begin{aligned}
& \int_0^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v) dv} dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \\
& \int_0^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-\theta x - \int_0^x \beta_1(v) dv} (1 - e^{-N\lambda_1 x}) dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \\
& \theta \int_0^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.15}$$

Similarly, from Eq (3.2h) and Theorem 2.1, it follows that

$$|H_2^1(t-x-g, 0)| \leq \bar{\beta}_2 \int_0^\infty |H_2^2(t-x-g, q)| dq \leq \bar{\beta}_2 \|\mathcal{E}(t-x-g)\varpi(\cdot)\|_X \leq \bar{\beta}_2 \|\varpi\|_X. \tag{3.16}$$

Use Eq (3.16) to estimate the fifth term of Eq (3.8) as follows:

$$\begin{aligned}
& \int_0^t |H_2^1(t-x-g, 0)| \left| e^{-N\lambda_1(x+g) - \int_0^{x+g} \beta_2(v) dv} - e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \right| dx \\
& \leq \bar{\beta}_2 \|\varpi\|_X \int_0^t \left| e^{-N\lambda_1(x+g) - \int_0^{x+g} \beta_2(v) dv} - e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \right| dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.17}$$

From Eq (3.2h) and Proposition 3.1 and applying the new integral variables $t - x - g - q = y$ and $t - x - q = y$, we get the following:

$$\begin{aligned}
& |H_2^1(t - x - h, 0) - H_2^1(t - x, 0)| \\
& \leq \left| \int_0^\infty H_2^2(t - x - g, q) \beta_2(q) dq - \int_0^\infty H_2^2(t - x, q) \beta_2(q) dq \right| \\
& \leq \left| \int_0^{t-x-g} H_2^2(t - x - g, t - x - g - y) \beta_2(t - x - g - y) dy \right. \\
& \quad \left. - \int_0^{t-x} H_2^2(t - x, t - x - y) \beta_2(t - x - y) dy \right| \\
& \quad + \left| \int_{t-x-g}^\infty \beta_2(q) \left\{ \varpi_{2,2}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_2(v) dv} + \theta e^{-\int_{q-t+x+g}^q \beta_2(v) dv - \theta(t-x-g-q)} \right. \right. \\
& \quad \times \int_{q-t+x+g}^q e^{-\int_{q-l}^l (\beta_1(v) - \beta_2(v)) dv - \theta l} [\varpi_{1,2}(q - t + x + g) - \varpi_{1,1}(q - t + x + g) \\
& \quad \times (1 - e^{-N\lambda_1(l-q+t-x-g)})] dl + N\lambda_1 \varpi_{2,1}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_2(v) dv} \\
& \quad \times (1 - e^{-N\lambda_1(t-x-g)}) + \theta \varpi_{1,1}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_2(v) dv + (N\lambda_1 + \theta)(t-x-g-q)} \\
& \quad \times \int_{q-t+x+g}^q e^{-\int_{q-t+x+g}^l (\beta_1(v) - \beta_2(v)) dv - \theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl \Big\} dq \\
& \quad - \int_{t-x}^\infty \beta_2(q) \left\{ \varpi_{2,2}(q - t + x) e^{-\int_{q-t+x}^q \beta_2(v) dv} + \theta e^{-\int_{q-t+x}^q \beta_2(v) dv - \theta(t-x-q)} \right. \\
& \quad \times \int_{q-t+x}^q e^{-\int_{q-l}^l (\beta_1(v) - \beta_2(v)) dv - \theta l} [\varpi_{1,2}(q - t + x) - \varpi_{1,1}(q - t + x) \\
& \quad \times (1 - e^{-N\lambda_1(l-q+t-x)})] dl + N\lambda_1 \varpi_{2,1}(q - t + x) e^{-\int_{q-t+x}^q \beta_2(v) dv} (1 - e^{-N\lambda_1(t-x)}) \\
& \quad + \theta \varpi_{1,1}(q - t + x) e^{-\int_{q-t+x}^q \beta_2(v) dv + (N\lambda_1 + \theta)(t-x-q)} \\
& \quad \times \int_{q-t+x}^q e^{-\int_{q-t+x}^l (\beta_1(v) - \beta_2(v)) dv - \theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl \Big\} dq \Big|. \tag{3.18}
\end{aligned}$$

Below, we evaluate each term in Eq (3.18). Setting $q - t + x + g = y$ and $q - t + x = y$, and using the Lipschitz continuity of $\beta_2(x)$, we find that the second term in Eq (3.18) satisfies the following:

$$\begin{aligned}
& \left| \int_{t-x-g}^\infty \beta_2(q) \left\{ \varpi_{2,2}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_2(v) dv} + \theta e^{-\int_{q-t+x+g}^q \beta_2(v) dv - \theta(t-x-g-q)} \right. \right. \\
& \quad \times \int_{q-t+x+g}^q e^{-\int_{q-l}^l (\beta_1(v) - \beta_2(v)) dv - \theta l} [\varpi_{1,2}(q - t + x + g) - \varpi_{1,1}(q - t + x + g) \\
& \quad \times (1 - e^{-N\lambda_1(l-q+t-x-g)})] dl + N\lambda_1 \varpi_{2,1}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_2(v) dv} \\
& \quad \times (1 - e^{-N\lambda_1(t-x-g)}) + \theta \varpi_{1,1}(q - t + x + g) e^{-\int_{q-t+x+g}^q \beta_2(v) dv + (N\lambda_1 + \theta)(t-x-g-q)} \\
& \quad \times \int_{q-t+x+g}^q e^{-\int_{q-t+x+g}^l (\beta_1(v) - \beta_2(v)) dv - \theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl \Big\} dq \\
& \quad - \int_{t-x}^\infty \beta_2(q) \left\{ \varpi_{2,2}(q - t + x) e^{-\int_{q-t+x}^q \beta_2(v) dv} + \theta e^{-\int_{q-t+x}^q \beta_2(v) dv - \theta(t-x-q)} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \int_{q-t+x}^q e^{-\int_{q-t}^l (\beta_1(v)-\beta_2(v))dv-\theta l} [\varpi_{1,2}(q-t+x) - \varpi_{1,1}(q-t+x) \\
& \times (1 - e^{-N\lambda_1(l-q+t-x)})] dl + N\lambda_1 \varpi_{2,1}(q-t+x) e^{-\int_{q-t+x}^q \beta_2(v)dv} (1 - e^{-N\lambda_1(t-x)}) \\
& + \theta \varpi_{1,1}(q-t+x) e^{-\int_{q-t+x}^q \beta_2(v)dv+(N\lambda_1+\theta)(t-x-q)} \\
& \times \int_{q-t+x}^q e^{-\int_{q-t}^l (\beta_1(v)-\beta_2(v))dv-\theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl \Big| dq \Big| \\
& \leq \|\varpi_{2,2}\|_{L^1[0,\infty)} \{ |g| \sup_{y \in [0,\infty)} e^{-\int_y^{y+t-x-g} \beta_2(v)dv} \\
& + \overline{\beta_2} \sup_{y \in [0,\infty)} |e^{-\int_y^{y+t-x-g} \beta_2(v)dv} - e^{-\int_y^{y+t-x} \beta_2(v)dv}| \} \\
& + \theta |g| \sup_{y \in [0,\infty)} e^{-\int_y^{y+t-x-g} \beta_2(v)dv+\theta y} \int_0^\infty \int_y^{y+t-x-g} e^{-\int_{y+t-x-g-l}^l (\beta_1(v)-\beta_2(v))dv-\theta l} \\
& \times [\varpi_{1,2}(y) - \varpi_{1,1}(y)(1 - e^{-N\lambda_1(l-y)})] dl dy + \theta \overline{\beta_2} \int_0^\infty |e^{-\int_y^{y+t-x-g} \beta_2(v)dv+\theta y} \\
& \times \int_y^{y+t-x-g} e^{-\int_{y+t-x-g-l}^l (\beta_1(v)-\beta_2(v))dv-\theta l} [\varpi_{1,2}(y) - \varpi_{1,1}(y)(1 - e^{-N\lambda_1(l-y)})] dl \\
& - e^{-\int_y^{y+t-x} \beta_2(v)dv+\theta y} \int_y^{y+t-x} e^{-\int_{y+t-x-l}^l (\beta_1(v)-\beta_2(v))dv-\theta l} [\varpi_{1,2}(y) - \varpi_{1,1}(y) \\
& \times (1 - e^{-N\lambda_1(l-y)})] dl dy + N\lambda_1 \|\varpi_{2,1}\|_{L^1[0,\infty)} \{ |g| \sup_{y \in [0,\infty)} e^{-\int_y^{y+t-x-g} \beta_2(v)dv} \\
& \times (1 - e^{-N\lambda_1(t-x-g)}) + \overline{\beta_2} \sup_{y \in [0,\infty)} |e^{-\int_y^{y+t-x-g} \beta_2(v)dv} (1 - e^{-N\lambda_1(t-x-g)}) \\
& - e^{-\int_y^{y+t-x} \beta_2(v)dv} (1 - e^{-N\lambda_1(t-x)}) dy| \} \\
& + \theta |g| \sup_{y \in [0,\infty)} e^{-\int_y^{y+t-x-g} \beta_2(v)dv-(N\lambda_1+\theta)y} \int_0^\infty |\varpi_{1,1}(y)| \int_y^{y+t-x-g} e^{-\int_y^l (\beta_1(v)-\beta_2(v))dv-\theta l} \\
& \times (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl dy + \theta \overline{\beta_2} \int_0^\infty |\varpi_{1,1}(y)| |e^{-\int_y^{y+t-x-g} \beta_2(v)dv-(N\lambda_1+\theta)y} \\
& \times \int_y^{y+t-x-g} e^{-\int_y^l (\beta_1(v)-\beta_2(v))dv-\theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl - e^{-\int_y^{y+t-x} \beta_2(v)dv-(N\lambda_1+\theta)y} \\
& \times \int_y^{y+t-x} e^{-\int_y^l (\beta_1(v)-\beta_2(v))dv-\theta l} (e^{-N\lambda_1 l} - e^{-N\lambda_1 x}) dl dy \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.19}$$

Using Eqs (3.9) and (3.10), Proposition 3.1, and the Lipschitz continuity of $\beta_2(x)$, we calculate the first term in Eq (3.18) as follows:

$$\begin{aligned}
& \left| \int_0^{t-x-g} H_2^2(t-x-g, t-x-g-y) \beta_2(t-x-g-y) dy \right. \\
& \left. - \int_0^{t-x} H_2^2(t-x, t-x-y) \beta_2(t-x-y) dy \right|
\end{aligned}$$

$$\begin{aligned}
& \leq \theta \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \bar{\beta}_2 \int_{t-x-g}^{t-x} (1 - e^{-N\lambda_1(t-x-g-y)}) e^{-\int_0^{t-x-g-y} \beta_2(v) dv} \\
& \quad \times \int_0^{t-x-g-y} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz dy + N\lambda_1 \bar{\beta}_2 \|\varpi\|_X \bar{\beta}_2 \\
& \quad \times \int_{t-x-g}^{t-x} (1 - e^{-N\lambda_1(t-x-g-y)}) e^{-\int_0^{t-x-g-y} \beta_2(v) dv} dy + \theta |g| \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \\
& \quad \times \int_0^{t-x} (1 - e^{-N\lambda_1(t-x-g-y)}) e^{-\int_0^{t-x-g-y} \beta_2(v) dv} \int_0^{t-x-g-y} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz dy \\
& \quad + N\lambda_1 |g| \bar{\beta}_2 \|\varpi\|_X \int_0^{t-x} (1 - e^{-N\lambda_1(t-x-g-y)}) e^{-\int_0^{t-x-g-y} \beta_2(v) dv} dy \\
& \quad + \theta \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \bar{\beta}_2 \int_0^{t-x} |(1 - e^{-N\lambda_1(t-x-g-y)}) e^{-\int_0^{t-x-g-y} \beta_2(v) dv} \\
& \quad \times \int_0^{t-x-g-y} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz - (1 - e^{-N\lambda_1(t-x-y)}) e^{-\int_0^{t-x-y} \beta_2(v) dv} \\
& \quad \times \int_0^{t-x-y} e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz| dy + N\lambda_1 \bar{\beta}_2 \|\varpi\|_X \bar{\beta}_2 \\
& \quad \times \int_0^{t-x} |(1 - e^{-N\lambda_1(t-x-g-y)}) e^{-\int_0^{t-x-g-y} \beta_2(v) dv} - (1 - e^{-N\lambda_1(t-x-y)}) e^{-\int_0^{t-x-y} \beta_2(v) dv}| dy \\
& \quad \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \text{ uniformly for } \varpi.
\end{aligned} \tag{3.20}$$

By combining Eqs (3.18)–(3.20), we derive the following:

$$|H_2^1(t-x-g, 0) - H_2^1(t-x, 0)| \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \tag{3.21}$$

From this equation, the sixth term of Eq (3.8) can be estimated as follows:

$$\begin{aligned}
& \int_0^t |H_2^1(t-x-g, 0) - H_2^1(t-x, 0)| e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} dx \rightarrow 0, \\
& \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.22}$$

By combining Eqs (3.10) and (3.15) with Eqs (3.17) and (3.22), it follows that, for $x \in [0, t]$, $g \in [0, t]$, $x+g \in [0, t]$,

$$\sum_{i=1}^2 \sum_{j=1}^2 \int_0^t |H_i^j(t, x+g) - H_i^j(t, x)| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \tag{3.23}$$

In the case where $g \in (-t, 0)$, $x \in [0, t]$, then $H_i^1(t, x+g) = H_i^2(t, x+g) = 0$, $i = 1, 2$, for $x+g < 0$ given

$$\begin{aligned}
& \sum_{i=1}^2 \sum_{j=1}^2 \int_0^t |H_i^j(t, x+g) - H_i^j(t, x)| dx \\
& = \int_{-g}^t |H_1^1(t, x+g) - H_1^1(t, x)| dx + \int_0^{-g} |H_1^1(t, x)| dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{-g}^t |H_1^2(t, x+g) - H_1^2(t, x)| dx + \int_0^{-g} |H_1^2(t, x)| dx \\
& + \int_{-g}^t |H_2^1(t, x+g) - H_2^1(t, x)| dx + \int_0^{-g} |H_2^1(t, x)| dx \\
& + \int_{-g}^t |H_2^2(t, x+g) - H_2^2(t, x)| dx + \int_0^{-g} |H_2^2(t, x)| dx.
\end{aligned} \tag{3.24}$$

For $x \in [0, t)$, $g \in (-t, 0)$, and $x+g \in [0, t)$, following the approach in Eq (3.23), we estimate the first term in Eq (3.24) as follows:

$$\begin{aligned}
& \int_{-g}^t |H_1^1(t, x+g) - H_1^1(t, x)| dx \\
& \leq \int_{-g}^t |H_1^1(t-x-g, 0)| \left| e^{-(N\lambda_1+\theta)(x+g)-\int_0^{x+g} \beta_1(v)dv} - e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} \right| dx \\
& + \int_{-g}^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} dx.
\end{aligned} \tag{3.25}$$

Use Eq (3.9) to estimate the first term of Eq (3.25) as follows:

$$\begin{aligned}
& \int_{-g}^t |H_1^1(t-x-g, 0)| \left| e^{-(N\lambda_1+\theta)(x+g)-\int_0^{x+g} \beta_1(v)dv} - e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} \right| dx \\
& \leq \max\{N\lambda_1 + \lambda_2, \overline{\beta_1}\} \|\varpi\|_X \int_{-g}^t \left| e^{-(N\lambda_1+\theta)(x+g)-\int_0^{x+g} \beta_1(v)dv} - e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} \right| dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.26}$$

Estimate the second term in Eq (3.25) with Eq (3.14) as follows:

$$\begin{aligned}
& \int_{-g}^t |H_1^1(t-x-g, 0) - H_1^1(t-x, 0)| e^{-(N\lambda_1+\theta)x-\int_0^x \beta_1(v)dv} dx \\
& \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.27}$$

Thus, from Eqs (3.26) and (3.27), we get the following:

$$\int_{-g}^t |H_1^1(t, x+g) - H_1^1(t, x)| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \tag{3.28}$$

As in the proof of Eq (3.28), we estimate the third, fifth, and seventh terms in Eq (3.24) as follows:

$$\begin{aligned}
& \int_{-g}^t |H_1^2(t, x+g) - H_1^2(t, x)| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \\
& \int_{-g}^t |H_2^1(t, x+g) - H_2^1(t, x)| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \\
& \int_{-g}^t |H_2^2(t, x+g) - H_2^2(t, x)| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi.
\end{aligned} \tag{3.29}$$

From Proposition 3.1 and Eqs (3.9) and (3.16), we obtain the following:

$$\begin{aligned} \int_0^{-g} |H_1^1(t, x)| dx &\leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^{-g} e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v) dv} dx \\ &\rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \end{aligned} \quad (3.30)$$

$$\begin{aligned} \int_0^{-g} |H_1^2(t, x)| dx &\leq \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^{-g} e^{-\theta x - \int_0^x \beta_1(v) dv} (1 - e^{-N\lambda_1 x}) dx \\ &\rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \end{aligned} \quad (3.31)$$

$$\begin{aligned} \int_0^{-g} |H_2^1(t, x)| dx &\leq \int_0^{-g} |H_2^1(t - x, 0) e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv}| dx \\ &\quad + \theta \int_0^{-g} |H_1^1(t - x, 0) e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz| dx \\ &\leq \bar{\beta}_2 \|\varpi\|_X \int_0^{-g} e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} dx + \theta \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \\ &\quad \times \int_0^{-g} e^{-N\lambda_1 x - \int_0^x \beta_2(v) dv} \int_0^x e^{-\theta z - \int_0^z (\beta_1(v) - \beta_2(v)) dv} dz dx \\ &\rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi, \end{aligned} \quad (3.32)$$

$$\begin{aligned} \int_0^{-g} |H_2^2(t, x)| dx &\leq \theta \int_0^{-g} |H_1^1(t - x, 0) e^{-\int_0^x \beta_2(v) dv} (1 - e^{-N\lambda_1 x}) \int_0^x e^{-\int_0^z (\beta_1(v) - \beta_2(v)) dv} \\ &\quad \times e^{-\theta z} dz| dx + N\lambda_1 \int_0^{-g} |H_2^1(t - x, 0) e^{-\int_0^x \beta_2(v) dv} (1 - e^{-N\lambda_1 x})| dx \\ &\leq \theta \max\{N\lambda_1 + \lambda_2, \bar{\beta}_1\} \|\varpi\|_X \int_0^{-g} e^{-\int_0^x \beta_2(v) dv} (1 - e^{-N\lambda_1 x}) \int_0^x e^{-\theta z} \\ &\quad \times e^{-\int_0^z (\beta_1(v) - \beta_2(v)) dv} dz dx + N\lambda_1 \bar{\beta}_2 \|\varpi\|_X \int_0^{-g} e^{-\int_0^x \beta_2(v) dv} \\ &\quad \times (1 - e^{-N\lambda_1 x}) dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \end{aligned} \quad (3.33)$$

By combining Eqs(3.25)–(3.33), we deduce, for $x \in [0, t)$, $g \in (-t, 0)$, $x + g \in [0, t)$,

$$\sum_{i=1}^2 \sum_{j=1}^2 \int_0^t |H_i^j(t, x + g) - H_i^j(t, x)| dx \rightarrow 0, \quad \text{as } |g| \rightarrow 0, \quad \text{uniformly for } \varpi. \quad (3.34)$$

The conclusion of this theorem follows from Eqs (3.23) and (3.34). $\square$

Theorem 3.2. Suppose $\bar{\beta}_i$ and $\underline{\beta}_i$ such that $0 < \underline{\beta}_i \leq \beta_i(x) \leq \bar{\beta}_i < \infty$ ($i = 1, 2$) and $\underline{\beta}_1 + \underline{\beta}_2 > \bar{\beta}_2$; then, $Q(t)$ satisfies

$$\begin{aligned} \|Q(t)\varpi\|_X &\leq e^{-(N\lambda_1 + \lambda_2)t} |\varpi_{1,0}| + e^{-\underline{\beta}_1 t} \{ \|\varpi_{1,1}\|_{L^1[0,\infty)} + \|\varpi_{1,2}\|_{L^1[0,\infty)} \} \\ &\quad + e^{-\underline{\beta}_2 t} \{ \|\varpi_{2,1}\|_{L^1[0,\infty)} + \|\varpi_{2,2}\|_{L^1[0,\infty)} \} \\ &\quad + e^{-(\underline{\beta}_1 + \underline{\beta}_2 - \bar{\beta}_2)t} \left\{ \frac{3\theta}{\bar{\beta}_2 - \underline{\beta}_1} \|\varpi_{1,1}\|_{L^1[0,\infty)} + \frac{\theta}{\bar{\beta}_2 - \underline{\beta}_1} \|\varpi_{1,2}\|_{L^1[0,\infty)} \right\}. \end{aligned} \quad (3.35)$$

Proof. For $\forall \varpi \in X$, by Eq (3.7a), $H(t, x) = (\mathcal{E}(t)\varpi)(x)$, $x \in [t, \infty)$, and the value of $H(t, x)$ at $x > t$ in Proposition 3.1, we estimate the following:

$$\begin{aligned}
& \|Q(t)\varpi(\cdot)\| \\
&= |e^{-(N\lambda_1+\lambda_2)t}\varpi_{1,0}| + \int_t^\infty |\varpi_{1,1}(x-t)e^{-(N\lambda_1+\theta)t-\int_{x-t}^x\beta_1(v)dv}|dx + \int_t^\infty |\varpi_{1,2}(x-t) \\
&\quad \times e^{-\theta t-\int_{x-t}^x\beta_1(v)dv} + \varpi_{1,1}(x-t)e^{-\theta t-\int_{x-t}^x\beta_1(v)dv}(1-e^{-N\lambda_1 t})|dx + \int_t^\infty |\varpi_{2,1}(x-t)e^{-N\lambda_1 t} \\
&\quad \times e^{-\int_{x-t}^x\beta_2(v)dv} + \theta\varpi_{1,1}(x-t)e^{-N\lambda_1 t-\int_{x-t}^x\beta_2(v)dv} \int_{x-t}^x e^{-\theta(l-x+t)-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv}dl|dx \\
&\quad + \int_t^\infty |\varpi_{2,2}(x-t)e^{-\int_{x-t}^x\beta_2(v)dv} + \theta e^{-\theta(t-x)-\int_{x-t}^x\beta_2(v)dv} \int_{x-t}^x e^{-\theta l-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv} \\
&\quad \times [\varpi_{1,2}(x-t) + \varpi_{1,1}(x-t)(1-e^{-N\lambda_1(l-x+t)})]dl \\
&\quad + N\lambda_1\varpi_{2,1}(x-t)e^{-\int_{x-t}^x\beta_2(v)dv}(1-e^{-N\lambda_1 t}) + \theta\varpi_{1,1}(x-t)e^{-(N\lambda_1+\theta)(t-x)-\int_{x-t}^x\beta_2(v)dv} \\
&\quad \times \int_{x-t}^x e^{-\theta l-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv}(e^{-N\lambda_1 l} - e^{-N\lambda_1 x})dl|dx \\
&\leq |\varpi_{1,0}|e^{-(N\lambda_1+\lambda_2)t} + \sup_{x \in [0, \infty)} e^{-(N\lambda_1+\theta)t-\int_{x-t}^x\beta_1(v)dv} \int_t^\infty |\varpi_{1,1}(x-t)|dx \\
&\quad + \sup_{x \in [0, \infty)} e^{-\theta t-\int_{x-t}^x\beta_1(v)dv} \int_t^\infty |\varpi_{1,2}(x-t)|dx + \sup_{x \in [0, \infty)} e^{-\theta t-\int_{x-t}^x\beta_1(v)dv}(1-e^{-N\lambda_1 t}) \\
&\quad \times \int_t^\infty |\varpi_{1,1}(x-t)|dx + \sup_{x \in [0, \infty)} e^{-N\lambda_1 t-\int_{x-t}^x\beta_2(v)dv} \int_t^\infty |\varpi_{2,1}(x-t)|dx \\
&\quad + \theta \sup_{x \in [0, \infty)} e^{-N\lambda_1 t-\int_{x-t}^x\beta_2(v)dv} \int_t^\infty |\varpi_{1,1}(x-t)| \int_{x-t}^x e^{-\theta(l-x+t)-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv} dldx \\
&\quad + \sup_{x \in [0, \infty)} e^{-\int_{x-t}^x\beta_2(v)dv} \int_t^\infty |\varpi_{2,2}(x-t)|dx + \theta \sup_{x \in [0, \infty)} e^{-\int_{x-t}^x\beta_2(v)dv} \int_t^\infty |\varpi_{1,2}(x-t)| \\
&\quad \times e^{-\theta(t-x)} \int_{x-t}^x e^{-\theta l-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv} dldx + \theta \sup_{x \in [0, \infty)} e^{-\int_{x-t}^x\beta_2(v)dv} \int_t^\infty |\varpi_{1,1}(x-t)| \\
&\quad \times e^{-\theta(t-x)} \int_{x-t}^x e^{-\theta l-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv}(1-e^{-N\lambda_1(l-x+t)})dldx + N\lambda_1 \sup_{x \in [0, \infty)} e^{-\int_{x-t}^x\beta_2(v)dv} \\
&\quad \times (1-e^{-N\lambda_1 t}) \int_t^\infty |\varpi_{2,1}(x-t)|dx + \theta \sup_{x \in [0, \infty)} e^{-\int_{x-t}^x\beta_2(v)dv} \int_t^\infty |\varpi_{1,1}(x-t)|e^{-(N\lambda_1+\theta)(t-x)} \\
&\quad \times \int_{x-t}^x e^{-\theta l-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv}|e^{-N\lambda_1 l} - e^{-N\lambda_1 x}|dldx \\
&\leq |\varpi_{1,0}|e^{-(N\lambda_1+\lambda_2)t} + e^{-(N\lambda_1+\theta+\underline{\beta}_1)t} \int_t^\infty |\varpi_{1,1}(x-t)|dx + e^{-(\theta+\underline{\beta}_1)t} \int_t^\infty |\varpi_{1,2}(x-t)|dx \\
&\quad + e^{-(\theta+\underline{\beta}_1)t}(1-e^{-N\lambda_1 t}) \int_t^\infty |\varpi_{1,1}(x-t)|dx + e^{-(N\lambda_1+\underline{\beta}_2)t} \int_t^\infty |\varpi_{2,1}(x-t)|dx \\
&\quad + \theta e^{-(N\lambda_1+\underline{\beta}_2)t} \int_t^\infty |\varpi_{1,1}(x-t)| \int_{x-t}^x e^{-\theta(l-x+t)-\int_{x-t}^l(\beta_1(v)-\beta_2(v))dv} dldx + e^{-\underline{\beta}_2 t}
\end{aligned}$$

$$\begin{aligned}
& \times \int_t^\infty |\varpi_{2,2}(x-t)|dx + \theta e^{-\underline{\beta}_2 t} \int_t^\infty |\varpi_{1,2}(x-t)|e^{-\theta(t-x)} \int_{x-t}^x e^{-\theta l - \int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} dldx \\
& + \theta e^{-\underline{\beta}_2 t} \int_t^\infty |\varpi_{1,1}(x-t)|e^{-\theta(t-x)} \int_{x-t}^x e^{-\theta l - \int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1(l-x+t)})dldx \\
& + N\lambda_1 e^{-\underline{\beta}_2 t} (1 - e^{-N\lambda_1 t}) \int_t^\infty |\varpi_{2,1}(x-t)|dx + \theta e^{-\underline{\beta}_2 t} \int_t^\infty |\varpi_{1,1}(x-t)|e^{-(N\lambda_1 + \theta)(t-x)} \\
& \times \int_{x-t}^x e^{-\theta l - \int_{x-t}^l (\beta_1(v) - \beta_2(v))dv} |e^{-N\lambda_1 l} - e^{-N\lambda_1 x}|dldx \\
& = |\varpi_{1,0}|e^{-(N\lambda_1 + \lambda_2)t} + e^{-(N\lambda_1 + \theta + \underline{\beta}_1)t} \int_0^\infty |\varpi_{1,1}(y)|dy + e^{-(\theta + \underline{\beta}_1)t} \int_0^\infty |\varpi_{1,2}(y)|dy \\
& + e^{-(\theta + \underline{\beta}_1)t} (1 - e^{-N\lambda_1 t}) \int_0^\infty |\varpi_{1,1}(y)|dy + e^{-(N\lambda_1 + \underline{\beta}_2)t} \int_0^\infty |\varpi_{2,1}(y)|dy \\
& + \theta e^{-(N\lambda_1 + \underline{\beta}_2)t} \int_0^\infty |\varpi_{1,1}(y)| \int_y^{y+t} e^{-\theta(l-y) - \int_y^l (\beta_1(v) - \beta_2(v))dv} dldy + e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{2,2}(y)|dy \\
& + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,2}(y)|e^{\theta y} \int_y^{y+t} e^{-\theta l - \int_y^l (\beta_1(v) - \beta_2(v))dv} dldy \\
& + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,1}(y)|e^{\theta y} \int_y^{y+t} e^{-\theta l - \int_y^l (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1(l-y)})dldy \\
& + N\lambda_1 e^{-\underline{\beta}_2 t} (1 - e^{-N\lambda_1 t}) \int_0^\infty |\varpi_{2,1}(y)|dy + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,1}(y)|e^{(N\lambda_1 + \theta)y} \\
& \times \int_y^{y+t} e^{-\theta l - \int_y^l (\beta_1(v) - \beta_2(v))dv} |e^{-N\lambda_1 l} - e^{-N\lambda_1(y+t)}|dldy \\
& = e^{-(N\lambda_1 + \lambda_2)t} |\varpi_{1,0}| + e^{-(\theta + \underline{\beta}_1)t} \|\varpi_{1,1}\|_{L^1[0,\infty)} + e^{-(\theta + \underline{\beta}_1)t} \|\varpi_{1,2}\|_{L^1[0,\infty)} \\
& + [(1 - N\lambda_1)e^{-(N\lambda_1 + \underline{\beta}_2)t} + N\lambda_1 e^{-\underline{\beta}_2 t}] \|\varpi_{2,1}\|_{L^1[0,\infty)} + e^{-\underline{\beta}_2 t} \|\varpi_{2,2}\|_{L^1[0,\infty)} \\
& + \theta e^{-(N\lambda_1 + \underline{\beta}_2)t} \int_0^\infty |\varpi_{1,1}(y)| \int_y^{y+t} e^{-\theta(l-y) - \int_y^l (\beta_1(v) - \beta_2(v))dv} dldy \\
& + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,1}(y)| \int_y^{y+t} e^{-\theta(l-y) - \int_y^l (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1(l-y)})dldy \\
& + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,1}(y)| \int_y^{y+t} e^{-\theta(l-y) - \int_y^l (\beta_1(v) - \beta_2(v))dv} |e^{-N\lambda_1(l-y)} - e^{-N\lambda_1 t}|dldy \\
& + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,2}(y)| \int_y^{y+t} e^{-\theta(l-y) - \int_y^l (\beta_1(v) - \beta_2(v))dv} dldy \\
& \leq e^{-(N\lambda_1 + \lambda_2)t} |\varpi_{1,0}| + e^{-\underline{\beta}_1 t} \|\varpi_{1,1}\|_{L^1[0,\infty)} + e^{-\underline{\beta}_1 t} \|\varpi_{1,2}\|_{L^1[0,\infty)} + e^{-\underline{\beta}_2 t} \|\varpi_{2,1}\|_{L^1[0,\infty)} \\
& + e^{-\underline{\beta}_2 t} \|\varpi_{2,2}\|_{L^1[0,\infty)} + 3\theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,1}(y)| \int_y^{y+t} e^{-\theta(l-y)} e^{-\int_y^l (\beta_1(v) - \beta_2(v))dv} dldy \\
& + \theta e^{-\underline{\beta}_2 t} \int_0^\infty |\varpi_{1,2}(y)| \int_y^{y+t} e^{-\theta(l-y) - \int_y^l (\beta_1(v) - \beta_2(v))dv} dldy \\
& \leq e^{-(N\lambda_1 + \lambda_2)t} |\varpi_{1,0}| + e^{-\underline{\beta}_1 t} \{\|\varpi_{1,1}\|_{L^1[0,\infty)} + \|\varpi_{1,2}\|_{L^1[0,\infty)}\} + e^{-\underline{\beta}_2 t} \{\|\varpi_{2,1}\|_{L^1[0,\infty)}
\end{aligned}$$

$$+ \|\varpi_{2,2}\|_{L^1[0,\infty)}\} + e^{-(\underline{\beta}_1 + \underline{\beta}_2 - \overline{\beta}_2)t} \left\{ \frac{3\theta}{\overline{\beta}_2 - \underline{\beta}_1} \|\varpi_{1,1}\|_{L^1[0,\infty)} + \frac{\theta}{\overline{\beta}_2 - \underline{\beta}_1} \|\varpi_{1,2}\|_{L^1[0,\infty)} \right\}. \quad (3.36)$$

Equation (3.36) shows that the conclusion of this theorem is true. $\square$

By applying Theorems 3.1 and 3.2, we arrive at the following conclusion: if $\underline{\beta}_1 + \underline{\beta}_2 > \overline{\beta}_2$, then it can be deduced that

$$\|\mathcal{E}(t) - \mathcal{O}(t)\| = \|\mathcal{Q}(t)\| \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Consequently, based on the definition of a quasi-compact C_0 -semigroup and Theorem 3.1, we can draw the subsequent conclusion.

Theorem 3.3. If $\beta_i(x)$ ($i = 1, 2$) is Lipschitz continuous and satisfies the conditions of Theorem 3.2, then $\mathcal{E}(t)$ is a quasi-compact C_0 -semigroup on X .

Clearly, $\mathfrak{K} : X \rightarrow \mathbb{R}^5$ is a bounded linear operator, and thus $\mathfrak{K}$ is a compact operator on X , and we can use the perturbation theory of the quasi-compact C_0 -semigroups and Theorem 3.3 to obtain the following result.

Corollary 3.1. If $\beta_i(x)$ ($i = 1, 2$) is Lipschitz continuous and fulfils the conditions of Theorem 3.2, then $\mathcal{T}(t)$ is a quasi-compact C_0 -semigroup on X .

Lemma 3.1. If $\beta_i(x)$ ($i = 1, 2$) is Lipschitz continuous and fulfils the conditions of Theorem 3.2, then 0 is an eigenvalue of $\mathfrak{A} + \mathfrak{G} + \mathfrak{K}$ with a geometric multiplicity of 1.

Proof. Taking the equation $(\mathfrak{A} + \mathfrak{G} + \mathfrak{K})H = 0$ into account, it is equivalent to the following:

$$(N\lambda_1 + \lambda_2)H_1^0 = \int_0^\infty H_1^1(x)\beta_1(x)dx + \int_0^\infty H_2^1(x)\beta_2(x)dx, \quad (3.37a)$$

$$\frac{dH_1^1(x)}{dx} = -(N\lambda_1 + \beta_1(x) + \theta)H_1^1(x), \quad (3.37b)$$

$$\frac{dH_1^2(x)}{dx} = -(\beta_1(x) + \theta)H_1^2(x) + N\lambda_1 H_1^1(x), \quad (3.37c)$$

$$\frac{dH_2^1(x)}{dx} = -(N\lambda_1 + \beta_2(x))H_2^1(x) + \theta H_1^1(x), \quad (3.37d)$$

$$\frac{dH_2^2(x)}{dx} = -\beta_2(x)H_2^2(x) + \theta H_1^2(x) + N\lambda_1 H_2^1(x), \quad (3.37e)$$

$$H_1^1(0) = (N\lambda_1 + \lambda_2)H_1^0 + \int_0^\infty H_1^1(x)\beta_1(x)dx, \quad (3.37f)$$

$$H_1^2(0) = 0, \quad (3.37g)$$

$$H_2^1(0) = \int_0^\infty H_2^1(x)\beta_2(x)dx, \quad (3.37h)$$

$$H_2^2(0) = 0. \quad (3.37i)$$

Solving Eqs (3.37b) and (3.37f), we get the following:

$$H_1^1(x) = a_{1,1}e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v)dv}, \quad (3.38a)$$

$$H_1^2(x) = a_{1,2}e^{-\theta x - \int_0^x \beta_1(v)dv} + N\lambda_1 e^{-\theta x - \int_0^x \beta_1(v)dv} \int_0^x H_1^1(s)e^{\theta s + \int_0^s \beta_1(v)dv} ds, \quad (3.38b)$$

$$H_2^1(x) = a_{2,1}e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} + \theta e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} \int_0^x H_1^1(\varsigma) e^{N\lambda_1 \varsigma + \int_0^\varsigma \beta_2(v)dv} d\varsigma, \quad (3.38c)$$

$$H_2^2(x) = a_{2,2}e^{-\int_0^x \beta_2(v)dv} + e^{-\int_0^x \beta_2(v)dv} \int_0^x [\theta H_1^2(\varsigma) + N\lambda_1 H_2^1(\varsigma)] e^{\int_0^\varsigma \beta_2(v)dv} d\varsigma. \quad (3.38d)$$

From Eqs (3.37g) and (3.38b), we have the following:

$$a_{1,2} = H_1^2(0) = 0 \Rightarrow H_1^2(x) = N\lambda_1 e^{-\theta x - \int_0^x \beta_1(v)dv} \int_0^x H_1^1(\varsigma) e^{\theta \varsigma + \int_0^\varsigma \beta_1(v)dv} d\varsigma. \quad (3.39)$$

Substituting Eq (3.38a) into the above equation yields the following:

$$\begin{aligned} H_1^2(x) &= a_{1,1} N\lambda_1 e^{-\theta x - \int_0^x \beta_1(v)dv} \int_0^x e^{-(N\lambda_1 + \theta)\varsigma - \int_0^\varsigma \beta_1(v)dv + \theta \varsigma + \int_0^\varsigma \beta_1(v)dv} d\varsigma \\ &= a_{1,1} N\lambda_1 e^{-\theta x - \int_0^x \beta_1(v)dv} \int_0^x e^{-N\lambda_1 \varsigma} d\varsigma \\ &= a_{1,1} e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}). \end{aligned} \quad (3.40)$$

Then, from Eqs (3.37f), (3.38a), and (3.40), we get the following:

$$\begin{aligned} a_{1,1} &= H_1^1(0) = (N\lambda_1 + \lambda_2)H_1^0 + \int_0^\infty H_1^2(x)\beta_1(x)dx \\ &= (N\lambda_1 + \lambda_2)H_1^0 + a_{1,1} \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx \\ \Rightarrow a_{1,1} (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx) &= (N\lambda_1 + \lambda_2)H_1^0 \\ \Rightarrow a_{1,1} &= \frac{(N\lambda_1 + \lambda_2)H_1^0}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \\ \Rightarrow |a_{1,1}| &< \infty. \end{aligned} \quad (3.41)$$

Substituting Eq (3.40) into Eqs (3.38a) and (3.40), we can deduce the following:

$$H_1^1(x) = \frac{(N\lambda_1 + \lambda_2) e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v)dv} H_1^0}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx}, \quad (3.42)$$

$$H_1^2(x) = \frac{(N\lambda_1 + \lambda_2) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) H_1^0}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx}, \quad (3.43)$$

respectively. From Eqs (3.37i) and (3.38d), we obtain the following:

$$\begin{aligned} a_{2,2} &= H_2^2(0) = 0 \\ \Rightarrow H_2^2(x) &= e^{-\int_0^x \beta_2(v)dv} \int_0^x [\theta H_1^2(\varsigma) + N\lambda_1 H_2^1(\varsigma)] e^{\int_0^\varsigma \beta_2(v)dv} d\varsigma \end{aligned}$$

$$= \theta e^{-\int_0^x \beta_2(v)dv} \int_0^x H_1^2(\varsigma) e^{\int_0^\varsigma \beta_2(v)dv} d\varsigma + N\lambda_1 e^{-\int_0^x \beta_2(v)dv} \int_0^x H_2^1(\varsigma) e^{\int_0^\varsigma \beta_2(v)dv} d\varsigma. \quad (3.44)$$

From Eqs (3.38c) and (3.42)–(3.44), we have the following:

$$\begin{aligned} H_2^2(x) = & \theta(N\lambda_1 + \lambda_2) H_1^0 \frac{\int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1\varsigma}) d\varsigma e^{-\int_0^x \beta_2(v)dv}}{1 - \int_0^\infty \beta_1(\varsigma) e^{-\theta\varsigma - \int_0^\varsigma \beta_1(v)dv} (1 - e^{-N\lambda_1\varsigma}) d\varsigma} \\ & + a_{2,1} e^{-\int_0^x \beta_2(v)dv} (1 - e^{-N\lambda_1 x}) \\ & + \theta N\lambda_1 (N\lambda_1 + \lambda_2) H_1^0 \frac{\int_0^x e^{-N\lambda_1\varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma e^{-\int_0^x \beta_2(v)dv}}{1 - \int_0^\infty \beta_1(\varsigma) e^{-\theta\varsigma - \int_0^\varsigma \beta_1(v)dv} (1 - e^{-N\lambda_1\varsigma}) d\varsigma}. \end{aligned} \quad (3.45)$$

From Eqs (3.37h), (3.38c), and (3.45), we gain the following:

$$\begin{aligned} a_{2,1} = & H_2^1(0) = \int_0^\infty H_2^2(x) \beta_2(x) dx, \\ = & \theta(N\lambda_1 + \lambda_2) H_1^0 \frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1\varsigma}) d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \\ & + a_{2,1} (1 - \int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx) + \theta N\lambda_1 (N\lambda_1 + \lambda_2) H_1^0 \\ & \times \frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1\varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx}, \\ \Rightarrow a_{2,1} = & \frac{1}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx} \left[\theta(N\lambda_1 + \lambda_2) H_1^0 \right. \\ & \times \frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1\varsigma}) d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} + \theta N\lambda_1 H_1^0 \\ & \left. \times (N\lambda_1 + \lambda_2) \frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1\varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \right], \quad (3.46) \\ \Rightarrow |a_{2,1}| < \infty. \end{aligned}$$

Substituting Eq (3.45) into Eqs (3.38c) and (3.45), and combining with Eq (3.42), we get the following:

$$\begin{aligned} H_2^1(x) = & \frac{\theta(N\lambda_1 + \lambda_2) H_1^0 e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv}}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx} \\ & \times \left[\frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1\varsigma}) d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \right. \\ & \left. + N\lambda_1 \frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1\varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \right] \end{aligned}$$

$$+ \theta(N\lambda_1 + \lambda_2)H_1^0 \frac{e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} d\varsigma}{1 - \int_0^\infty \beta_1(\varsigma) e^{-\theta\varsigma - \int_0^\varsigma \beta_1(v)dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma}, \quad (3.47)$$

$$\begin{aligned} H_2^2(x) = & \theta(N\lambda_1 + \lambda_2)H_1^0 \frac{e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma}{1 - \int_0^\infty \beta_1(\varsigma) e^{-\theta\varsigma - \int_0^\varsigma \beta_1(v)dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma} \\ & + \theta N\lambda_1(N\lambda_1 + \lambda_2)H_1^0 \frac{e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1 \varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma}{1 - \int_0^\infty \beta_1(\varsigma) e^{-\theta\varsigma - \int_0^\varsigma \beta_1(v)dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma} \\ & + \frac{\theta(N\lambda_1 + \lambda_2)H_1^0 e^{-\int_0^x \beta_2(v)dv} (1 - e^{-N\lambda_1 x})}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx} \\ & \times \left[\frac{\int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \right. \\ & \left. + \frac{N\lambda_1 \int_0^\infty \beta_2(x) e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1 \varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} \right]. \quad (3.48) \end{aligned}$$

Therefore, from Eqs (3.42), (3.43), (3.47), and (3.48), we can estimate the following:

$$\begin{aligned} \|H\| = & |H_1^0| + \|H_1^1\|_{L^1[0,\infty)} + \|H_1^2\|_{L^1[0,\infty)} + \|H_2^1\|_{L^1[0,\infty)} + \|H_2^2\|_{L^1[0,\infty)} \\ \leq & |H_1^0| + \frac{(N\lambda_1 + \lambda_2) \int_0^\infty e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v)dv} dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} |H_1^0| \\ & + \frac{(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} |H_1^0| \\ & + \theta(N\lambda_1 + \lambda_2) |H_1^0| \int_0^\infty e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx \\ & \times \frac{\int_0^\infty \beta_2(y) e^{-\int_0^y \beta_2(v)dv} \int_0^y e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma dy}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx)} \\ & + \frac{\int_0^\infty \beta_2(y) e^{-\int_0^y \beta_2(v)dv} \int_0^y e^{-N\lambda_1 \varsigma} \int_0^\varsigma e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\varsigma dy}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx)} \\ & \times \theta N\lambda_1(N\lambda_1 + \lambda_2) |H_1^0| \int_0^\infty e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx \\ & + \frac{\theta(N\lambda_1 + \lambda_2) \int_0^\infty e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} |H_1^0| \\ & + \frac{\theta(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\varsigma - \int_0^\varsigma (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1 \varsigma}) d\varsigma dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} |H_1^0| \end{aligned}$$

$$\begin{aligned}
& + \frac{\theta N\lambda_1(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1 s} \int_0^s e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\zeta dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx} |H_1^0| \\
& + \theta(N\lambda_1 + \lambda_2) |H_1^0| \int_0^\infty e^{-\int_0^x \beta_2(v)dv} \\
& \times (1 - e^{-N\lambda_1 x}) \int_0^\infty \beta_2(y) e^{-\int_0^y \beta_2(v)dv} \int_0^y e^{-\theta\zeta - \int_0^\zeta (\beta_1(v) - \beta_2(v))dv} (1 - e^{-N\lambda_1 \zeta}) d\zeta dy dx \\
& \times \frac{1}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx)} \\
& + \frac{1}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} (1 - e^{-N\lambda_1 x}) dx)} \\
& \times \theta N\lambda_1(N\lambda_1 + \lambda_2) |H_1^0| \int_0^\infty e^{-\int_0^x \beta_2(v)dv} (1 - e^{-N\lambda_1 x}) \int_0^\infty \beta_2(y) e^{-\int_0^y \beta_2(v)dv} \\
& \times \int_0^y e^{-N\lambda_1 \zeta} \int_0^\zeta e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\zeta dy dx \\
& \leq \left(1 + \frac{(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\theta x - \int_0^x \beta_1(v)dv} dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx} \right. \\
& + \frac{2\theta(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\zeta - \int_0^\zeta (\beta_1(v) - \beta_2(v))dv} d\zeta dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx} \\
& + \frac{\theta(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} dx \int_0^\infty \beta_2(y) e^{-\int_0^y \beta_2(v)dv} \int_0^y e^{-\theta\zeta - \int_0^\zeta (\beta_1(v) - \beta_2(v))dv} d\zeta dy}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx)} \\
& + \frac{\theta N\lambda_1(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-N\lambda_1 \zeta} \int_0^\zeta e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\zeta dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx} \\
& + \frac{\int_0^\infty e^{-\int_0^x \beta_2(v)dv} dx \int_0^\infty \beta_2(y) e^{-\int_0^y \beta_2(v)dv} \int_0^y e^{-N\lambda_1 \zeta} \int_0^\zeta e^{-\theta\eta - \int_0^\eta (\beta_1(v) - \beta_2(v))dv} d\eta d\zeta dy}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx)} \\
& \left. \times \theta N\lambda_1(N\lambda_1 + \lambda_2) \right) |H_1^0| \\
& \leq \left(1 + \frac{(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\theta x - \int_0^x \beta_1(v)dv} dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx} \right. \\
& + \frac{3\theta(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} \int_0^x e^{-\theta\zeta - \int_0^\zeta (\beta_1(v) - \beta_2(v))dv} d\zeta dx}{1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx} \\
& + \frac{2\theta(N\lambda_1 + \lambda_2) \int_0^\infty e^{-\int_0^x \beta_2(v)dv} dx \int_0^\infty e^{-\theta y - \int_0^y \beta_1(v)dv} dy}{\int_0^\infty \beta_2(x) e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv} dx (1 - \int_0^\infty \beta_1(x) e^{-\theta x - \int_0^x \beta_1(v)dv} dx)} \left. \right) |H_1^0| \\
& < \infty.
\end{aligned}$$

The above equation shows that 0 is an eigenvalue of $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ with a geometric multiplicity of 1,

as follows from Eqs (3.42), (3.43), (3.47), and (3.48). $\square$

Lemma 3.2. The adjoint operator of $\mathfrak{A} + \mathfrak{G} + \mathfrak{R}$ is $(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})^*$, it has the following form:

$$(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})^* G^* = (\mathfrak{H} + \mathfrak{F}) G^*, \quad \forall G^* \in D(\mathfrak{H}),$$

where

$$\begin{aligned} \mathfrak{H} G^*(x) &= B_1 G^*(x) + B_2 G^*(0), \quad \mathfrak{F} G^*(x) = B_3 G^*(0) + B_4 G^*(x), \\ D(\mathfrak{H}) &= \left\{ G^* \in X^* \mid \frac{dG_i^{j*}(x)}{dx} \text{ exist and } G_i^{j*}(\infty) = \ell, i, j = 1, 2 \right\}, \quad D(\mathfrak{F}) = X^*, \\ B_1 &= \begin{pmatrix} -(N\lambda_1 + \lambda_2) & & & & \\ & \frac{d}{dx} - b & & & \\ & & \frac{d}{dx} - (\beta_1(x) + \theta) & & \\ & & & \frac{d}{dx} - (\beta_2(x) + N\lambda_1) & \\ & & & & \frac{d}{dx} - \beta_2(x) \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 0 & & & & \\ \beta_1(x) & 0 & & & \\ 0 & \beta_1(x) & 0 & & \\ \beta_2(x) & 0 & 0 & 0 & \\ 0 & 0 & 0 & \beta_2(x) & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & N\lambda_1 + \lambda_2 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & 0 & \\ 0 & & & & 0 \end{pmatrix}, \\ B_4 &= \begin{pmatrix} 0 & 0 & 0 & & \\ & 0 & N\lambda_1 & \theta & \\ & & 0 & 0 & \theta \\ & & & 0 & N\lambda_1 \\ & & & & 0 \end{pmatrix}, \end{aligned}$$

in which $b = N\lambda_1 + \beta_1(x) + \theta$, ℓ in $D(\mathfrak{H})$ is a constant unrelated to i and j .

Lemma 3.3. 0 is the eigenvalue of $(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})^*$ with a geometric multiplicity of 1.

Proof. Consider $(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})^* G^* = (\mathfrak{H} + \mathfrak{F})^* G^* = 0$, i.e.,

$$-(N\lambda_1 + \lambda_2) G_1^{0*} + (N\lambda_1 + \lambda_2) G_1^{1*}(0) = 0, \quad (3.49a)$$

$$\frac{dG_1^{1*}(x)}{dx} = (N\lambda_1 + \beta_1(x) + \theta) G_1^{1*}(x) - \beta_1(x) G_1^{0*} - N\lambda_1 G_1^{2*}(x) - \theta G_2^{1*}(x), \quad (3.49b)$$

$$\frac{dG_1^{2*}(x)}{dx} = (\beta_1(x) + \theta) G_1^{2*}(x) - \beta_1(x) G_1^{1*}(0) - \theta G_2^{2*}(x), \quad (3.49c)$$

$$\frac{dG_2^{1*}(x)}{dx} = (N\lambda_1 + \beta_2(x)) G_2^{1*}(x) - \beta_2(x) G_1^{0*} - N\lambda_1 G_2^{2*}(x), \quad (3.49d)$$

$$\frac{dG_2^{2*}(x)}{dx} = \beta_2(x) G_2^{2*}(x) - \beta_2(x) G_2^{1*}(0), \quad (3.49e)$$

$$G_1^{1*}(\infty) = G_1^{2*}(\infty) = G_2^{1*}(\infty) = G_2^{2*}(\infty) = \ell. \quad (3.49f)$$

Solving Eqs (3.49a)–(3.49f) leads to the following:

$$G_1^{1*}(0) = G_1^{0*}, \quad (3.50a)$$

$$G_1^{1*}(x) = b_{1,1} e^{(N\lambda_1 + \theta)x + \int_0^x \beta_1(v)dv} - e^{(N\lambda_1 + \theta)x + \int_0^x \beta_1(v)dv} \int_0^x [\beta_1(v)G_1^{0*} + N\lambda_1 G_1^{2*}(v) + \theta G_2^{1*}(v)] e^{-(N\lambda_1 + \theta)v - \int_0^v \beta_1(s)ds} dv, \quad (3.50b)$$

$$G_1^{2*}(x) = b_{1,2} e^{\theta x + \int_0^x \beta_1(v)dv} - e^{\theta x + \int_0^x \beta_1(v)dv} \int_0^x [\beta_1(v)G_1^{1*}(0) + \theta G_2^{2*}(v)] e^{-\theta v - \int_0^v \beta_1(s)ds} dv, \quad (3.50c)$$

$$G_2^{1*}(x) = b_{2,1} e^{N\lambda_1 x + \int_0^x \beta_2(v)dv} - e^{N\lambda_1 x + \int_0^x \beta_2(v)dv} \int_0^x [\beta_2(v)G_1^{0*} + N\lambda_1 G_2^{2*}(v)] \times e^{-N\lambda_1 v - \int_0^v \beta_2(s)ds} dv, \quad (3.50d)$$

$$G_2^{2*}(x) = b_{2,2} e^{\int_0^x \beta_2(v)dv} - e^{\int_0^x \beta_2(v)dv} \int_0^x \beta_2(v)G_2^{1*}(0) e^{-\int_0^v \beta_2(s)ds} dv. \quad (3.50e)$$

Simultaneously multiply both sides of Eq (3.50b) by $e^{-(N\lambda_1 + \theta)x - \int_0^x \beta_1(v)dv}$, simultaneously multiply both sides of Eq (3.50c) by $e^{-\theta x - \int_0^x \beta_1(v)dv}$, simultaneously multiply both sides of Eq (3.50d) by $e^{-N\lambda_1 x - \int_0^x \beta_2(v)dv}$, simultaneously multiply both sides of Eq (3.50e) by $e^{-\int_0^x \beta_2(v)dv}$; then, by taking the limit at $x \rightarrow \infty$ and using Eq (3.49f), we get the following:

$$b_{1,1} = \int_0^\infty [\beta_1(v)G_1^{0*} + N\lambda_1 G_1^{2*}(v) + \theta G_2^{1*}(v)] e^{-(N\lambda_1 + \theta)v - \int_0^v \beta_1(s)ds} dv, \quad (3.51a)$$

$$b_{1,2} = \int_0^\infty [\beta_1(v)G_1^{1*}(0) + \theta G_2^{2*}(v)] e^{-\theta v - \int_0^v \beta_1(s)ds} dv, \quad (3.51b)$$

$$b_{2,1} = \int_0^\infty [\beta_2(v)G_1^{0*} + N\lambda_1 G_2^{2*}(v)] e^{-N\lambda_1 v - \int_0^v \beta_2(s)ds} dv, \quad (3.51c)$$

$$b_{2,2} = \int_0^\infty \beta_2(v)G_2^{1*}(0) e^{-\int_0^v \beta_2(s)ds} dv. \quad (3.51d)$$

By substituting Eqs (3.51a)–(3.51d) into Eqs (3.50b)–(3.50e), respectively, and noting that $\beta_i(x) \geq 0$, $\int_0^\infty \beta_i(x)dx = \infty$, $i = 1, 2$, we can deduce the following:

$$G_1^{1*}(x) = e^{(N\lambda_1 + \theta)x + \int_0^x \beta_1(v)dv} \int_x^\infty [\beta_1(v)G_1^{0*} + N\lambda_1 G_1^{2*}(v) + \theta G_2^{1*}(v)] e^{-(N\lambda_1 + \theta)v - \int_0^v \beta_1(s)ds} dv, \quad (3.52a)$$

$$G_1^{2*}(x) = e^{\theta x + \int_0^x \beta_1(v)dv} \int_x^\infty [\beta_1(v)G_1^{1*}(0) + \theta G_2^{2*}(v)] e^{-\theta v - \int_0^v \beta_1(s)ds} dv, \quad (3.52b)$$

$$G_2^{1*}(x) = e^{N\lambda_1 x + \int_0^x \beta_2(v)dv} \int_x^\infty [\beta_2(v)G_1^{0*} + N\lambda_1 G_2^{2*}(v)] e^{-N\lambda_1 v - \int_0^v \beta_2(s)ds} dv, \quad (3.52c)$$

$$G_2^{2*}(x) = e^{\int_0^x \beta_2(v)dv} \int_x^\infty \beta_2(v)G_2^{1*}(0) e^{-\int_0^v \beta_2(s)ds} dv \\ = e^{\int_0^x \beta_2(v)dv} G_2^{1*}(0) (-e^{-\int_0^v \beta_2(s)ds} \Big|_x^\infty) = G_2^{1*}(0). \quad (3.52d)$$

Let $x = 0$; by substituting Eq (3.52d) into Eq (3.52c), we get the following:

$$G_2^{1*}(0) = \int_0^\infty [\beta_2(v)G_1^{0*} + N\lambda_1 G_2^{2*}(0)] e^{-N\lambda_1 v - \int_0^v \beta_2(s)ds} dv$$

$$\begin{aligned}
\Rightarrow G_2^{1*}(0) &= \frac{G_1^{0*} \int_0^\infty \beta_2(v) e^{-N\lambda_1 v - \int_0^v \beta_2(s) ds} dv}{1 - N\lambda_1 \int_0^\infty e^{-N\lambda_1 v - \int_0^v \beta_2(s) ds} dv} = \frac{G_1^{0*} \int_0^\infty \beta_2(v) e^{-N\lambda_1 v - \int_0^v \beta_2(s) ds} dv}{\int_0^\infty \beta_2(v) e^{-N\lambda_1 v - \int_0^v \beta_2(s) ds} dv} = G_1^{0*} \\
\Rightarrow G_2^{2*}(x) &= G_1^{0*}.
\end{aligned} \tag{3.53}$$

Then, by substituting Eq (3.53) into Eq (3.52b) and (3.52c), and using Eq (3.50a), we have the following:

$$G_1^{2*}(x) = e^{\theta x + \int_0^x \beta_1(v) dv} \int_x^\infty [\theta G_1^{0*} + \beta_1(v) G_1^{0*}] e^{-\theta v - \int_0^v \beta_1(s) ds} dv = G_1^{0*}, \tag{3.54}$$

$$G_2^{1*}(x) = e^{N\lambda_1 x + \int_0^x \beta_2(v) dv} \int_x^\infty [\beta_2(v) G_1^{0*} + N\lambda_1 G_1^{0*}] e^{-N\lambda_1 v - \int_0^v \beta_2(s) ds} dv = G_1^{0*}. \tag{3.55}$$

Furthermore, by substituting Eqs (3.54) and (3.55) into Eq (3.52a), we get the following:

$$G_1^{1*}(x) = e^{(N\lambda_1 + \theta)x + \int_0^x \beta_1(v) dv} G_1^{0*} \int_x^\infty [\beta_1(v) + N\lambda_1 + \theta] e^{-(N\lambda_1 + \theta)v - \int_0^v \beta_1(s) ds} dv = G_1^{0*}. \tag{3.56}$$

By combining Eqs (3.50a) and (3.53)–(3.56), we have the following:

$$|||G^*||| = \max \{ |G_1^{0*}|, \|G_1^{1*}\|_{L^\infty[0,\infty)}, \|G_1^{2*}\|_{L^\infty[0,\infty)}, \|G_2^{1*}\|_{L^\infty[0,\infty)}, \|G_2^{2*}\|_{L^\infty[0,\infty)} \} = |G_1^{0*}| < \infty. \tag{3.57}$$

Thus, 0 is an eigenvalue of $(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})^*$ with a geometric multiplicity of 1, by Eqs (3.50a) and (3.53)–(3.57). $\square$

Based on Lemmas 3.1 and 3.3, Theorem 2.1, Corollary 3.1, and [20, Theorem 1.90], we now state the main results of this section.

Theorem 3.4. If $\beta_i(x)$ ($i = 1, 2$) is Lipschitz continuous and satisfies the conditions of Theorem 3.2, then there exists a positive projection operator $\mathbb{P}$ and a positive constant $\varrho > 0, \bar{N} \geq 0$ to make the following:

$$\|\mathcal{T}(t) - \mathbb{P}\| \leq \bar{N} e^{-\varrho t},$$

in which $\mathbb{P} = \frac{1}{2\pi i} \int_\Gamma (\ell I - \mathfrak{A} - \mathfrak{G} - \mathfrak{R})^{-1} d\ell$, Γ is a circle with an adequately tiny radius of 0 as its centre.

Theorem 2.1, Lemma 3.1, Corollary 3.1, and [20, Theorem 1.88] yield the following:

$$\sigma(\mathfrak{A} + \mathfrak{G} + \mathfrak{R}) \cap i\mathbb{R} = \{0\}.$$

Thus, every nonzero point on the imaginary axis lies in $\rho(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})$, and by [20, Theorem 1.96], we obtain the following:

Theorem 3.5. In the case where $\beta_i(x)$ ($i = 1, 2$) is Lipschitz continuous and satisfies the conditions of Theorem 3.2, then the T-DS $H(t, x)$ of System (1.11) strongly converges to its S-SS $H(x)$, i.e.,

$$\lim_{t \rightarrow \infty} H(t, x) = \langle H(0), G^* \rangle H(x),$$

where $H(x)$ is the eigenvector belonging to 0 (by Lemma 3.1), $H(0)$ is the original value of System (2.1), and G^* is the eigenvector of $(\mathfrak{A} + \mathfrak{G} + \mathfrak{R})^*$ belonging to 0 (by Lemma 3.3).

4. Numerical examples

In this section, an example of a power system is taken to further validate the availability of the paper's results. Assume that the power system consists of three identical generators connected in series, jointly supplying power to the grid. In order to enhance the system's reliability and its ability to respond to sudden failures, the system is also equipped with a standby generator, which can quickly replace any faulty generator to minimize the power outage time. These generators are typically installed in power plants or substations and are connected to the grid via transmission lines. When two generators fail, the power system becomes inoperable. The remaining two generators do not experience any further faults. In turn, the Repairman immediately repairs the faulty units in turn. Meanwhile, it is assumed that the repairman follows the MWV strategy.

Let λ_1, λ_2 represent the failure rates of the working generator and standby generator, respectively. β_1 and β_2 represent the repair rates of the generators during the WV and normal busy period, respectively. The repairman's vacation time is exponentially distributed with parameter θ . Assume that the initial parameter values are $\lambda_i = 0.02i$ ($i = 1, 2$), $\beta_i = 0.2i$ ($i = 1, 2$), $\theta = 0.04$, and $N = 3$. Then, the impacts of the variations in each parameter on the system's T-DRI are discussed.

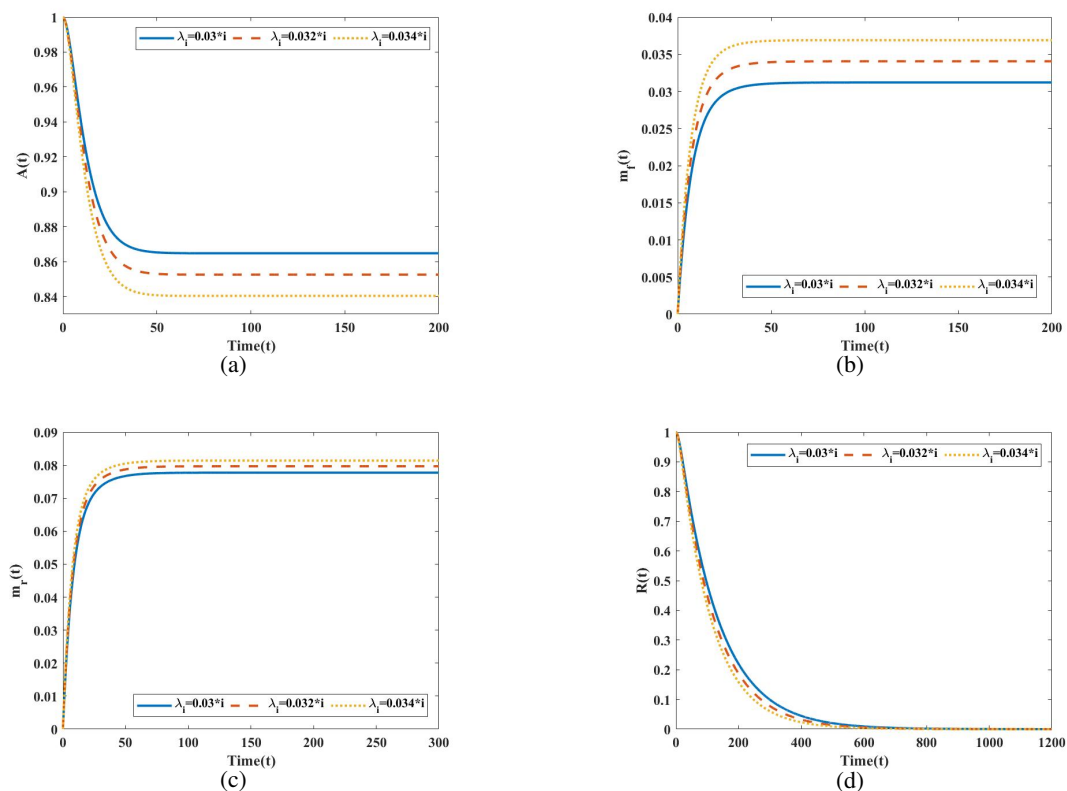


Figure 1. T-DRI for various values of λ_i ($i = 1, 2$). (a) Availability for various λ_i ($i = 1, 2$); (b) Failure frequency for various λ_i ($i = 1, 2$); (c) Renewal frequency for various λ_i ($i = 1, 2$); (d) Reliability for various λ_i ($i = 1, 2$).

As shown in Figure 1, the (a) and (d) indicate that as the parameter λ_i ($i = 1, 2$) increases, the

system's availability and reliability actually decrease. However, in Figure 1 (b) and (c), the system's failure frequency and renewal frequency increase with the rise of λ_i ($i = 1, 2$).

It is obvious from Figure 2 that as β_i ($i = 1, 2$) increases, the range of increase in availability (Figure 2(a)) decreases. Compared to $\beta_i = 0.3i$, $0.4i$ ($i = 1, 2$), when $\beta_i = 0.2i$ ($i = 1, 2$), the decrease in availability (Figure 2(a)) is more pronounced. Figure 3 illustrates that the range of increase in the system's availability (Figure 3(a)), renewal frequency (Figure 3(c)) and reliability (Figure 3(d)), as well as the range of decrease in failure frequency (Figure 3(b)) all increase as θ increases. Moreover, it can be observed that θ has little impact on the system's reliability.

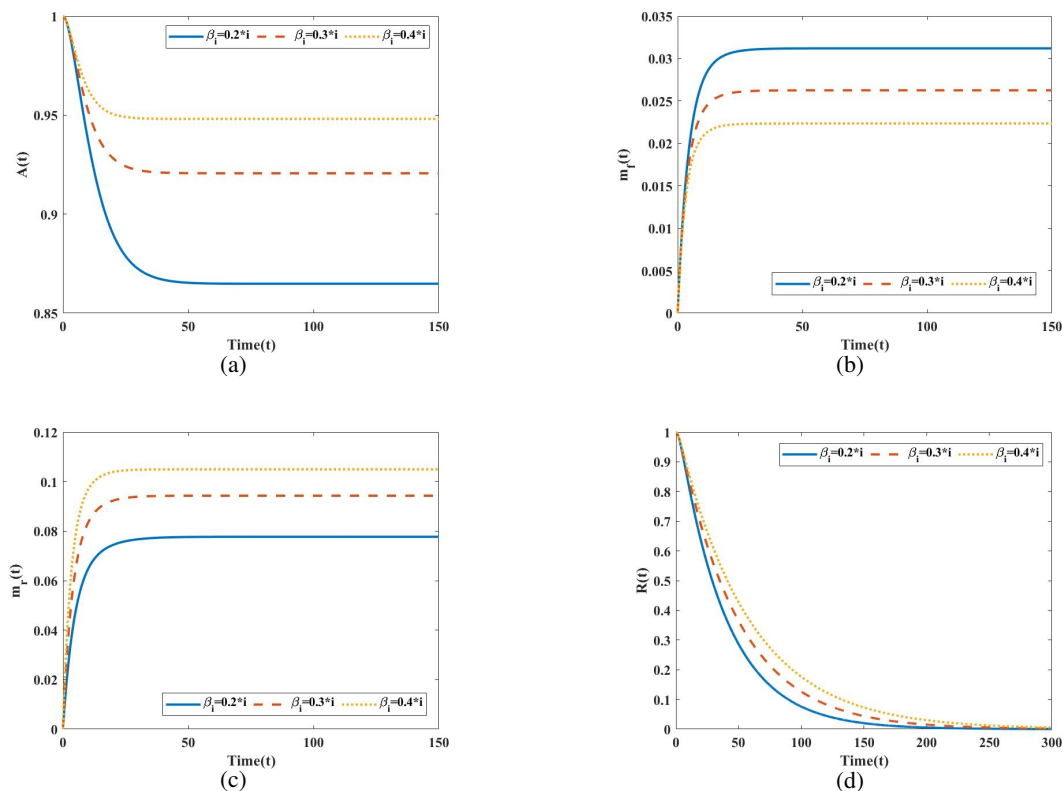


Figure 2. T-DRI for various values of β_i ($i = 1, 2$). (a) Availability for various β_i ($i = 1, 2$); (b) Failure frequency for various β_i ($i = 1, 2$); (c) Renewal frequency for various β_i ($i = 1, 2$); (d) Reliability for various β_i ($i = 1, 2$).

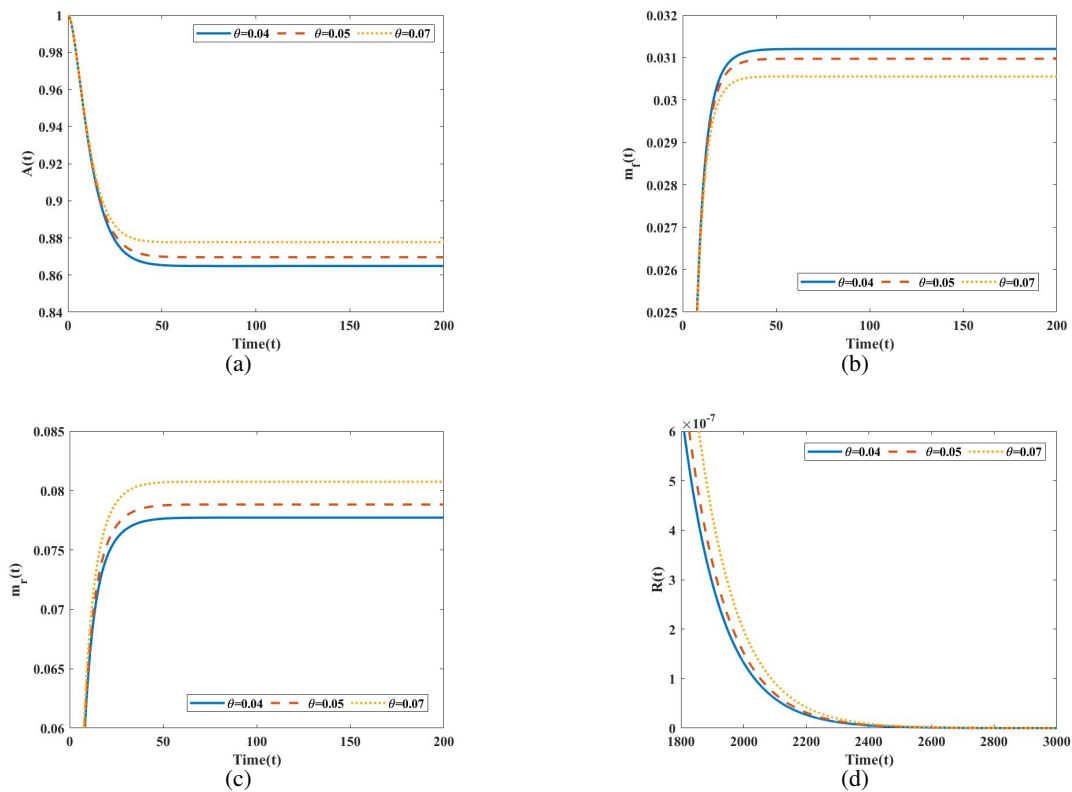


Figure 3. T-DRI for various values of θ . (a) Availability for various θ ; (b) Failure frequency for various θ ; (c) Renewal frequency for various θ ; (d) Reliability for various θ .

Figure 4 (a) and (b) show that the availability and reliability of the system decrease as N increases. On the contrary, the failure frequency and renewal frequency of the system in Figure 4 (b) and (d) increase with N , and the curve relating to the reliability reaches its fixed value earlier at $N=5$. The mean time to first failure (MTTFF) is defined as the expected time elapsed before the first system failure occurs. As shown in Figure 5, the MTTFF decreases with the increase of λ_1 .

More importantly, the numerical results in this Section are intended to verify the theoretical convergence established in Section 3, namely that the T-DS converges to its S-SS. As shown in Figures 1–4, the failure frequency and renewal frequency increase with time and eventually converge to fixed values, whereas the availability and reliability decrease over time and also approach their respective steady-state values after a sufficiently long period. In particular, the reliability tends to 0 as $t \rightarrow \infty$ under different parameter settings. These observations not only confirm that the T-DRI converge to their steady-state counterparts but also reveal the sensitivity of these transient indices to the system parameters.

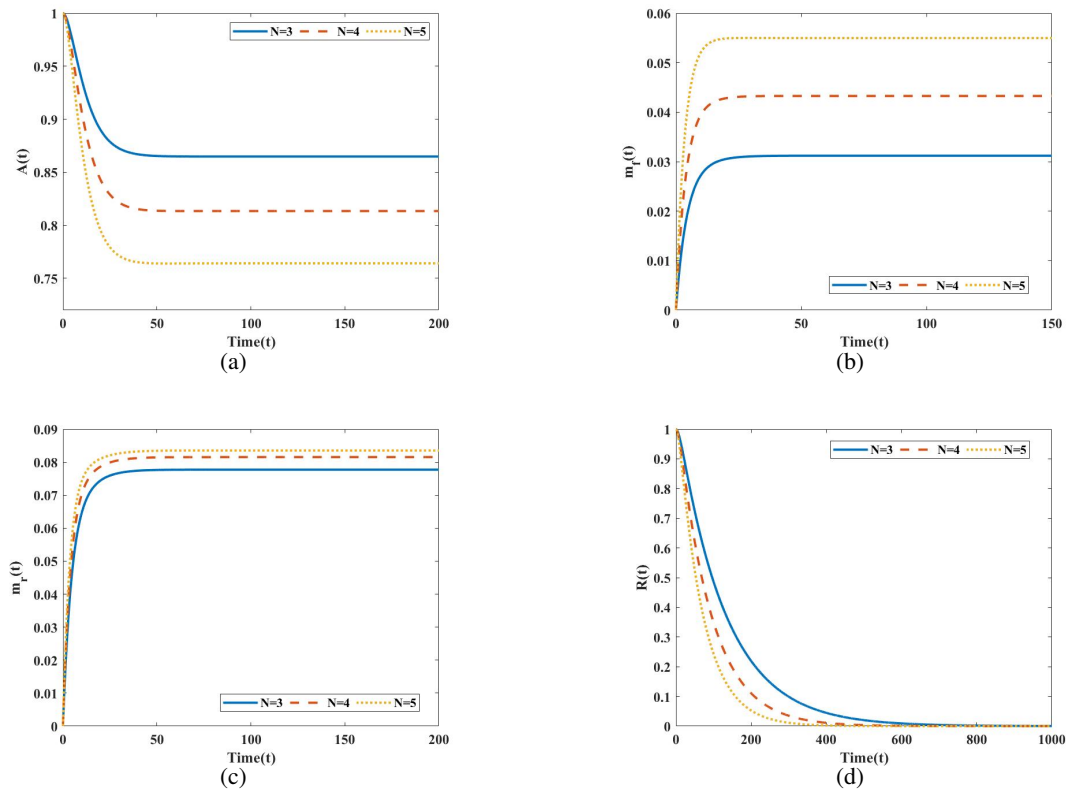


Figure 4. T-DRI for various values of N . (a) Availability for various N ; (b) Failure frequency for various N ; (c) Renewal frequency for various N ; (d) Reliability for various N .

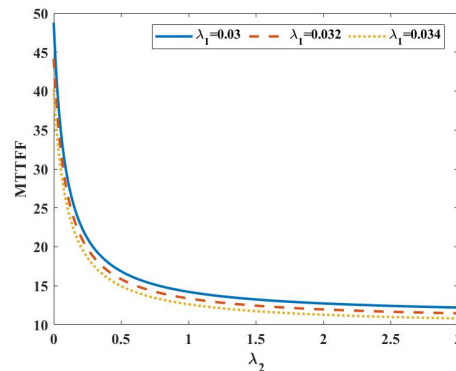


Figure 5. MTTF with different λ_1 .

5. Conclusions

In this paper, we have performed a comprehensive dynamic analysis of the Gnedenko repairable system with MWVs, based on the corrected model established in our previous work [19]. Using the theory of C_0 -semigroups, we proved the well-posedness of the system and demonstrated the strong convergence of the T-DS to its S-SS. Our results provide a rigorous mathematical foundation for the

system's dynamic behavior.

It is also worth noting that the models studied by Yunus and Haji [29] and Hoxur et al. [30] are special cases of the one discussed here. As mentioned in [19], their conclusion that no S-SS exist was based on an incorrect model formulation. In contrast, this study proves the existence of S-SS, thereby correcting their result. This highlights the importance of accurate modeling in reliability analyses and the value of the framework introduced in [19].

In addition, through specific numerical examples, we verified that the T-DRI of the system gradually converge to their steady-state values and discussed, in detail, the influence of each parameter on the reliability indices of the system. As future research directions, one may investigate whether the T-DS of the model exponentially converges to its S-SS, and explore the asymptotic expansion of the T-DS in greater depth.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Conceptualization, E.K. and R.E.; methodology, E.K. and R.E.; validation, E.K. and R.E.; writing-original draft preparation, E.K.; writing-review and editing, E.K. and R.E. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare there is no conflict of interest.

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