



Research article

Some properties of mixed-type hypergeometric Bernoulli/Euler r -Dowling polynomials and their zero distribution patterns

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Abstract: In this paper, we introduce a new class of polynomials called the mixed-type hypergeometric Bernoulli and Euler r -Dowling polynomials. These polynomials extend the classical Bernoulli and Euler polynomials, together with their hypergeometric (level- m) versions, while incorporating the combinatorial structure of the r -Dowling framework within a single generating function. Using exponential generating functions, we derive their main properties such as explicit formulas, recurrence relations, differential identities, and integral representations. In addition, we examine the zeros of these polynomials numerically and graphically. The results show that the zeros follow symmetric and structured patterns in the complex plane and display regular behavior as the degree increases. Overall, this study provides a natural extension of classical polynomial families and offers new insights for further research in approximation theory and related areas.

Keywords: r -Dowling polynomials; hypergeometric Bernoulli and Euler polynomials; mixed-type polynomials

1. Introduction

Bernoulli and Euler polynomials constitute two of the most fundamental classes of special polynomials in mathematics, with wide-ranging applications in number theory, approximation theory,

numerical analysis, and mathematical physics. Their importance is particularly evident in the classical Euler–Maclaurin summation formula, numerical quadrature rules, and the study of special values of zeta-type functions [1, 2]. Moreover, these polynomials play a central role in the expansion of analytic functions and in the representation of periodic functions, as well as in various problems arising in applied mathematics [3]. Owing to their Appell-type structure, they satisfy elegant differential, integral, and convolution identities, making them powerful tools in both theoretical investigations and computational applications.

In recent decades, considerable attention has been devoted to the generalization of Bernoulli and Euler polynomials. Among these developments, hypergeometric extensions and level-based generalizations have proven particularly fruitful. In this direction, hypergeometric Bernoulli and Euler polynomials of level m have been introduced and systematically studied, preserving the Appell property while enriching their structural and functional behavior [4, 5]. These generalized families have found applications in the construction of operational matrices, generalized Euler–Maclaurin-type quadrature formulas, and symbolic techniques for solving differential and integral equations [6, 7]. Furthermore, they have been connected to special functions and zeta-type identities, revealing deeper analytic structures [8].

On the other hand, Dowling polynomials and their r -extensions arise naturally in combinatorics due to their close relationships with partition theory, r -Stirling numbers, and Dowling lattices [9, 10]. These polynomials provide a unifying combinatorial framework that extends classical Bell polynomials and enables refined counting techniques. In particular, the r -Dowling structure has been extensively studied in connection with Whitney numbers and generalized partition lattices, leading to significant advances in algebraic combinatorics [11].

Motivated by these developments, the present paper introduces and investigates a new class of special polynomials, referred to as the *mixed-type hypergeometric Bernoulli and Euler r -Dowling polynomials*. These polynomials are constructed by combining hypergeometric Bernoulli–Euler polynomials of level m with the exponential generating structure associated with the r -Dowling framework.

What gives this construction its content is that its two sides are governed by separate parameters. The level m controls the analytic side, since the factor $z^m/(e^z - \sum_{h=0}^{m-1} z^h/h!)$ measures how strongly the exponential is corrected near the origin and, with it, how deep the underlying Appell structure runs. The combinatorial side is carried by μ and r : The exponent $\Phi_\mu(z) = z + (e^{\mu z} - 1)/\mu$ deforms the ordinary shift e^{xz} by an exponential term whose expansion is governed by set partitions, so that μ moves the family between a purely additive regime, recovered as $\mu \rightarrow 0$ where $\Phi_\mu(z)$ reduces to $2z$, and a genuinely combinatorial one, while the shift r places the whole family inside the r -Whitney lattice and connects it with the r -Stirling and r -Bell numbers. It is this joint action of the three parameters that we set out to study, since none of them acting alone reproduces the interplay between hypergeometric and Dowling-type structures that the new families display.

The main contributions of this work can be summarized as follows:

- We introduce a new family of polynomials defined by suitable exponential generating functions.
- We study their basic properties, including their algebraic structure, recurrence relations, differential identities, and integral formulas.
- We establish a convolution decomposition that links each family to the hypergeometric Bernoulli and Euler polynomials and to Dowling-type polynomials with explicit Stirling-number coefficients, thereby making their combinatorial structure explicit.
- We show that these polynomials can be used as a basis for polynomial spaces and describe how

they relate to standard monomials.

- We examine the behavior of their zeros by numerical and graphical means, and complement these experiments with a theoretical discussion that explains their symmetry and relates their limiting distribution to that of the classical Bernoulli and Euler polynomials.

The hypergeometric Bernoulli and Euler polynomials of level m are Appell sequences, generated by $A(z)e^{xz}$ with a linear exponent; replacing e^{xz} by $e^{x\Phi_\mu(z)}$ makes the exponent nonlinear, so the present families are no longer Appell but of Sheffer type, the derivative in x being a binomial convolution against the coefficients of Φ_μ (Section 3) rather than a single-term shift. It is instructive to record how the known families are recovered from this construction. Putting $x = 0$ gives the hypergeometric Bernoulli and Euler polynomials evaluated at r ; letting $\mu \rightarrow 0$ with $r = 0$ yields the level- m polynomials $B_n^{[m-1]}(2x)$ and $E_n^{[m-1]}(2x)$; and taking in addition $m = 1$ returns the classical Bernoulli and Euler polynomials $B_n(2x)$ and $E_n(2x)$. In this sense the new families extend, rather than merely restate, the classical and level- m polynomials.

The novelty of the present work lies in simultaneously integrating two previously distinct constructions. Earlier studies on hypergeometric Bernoulli and Euler polynomials focused primarily on their Appell structure, operational properties, differential relations, and applications [4, 5], while studies on r -Dowling polynomials were developed mainly from the combinatorial perspective of r -Whitney numbers, set partitions, and Dowling lattices [9–11]. To the best of our knowledge, this particular combination and the resulting structural formulas have not previously been considered in the literature.

Of the contributions listed above, the mixed generating functions and their less routine consequences, above all the decomposition into a hypergeometric part and a Dowling part, the Sheffer-type differential identity that accompanies it, and the theoretical discussion of the symmetry and limiting behavior of the zeros, are what we regard as genuinely new. The basis property, the integral representations, and the elementary summation identities, on the other hand, are the natural analogues of well-known facts for classical Appell families, and we include them because the later sections rely on them rather than as claims of novelty.

Both of the directions outlined above have continued to develop in recent years, and it is worth describing the current state of the art in a little more detail. On the analytic side, the level- m hypergeometric Bernoulli and Euler polynomials have been absorbed into broader Apostol- and Hermite-type frameworks: Degenerate unified Bernoulli–Euler Hermite polynomials of Apostol type have been introduced [12], the monomiality principle has been applied to extensions of Apostol-type Hermite polynomials [13], and q -operational methods have been developed for Gould–Hopper-type families [14], alongside continued work on operational matrices and Euler–Maclaurin quadrature built from level- m polynomials [5, 7]. On the combinatorial side, the theory of r -Whitney and r -Dowling numbers has been placed on a firm footing, in particular through the comprehensive study of r -Dowling polynomials by Gyimesi and Nyul [11], which rests on the r -Stirling numbers of Broder [9] and the r -Whitney numbers of Cheon and Jung [10].

Furthermore, recent developments have further demonstrated the effectiveness of generating-function and monomiality-based methods in the construction of generalized polynomial families. In particular, these techniques have been applied to Bell-based Apostol–Bernoulli-type polynomials, multivariate q -Hermite-based Appell polynomials, and multivariable Hermite extensions associated with Apostol-type Frobenius–Genocchi polynomials; see [15, 16].

Recent developments in numerical analysis have emphasized the effectiveness of kernel-based and polynomially augmented approximation methods for differential equations and nonlinear systems. Radial basis function neural networks have been used to construct efficient approximations for time-dependent diffusion equations [17], while local RBF–FD discretizations with polynomial augmentation have shown favorable accuracy and stability properties for partial integro-differential equations arising in option-pricing models [18]. High-order compact finite-difference formulas constructed from smooth or piecewise-smooth radial basis functions and polynomials have also demonstrated the importance of the choice of basis for conditioning, truncation error, and convergence order [19]. Related high-order iterative procedures illustrate the role of efficiently computable polynomial structures in the numerical solution of nonlinear systems [20]. These developments motivate the study of polynomial families possessing explicit recurrence, differential, and basis properties, since such structures may support the construction of operational matrices, collocation approximations, and structured nonlinear solvers.

The organization of the paper is as follows. In Section 2, we recall the fundamental definitions and properties of hypergeometric Bernoulli and Euler polynomials of level m , together with the necessary background on r -Dowling structures. In Section 3, we introduce the mixed-type hypergeometric Bernoulli and Euler r -Dowling polynomials via their generating functions and establish their main structural properties, including their degree and basis characterizations, a convolution decomposition, differential identities, recurrence relations, and integral representations. Section 4 is devoted to the analysis of their zeros, supported by numerical and graphical illustrations. Finally, we present concluding remarks and discuss possible directions for future research.

2. Notations and basic facts

Throughout this paper, let $\mathbb{N}, \mathbb{N}_0, \mathbb{R}^+$, and $\mathbb{C}$ denote, respectively, the set of all natural numbers, the set of all nonnegative integers, the set of all positive real numbers, and the set of all complex numbers, and $\mathbb{P}_n$ denote the linear space of polynomials with real coefficients and a degree less than or equal to n .

2.1. Hypergeometric Bernoulli polynomials

For a fixed $m \in \mathbb{N}$, the hypergeometric Bernoulli polynomials are defined by means of the following generating function [3–5, 21]:

$$\frac{z^m e^{xz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < 2\pi \quad (2.1)$$

and the hypergeometric Bernoulli numbers are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$ for all $n \geq 0$. The hypergeometric Bernoulli polynomials also are called generalized Bernoulli polynomials of level m [5, 8]. It is clear that if $m = 1$ in (2.1), then we obtain the definition of the classical Bernoulli polynomials $B_n(x)$ and classical Bernoulli numbers, respectively, i.e., $B_n(x) = B_n^{[0]}(x)$ and $B_n = B_n^{[0]}$, respectively, for all $n \geq 0$. The first four hypergeometric Bernoulli polynomials are as follows:

$$\begin{aligned}
B_0^{[m-1]}(x) &= m!, \\
B_1^{[m-1]}(x) &= m! \left(x - \frac{1}{m+1} \right), \\
B_2^{[m-1]}(x) &= m! \left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)} \right), \\
B_3^{[m-1]}(x) &= m! \left(x^3 - \frac{3}{m+1}x^2 + \frac{6}{(m+1)^2(m+2)}x + \frac{6(m-1)}{(m+1)^3(m+2)(m+3)} \right).
\end{aligned}$$

The following results show some properties satisfied by the hypergeometric Bernoulli polynomials and the associated numbers (cf., [3]).

Proposition 2.1. For a fixed $m \in \mathbb{N}$, let $\{B_n^{[m-1]}(x)\}_{n \geq 0}$ be the hypergeometric Bernoulli polynomials. Then the following statements hold:

a) *Summation formula.* For every $n \geq 0$,

$$B_n^{[m-1]}(x) = \sum_{k=0}^n \binom{n}{k} B_k^{[m-1]} x^{n-k}.$$

b) *Differential relations (Appell polynomial sequences).* For $n, j \geq 0$ with $0 \leq j \leq n$, we have

$$[B_n^{[m-1]}(x)]^{(j)} = \frac{n!}{(n-j)!} B_{n-j}^{[m-1]}(x).$$

c) *Inversion formula.* [3, Eq (2.6)] For every $n \geq 0$,

$$x^n = \sum_{k=0}^n \binom{n}{k} \frac{k!}{(m+k)!} B_{n-k}^{[m-1]}(x).$$

d) *Recurrence relation.* [3, Lemma 3.2] For every $n \geq 1$,

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1} \right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x).$$

e) *Integral formulas.*

$$\begin{aligned}
\int_{x_0}^{x_1} B_n^{[m-1]}(x) dx &= \frac{1}{n+1} [B_{n+1}^{[m-1]}(x_1) - B_{n+1}^{[m-1]}(x_0)] \\
&= \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} B_k^{[m-1]} ((x_1)^{n-k+1} - (x_0)^{n-k+1}).
\end{aligned}$$

$$B_n^{[m-1]}(x) = n \int_0^x B_{n-1}^{[m-1]}(t) dt + B_n^{[m-1]}.$$

f) [3, Theorem 3.1] *Differential equation.* For every $n \geq 1$, the polynomial $B_n^{[m-1]}(x)$ satisfies the following differential equation:

$$\frac{B_n^{[m-1]}}{n!}y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!}y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!}y'' + (m-1)!\left(\frac{1}{m+1} - x\right)y' + n(m-1)!y = 0.$$

Proposition 2.2. For a fixed $m \in \mathbb{N}$ and each $n \geq 0$, the set $\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \dots, B_n^{[m-1]}(x)\}$ is a basis for $\mathbb{P}_n$, i.e., (cf., [5]),

$$\mathbb{P}_n = \mathfrak{B}_n^{[m-1]} = \text{span}\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \dots, B_n^{[m-1]}(x)\}.$$

2.2. Hypergeometric Euler polynomials

For a fixed $m \in \mathbb{N}$, the hypergeometric Euler polynomials are defined by means of the following generating function (see, for instance, [7]):

$$\frac{2^m e^{xt}}{e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} E_n^{[m-1]}(x) \frac{t^n}{n!}, \quad |t| < \pi, \quad (2.2)$$

and the hypergeometric Euler numbers are defined by $E_n^{[m-1]} := E_n^{[m-1]}(0)$, for all $n \geq 0$. The hypergeometric Euler polynomials also are called generalized Euler polynomials of level m [8]. Again, if $m = 1$ in Eq (2.2), then we recover the definition of the classical Euler polynomials $E_n(x)$, and classical Euler numbers, respectively, i.e., $E_n(x) = E_n^{[0]}(x)$, and $\mathcal{E}_n = 2^n E_n^{[0]}(\frac{1}{2}) = 2^n E_n(\frac{1}{2})$, respectively, for all $n \geq 0$. For instance, if $m = 3$, then the first six hypergeometric Euler polynomials are:

$$\begin{aligned} E_0^{[2]}(x) &= 4, \\ E_1^{[2]}(x) &= 4(x-1), \\ E_2^{[2]}(x) &= 4(x-1)^2, \\ E_3^{[2]}(x) &= 4x^3 - 12x^2 + 12x - 2, \\ E_4^{[2]}(x) &= 4x^4 - 16x^3 + 24x^2 - 8x - 10, \\ E_5^{[2]}(x) &= 4x^5 - 20x^4 + 40x^3 - 20x^2 - 50x + 58. \end{aligned}$$

The following results collect some properties satisfied by the hypergeometric Euler polynomials and the associated numbers (cf., [3, 7]).

Proposition 2.3. For a fixed $m \in \mathbb{N}$, let $\{E_n^{[m-1]}(x)\}_{n \geq 0}$ be the hypergeometric Euler polynomials. Then the following statements hold:

a) *Summation formulas.* For every $n \geq 0$,

$$E_n^{[m-1]}(x+y) = \sum_{k=0}^n \binom{n}{k} y^k E_{n-k}^{[m-1]}(x) = \sum_{k=0}^n \binom{n}{k} E_k^{[m-1]}(y) x^{n-k}.$$

In particular,

$$E_n^{[m-1]}(x) = \sum_{k=0}^n \binom{n}{k} E_k^{[m-1]} x^{n-k}.$$

b) *Differential relations (Appell polynomial sequences).* For $n, j \geq 0$ with $0 \leq j \leq n$, we have

$$\left[E_n^{[m-1]}(x) \right]^{(j)} = \frac{n!}{(n-j)!} E_{n-j}^{[m-1]}(x).$$

c) *Inversion formula.* For every $n \geq 0$,

$$2^m x^n = \sum_{k=0}^n \binom{n}{k} (1 + a_{k,m}) E_{n-k}^{[m-1]}(x),$$

where

$$a_{k,m} = \begin{cases} 1, & 0 \leq k < m, \\ 0, & k \geq m. \end{cases}$$

Consequently, the set $\{E_0^{[m-1]}(x), E_1^{[m-1]}(x), \dots, E_n^{[m-1]}(x)\}$ is a basis for $\mathbb{P}_n$ (cf., [7]).

d) *Integral formulas.*

$$\begin{aligned} \int_{x_0}^{x_1} E_n^{[m-1]}(x) dx &= \frac{1}{n+1} [E_{n+1}^{[m-1]}(x_1) - E_{n+1}^{[m-1]}(x_0)] \\ &= \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} E_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}). \\ E_n^{[m-1]}(x) &= n \int_0^x E_{n-1}^{[m-1]}(t) dt + E_n^{[m-1]}. \end{aligned}$$

e) *Recurrence relation.* For any $m \geq 2$ and $n \geq 0$, the following recurrence relation holds:

$$\begin{aligned} E_{n+1}^{[m-1]}(x) &= (2xE_n^{[m-2]} - E_n^{[m-1]}) \\ &\quad + \frac{1}{2^{m-1}} \sum_{k=1}^n \left[\binom{n}{k} (2xE_{n-k}^{[m-2]} - E_{n-k}^{[m-1]}) \right. \\ &\quad \left. - 2 \binom{n}{k-1} E_{n-k+1}^{[m-2]} \right] E_k^{[m-1]}(x). \end{aligned}$$

f) *Differential equation.* For any $m \geq 2$ and $n \geq 1$, the hypergeometric Euler polynomials $y = E_n^{[m-1]}(x)$ satisfy the differential equation

$$\sum_{k=m}^n \frac{E_{k-m}^{[m-1]}}{2^m (m-1)! (k-m)!} y^{(k)} + (x-1)y' - ny = 0,$$

where $E_j^{[m-1]} = E_j^{[m-1]}(0)$ denote the hypergeometric Euler numbers, and the sum is understood to be empty whenever $n < m$.

3. Mixed-type hypergeometric Bernoulli/Euler r -Dowling polynomials

This section is devoted to the introduction of two new classes of special polynomials, namely the mixed-type hypergeometric Bernoulli r -Dowling polynomials and the mixed-type hypergeometric Euler r -Dowling polynomials. These families are constructed by means of appropriately designed generating

functions, which combine features of hypergeometric-type structures with r -Dowling polynomial frameworks. Motivated by the rich interplay between generating functions and special polynomial sequences, we employ these representations as a fundamental tool to systematically derive various structural properties. In particular, we establish degree characterizations, differential identities, recurrence relations, and integral representations associated with these polynomials. To proceed with this development, we first introduce the following definitions.

Definition 3.1. *The mixed-type hypergeometric Bernoulli r -Dowling polynomials ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ and the mixed-type hypergeometric Euler r -Dowling polynomials ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ are defined through the generating functions*

$$\left(\frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}} \right) e^{x\Phi_\mu(z)} = \sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!}, \quad (3.1)$$

and

$$\left(\frac{2^m e^{rz}}{e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}} \right) e^{x\Phi_\mu(z)} = \sum_{n=0}^{\infty} {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!}, \quad (3.2)$$

where the function $\Phi_\mu(z)$ is given by

$$\Phi_\mu(z) := z + \frac{e^{\mu z} - 1}{\mu}. \quad (3.3)$$

It is worth noting that the function $\Phi_\mu(z)$ plays a central role in encoding the hypergeometric-type deformation within the generating structure. Consequently, the above definitions unify and extend several known polynomial families by incorporating both exponential and combinatorial components in a single framework. In the subsequent results, we will exploit these generating functions to derive explicit identities and structural relations for the introduced polynomial families. To determine the algebraic nature of the newly introduced polynomial families, we next examine their degree and leading coefficients with respect to the variable x . The following result shows that both sequences indeed consist of polynomials of exact degree n , and it also identifies their dominant terms explicitly.

Theorem 3.1. *For each $n \geq 0$, the mixed-type hypergeometric Bernoulli r -Dowling polynomial ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ and the mixed-type hypergeometric Euler r -Dowling polynomial ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ are polynomials in x of degree exactly n . More precisely,*

$${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) = m! 2^n x^n + P_{n-1}(x),$$

and

$${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) = 2^{m+n-1} x^n + Q_{n-1}(x),$$

where $P_{n-1}(x)$ and $Q_{n-1}(x)$ are polynomials of degree at most $n-1$.

Proof. We first consider the mixed-type hypergeometric Bernoulli r -Dowling polynomials. From Eq (3.1), we write

$$\sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = A_B(z) e^{x\Phi_\mu(z)},$$

where

$$A_B(z) = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}},$$

which is independent of x . Since

$$e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!} = \sum_{h=m}^{\infty} \frac{z^h}{h!} = z^m \left(\frac{1}{m!} + \frac{z}{(m+1)!} + \cdots \right),$$

it follows that

$$A_B(z) = \frac{e^{rz}}{\frac{1}{m!} + \frac{z}{(m+1)!} + \cdots},$$

and hence

$$A_B(0) = m!.$$

On the other hand, from Eq (3.3),

$$\Phi_\mu(z) = z + \frac{e^{\mu z} - 1}{\mu} = 2z + O(z^2).$$

Therefore,

$$e^{x\Phi_\mu(z)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Phi_\mu(z)^k,$$

and the lowest-degree term of $\Phi_\mu(z)^n$ is $(2z)^n = 2^n z^n$. Consequently, the coefficient of $z^n/n!$ in

$$A_B(z) e^{x\Phi_\mu(z)}$$

is a polynomial in x of degree exactly n , whose leading term is

$$A_B(0) 2^n x^n = m! 2^n x^n.$$

Thus,

$${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) = m! 2^n x^n + P_{n-1}(x),$$

where $P_{n-1}(x)$ has degree at most $n-1$. Similarly, from Eq (3.2),

$$\sum_{n=0}^{\infty} {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = A_E(z) e^{x\Phi_\mu(z)},$$

where

$$A_E(z) = \frac{2^m e^{rz}}{e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}}.$$

Evaluating at $z = 0$, we obtain

$$A_E(0) = \frac{2^m}{1+1} = 2^{m-1}.$$

Using again the expansion

$$\Phi_\mu(z) = 2z + O(z^2),$$

the same argument shows that the coefficient of $z^n/n!$ in

$$A_E(z)e^{x\Phi_\mu(z)}$$

is a polynomial in x of degree exactly n , whose leading term is

$$A_E(0) 2^n x^n = 2^{m-1} 2^n x^n = 2^{m+n-1} x^n.$$

Hence,

$${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) = 2^{m+n-1} x^n + Q_{n-1}(x),$$

where $Q_{n-1}(x)$ has degree at most $n-1$. This completes the proof. $\square$

To further investigate the structural properties of the introduced polynomial families, we now show that they form bases for the space of polynomials of bounded degree. This result follows naturally from the triangular structure established in the previous theorem.

Theorem 3.2. *For each $n \geq 0$, the sets*

$$\left\{ {}_B\mathcal{D}_0^{[m-1]}(x; \mu, r), {}_B\mathcal{D}_1^{[m-1]}(x; \mu, r), \dots, {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \right\}$$

and

$$\left\{ {}_E\mathcal{D}_0^{[m-1]}(x; \mu, r), {}_E\mathcal{D}_1^{[m-1]}(x; \mu, r), \dots, {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) \right\}$$

form bases of $\mathbb{P}_n$.

Proof. We first consider the mixed-type hypergeometric Bernoulli r -Dowling polynomials. By the previous theorem, for each $k \in \{0, 1, \dots, n\}$, we have

$${}_B\mathcal{D}_k^{[m-1]}(x; \mu, r) = c_k x^k + P_{k-1}(x),$$

where $c_k \neq 0$ and $P_{k-1}(x) \in \mathbb{P}_{k-1}$. This shows that the sequence forms a triangular system with respect to the monomial basis $\{1, x, x^2, \dots, x^n\}$. Now, suppose that

$$\sum_{k=0}^n \lambda_k {}_B\mathcal{D}_k^{[m-1]}(x; \mu, r) = 0.$$

By comparing the coefficient of the highest-degree term x^n , we obtain

$$\lambda_n c_n = 0,$$

which implies $\lambda_n = 0$ since $c_n \neq 0$. Proceeding inductively and comparing the coefficients of $x^{n-1}, x^{n-2}, \dots, x^0$, we obtain

$$\lambda_{n-1} = \lambda_{n-2} = \dots = \lambda_0 = 0.$$

Hence, the set is linearly independent. Since it consists of $n+1$ polynomials in the $(n+1)$ -dimensional vector space $\mathbb{P}_n$, it follows that it forms a basis of $\mathbb{P}_n$. The proof for the mixed-type hypergeometric Euler r -Dowling polynomials is entirely analogous, and is therefore omitted. $\square$

Corollary 3.1. Let $p(x) \in \mathbb{P}_n$. Then there exist unique scalars $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_n$ such that

$$p(x) = \sum_{k=0}^n a_k {}_B\mathcal{D}_k^{[m-1]}(x; \mu, r), \quad p(x) = \sum_{k=0}^n b_k {}_E\mathcal{D}_k^{[m-1]}(x; \mu, r).$$

Proof. The result follows immediately from the fact that the corresponding polynomial families form bases of $\mathbb{P}_n$. $\square$

Theorem 3.3. For every $n \geq 0$,

$$\begin{aligned} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) &= \sum_{k=0}^n \binom{n}{k} B_k^{[m-1]}(r) \mathfrak{D}_{n-k}(x; \mu), \\ {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) &= \sum_{k=0}^n \binom{n}{k} E_k^{[m-1]}(r) \mathfrak{D}_{n-k}(x; \mu), \end{aligned} \quad (3.4)$$

where $B_k^{[m-1]}$ and $E_k^{[m-1]}$ are the hypergeometric Bernoulli and Euler polynomials of level m ; the Dowling-type polynomials $\mathfrak{D}_j(x; \mu)$, generated by

$$e^{x\Phi_\mu(z)} = \sum_{j \geq 0} \mathfrak{D}_j(x; \mu) \frac{z^j}{j!}, \quad (3.5)$$

are given explicitly by

$$\mathfrak{D}_j(x; \mu) = \sum_{i=0}^j \binom{j}{i} x^{j-i} \sum_{k=0}^i \mu^{i-k} S(i, k) x^k; \quad (3.6)$$

and $S(i, k)$ are the Stirling numbers of the second kind, determined by

$$\frac{(e^t - 1)^k}{k!} = \sum_{i \geq k} S(i, k) \frac{t^i}{i!} \quad (k \geq 0); \quad (3.7)$$

see [22].

Proof. By Eq (3.1),

$$\sum_{n \geq 0} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = A_B(z) e^{x\Phi_\mu(z)}, \quad A_B(z) = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} z^h/h!}.$$

Setting $x = r$ in Eq (2.1) identifies A_B with the generating function of the hypergeometric Bernoulli polynomials at r ,

$$A_B(z) = \sum_{k \geq 0} B_k^{[m-1]}(r) \frac{z^k}{k!}.$$

Multiplying this by Eq (3.5) and regrouping through the Cauchy product, we obtain

$$\begin{aligned} A_B(z) e^{x\Phi_\mu(z)} &= \left(\sum_{k \geq 0} B_k^{[m-1]}(r) \frac{z^k}{k!} \right) \left(\sum_{j \geq 0} \mathfrak{D}_j(x; \mu) \frac{z^j}{j!} \right) \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k} B_k^{[m-1]}(r) \mathfrak{D}_{n-k}(x; \mu) \right) \frac{z^n}{n!}, \end{aligned}$$

so that comparing the coefficient of $z^n/n!$ proves the first line of Eq (3.4). The second line follows in the same way from Eqs (3.2) and (2.2), which give $A_E(z) = \sum_{k \geq 0} E_k^{[m-1]}(r) z^k/k!$. For Eq (3.6), split the exponent by means of Eq (3.3),

$$e^{x\Phi_\mu(z)} = e^{xz} e^{\frac{x}{\mu}(e^{\mu z}-1)},$$

and expand the second factor using the Eq (3.7):

$$\begin{aligned} e^{\frac{x}{\mu}(e^{\mu z}-1)} &= \sum_{k \geq 0} \frac{x^k}{\mu^k} \frac{(e^{\mu z} - 1)^k}{k!} \\ &= \sum_{k \geq 0} \frac{x^k}{\mu^k} \sum_{i \geq k} S(i, k) \mu^i \frac{z^i}{i!} \\ &= \sum_{i \geq 0} \left(\sum_{k=0}^i \mu^{i-k} S(i, k) x^k \right) \frac{z^i}{i!}. \end{aligned}$$

A further Cauchy product with $e^{xz} = \sum_{l \geq 0} x^l z^l/l!$ then gives

$$e^{x\Phi_\mu(z)} = \sum_{j \geq 0} \left(\sum_{i=0}^j \binom{j}{i} x^{j-i} \sum_{k=0}^i \mu^{i-k} S(i, k) x^k \right) \frac{z^j}{j!},$$

and matching with Eq (3.5) yields Eq (3.6). $\square$

Remark 3.1. The two factors in Eq (3.4) keep the two sides of the construction apart: the hypergeometric Bernoulli or Euler polynomials at the shift r , and the Stirling-number part (3.6). As every term of $\mathfrak{D}_j$ with $j \geq 1$ vanishes at $x = 0$,

$${}_B\mathcal{D}_n^{[m-1]}(0; \mu, r) = B_n^{[m-1]}(r), \quad {}_E\mathcal{D}_n^{[m-1]}(0; \mu, r) = E_n^{[m-1]}(r).$$

We next derive a differential identity with respect to the variable x , which exhibits the Sheffer-type structure of the introduced polynomial families, governed by the auxiliary function $\Phi_\mu(z)$.

Theorem 3.4. The mixed-type hypergeometric Bernoulli r -Dowling polynomials satisfy

$$\frac{\partial}{\partial x} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) = \sum_{k=0}^{n-1} \binom{n}{k} \phi_{n-k} {}_B\mathcal{D}_k^{[m-1]}(x; \mu, r), \quad n \geq 1. \quad (3.8)$$

Similarly, the mixed-type hypergeometric Euler r -Dowling polynomials satisfy

$$\frac{\partial}{\partial x} {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) = \sum_{k=0}^{n-1} \binom{n}{k} \phi_{n-k} {}_E\mathcal{D}_k^{[m-1]}(x; \mu, r), \quad n \geq 1, \quad (3.9)$$

where

$$\Phi_\mu(z) = \sum_{j=1}^{\infty} \phi_j \frac{z^j}{j!}$$

is the auxiliary function defined in Eq (3.3).

Proof. We prove Eq (3.8); the second identity follows analogously. From Eq (3.1), we have

$$\sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = A_B(z) e^{x\Phi_\mu(z)},$$

where $A_B(z)$ is independent of x . Differentiating both sides with respect to x , we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = \Phi_\mu(z) \sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!}.$$

Next, expanding $\Phi_\mu(z)$ into its power series

$$\Phi_\mu(z) = z + \frac{e^{\mu z} - 1}{\mu} = \sum_{j=1}^{\infty} \phi_j \frac{z^j}{j!},$$

we identify the coefficients

$$\phi_1 = 2, \quad \phi_j = \mu^{j-1}, \quad j \geq 2.$$

Applying the Cauchy product of the two series, we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} \phi_{n-k} {}_B\mathcal{D}_k^{[m-1]}(x; \mu, r) \right) \frac{z^n}{n!}.$$

Finally, by comparing the coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the Eq (3.8). The proof of Eq (3.9) follows by the same argument using the Eq (3.2). $\square$

We now derive a recurrence relation for the introduced polynomial families by exploiting the structure of their generating functions. The result reveals that both families satisfy a Sheffer-type recurrence governed by the logarithmic derivative of the generating function and the auxiliary function $\Phi_\mu(z)$.

Theorem 3.5. *There exist scalar sequences $\{\alpha_j^{(B)}\}_{j \geq 0}$, $\{\alpha_j^{(E)}\}_{j \geq 0}$, and $\{\beta_j\}_{j \geq 0}$ such that*

$${}_B\mathcal{D}_{n+1}^{[m-1]}(x; \mu, r) = \sum_{k=0}^n \binom{n}{k} (\alpha_{n-k}^{(B)} + x\beta_{n-k}) {}_B\mathcal{D}_k^{[m-1]}(x; \mu, r), \quad n \geq 0, \quad (3.10)$$

and

$${}_E\mathcal{D}_{n+1}^{[m-1]}(x; \mu, r) = \sum_{k=0}^n \binom{n}{k} (\alpha_{n-k}^{(E)} + x\beta_{n-k}) {}_E\mathcal{D}_k^{[m-1]}(x; \mu, r), \quad n \geq 0. \quad (3.11)$$

Proof. We first establish the Eq (3.10). Let

$$G_B(x, z) := \sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = A_B(z) e^{x\Phi_\mu(z)},$$

where

$$A_B(z) = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}}.$$

Differentiating $G_B(x, z)$ with respect to z , and applying the product rule, we obtain

$$\frac{\partial}{\partial z} G_B(x, z) = \left(\frac{A'_B(z)}{A_B(z)} + x\Phi'_\mu(z) \right) G_B(x, z) = \left(\frac{d}{dz} \log A_B(z) + x\Phi'_\mu(z) \right) G_B(x, z).$$

We now introduce the sequences $\{\alpha_j^{(B)}\}_{j \geq 0}$ and $\{\beta_j\}_{j \geq 0}$ via the expansions

$$\frac{d}{dz} \log A_B(z) = \sum_{j=0}^{\infty} \alpha_j^{(B)} \frac{z^j}{j!}, \quad \Phi'_\mu(z) = \sum_{j=0}^{\infty} \beta_j \frac{z^j}{j!}.$$

Thus,

$$\frac{\partial}{\partial z} G_B(x, z) = \left(\sum_{j=0}^{\infty} \alpha_j^{(B)} \frac{z^j}{j!} + x \sum_{j=0}^{\infty} \beta_j \frac{z^j}{j!} \right) \left(\sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} \right).$$

On the other hand, differentiating termwise yields

$$\frac{\partial}{\partial z} G_B(x, z) = \sum_{n=0}^{\infty} {}_B\mathcal{D}_{n+1}^{[m-1]}(x; \mu, r) \frac{z^n}{n!}.$$

Applying the Cauchy product formula to the right-hand side, we obtain

$$\sum_{n=0}^{\infty} {}_B\mathcal{D}_{n+1}^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^n \binom{n}{k} (\alpha_{n-k}^{(B)} + x\beta_{n-k}) {}_B\mathcal{D}_k^{[m-1]}(x; \mu, r) \right] \frac{z^n}{n!}.$$

Finally, by comparing coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the Eq (3.10). The proof of Eq (3.11) follows in a completely analogous manner by replacing $A_B(z)$ with

$$A_E(z) = \frac{2^m e^{rz}}{e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}},$$

and defining the sequence $\{\alpha_j^{(E)}\}_{j \geq 0}$ accordingly. $\square$

We now derive integral representations for the introduced polynomial families by integrating their generating functions over the unit interval. These identities provide an alternative characterization and will be useful in further analytical investigations.

Remark 3.2. *The coefficients appearing in Theorem 3.5 can be computed explicitly. Indeed, write*

$$S_m(z) = \sum_{j=0}^{\infty} c_j z^j, \quad c_j = \frac{m!}{(m+j)!}, \quad c_0 = 1.$$

Since

$$A_B(z) = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}} = m! \frac{e^{rz}}{S_m(z)},$$

we obtain

$$\frac{d}{dz} \log A_B(z) = r - \frac{S'_m(z)}{S_m(z)} = \sum_{j=0}^{\infty} \alpha_j^{(B)} \frac{z^j}{j!}.$$

If

$$\frac{S'_m(z)}{S_m(z)} = \sum_{j=0}^{\infty} Q_j(c_1, \dots, c_{j+1}) z^j,$$

then

$$\alpha_j^{(B)} = r\delta_{j,0} - j!Q_j(c_1, \dots, c_{j+1}).$$

The first coefficients are therefore given in Table 1.

Table 1. Explicit coefficients in the recurrence relation of Theorem 3.5.

j	$\alpha_j^{(B)}$	β_j
0	$r - c_1$	2
1	$c_1^2 - 2c_2$	μ
2	$-2(c_1^3 - 3c_1c_2 + 3c_3)$	μ^2
3	$6(c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4)$	μ^3
4	$-24(c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5)$	μ^4
5	$-120(-c_1^6 + 6c_1^4c_2 - 9c_1^2c_2^2 - 6c_1^3c_3 + 2c_2^3 + 12c_1c_2c_3 + 6c_1^2c_4 - 3c_3^2 - 6c_2c_4 - 6c_1c_5 + 6c_6)$	μ^5

Theorem 3.6. The mixed-type hypergeometric Bernoulli r -Dowling polynomials and mixed-type hypergeometric Euler r -Dowling polynomials satisfy the integral generating relations

$$\sum_{n=0}^{\infty} \left(\int_0^1 {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) dx \right) \frac{z^n}{n!} = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}} \frac{\exp\left(z + \frac{e^{\mu z} - 1}{\mu}\right) - 1}{z + \frac{e^{\mu z} - 1}{\mu}}, \quad (3.12)$$

and

$$\sum_{n=0}^{\infty} \left(\int_0^1 {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) dx \right) \frac{z^n}{n!} = \frac{2^m e^{rz}}{e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}} \frac{\exp\left(z + \frac{e^{\mu z} - 1}{\mu}\right) - 1}{z + \frac{e^{\mu z} - 1}{\mu}}. \quad (3.13)$$

Proof. We first establish the Eq (3.12). From Eq (3.1), we have

$$\sum_{n=0}^{\infty} {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) \frac{z^n}{n!} = A_B(z) e^{x\Phi_{\mu}(z)},$$

where

$$A_B(z) = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}}, \quad \Phi_\mu(z) = z + \frac{e^{\mu z} - 1}{\mu}.$$

Integrating both sides over the interval $[0, 1]$ with respect to x , and exchanging the order of summation and integration, we obtain

$$\sum_{n=0}^{\infty} \left(\int_0^1 {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) dx \right) \frac{z^n}{n!} = A_B(z) \int_0^1 e^{x\Phi_\mu(z)} dx.$$

Evaluating the integral explicitly, we get

$$\int_0^1 e^{x\Phi_\mu(z)} dx = \frac{e^{\Phi_\mu(z)} - 1}{\Phi_\mu(z)}.$$

Substituting this expression into the previous identity yields Eq (3.12). The proof of Eq (3.13) follows in a completely analogous manner by starting from Eq (3.2) and replacing $A_B(z)$ with

$$A_E(z) = \frac{2^m e^{rz}}{e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}}.$$

□

We now express the integral means of the introduced polynomial families in terms of a convolution structure. This representation follows naturally from the generating function identities obtained in the previous theorem.

Theorem 3.7. *The mixed-type hypergeometric Bernoulli r -Dowling polynomials satisfy*

$$\int_0^1 {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) dx = \sum_{k=0}^n \binom{n}{k} {}_B\mathcal{D}_k^{[m-1]}(0; \mu, r) \gamma_{n-k}, \quad n \geq 0, \quad (3.14)$$

while the mixed-type hypergeometric Euler r -Dowling polynomials satisfy

$$\int_0^1 {}_E\mathcal{D}_n^{[m-1]}(x; \mu, r) dx = \sum_{k=0}^n \binom{n}{k} {}_E\mathcal{D}_k^{[m-1]}(0; \mu, r) \gamma_{n-k}, \quad n \geq 0, \quad (3.15)$$

where the sequence $\{\gamma_j\}_{j \geq 0}$ is defined by the generating function

$$\frac{\exp\left(z + \frac{e^{\mu z} - 1}{\mu}\right) - 1}{z + \frac{e^{\mu z} - 1}{\mu}} = \sum_{j=0}^{\infty} \gamma_j \frac{z^j}{j!}. \quad (3.16)$$

Proof. We prove the Eq (3.14). From Eq (3.12), we have

$$\sum_{n=0}^{\infty} \left(\int_0^1 {}_B\mathcal{D}_n^{[m-1]}(x; \mu, r) dx \right) \frac{z^n}{n!} = A_B(z) \frac{\exp(\Phi_\mu(z)) - 1}{\Phi_\mu(z)},$$

where

$$A_B(z) = \frac{z^m e^{rz}}{e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}}, \quad \Phi_\mu(z) = z + \frac{e^{\mu z} - 1}{\mu}.$$

By the Eq (3.16), the second factor admits the expansion

$$\frac{\exp(\Phi_\mu(z)) - 1}{\Phi_\mu(z)} = \sum_{j=0}^{\infty} \gamma_j \frac{z^j}{j!}.$$

On the other hand, from Eq (3.1) with $x = 0$, we obtain

$$\sum_{k=0}^{\infty} {}_B\mathcal{D}_k^{[m-1]}(0; \mu, r) \frac{z^k}{k!} = A_B(z).$$

Therefore, the right-hand side becomes the product of two exponential generating functions:

$$\left(\sum_{k=0}^{\infty} {}_B\mathcal{D}_k^{[m-1]}(0; \mu, r) \frac{z^k}{k!} \right) \left(\sum_{j=0}^{\infty} \gamma_j \frac{z^j}{j!} \right).$$

Applying the Cauchy product formula, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} {}_B\mathcal{D}_k^{[m-1]}(0; \mu, r) \gamma_{n-k} \right) \frac{z^n}{n!}.$$

Finally, by comparing coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the Eq (3.14). The proof of Eq (3.15) follows in the same way by using the Eq (3.13) and the corresponding generating function

$$A_E(z) = \frac{2^m e^{rz}}{e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}}.$$

□

4. Some values with graphical representations and zeros of the mixed-type hypergeometric Bernoulli/Euler r -Dowling polynomials

In this section we study the zeros of the mixed-type hypergeometric Bernoulli r -Dowling polynomials ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ and the mixed-type hypergeometric Euler r -Dowling polynomials ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$, both numerically and graphically. We fix the parameters $m = 3$, $\mu = 1$, and $r = 2$, and compute approximate solutions of the corresponding equations for $n = 1, 2, \dots, 10$. This keeps all three parameters active at once; other admissible values were found to give qualitatively the same picture, so the tables below are representative rather than tied to one configuration. The results are collected in Tables 2 and 3.

Table 2. Approximate zeros of the mixed-type hypergeometric Bernoulli r -Dowling polynomials ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$, for $n = 1, \dots, 10$.

Degree n	Approximate zeros x
1	-0.875
2	-1.49371; -0.50629
3	-2.02744; -1.04266; -0.304896
4	-2.49751; -1.57259; -0.745179; -0.184724
5	-2.90724; -2.09454; -1.22504; -0.537007; -0.111169
6	-3.24452; -2.62232; -1.72009; -0.959605; -0.387420; -0.0660423
7	-3.39965; -3.26598; -2.22225; -1.41639; -0.753410; -0.278732; -0.0385850
8	-3.69971 $\pm$ 0.351635 i; -2.72706; -1.89184; -1.16848; -0.591536; -0.199533; -0.0221104
9	-4.03905 $\pm$ 0.558710 i; -3.22321; -2.37835; -1.61260; -0.964642; -0.463757; -0.141935; -0.0123973
10	-4.36472 - 0.772495 i; -4.36472 + 0.772495 i; -3.68209; -2.87193; -2.07504; -1.3756; -0.796171; -0.362685; -0.10025; -0.00678676

Table 3. Approximate zeros of the mixed-type hypergeometric Euler r -Dowling polynomials ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$, for $n = 1, \dots, 10$.

Degree n	Approximate zeros x
1	-0.5
2	-1.0; -0.25
3	-1.5; -0.5; -0.25
4	-2.00926; -0.705041 $\pm$ 0.285304 i; -0.0806577
5	-2.52822; -1.03989 $\pm$ 0.527271 i; -0.353411; -0.0385927
6	-3.05291; -1.35788 $\pm$ 0.769462 i; -0.779819; -0.100753 $\pm$ 0.0858021 i
7	-3.57913; -1.66422 $\pm$ 1.03615 i; -1.30113; -0.293585 $\pm$ 0.257048 i; 0.0458684
8	-4.10379; -1.97863 $\pm$ 1.32823 i; -1.84886; -0.539319 $\pm$ 0.433022 i; -0.00572307 $\pm$ 0.0942016 i
9	-4.62486; -2.39276; -2.30768 $\pm$ 1.63200 i; -0.800814 $\pm$ 0.610637 i; -0.215211; -0.0250913 $\pm$ 0.105891 i
10	-5.14097; -2.93081; -2.64767 $\pm$ 1.93992 i; -1.05826 $\pm$ 0.801566 i; -0.689735; -0.102308 $\pm$ 0.290142 i; 0.128001

The tables already show a clear pattern. At low degrees all the zeros are real; non-real zeros appear only once the degree is large enough, and then always in conjugate pairs, so the zero sets are symmetric about the real axis, as the real coefficients require. This transition begins earlier for the Euler family, at $n = 4$, than for the Bernoulli family, at $n = 8$. To see how the zeros are arranged, we plot them in the complex plane, with the real part on the horizontal axis and the imaginary part on the vertical axis; different colors correspond to different degrees n .

As n increases, the zeros in these plots become denser and line up along well-defined curves. This can be related to a classical situation. For the ordinary Bernoulli and Euler polynomials the limiting distribution of the suitably scaled zeros is well understood, and our families reduce to those classical

polynomials when $m = 1$, $\mu \rightarrow 0$, and $r = 0$. Guided by this analogy and by the form of the generating functions (3.1) and (3.2), one expects that, after a suitable rescaling of the variable, the zeros tend to a fixed limiting curve as $n \rightarrow \infty$, determined by the factor $e^{x\Phi_\mu(z)}$ together with the singularities of the hypergeometric factor. A rigorous proof of this is beyond the scope of the present paper and is left as an open problem, although the data in Tables 2 and 3 and Figures 1 and 2 are consistent with it. We next view the same zeros through their mesh structure.

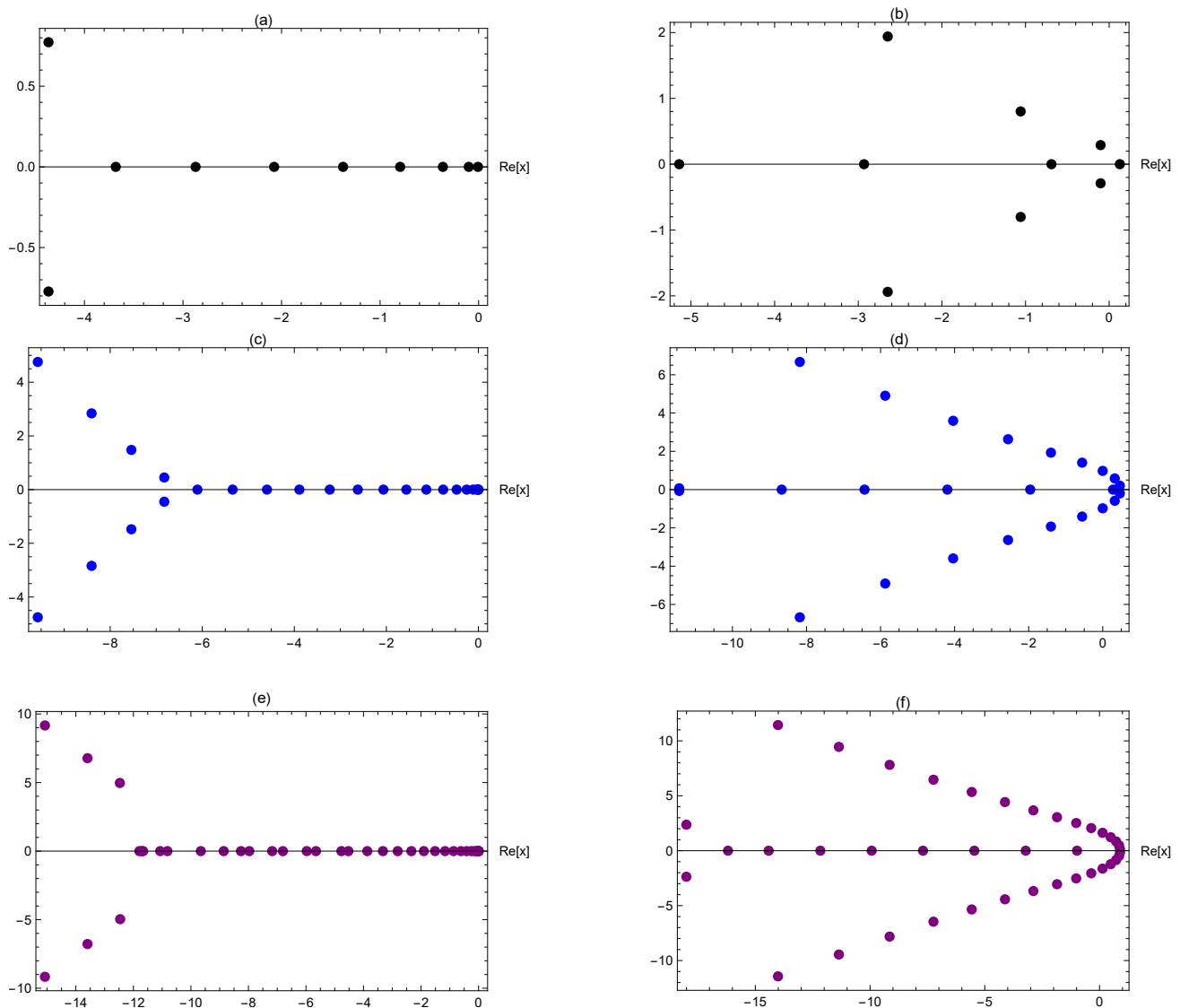


Figure 1. Distributions of the zeros of the mixed-type hypergeometric Bernoulli r -Dowling polynomials ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ and the mixed-type hypergeometric Euler r -Dowling polynomials ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ corresponding to $n = 10$, $n = 25$, and $n = 40$. (a) ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 10$, (b) ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 10$, (c) ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 25$, (d) ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 25$, (e) ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 40$, and (f) ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 40$.

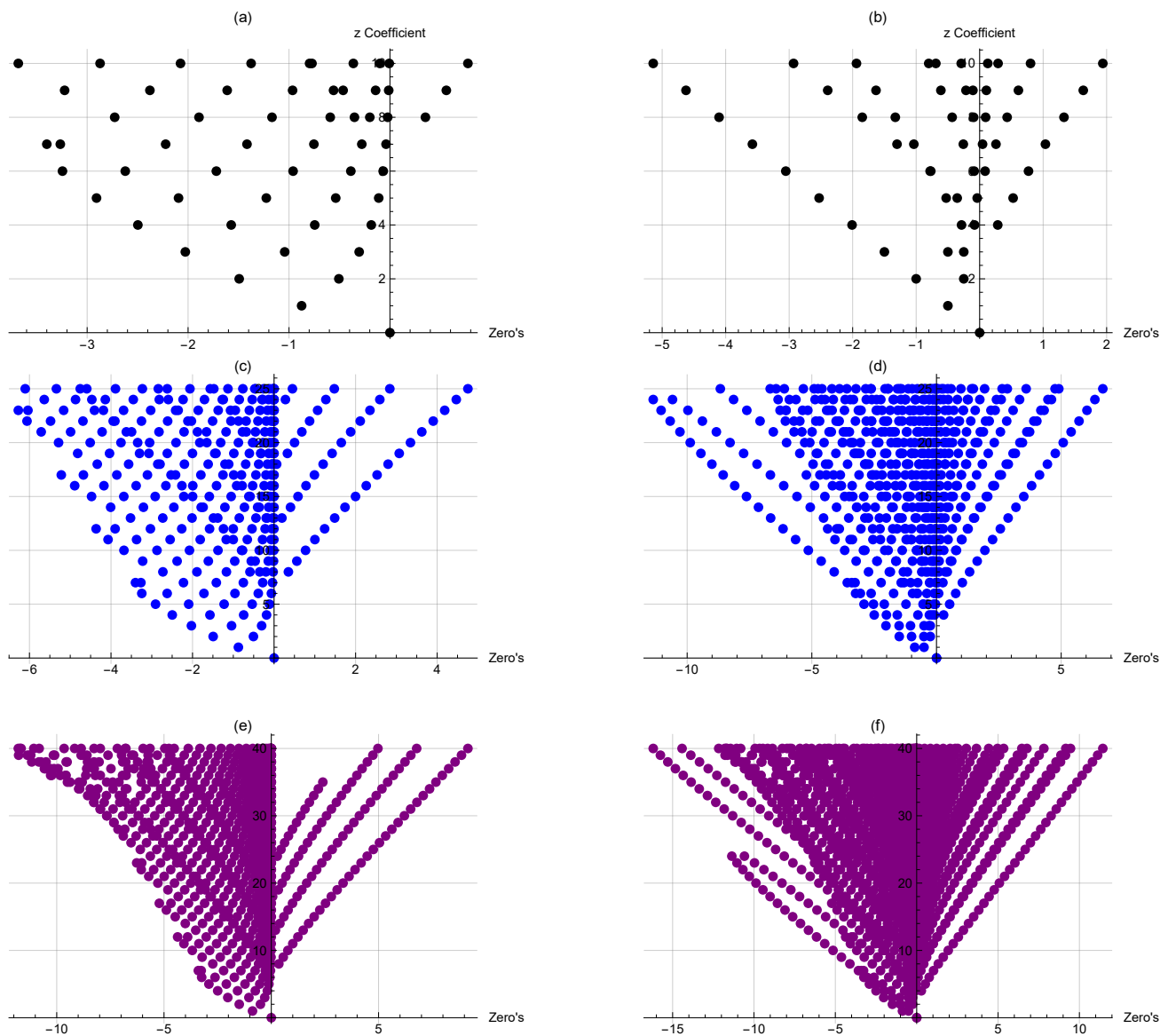


Figure 2. Mesh distributions of the zeros of the mixed-type hypergeometric Bernoulli r -Dowling polynomials ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ and the mixed-type hypergeometric Euler r -Dowling polynomials ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ corresponding to $n = 10$, $n = 25$, and $n = 40$. (a) ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 10$, (b) ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 10$, (c) ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 25$, (d) ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 25$, (e) ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 40$, and (f) ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$ for $n = 40$.

In these plots each layer of the mesh corresponds to a coefficient index of the truncated generating function, so the mesh records how the zeros develop as the degree grows. The patterns at $n = 10$, $n = 25$, and $n = 40$ closely resemble one another, which indicates that this structure is stable rather than an artifact of a particular degree. The two figures are complementary: Figure 1 shows where the zeros lie and their symmetry, while Figure 2 shows how their arrangement changes with the degree. The two families behave similarly in these respects but differ in how concentrated and dispersed their zeros are, most noticeably at higher degrees. Finally, Figure 3 presents the three-dimensional mesh distributions

of the mixed-type hypergeometric Bernoulli and Euler r -Dowling polynomials for $n = 40$, providing a visual comparison of the patterns exhibited by the two families.

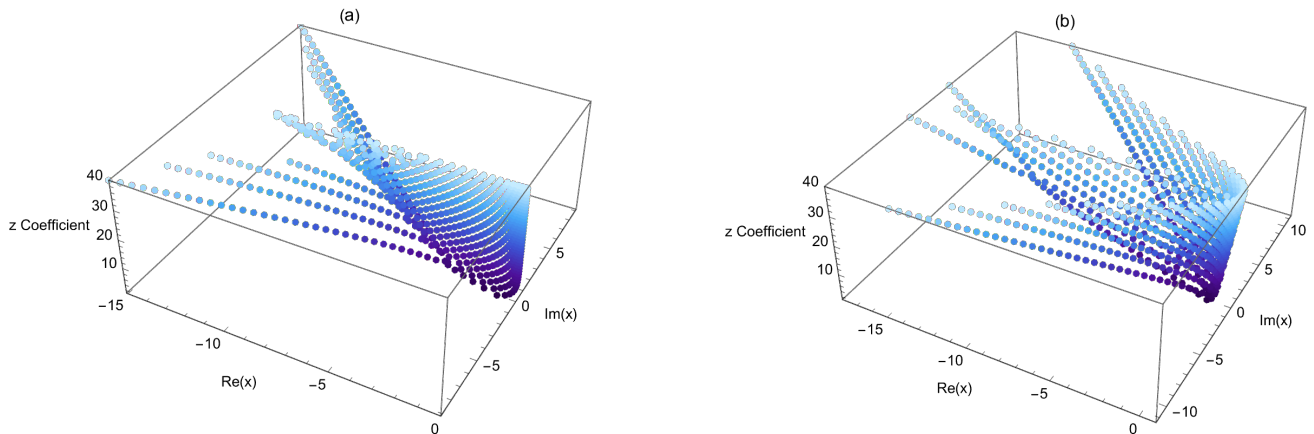


Figure 3. Three-dimensional mesh distribution of (a) the mixed-type hypergeometric Bernoulli r -Dowling polynomials ${}_B\mathcal{D}_n^{[m-1]}(x; \mu, r)$ and (b) the mixed-type hypergeometric Euler r -Dowling polynomials ${}_E\mathcal{D}_n^{[m-1]}(x; \mu, r)$, corresponding to $n = 40$.

5. Conclusions

In this paper, we introduced and studied the mixed-type hypergeometric Bernoulli and Euler r -Dowling polynomials, which extend several classical polynomial families within a unified framework. Their main structural properties were obtained using exponential generating functions, including recurrence relations, differential identities, and integral representations. It was also shown that these polynomials form the bases of polynomial spaces. In addition, numerical and graphical analyses of their zeros indicate symmetric and regular distribution patterns in the complex plane, suggesting an underlying structured behavior. Overall, the results provide a natural extension of existing polynomial families and contribute to a better understanding of their algebraic and analytic properties. Possible future work includes a deeper analysis of zero distributions, as well as the investigation of approximation properties of these polynomials via suitable operators.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Clemente Cesarano is an editorial board member for *Networks and Heterogeneous Media* and was not involved in the editorial review or the decision to publish this article. All authors declare that they have no competing interests.

Author contributions

W.R., C.K., C.C., M.-F.H.-M., and T.U. contributed to the conceptualization, theoretical development, and methodology of the study. All authors participated in deriving and verifying the mathematical results, analyzing the zero distribution patterns, and preparing and revising the manuscript. All authors read and approved the final manuscript.

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