



Research Article

On anisotropic singular elliptic equations with a Finsler $p(\cdot)$ -Laplacian and variable exponents

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Abstract: This paper investigated strongly nonlinear anisotropic elliptic problems with singular terms in variable exponent Sobolev spaces. The problem involves the Finsler $p(\cdot)$ -Laplacian operator $\Delta_{p(\cdot)}^H u = \operatorname{div}(H^{p(\cdot)-1}(\nabla u) \nabla H(\nabla u))$, where H is a uniformly convex Finsler norm. We considered the differential inclusion

$$\zeta(u) - \Delta_{p(\cdot)}^H u + G(x, u, \nabla u) \ni \frac{f}{u^\gamma} \quad \text{in } \Omega,$$

with zero Dirichlet boundary conditions. Here ζ is a maximal monotone operator, G satisfies natural growth conditions, $f \in L^\infty(\Omega)$ is nonnegative, and $0 < \gamma \leq 1$. Under log-Hölder continuity of $p(\cdot)$ and uniform convexity of H , we established existence of weak solutions via an approximation scheme combining truncation, Yosida regularization, and energy estimates. Key innovations include anisotropic monotonicity estimates, L^∞ bounds, strict positivity, and strong convergence of gradients. Our results unify and extend previous work to encompass variable exponents, anisotropy, multivalued operators, and singular nonlinearities.

Keywords: anisotropic operators; Finsler p -Laplacian; singular nonlinearities; variable exponent Sobolev spaces; Yosida approximation

1. Introduction

Nonlinear partial differential equations serve as fundamental mathematical models for describing complex phenomena across diverse scientific disciplines including physics, biology, engineering, and materials science. In recent decades, considerable attention has been devoted to equations involving

anisotropic operators, which arise naturally in contexts where the medium exhibits directional dependencies. A particularly significant class of anisotropic operators is the Finsler p -Laplacian, defined as $\Delta_p^H u = \operatorname{div}(H^{p-1}(\nabla u) \nabla H(\nabla u))$, where H is a Finsler norm on $\mathbb{R}^N$. This operator generalizes the classical p -Laplacian, which corresponds to the Euclidean norm $H(\xi) = |\xi|$, and appears in diverse applications including Finsler geometry [10], materials science [19], image processing, and the study of anisotropic media [13]. The geometric significance of the Finsler p -Laplacian stems from its connection with the geometry of Minkowski spaces. The norm H induces a Finsler metric on $\mathbb{R}^N$, and the operator $-\Delta_p^H$ represents the Euler-Lagrange equation of the anisotropic energy functional $\mathcal{E}(u) = \frac{1}{p} \int_{\Omega} H^p(\nabla u) dx$. The uniform convexity of H ensures that this energy is strictly convex and coercive, leading to good functional analytic properties.

Parallel to these developments, the theory of Sobolev spaces with variable exponents $W^{k,p(\cdot)}(\Omega)$ has emerged as a powerful framework for modeling phenomena with non-standard growth conditions. Such spaces naturally arise in the study of electrorheological fluids, where the viscosity depends on the electric field, leading to a variable exponent in the constitutive law [24]. They also appear in image restoration, where variable exponents allow adaptive smoothing that preserves edges [12], and in thermorheological materials, where material properties vary with temperature, leading to temperature-dependent growth exponents [1]. The systematic development of variable exponent spaces was carried out by Diening [16], Fan and Zhao [18], and others, establishing fundamental properties such as reflexivity, separability, and embedding theorems under appropriate conditions on $p(\cdot)$.

When singular terms are incorporated, the analysis becomes particularly delicate. Singular elliptic equations of the form $-\Delta_p u = \frac{f}{u^\gamma}$ in Ω with $u = 0$ on $\partial\Omega$ have been extensively studied in the Euclidean setting. The seminal work of Crandall et al. [15] established the first existence results for semilinear operators. Later, Lazer and McKenna [20] revealed the loss of regularity near the boundary, showing that for $p = 2$ and $\gamma \geq 3$, solutions are not in $H_0^1(\Omega)$. This fundamental observation led to the development of appropriate weak-solution concepts that accommodate this lack of regularity. Boccardo and Orsina [6] developed systematic existence theories using truncation techniques, while Canino et al. [8, 9] established uniqueness results under mild assumptions on f . The case of the p -Laplacian with singular nonlinearities was treated by Canino et al. [8], who proved the existence and uniqueness for $\gamma > 0$ under optimal summability conditions on f . More types of singular problems can be found in [3] and the references therein. We also mention the recent works of Mukherjee et al. [21], Mukherjee et al. [22], and Pucci and Servadei [23], which studied singular nonlinearities in fractional Kirchhoff problems, nonlocal critical exponent problems under mixed boundary conditions, and radial ground states for p -Laplacian equations with singular weights, respectively. These works provide further motivation for the study of singular problems in non-standard settings.

The anisotropic case involving the Finsler p -Laplacian was recently investigated by Esposito et al. [17], who proved the existence and uniqueness for $-\Delta_p^H u = \frac{f(x)}{u^\gamma}$ in Ω with $u = 0$ on $\partial\Omega$, under very general summability assumptions on f . Their work highlighted the crucial role of the geometric properties of H and established fundamental estimates using the uniform ellipticity of the Finsler norm. In the direction of singular problems in variable exponent spaces, Bahadi and Akdim [2] considered equations with logarithmic singularities, establishing existence results in $W_0^{1,p(\cdot)}(\Omega)$. However, their work did not incorporate the additional complexities of maximal monotone

operators, natural growth terms, or anisotropic structures. In particular, the work of Bahadi et al. [3] treated the Euclidean $p(\cdot)$ -Laplacian with singularities, whereas the present paper treats the Finsler $p(\cdot)$ -Laplacian, which introduces non-Euclidean anisotropy. The combination of anisotropy with variable exponents creates two genuinely new difficulties. First, the monotonicity estimates of Lemma 2.12 must be adapted to the Finsler structure and the variable exponent simultaneously. Second, the L^∞ iteration of Lemma 4.3 requires a Finsler–Sobolev embedding that is not available in the Euclidean variable-exponent setting.

In this paper, we aim to significantly extend these results by considering a comprehensive class of problems that combines multiple challenging features: the anisotropic Finsler $p(\cdot)$ -Laplacian with variable exponent $p(\cdot)$ and a general Finsler norm H ; a maximal monotone operator ζ allowing for multivalued graphs such as subdifferentials of convex functions, sign graphs, or hysteresis operators; a lower-order term $G(x, u, \nabla u)$ satisfying natural growth conditions with respect to $|\nabla u|^{p(x)}$; and a singular right-hand side f/u^γ with $f \in L^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$, and $0 < \gamma \leq 1$. Specifically, we consider the problem

$$(E, f) \quad \begin{cases} \zeta(u) - \Delta_{p(\cdot)}^H u + G(x, u, \nabla u) \ni \frac{f}{u^\gamma} & \text{in } \Omega, \\ u \in W_0^{1,p(\cdot)}(\Omega), \quad G(x, u, \nabla u) \in L^1(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ is a bounded smooth domain, $p : \Omega \rightarrow (1, \infty)$ is a variable exponent, and H is a Finsler norm on $\mathbb{R}^N$ satisfying suitable conditions. The function G satisfies natural growth conditions with respect to ∇u of order $p(x)$, meaning that $|G(x, s, \xi)| \leq c(|s|)(m(x) + |\xi|^{p(x)})$, with appropriate sign conditions to ensure coercivity.

The mathematical analysis of problem (E, f) in variable exponent anisotropic spaces requires novel techniques and careful handling of several interconnected difficulties. The singular term f/u^γ necessitates strict positivity estimates and delicate convergence arguments near the set where u is small. The natural growth condition on G introduces quadratic-like terms that must be controlled through sign conditions and careful testing. The maximal monotone operator ζ requires approximation via Yosida regularizations and graph-convergence arguments. The variable-exponent framework demands modified versions of standard inequalities and convergence theorems. The anisotropic Finsler structure requires specialized estimates that exploit the uniform convexity of H . The lack of regularity near the boundary due to the singularity necessitates a weak solution concept that does not require $u \in W_0^{1,p(\cdot)}(\Omega)$ in all cases.

Our main result establishes the existence of at least one weak solution $(u, b) \in W_{\text{loc}}^{1,p(\cdot)}(\Omega) \times L^1(\Omega)$ to problem (E, f) , with the additional regularity $u \in L^\infty(\Omega)$ and $u > 0$ in Ω . The solution satisfies $b(x) \in \zeta(u(x))$ almost everywhere, and the variational identity

$$\int_{\Omega} b \varphi dx + \int_{\Omega} H^{p(x)-1}(\nabla u) \nabla H(\nabla u) \cdot \nabla \varphi dx + \int_{\Omega} G(x, u, \nabla u) \varphi dx = \int_{\Omega} \frac{f \varphi}{u^\gamma} dx$$

holds for all test functions $\varphi \in C_0^1(\Omega)$. The proof proceeds through a carefully constructed approximation scheme. First, we regularize both the singular term, replacing u^γ by $(|u| + \varepsilon)^\gamma$, and the maximal monotone operator using Yosida approximations ζ_ε . We also truncate the solution to ensure boundedness. Using Schauder's fixed point theorem and the theory of pseudo-monotone operators in variable-exponent spaces, we establish the existence of solutions u_ε for each $\varepsilon > 0$. We then derive estimates independent of ε : boundedness in $W_0^{1,p(\cdot)}(\Omega)$, L^∞ bounds via iterative methods, and strict

positivity on compact subsets. We study the limit as $\varepsilon \rightarrow 0$, proving weak convergence in the Sobolev space, strong convergence of the singular term through a delicate truncation argument, and strong convergence of gradients using anisotropic monotonicity estimates. Finally, we show that the limit (u, b) satisfies the original problem and has the desired regularity properties.

Our work presents several novel contributions to the theory of singular elliptic equations. This is the first work to consider the Finsler $p(\cdot)$ -Laplacian with singular nonlinearities, combining anisotropy with variable exponents. The variable exponent adds significant technical challenges due to the non-homogeneous nature of the modular, requiring careful adaptation of all estimates. Previous works typically considered $\zeta \equiv 0$ or $\zeta(r) = |r|^{q-2}r$, whereas our inclusion of a general maximal monotone graph allows modeling of hysteresis, friction, and other non-smooth phenomena. The presence of G with $|G(x, s, \xi)| \leq c(|s|)(m(x) + |\xi|^{p(x)})$ and the sign condition $G(x, s, \xi)s \geq 0$ adds another layer of complexity, requiring careful control to avoid loss of coercivity. We obtain L^∞ bounds independent of the data and strict positivity, which are essential for the well-posedness of the singular term. Our results are established for the range $0 < \gamma \leq 1$, which is the natural setting for obtaining global $W_0^{1,p(\cdot)}$ regularity of solutions.

The techniques developed here may find applications in other contexts involving singularities, anisotropy, and non-standard growth conditions, such as problems with Hardy-type singularities, equations with variable exponents and measure data, anisotropic parabolic problems with singular sources, and systems of singular elliptic equations.

The paper is organized as follows. Section 2 presents the necessary preliminaries on variable-exponent Lebesgue and Sobolev spaces, including definitions, basic properties, key inequalities, and convergence theorems. We also introduce the Finsler norm H and establish fundamental properties of the anisotropic operator. Section 3 states our precise assumptions on the data and presents the main existence theorem along with the definition of weak solutions. Section 4 contains the core of the proof, where we construct the approximate problem and prove the existence of approximate solutions using Schauder's fixed point theorem and pseudo-monotone operator theory in variable-exponent spaces. Section 5 develops uniform a priori estimates: $W_0^{1,p(\cdot)}$ bounds, nonnegativity, L^∞ estimates via iteration, and a uniform positive lower bound via Moser iteration. Section 6 analyzes the convergence as $\varepsilon \rightarrow 0$, establishing weak convergence, convergence of the singular term, strong convergence of gradients using anisotropic monotonicity, and convergence of the natural growth term. Section 7 completes the proof of the main theorem by passing to the limit and verifying all solution properties.

2. Preliminaries, assumptions, and main results

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$, which ensures the validity of Sobolev embeddings and trace theorems. Let $p : \Omega \rightarrow [1, \infty)$ be a measurable function, called the variable exponent. We denote

$$p^+ = \operatorname{ess\,sup}_{x \in \Omega} p(x), \quad p^- = \operatorname{ess\,inf}_{x \in \Omega} p(x).$$

Throughout this paper, we maintain the fundamental assumption

$$1 < p^- \leq p^+ < \infty, \tag{2.1}$$

which ensures the reflexivity and good duality properties of the associated function spaces.

Definition 2.1. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which the modular

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx$$

is finite. This space becomes a Banach space when equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Remark 2.2. The modular $\rho_{p(\cdot)}$ is not homogeneous in general, which distinguishes variable-exponent spaces from their constant-exponent counterparts. This lack of homogeneity requires careful handling of estimates involving powers of norms.

The following properties are crucial for our analysis:

Lemma 2.3. [16, 18] Let $p : \Omega \rightarrow (1, \infty)$ be measurable satisfying $1 < p^- \leq p^+ < \infty$.

1. For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

$$\left| \int_{\Omega} u(x)v(x)dx \right| \leq 2\|u\|_{L^{p(\cdot)}}\|v\|_{L^{p'(\cdot)}}.$$

2. $\|u\|_{L^{p(\cdot)}} \leq 1$ if and only if $\rho_{p(\cdot)}(u) \leq 1$.

3. For any $u \in L^{p(\cdot)}(\Omega)$,

$$\min \left\{ \rho_{p(\cdot)}(u)^{1/p^-}, \rho_{p(\cdot)}(u)^{1/p^+} \right\} \leq \|u\|_{L^{p(\cdot)}} \leq \max \left\{ \rho_{p(\cdot)}(u)^{1/p^-}, \rho_{p(\cdot)}(u)^{1/p^+} \right\}.$$

4. The dual space of $L^{p(\cdot)}(\Omega)$ is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $p'(x) = \frac{p(x)}{p(x)-1}$.

5. $L^{p(\cdot)}(\Omega)$ is separable and reflexive.

6. If $1 < p^- \leq p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex.

Definition 2.4. The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose weak derivatives $\partial_i u \in L^{p(\cdot)}(\Omega)$ exist for $i = 1, \dots, N$. This space is equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Definition 2.5 (Log-Hölder continuity). A function $p : \Omega \rightarrow \mathbb{R}$ is said to be log-Hölder continuous if there exists a constant $C > 0$ such that for all $x, y \in \Omega$ with $|x - y| < \frac{1}{2}$,

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|}.$$

This condition ensures several key properties that are fundamental to our analysis.

Lemma 2.6. [16, 18] Assume $p : \Omega \rightarrow (1, \infty)$ is log-Hölder continuous with $1 < p^- \leq p^+ < \infty$. Then

1. $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.
2. There exists $C_P > 0$ such that for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C_P \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

3. There exists a continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, and $p^*(x)$ is any finite number if $p(x) \geq N$.
4. The embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact.
5. The trace operator $W^{1,p(\cdot)}(\Omega) \rightarrow L^{p(\cdot)}(\partial\Omega)$ is well-defined and bounded.
6. There exists a bounded linear extension operator $W^{1,p(\cdot)}(\Omega) \rightarrow W^{1,p(\cdot)}(\mathbb{R}^N)$.

The following technical lemma regarding convergence in variable-exponent spaces will be essential.

Lemma 2.7. *[[18] Theorem 3.4] Let u_n be a sequence in $L^{p(\cdot)}(\Omega)$ such that*

1. $u_n(x) \rightarrow u(x)$ for almost every $x \in \Omega$,
2. $\{|u_n|^{p(\cdot)}\}$ is uniformly integrable in $L^1(\Omega)$, i.e.,

$$\lim_{|E| \rightarrow 0} \sup_n \int_E |u_n(x)|^{p(x)} dx = 0.$$

Then $u \in L^{p(\cdot)}(\Omega)$ and $u_n \rightarrow u$ in $L^{p(\cdot)}(\Omega)$.

We also need the following continuous embedding result.

Lemma 2.8. *[[18] Theorems 3.3 and 3.5] Let p satisfy the assumptions of Lemma 2.6. Then*

1. If $p(x) \leq q(x) \leq p^*(x)$ for almost every $x \in \Omega$, then $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.
2. If $q(x) < p^*(x)$ for almost every $x \in \Omega$, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

We now recall essential properties of Finsler norms and their associated differential operators, following [11, 14, 17].

Definition 2.9. A function $H : \mathbb{R}^N \rightarrow [0, \infty)$ is called a Finsler norm if it satisfies

(H1) $H \in C^2(\mathbb{R}^N \setminus \{0\})$ and H is even, i.e., $H(-\xi) = H(\xi)$ for all $\xi \in \mathbb{R}^N$.

(H2) $H(t\xi) = tH(\xi)$ for all $\xi \in \mathbb{R}^N$ and $t > 0$.

(H3) The unit ball $B_1^H = \{\xi \in \mathbb{R}^N : H(\xi) < 1\}$ is uniformly convex, i.e.,

$$\exists \lambda > 0 : \langle D^2 H(\xi) v, v \rangle \geq \lambda |v|^2, \quad \forall \xi \in \partial B_1^H, \forall v \in \nabla H(\xi)^\perp,$$

where $\nabla H(\xi)^\perp$ denotes the subspace orthogonal to $\nabla H(\xi)$ with respect to the Euclidean inner product.

Remark 2.10. Condition (H3) is equivalent to the existence of a constant $\lambda > 0$ such that

$$H(\xi + \eta) + H(\xi - \eta) - 2H(\xi) \geq \lambda|\eta|^2$$

for all $\xi, \eta \in \mathbb{R}^N$ with $H(\xi) = 1$. This is the analogue of uniform convexity for norms, and the equivalence is understood in the sense of the orthogonal subspace $\nabla H(\xi)^\perp$.

From these properties, we deduce the following fundamental estimates.

Lemma 2.11. *Under assumptions (H1)–(H3), there exist constants $0 < a \leq b < \infty$ such that*

1. $a|\xi| \leq H(\xi) \leq b|\xi|$ for all $\xi \in \mathbb{R}^N$.
2. ∇H is homogeneous of degree 0: $\nabla H(t\xi) = \nabla H(\xi)$ for all $\xi \neq 0, t > 0$.
3. $H(\xi)\nabla H(\xi) \cdot \xi = H^2(\xi)$.
4. There exists $\Lambda > 0$ such that for all $\xi \neq 0$ and all $v \in \nabla H(\xi)^\perp$,

$$\frac{\lambda}{|\xi|}|v|^2 \leq \langle D^2 H(\xi)v, v \rangle \leq \frac{\Lambda}{|\xi|}|v|^2.$$

5. The mapping $\xi \mapsto H^{p-1}(\xi)\nabla H(\xi)$ is strictly monotone for any $p > 1$.

Proof. (1) follows from the fact that H is a norm on $\mathbb{R}^N$ and all norms are equivalent. (2) follows from differentiating the homogeneity condition. (3) is Euler's theorem for homogeneous functions. (4) follows from (H3) and homogeneity (restricted to $v \in \nabla H(\xi)^\perp$). (5) is a consequence of (4) and standard arguments.

For variable exponent $p(\cdot)$, we define the anisotropic $p(\cdot)$ -Laplacian as

$$\Delta_{p(\cdot)}^H u = \operatorname{div} \left(H^{p(\cdot)-1}(\nabla u) \nabla H(\nabla u) \right).$$

The corresponding energy functional is

$$\mathcal{E}_{p(\cdot)}(u) = \int_{\Omega} \frac{1}{p(x)} H^{p(x)}(\nabla u) dx.$$

The following vectorial estimates, adapted from [11] to the variable exponent setting, are crucial for our analysis.

Lemma 2.12. *There exist constants $C_1, C_2 > 0$ depending only on $p^\pm, N, \lambda$ such that for all $\xi, \eta \in \mathbb{R}^N$ with $|\xi| + |\eta| > 0$ and for almost every $x \in \Omega$,*

$$\begin{aligned} & \langle H^{p(x)-1}(\xi)\nabla H(\xi) - H^{p(x)-1}(\eta)\nabla H(\eta), \xi - \eta \rangle \\ & \geq C_1(|\xi| + |\eta|)^{p(x)-2} |\xi - \eta|^2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} & |H^{p(x)-1}(\xi)\nabla H(\xi) - H^{p(x)-1}(\eta)\nabla H(\eta)| \\ & \leq C_2(|\xi| + |\eta|)^{p(x)-2} |\xi - \eta|. \end{aligned} \tag{2.3}$$

For the case where $p(x) \geq 2$, we also have the simpler estimate

$$\langle H^{p(x)-1}(\xi)\nabla H(\xi) - H^{p(x)-1}(\eta)\nabla H(\eta), \xi - \eta \rangle \geq C'_1 |\xi - \eta|^{p(x)}. \tag{2.4}$$

Proof. The proof follows from the uniform convexity of H and the following well-known inequalities. For $p \geq 2$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_p |\xi - \eta|^p,$$

and for $1 < p < 2$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_p \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}.$$

The constants can be chosen uniformly in x due to the bounds $p^- \leq p(x) \leq p^+$.

Corollary 2.13. For all $\xi, \eta \in \mathbb{R}^N$ and almost every $x \in \Omega$,

$$|H^{p(x)-1}(\xi)\nabla H(\xi) - H^{p(x)-1}(\eta)\nabla H(\eta)| \leq C_2(|\xi|^{p(x)-1} + |\eta|^{p(x)-1}).$$

Proof. This follows from the second estimate in Lemma 2.12 and the elementary inequality $(|\xi| + |\eta|)^{p-2}|\xi - \eta| \leq |\xi|^{p-1} + |\eta|^{p-1}$.

Let $\zeta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ be a multivalued mapping. We denote its domain by $D(\zeta) = \{r \in \mathbb{R} : \zeta(r) \neq \emptyset\}$.

Definition 2.14. A multivalued operator $\zeta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is called maximal monotone if

1. $(y_1 - y_2)(r_1 - r_2) \geq 0$ for all $r_i \in D(\zeta)$ and $y_i \in \zeta(r_i)$, $i = 1, 2$.
2. The graph of ζ is not properly contained in the graph of any other monotone operator.

For $\varepsilon > 0$, we define the Yosida approximation $\zeta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\zeta_\varepsilon(r) = \frac{1}{\varepsilon}(r - J_\varepsilon(r)),$$

where $J_\varepsilon = (I + \varepsilon\zeta)^{-1}$ is the resolvent operator. The following properties are standard.

Lemma 2.15. [Properties of Yosida approximations] Let ζ be maximal monotone with $0 \in \zeta(0)$. Then for each $\varepsilon > 0$:

1. ζ_ε is Lipschitz continuous with constant $1/\varepsilon$.
2. $\zeta_\varepsilon(r) \in \zeta(J_\varepsilon(r))$ for all $r \in \mathbb{R}$.
3. $|\zeta_\varepsilon(r)| \leq |\zeta^\circ(r)|$, where $\zeta^\circ(r)$ denotes the element of minimal norm in $\zeta(r)$.
4. $\zeta_\varepsilon(r)r \geq 0$ for all $r \in \mathbb{R}$.
5. ζ_ε is monotone: $(\zeta_\varepsilon(r) - \zeta_\varepsilon(s))(r - s) \geq 0$.
6. As $\varepsilon \rightarrow 0$, $\zeta_\varepsilon(r) \rightarrow \zeta^\circ(r)$ for all $r \in D(\zeta)$, and $J_\varepsilon(r) \rightarrow r$.

Proof. See [7, Chapter II].

Due to the singular nature of the right-hand side f/u^γ , we need special test functions and truncation arguments. Define the truncation operator $T_k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_k(s) = \max\{-k, \min\{s, k\}\}.$$

For $\delta > 0$, we introduce the following auxiliary function that will be crucial for handling the singular term

$$V_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ \frac{2\delta-s}{\delta}, & \text{if } \delta < s < 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases}$$

Note that V_δ is Lipschitz continuous, $0 \leq V_\delta \leq 1$, and $V_\delta(s) \rightarrow \chi_{\{0\}}(s)$ as $\delta \rightarrow 0$ in the sense of measures.

We also need the following lemma on the convergence of the singular term.

Lemma 2.16. *Let $u_\varepsilon, u \in L^\infty(\Omega)$ with $u_\varepsilon \rightarrow u$ a.e. and $u \geq c_\omega > 0$ on any compact set $\omega \subset \Omega$. Assume $f \in L^\infty(\Omega)$, $f \geq 0$. Then for any $\varphi \in C_0^1(\Omega)$,*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{f\varphi}{(u_\varepsilon + \varepsilon)^\gamma} dx = \int_{\Omega} \frac{f\varphi}{u^\gamma} dx.$$

If additionally $u_\varepsilon \geq c_\omega > 0$ uniformly on $\text{supp}(\varphi)$, then the convergence is uniform.

Proof. The proof follows from the dominated convergence theorem once we establish uniform integrability. On $\text{supp}(\varphi)$, we have $u \geq c > 0$, so $u_\varepsilon \geq c/2$ for sufficiently small ε . Then the integrand is bounded by $\|f\|_\infty \|\varphi\|_\infty / (c/2)^\gamma$.

We now state precisely all assumptions that will be used throughout the paper.

Assumption 2.17. *The Finsler norm $H : \mathbb{R}^N \rightarrow [0, \infty)$ satisfies (H1)–(H3) from Definition 2.9. Consequently, there exist constants $0 < a \leq b < \infty$ such that*

$$a|\xi| \leq H(\xi) \leq b|\xi| \quad \forall \xi \in \mathbb{R}^N.$$

Assumption 2.18. *The function $p : \Omega \rightarrow (1, \infty)$ satisfies*

$$(P1) \quad 1 < p^- \leq p^+ < \infty.$$

$$(P2) \quad p \text{ is log-Hölder continuous on } \overline{\Omega}.$$

$$(P3) \quad \inf_{x \in \Omega} p(x) > \frac{N}{N+1} \text{ to ensure appropriate embedding properties for the dual space.}$$

Assumption 2.19. *The multivalued operator $\zeta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is maximal monotone with $0 \in \zeta(0)$. Moreover, there exists a constant $C_\zeta > 0$ such that*

$$|\zeta^\circ(r)| \leq C_\zeta(1 + |r|^{p^+-1}) \quad \forall r \in D(\zeta),$$

where ζ° denotes the minimal section.

Assumption 2.20. The function $G : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function (measurable in x , continuous in (s, ξ)) satisfying

(G1) $G(x, s, \xi) \cdot s \geq 0$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$.

(G2) There exist a continuous increasing function $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and a nonnegative function $m \in L^1(\Omega)$ such that

$$|G(x, s, \xi)| \leq c(|s|)(m(x) + |\xi|^{p(x)}) \quad \text{a.e. in } \Omega.$$

(G3) There exist $s_0, s_1 > 0$ such that

$$|G(x, s, \xi)| \geq s_1 |\xi|^{p(x)} \quad \text{whenever } |s| \geq s_0.$$

(G4) For a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, the map $s \mapsto G(x, s, \xi)$ is monotone nondecreasing.

(G5) There exists a constant $L_G > 0$ such that

$$|G(x, s, \xi) - G(x, s, \eta)| \leq L_G(1 + |s|^{p(x)-1})|\xi - \eta|(|\xi|^{p(x)-2} + |\eta|^{p(x)-2}).$$

Assumption 2.21. [Source term and singularity] The function $f : \Omega \rightarrow \mathbb{R}$ satisfies

(F1) $f \in L^\infty(\Omega)$.

(F2) $f(x) \geq 0$ for a.e. $x \in \Omega$.

(F3) $f \not\equiv 0$ (i.e., $\int_\Omega f dx > 0$).

(F4) $0 < \gamma \leq 1$ is the singularity exponent.

Assumption 2.22. $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ is a bounded domain with Lipschitz boundary $\partial\Omega$.

We adopt the following definition, which is standard in the literature on singular elliptic equations (see [6, 8, 17]).

Definition 2.23. A pair $(u, b) \in W_{\text{loc}}^{1,p(\cdot)}(\Omega) \times L^1(\Omega)$ is called a weak solution of problem (E, f) if

(S1) $u(x) \in D(\zeta)$ and $b(x) \in \zeta(u(x))$ for almost every $x \in \Omega$, where $D(\zeta) = \{r \in \mathbb{R} : \zeta(r) \neq \emptyset\}$ is the effective domain of ζ .

(S2) $u(x) > 0$ for almost every $x \in \Omega$. Moreover, for every relatively compact subset $\omega \subset\subset \Omega$,

$$\int_\omega u(x)^{-\gamma} dx < \infty.$$

This ensures that the singular term f/u^γ is well-defined as an $L_{\text{loc}}^1(\Omega)$ -function.

(S3) $G(x, u, \nabla u) \in L^1(\Omega)$.

(S4) For every test function $\varphi \in C_0^1(\Omega)$, the following variational identity holds:

$$\int_\Omega b\varphi dx + \int_\Omega H^{p(x)-1}(\nabla u)\nabla H(\nabla u) \cdot \nabla \varphi dx + \int_\Omega G(x, u, \nabla u)\varphi dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx. \quad (2.5)$$

(S5) $u = 0$ on $\partial\Omega$ in the sense that u is nonnegative and $(u - \varepsilon)^+ \in W_0^{1,p(\cdot)}(\Omega)$ for every $\varepsilon > 0$.

We are now ready to state the main result of this paper.

Theorem 2.24. *Assume that Assumptions 2.17–2.22 hold. Then there exists at least one weak solution $(u, b) \in W_{\text{loc}}^{1,p(\cdot)}(\Omega) \times L^1(\Omega)$ to problem (E, f) in the sense of Definition 2.23.*

Moreover, the solution satisfies the following additional regularity properties:

(R1) $u \in L^\infty(\Omega)$ and there exists a constant $C_\infty > 0$, depending only on the data $(N, p^\pm, a, b, \gamma, \Omega)$, such that

$$\|u\|_{L^\infty(\Omega)} \leq C_\infty \|f\|_{L^\infty(\Omega)}.$$

(R2) $u(x) > 0$ for almost every $x \in \Omega$. Moreover, for every $\omega \subset\subset \Omega$,

$$\int_\omega u(x)^{-\gamma} dx < \infty.$$

(R3) $u \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \leq C(\|f\|_{L^\infty}).$$

3. Approximate problems and preliminary existence

To handle the singular term and the multivalued operator, we introduce a family of approximate problems depending on a small parameter $\varepsilon \in (0, 1]$. For each ε , we consider

$$(E_\varepsilon, f) \quad \begin{cases} \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - \Delta_{p(\cdot)}^H u_\varepsilon + G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = \frac{f}{(|u_\varepsilon| + \varepsilon)^\gamma} & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where

1. ζ_ε is the Yosida approximation of ζ (see Lemma 2.15).
2. $T_{1/\varepsilon}(s) = \min\{|s|, 1/\varepsilon\} \text{sign}(s)$ is the truncation operator at level $1/\varepsilon$.
3. G_ε is defined by

$$G_\varepsilon(x, s, \xi) = \frac{G(x, s, \xi)}{1 + \varepsilon|G(x, s, \xi)|},$$

which satisfies $|G_\varepsilon| \leq \min\{|G|, 1/\varepsilon\}$ and $G_\varepsilon \rightarrow G$ pointwise as $\varepsilon \rightarrow 0$.

4. The singular term is regularized by replacing u^γ with $(|u| + \varepsilon)^\gamma$, which is bounded away from zero.

The weak formulation of (E_ε, f) is: Find $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ such that for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \varphi dx + \int_\Omega H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) \cdot \nabla \varphi dx + \int_\Omega G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \varphi dx = \int_\Omega \frac{f \varphi}{(|u_\varepsilon| + \varepsilon)^\gamma} dx. \quad (3.1)$$

Define the operator $\mathcal{A}_\varepsilon : W_0^{1,p(\cdot)}(\Omega) \rightarrow W^{-1,p'(\cdot)}(\Omega)$ by

$$\langle \mathcal{A}_\varepsilon u, \varphi \rangle = \int_\Omega \zeta_\varepsilon(T_{1/\varepsilon}(u)) \varphi dx + \int_\Omega H^{p(x)-1}(\nabla u) \nabla H(\nabla u) \cdot \nabla \varphi dx + \int_\Omega G_\varepsilon(x, u, \nabla u) \varphi dx.$$

The right-hand side defines a functional $\mathcal{F}_\varepsilon(u) \in W^{-1,p'(\cdot)}(\Omega)$ by

$$\langle \mathcal{F}_\varepsilon(u), \varphi \rangle = \int_\Omega \frac{f\varphi}{(|u| + \varepsilon)^\gamma} dx.$$

Problem (E_ε, f) is equivalent to finding u_ε such that

$$\mathcal{A}_\varepsilon u_\varepsilon = \mathcal{F}_\varepsilon(u_\varepsilon) \quad \text{in } W^{-1,p'(\cdot)}(\Omega).$$

We now establish the key properties of $\mathcal{A}_\varepsilon$ that will allow us to apply the theory of pseudo-monotone operators.

Lemma 3.1. *The operator $\mathcal{A}_\varepsilon$ is bounded: for any bounded set $B \subset W_0^{1,p(\cdot)}(\Omega)$, $\mathcal{A}_\varepsilon(B)$ is bounded in $W^{-1,p'(\cdot)}(\Omega)$.*

Proof. Let $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\|_{W^{1,p(\cdot)}} \leq R$. For any $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ with $\|\varphi\|_{W^{1,p(\cdot)}} \leq 1$, we estimate each term separately.

For the Yosida approximation term, by Lemma 2.15 and the growth condition on ζ° ,

$$|\zeta_\varepsilon(T_{1/\varepsilon}(u))| \leq C_\zeta(1 + |T_{1/\varepsilon}(u)|^{p^+-1}) \leq C_\zeta(1 + \varepsilon^{-(p^+-1)}).$$

Thus,

$$\left| \int_\Omega \zeta_\varepsilon(T_{1/\varepsilon}(u)) \varphi dx \right| \leq C(\varepsilon) \|\varphi\|_{L^1(\Omega)} \leq C(\varepsilon) \|\varphi\|_{L^{p(\cdot)}(\Omega)}.$$

For the principal part, using Corollary 2.13 and the norm-modular equivalence,

$$\begin{aligned} \left| \int_\Omega H^{p(x)-1}(\nabla u) \nabla H(\nabla u) \cdot \nabla \varphi dx \right| &\leq \int_\Omega |H^{p(x)-1}(\nabla u) \nabla H(\nabla u)| |\nabla \varphi| dx \\ &\leq C_2 \int_\Omega |\nabla u|^{p(x)-1} |\nabla \varphi| dx. \end{aligned}$$

By Hölder's inequality in variable-exponent spaces,

$$\int_\Omega |\nabla u|^{p(x)-1} |\nabla \varphi| dx \leq 2 \| |\nabla u|^{p(\cdot)-1} \|_{L^{p'(\cdot)}} \| \nabla \varphi \|_{L^{p(\cdot)}}.$$

Now, $\| |\nabla u|^{p(\cdot)-1} \|_{L^{p'(\cdot)}} \leq C \| \nabla u \|_{L^{p(\cdot)}}^{p^+-1}$ by standard properties of variable-exponent spaces. Hence,

$$\text{Term 2} \leq CR^{p^+-1} \|\varphi\|_{W^{1,p(\cdot)}}.$$

For the natural-growth term, by the definition of G_ε , we have $|G_\varepsilon(x, s, \xi)| \leq \min\{|G(x, s, \xi)|, 1/\varepsilon\} \leq 1/\varepsilon$ pointwise. Thus,

$$\left| \int_\Omega G_\varepsilon(x, u, \nabla u) \varphi dx \right| \leq \frac{1}{\varepsilon} \int_\Omega |\varphi| dx.$$

By the continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ (which holds because Ω is bounded and $p(\cdot) \geq p^- > 1$), there exists a constant $C > 0$ such that $\|\varphi\|_{L^1} \leq C\|\varphi\|_{W^{1,p(\cdot)}}$. Therefore,

$$\left| \int_{\Omega} G_{\varepsilon}(x, u, \nabla u) \varphi dx \right| \leq \frac{C}{\varepsilon} \|\varphi\|_{W^{1,p(\cdot)}}.$$

Combining all estimates, we obtain

$$|\langle \mathcal{A}_{\varepsilon} u, \varphi \rangle| \leq C(\varepsilon, R) \|\varphi\|_{W^{1,p(\cdot)}},$$

showing that $\mathcal{A}_{\varepsilon}$ is bounded.

Lemma 3.2. *The operator $\mathcal{A}_{\varepsilon}$ is coercive: there exists $\alpha_0 > 0$ such that for all $u \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\frac{\langle \mathcal{A}_{\varepsilon} u, u \rangle}{\|u\|_{W^{1,p(\cdot)}}} \geq \alpha_0 \frac{\min\{\|\nabla u\|_{L^{p(\cdot)}}^{p^-}, \|\nabla u\|_{L^{p(\cdot)}}^{p^+}\}}{\|u\|_{W^{1,p(\cdot)}}}.$$

Consequently,

$$\frac{\langle \mathcal{A}_{\varepsilon} u, u \rangle}{\|u\|_{W^{1,p(\cdot)}}} \rightarrow \infty \quad \text{as } \|u\|_{W^{1,p(\cdot)}} \rightarrow \infty.$$

Proof. For any $u \in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\langle \mathcal{A}_{\varepsilon} u, u \rangle = \int_{\Omega} \zeta_{\varepsilon}(T_{1/\varepsilon}(u)) u dx + \int_{\Omega} H^{p(x)}(\nabla u) dx + \int_{\Omega} G_{\varepsilon}(x, u, \nabla u) u dx.$$

All three terms are nonnegative. By Assumption 2.17, $H^{p(x)}(\nabla u) \geq a^{p(x)} |\nabla u|^{p(x)} \geq \alpha |\nabla u|^{p(x)}$ with $\alpha = a^{p^+}$, and by Assumption 2.20(G1), $G_{\varepsilon}(x, u, \nabla u) u \geq 0$ since G_{ε} preserves the sign of G .

Thus,

$$\langle \mathcal{A}_{\varepsilon} u, u \rangle \geq \alpha \int_{\Omega} |\nabla u|^{p(x)} dx.$$

Using the norm-modular equivalence (Lemma 2.3(3)),

$$\int_{\Omega} |\nabla u|^{p(x)} dx \geq \min\{\|\nabla u\|_{L^{p(\cdot)}}^{p^-}, \|\nabla u\|_{L^{p(\cdot)}}^{p^+}\}.$$

By the Poincaré inequality (Lemma 2.6(2)),

$$\|u\|_{W^{1,p(\cdot)}} \leq (C_P + 1) \|\nabla u\|_{L^{p(\cdot)}}.$$

Therefore,

$$\frac{\langle \mathcal{A}_{\varepsilon} u, u \rangle}{\|u\|_{W^{1,p(\cdot)}}} \geq \frac{\alpha}{C_P + 1} \frac{\min\{\|\nabla u\|_{L^{p(\cdot)}}^{p^-}, \|\nabla u\|_{L^{p(\cdot)}}^{p^+}\}}{\|\nabla u\|_{L^{p(\cdot)}}}.$$

As $\|\nabla u\|_{L^{p(\cdot)}} \rightarrow \infty$, this ratio tends to ∞ since $p^- > 1$.

Lemma 3.3. *The operator $\mathcal{A}_{\varepsilon} : W_0^{1,p(\cdot)}(\Omega) \rightarrow W^{-1,p'(\cdot)}(\Omega)$ is pseudo-monotone.*

Proof. Let $(u_m) \subset W_0^{1,p(\cdot)}(\Omega)$ be such that $u_m \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)}(\Omega)$ and

$$\limsup_{m \rightarrow \infty} \langle \mathcal{A}_\varepsilon u_m, u_m - u \rangle \leq 0. \quad (3.2)$$

By the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ (Lemma 2.6), we have $u_m \rightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$, and hence in $L^1(\Omega)$.

Step 1: Convergence of lower-order terms. For the Yosida term, since ζ_ε and $T_{1/\varepsilon}$ are Lipschitz,

$$\zeta_\varepsilon(T_{1/\varepsilon}(u_m)) \rightarrow \zeta_\varepsilon(T_{1/\varepsilon}(u)) \quad \text{in } L^q(\Omega) \quad \forall q < \infty,$$

which implies

$$\int_{\Omega} \zeta_\varepsilon(T_{1/\varepsilon}(u_m))(u_m - u) \, dx \rightarrow 0. \quad (3.3)$$

For the truncated nonlinearity G_ε , by construction, $|G_\varepsilon(x, s, \xi)| \leq 1/\varepsilon$ for all x, s, ξ . Hence

$$\left| \int_{\Omega} G_\varepsilon(x, u_m, \nabla u_m)(u_m - u) \, dx \right| \leq \frac{1}{\varepsilon} \int_{\Omega} |u_m - u| \, dx \rightarrow 0. \quad (3.4)$$

Combining Eqs (3.2)–(3.4) yields

$$\limsup_{m \rightarrow \infty} \langle P(u_m), u_m - u \rangle \leq 0, \quad (3.5)$$

where $P(u) := -\operatorname{div}(H^{p(\cdot)-1}(\nabla u)\nabla H(\nabla u))$ is the principal part operator.

Step 2: Strong convergence of gradients. Since $u_m \rightharpoonup u$, we have $\langle P(u), u_m - u \rangle \rightarrow 0$. From Eq (3.5),

$$0 \leq \langle P(u_m) - P(u), u_m - u \rangle = \langle P(u_m), u_m - u \rangle - \langle P(u), u_m - u \rangle,$$

so taking the limsup gives

$$\lim_{m \rightarrow \infty} \langle P(u_m) - P(u), u_m - u \rangle = 0. \quad (3.6)$$

We prove that

$$\nabla u_m \rightarrow \nabla u \quad \text{strongly in } (L^{p(\cdot)}(\Omega))^N. \quad (3.7)$$

Denote $z_m := \nabla u_m - \nabla u$ and $a_m := |\nabla u_m| + |\nabla u|$. By the monotonicity estimate (Lemma 2.12), there is $C_1 > 0$ such that

$$\langle P(u_m) - P(u), u_m - u \rangle \geq C_1 \int_{\Omega} a_m^{p(x)-2} |z_m|^2 \, dx. \quad (3.8)$$

We split into two cases.

Case 1: $p(x) \geq 2$ a.e. Lemma 2.12 also gives

$$\langle P(u_m) - P(u), u_m - u \rangle \geq C'_1 \int_{\Omega} |z_m|^{p(x)} \, dx.$$

Together with Eq (3.6), this yields $\int_{\Omega} |z_m|^{p(x)} \, dx \rightarrow 0$, which establishes Eq (3.7).

Case 2: $1 < p(x) < 2$ a.e. From Eqs (3.6) and (3.8), we get

$$X_m := \int_{\Omega} a_m^{p(x)-2} |z_m|^2 dx \rightarrow 0. \quad (3.9)$$

We claim that Eq (3.9) implies $\int_{\Omega} |z_m|^{p(x)} dx \rightarrow 0$. Indeed, for every $\eta > 0$, the elementary inequality

$$t^p \leq C_{\eta} t^2 + \eta \quad \forall t \geq 0, \quad 1 < p < 2,$$

with C_{η} depending only on η and $p^{\pm}$, yields (setting $t = |z_m|/a_m$ when $a_m > 0$):

$$|z_m|^{p(x)} \leq C_{\eta} a_m^{p(x)-2} |z_m|^2 + \eta a_m^{p(x)} \quad \text{a.e.}$$

Integrating,

$$\int_{\Omega} |z_m|^{p(x)} dx \leq C_{\eta} X_m + \eta \int_{\Omega} a_m^{p(x)} dx.$$

By the uniform energy bound (Lemma 4.1), $\int_{\Omega} a_m^{p(x)} dx \leq C$ uniformly in m . Thus

$$\limsup_{m \rightarrow \infty} \int_{\Omega} |z_m|^{p(x)} dx \leq \eta C.$$

Letting $\eta \downarrow 0$ gives $\int_{\Omega} |z_m|^{p(x)} dx \rightarrow 0$, and by the norm-modular equivalence (Lemma 2.3), we obtain Eq (3.7).

Step 3: Passing to the limit. From Eq (3.7), we have $\nabla u_m \rightarrow \nabla u$ in measure, so, up to a subsequence, $\nabla u_m \rightarrow \nabla u$ a.e. and $u_m \rightarrow u$ a.e. By the Carathéodory property of G ,

$$G_{\varepsilon}(x, u_m, \nabla u_m) \rightarrow G_{\varepsilon}(x, u, \nabla u) \quad \text{a.e.}$$

Moreover, since $|G_{\varepsilon}| \leq 1/\varepsilon$, the dominated convergence theorem gives

$$G_{\varepsilon}(\cdot, u_m, \nabla u_m) \rightarrow G_{\varepsilon}(\cdot, u, \nabla u) \quad \text{in } L^1(\Omega). \quad (3.10)$$

Also, the strong convergence Eq (3.7) implies the principal part operator is continuous in the strong topology, so for every $v \in W_0^{1,p(\cdot)}(\Omega)$,

$$\langle P(u_m), u_m - v \rangle \rightarrow \langle P(u), u - v \rangle.$$

Combining this with the convergences of the ζ_{ε} -term and Eq (3.10), we obtain

$$\langle \mathcal{A}_{\varepsilon} u_m, u_m - v \rangle \rightarrow \langle \mathcal{A}_{\varepsilon} u, u - v \rangle.$$

Hence

$$\liminf_{m \rightarrow \infty} \langle \mathcal{A}_{\varepsilon} u_m, u_m - v \rangle = \langle \mathcal{A}_{\varepsilon} u, u - v \rangle,$$

which implies the pseudo-monotonicity condition.

With the properties established above, we can now prove the existence of solutions to (E_{ε}, f) .

Proposition 3.4. *For each $\varepsilon \in (0, 1]$, the approximate problem (E_ε, f) admits at least one weak solution $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Define the Nemytskii operator $\mathcal{F}_\varepsilon : W_0^{1,p(\cdot)}(\Omega) \rightarrow W^{-1,p'(\cdot)}(\Omega)$ by

$$\langle \mathcal{F}_\varepsilon(u), \varphi \rangle = \int_\Omega \frac{f\varphi}{(|u| + \varepsilon)^\gamma} dx.$$

We first show that $\mathcal{F}_\varepsilon$ is compact (i.e., it maps weakly convergent sequences to strongly convergent ones). Indeed, since the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact, if $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$, then $u_n \rightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$. The function $g_\varepsilon(t) = 1/(|t| + \varepsilon)^\gamma$ is Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant $L_\varepsilon \leq \gamma \varepsilon^{-\gamma-1}$. Thus,

$$\|g_\varepsilon(u_n) - g_\varepsilon(u)\|_{L^{p(\cdot)}(\Omega)} \leq C\|g_\varepsilon(u_n) - g_\varepsilon(u)\|_{L^{p^-}(\Omega)} \leq CL_\varepsilon\|u_n - u\|_{L^{p^-}(\Omega)} \rightarrow 0.$$

Since multiplication by $f \in L^\infty(\Omega)$ is continuous on $L^{p^-}(\Omega)$, we get

$$\frac{f}{(|u_n| + \varepsilon)^\gamma} \rightarrow \frac{f}{(|u| + \varepsilon)^\gamma} \quad \text{in } L^{p^-}(\Omega) \subset W^{-1,p'(\cdot)}(\Omega).$$

Hence $\mathcal{F}_\varepsilon$ is compact (indeed, completely continuous).

Now consider the operator $\mathcal{B}_\varepsilon := \mathcal{A}_\varepsilon - \mathcal{F}_\varepsilon$. By Lemma 3.1, $\mathcal{A}_\varepsilon$ is bounded, and $\mathcal{F}_\varepsilon$ is bounded because $\|\mathcal{F}_\varepsilon(u)\|_{W^{-1,p'(\cdot)}} \leq C\|f\|_\infty/\varepsilon^\gamma$. Hence $\mathcal{B}_\varepsilon$ is bounded. By Lemma 3.2 and the boundedness of $\mathcal{F}_\varepsilon$, $\mathcal{B}_\varepsilon$ is coercive: indeed,

$$\frac{\langle \mathcal{B}_\varepsilon u, u \rangle}{\|u\|_{W^{1,p}}} \geq \frac{\langle \mathcal{A}_\varepsilon u, u \rangle}{\|u\|_{W^{1,p}}} - \frac{\|\mathcal{F}_\varepsilon(u)\|_{W^{-1,p'}}\|u\|_{W^{1,p}}}{\|u\|_{W^{1,p}}} \rightarrow \infty$$

as $\|u\|_{W^{1,p}} \rightarrow \infty$ because the first term tends to infinity and the second is bounded.

It remains to show that $\mathcal{B}_\varepsilon$ is pseudo-monotone. Let $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$ and suppose

$$\limsup_{n \rightarrow \infty} \langle \mathcal{B}_\varepsilon u_n, u_n - u \rangle \leq 0.$$

Since $\mathcal{F}_\varepsilon$ is compact, $\mathcal{F}_\varepsilon(u_n) \rightarrow \mathcal{F}_\varepsilon(u)$ strongly in $W^{-1,p'(\cdot)}(\Omega)$. Hence

$$\langle \mathcal{F}_\varepsilon(u_n), u_n - u \rangle \rightarrow 0$$

because $u_n - u \rightharpoonup 0$. Therefore,

$$\limsup_{n \rightarrow \infty} \langle \mathcal{A}_\varepsilon u_n, u_n - u \rangle = \limsup_{n \rightarrow \infty} \langle \mathcal{B}_\varepsilon u_n, u_n - u \rangle + \lim_{n \rightarrow \infty} \langle \mathcal{F}_\varepsilon(u_n), u_n - u \rangle \leq 0.$$

By the pseudo-monotonicity of $\mathcal{A}_\varepsilon$ (Lemma 3.3), for every $v \in W_0^{1,p(\cdot)}(\Omega)$,

$$\liminf_{n \rightarrow \infty} \langle \mathcal{A}_\varepsilon u_n, u_n - v \rangle \geq \langle \mathcal{A}_\varepsilon u, u - v \rangle.$$

Since $\langle \mathcal{F}_\varepsilon(u_n), u_n - v \rangle \rightarrow \langle \mathcal{F}_\varepsilon(u), u - v \rangle$, it follows that

$$\liminf_{n \rightarrow \infty} \langle \mathcal{B}_\varepsilon u_n, u_n - v \rangle \geq \langle \mathcal{B}_\varepsilon u, u - v \rangle.$$

Thus $\mathcal{B}_\varepsilon$ is pseudo-monotone.

Applying the standard existence theorem for bounded, coercive, pseudo-monotone operators (see, e.g., [7, Theorem 2.1]), we obtain $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ such that $\mathcal{B}_\varepsilon u_\varepsilon = 0$, i.e.,

$$\mathcal{A}_\varepsilon u_\varepsilon = \mathcal{F}_\varepsilon(u_\varepsilon),$$

which is exactly the weak form of (E_ε, f) . The L^∞ -bound follows from Lemma 4.3 (which will be proved independently of this existence result. Note that the estimate there does not use the existence of u_ε , only the equation itself). Thus the proof is complete.

4. Uniform a priori estimates

In this section, we derive estimates for the approximate solutions u_ε that are independent of ε . These estimates are crucial for the convergence analysis as $\varepsilon \rightarrow 0$.

Lemma 4.1. *The family $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is uniformly bounded in $W_0^{1,p(\cdot)}(\Omega)$. More precisely, there exists a constant $C_1 > 0$, independent of ε , such that*

$$\|u_\varepsilon\|_{W^{1,p(\cdot)}(\Omega)} \leq C_1.$$

Proof. Test the approximate equation (3.1) with $\varphi = u_\varepsilon$:

$$\int_{\Omega} \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) u_\varepsilon dx + \int_{\Omega} H^{p(x)}(\nabla u_\varepsilon) dx + \int_{\Omega} G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon dx = \int_{\Omega} \frac{f u_\varepsilon}{(|u_\varepsilon| + \varepsilon)^\gamma} dx. \quad (4.1)$$

All terms on the left-hand side are nonnegative. Using the coercivity of the principal part,

$$\int_{\Omega} H^{p(x)}(\nabla u_\varepsilon) dx \geq \alpha \int_{\Omega} |\nabla u_\varepsilon|^{p(x)} dx,$$

with $\alpha = a^{p^+} > 0$.

For the right-hand side, note that $u_\varepsilon \geq 0$ (this will be proved in Lemma 4.2 below). Thus,

$$\frac{f u_\varepsilon}{(|u_\varepsilon| + \varepsilon)^\gamma} \leq f u_\varepsilon^{1-\gamma} \chi_{\{u_\varepsilon > 0\}}.$$

Since $0 < \gamma \leq 1$, we have $u_\varepsilon^{1-\gamma} \leq 1 + u_\varepsilon$. Therefore,

$$\frac{f u_\varepsilon}{(|u_\varepsilon| + \varepsilon)^\gamma} \leq \|f\|_\infty (1 + u_\varepsilon).$$

Integrating,

$$\int_{\Omega} \frac{f u_\varepsilon}{(|u_\varepsilon| + \varepsilon)^\gamma} dx \leq \|f\|_\infty (|\Omega| + \|u_\varepsilon\|_{L^1}).$$

By Hölder's inequality and the Poincaré inequality,

$$\|u_\varepsilon\|_{L^1} \leq C \|\nabla u_\varepsilon\|_{L^{p(\cdot)}}.$$

Thus,

$$\alpha \int_{\Omega} |\nabla u_{\varepsilon}|^{p(x)} dx \leq \|f\|_{\infty} |\Omega| + C \|f\|_{\infty} \|\nabla u_{\varepsilon}\|_{L^{p(\cdot)}}.$$

Using the norm-modular equivalence, let $X = \|\nabla u_{\varepsilon}\|_{L^{p(\cdot)}}$. Then

$$\alpha \min\{X^{p^-}, X^{p^+}\} \leq C_1 + C_2 X.$$

If $X \leq 1$, then X is trivially bounded. If $X \geq 1$, then $\min\{X^{p^-}, X^{p^+}\} = X^{p^-}$, so

$$\alpha X^{p^-} \leq C_1 + C_2 X.$$

Since $X \geq 1$, we have $C_1 \leq C_1 X$, hence

$$\alpha X^{p^-} \leq (C_1 + C_2) X.$$

Dividing by $X > 0$ yields

$$X^{p^- - 1} \leq \frac{C_1 + C_2}{\alpha},$$

and because $p^- > 1$, this implies

$$X \leq \left(\frac{C_1 + C_2}{\alpha} \right)^{1/(p^- - 1)}.$$

Thus X is bounded independently of ε . Hence, there exists $C_1 > 0$ independent of ε such that

$$\int_{\Omega} |\nabla u_{\varepsilon}|^{p(x)} dx \leq C_1.$$

By the Poincaré inequality and norm-modular equivalence,

$$\|u_{\varepsilon}\|_{W^{1,p(\cdot)}} \leq C_P \|\nabla u_{\varepsilon}\|_{L^{p(\cdot)}} + \|\nabla u_{\varepsilon}\|_{L^{p(\cdot)}} \leq (C_P + 1) \max\{C_1^{1/p^-}, C_1^{1/p^+}\} =: C'_1.$$

Lemma 4.2. *The approximate solutions satisfy $u_{\varepsilon}(x) \geq 0$ for almost every $x \in \Omega$ and all $\varepsilon \in (0, 1]$.*

Proof. Let $u_{\varepsilon}^- = \max\{-u_{\varepsilon}, 0\} \in W_0^{1,p(\cdot)}(\Omega)$. Note that $u_{\varepsilon}^- \geq 0$ and $\nabla u_{\varepsilon}^- = -\nabla u_{\varepsilon} \chi_{\{u_{\varepsilon} < 0\}}$.

Test the approximate equation (3.1) with $\varphi = u_{\varepsilon}^-$:

$$\begin{aligned} \int_{\Omega} \zeta_{\varepsilon}(T_{1/\varepsilon}(u_{\varepsilon})) u_{\varepsilon}^- dx + \int_{\Omega} H^{p(x)-1}(\nabla u_{\varepsilon}) \nabla H(\nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^- dx \\ + \int_{\Omega} G_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) u_{\varepsilon}^- dx = \int_{\Omega} \frac{f u_{\varepsilon}^-}{(|u_{\varepsilon}| + \varepsilon)^{\gamma}} dx. \end{aligned}$$

On the set $\{u_{\varepsilon} < 0\}$, we have $u_{\varepsilon}^- = -u_{\varepsilon} > 0$ and $\nabla u_{\varepsilon}^- = -\nabla u_{\varepsilon}$. Therefore,

$$H^{p(x)-1}(\nabla u_{\varepsilon}) \nabla H(\nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^- = -H^{p(x)}(\nabla u_{\varepsilon}) \leq 0.$$

Since $\zeta_{\varepsilon}(r)r \geq 0$ for all r (Lemma 2.15(4)), and on $\{u_{\varepsilon} < 0\}$, we have $T_{1/\varepsilon}(u_{\varepsilon}) = u_{\varepsilon} < 0$, we obtain

$$\zeta_{\varepsilon}(T_{1/\varepsilon}(u_{\varepsilon})) u_{\varepsilon}^- = \zeta_{\varepsilon}(u_{\varepsilon})(-u_{\varepsilon}) = -\zeta_{\varepsilon}(u_{\varepsilon}) u_{\varepsilon} \leq 0.$$

For the natural growth term, Assumption 2.20(G1) gives $G(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon \geq 0$. Since G_ε preserves the sign of G , we have $G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon \geq 0$. On $\{u_\varepsilon < 0\}$, this implies $G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \leq 0$. Hence,

$$G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon^- = G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(-u_\varepsilon) \leq 0.$$

The right-hand side satisfies

$$\frac{fu_\varepsilon^-}{(|u_\varepsilon| + \varepsilon)^\gamma} \geq 0,$$

since $f \geq 0$ and $u_\varepsilon^- \geq 0$.

Thus, on the set $\{u_\varepsilon < 0\}$, every term on the left-hand side is nonpositive, while the right-hand side is nonnegative. Therefore, both sides must be zero almost everywhere on $\{u_\varepsilon < 0\}$. In particular,

$$\int_{\{u_\varepsilon < 0\}} H^{p(x)}(\nabla u_\varepsilon) dx = 0,$$

which implies $\nabla u_\varepsilon = 0$ a.e. on $\{u_\varepsilon < 0\}$. Since $u_\varepsilon^- \in W_0^{1,p(\cdot)}(\Omega)$, the Poincaré inequality (Lemma 2.6) yields $u_\varepsilon^- = 0$ a.e., hence $u_\varepsilon \geq 0$ a.e. in Ω .

We now prove uniform L^∞ bounds for u_ε . The proof adapts the classical method of Boccardo-Giachetti [5] to the anisotropic variable-exponent setting.

Lemma 4.3. *The family $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, there exists a constant $C_\infty > 0$, independent of ε , such that*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C_\infty \|f\|_{L^\infty(\Omega)}.$$

Proof. For $k > 0$, define the truncation $G_k(s) = (s - k)^+$ (since $u_\varepsilon \geq 0$ by Lemma 4.2). Test the approximate equation (3.1) with $\varphi = G_k(u_\varepsilon)$:

$$\begin{aligned} \int_{\Omega} \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))G_k(u_\varepsilon)dx + \int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon)\nabla H(\nabla u_\varepsilon) \cdot \nabla G_k(u_\varepsilon)dx \\ + \int_{\Omega} G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)G_k(u_\varepsilon)dx = \int_{\Omega} \frac{fG_k(u_\varepsilon)}{(|u_\varepsilon| + \varepsilon)^\gamma}dx. \end{aligned} \quad (4.2)$$

All terms on the left are nonnegative. Using the coercivity of the principal part,

$$\int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon)\nabla H(\nabla u_\varepsilon) \cdot \nabla G_k(u_\varepsilon)dx = \int_{A_k} H^{p(x)}(\nabla u_\varepsilon)dx \geq \alpha \int_{A_k} |\nabla u_\varepsilon|^{p(x)}dx,$$

where $A_k = \{x \in \Omega : u_\varepsilon(x) > k\}$.

On A_k , we have $u_\varepsilon + \varepsilon > k$, so

$$\frac{fG_k(u_\varepsilon)}{(|u_\varepsilon| + \varepsilon)^\gamma} \leq \frac{f(u_\varepsilon - k)}{k^\gamma} \leq \frac{\|f\|_\infty}{k^\gamma}(u_\varepsilon - k).$$

Thus,

$$\alpha \int_{A_k} |\nabla u_\varepsilon|^{p(x)}dx \leq \frac{\|f\|_\infty}{k^\gamma} \int_{A_k} (u_\varepsilon - k)dx.$$

Let $w_k = (u_\varepsilon - k)^+$. Then $w_k \in W_0^{1,p(\cdot)}(A_k)$ and $w_k = 0$ on ∂A_k . By the Sobolev embedding (Lemma 2.6(3)),

$$\|w_k\|_{L^{p^*(\cdot)}(A_k)} \leq C_S \|\nabla w_k\|_{L^{p(\cdot)}(A_k)}.$$

Note that $\nabla w_k = \nabla u_\varepsilon$ on A_k and $w_k = u_\varepsilon - k$ on A_k . Let $q = \inf_{x \in A_k} p^*(x)$. Then $q \geq p^{*, -}$, where $p^{*, -} = \frac{Np^-}{N-p^-}$ if $p^- < N$, and q can be chosen arbitrarily large if $p^- \geq N$. By Hölder's inequality,

$$\int_{A_k} w_k dx \leq |A_k|^{1-1/q} \|w_k\|_{L^q(A_k)} \leq |A_k|^{1-1/q} C_S \|\nabla u_\varepsilon\|_{L^{p(\cdot)}(A_k)}.$$

Now, using the norm-modular equivalence,

$$\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(A_k)} \leq C \left(\int_{A_k} |\nabla u_\varepsilon|^{p(x)} dx \right)^{1/p^-} + C \left(\int_{A_k} |\nabla u_\varepsilon|^{p(x)} dx \right)^{1/p^+}.$$

Combining these estimates, we obtain a differential inequality for $y(k) = \int_{A_k} w_k dx$. A detailed iteration (see [4, Theorem 1.1] for the variable exponent case) gives

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)},$$

with C independent of ε . The linear dependence on $\|f\|_{L^\infty}$ arises from the structure of the iteration: at each step k , the inequality produces a factor $\|f\|_\infty^{1/p^-}$ and the Moser iteration yields the final bound. The constant C depends only on $N, p^\pm, a, b, \gamma$, and Ω .

The following lemma provides the necessary control on the regularized singular term, uniformly in ε . It replaces the standard but technically demanding pointwise lower bound and is sufficient for the compactness argument.

Lemma 4.4. *For every relatively compact subset $\omega \subset\subset \Omega$, there exists a constant $C_\omega > 0$, independent of ε , such that*

$$\int_\omega (u_\varepsilon(x) + \varepsilon)^{-\gamma} dx \leq C_\omega. \quad (4.3)$$

Consequently, for the almost-everywhere limit u , we have $\int_\omega u^{-\gamma} dx \leq C_\omega$.

Proof. Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on ω , and $\text{supp } \eta \subset \Omega' \subset\subset \Omega$. We use the admissible test function

$$\varphi_\varepsilon = (u_\varepsilon + \varepsilon)^{-\gamma} \eta^2 \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

Inserting φ_ε into the weak formulation of (E_ε, f) and dropping the nonnegative terms involving ζ_ε and G_ε yields

$$\int_\Omega H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) \cdot \nabla \varphi_\varepsilon dx \leq \int_\Omega \frac{f \varphi_\varepsilon}{(u_\varepsilon + \varepsilon)^\gamma} dx.$$

A direct computation of the left-hand side gives

$$-\gamma \int_\Omega (u_\varepsilon + \varepsilon)^{-\gamma-1} \eta^2 H^{p(\cdot)}(\nabla u_\varepsilon) dx + 2 \int_\Omega (u_\varepsilon + \varepsilon)^{-\gamma} \eta H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) \cdot \nabla \eta dx \leq \int_\Omega f (u_\varepsilon + \varepsilon)^{-2\gamma} \eta^2 dx. \quad (4.4)$$

By Young's inequality with parameter $\theta > 0$, we estimate the second term on the left:

$$\begin{aligned} & 2 \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma} \eta H^{p(\cdot)}(\nabla u_{\varepsilon}) \nabla H(\nabla u_{\varepsilon}) \cdot \nabla \eta \, dx \\ & \leq \frac{\gamma}{2} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 H^{p(\cdot)}(\nabla u_{\varepsilon}) \, dx + C_{\gamma} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma+1} H^{p(\cdot)}(\nabla \eta) \, dx. \end{aligned}$$

Absorbing the first term above into the left-hand side, we obtain

$$\frac{\gamma}{2} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 H^{p(\cdot)}(\nabla u_{\varepsilon}) \, dx \leq C_{\gamma} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma+1} H^{p(\cdot)}(\nabla \eta) \, dx + \|f\|_{\infty} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-2\gamma} \eta^2 \, dx.$$

Since $u_{\varepsilon} \in L^{\infty}(\Omega)$ with $\|u_{\varepsilon}\|_{\infty} \leq M$ (Lemma 4.3), and $0 < \gamma \leq 1$, we have $(u_{\varepsilon} + \varepsilon)^{-\gamma+1} \leq (M+1)^{-\gamma+1}$ and

$$(u_{\varepsilon} + \varepsilon)^{-2\gamma} = (u_{\varepsilon} + \varepsilon)^{-\gamma-1} (u_{\varepsilon} + \varepsilon)^{1-\gamma} \leq (M+1)^{1-\gamma} (u_{\varepsilon} + \varepsilon)^{-\gamma-1}.$$

Therefore,

$$\frac{\gamma}{2} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 H^{p(\cdot)}(\nabla u_{\varepsilon}) \, dx \leq C_1 \int_{\Omega} H^{p(\cdot)}(\nabla \eta) \, dx + C_2 \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 \, dx,$$

where C_1 and C_2 depend only on M, γ , and $\|f\|_{\infty}$. Now choose $\theta > 0$ so small that $C_2\theta \leq \gamma/4$. By Young's inequality,

$$C_2(u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 \leq \frac{\gamma}{4} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 H^{p(\cdot)}(\nabla u_{\varepsilon}) + C_3 \eta^2.$$

Substituting this back and rearranging, we get

$$\int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 H^{p(\cdot)}(\nabla u_{\varepsilon}) \, dx \leq C_4 \left(\int_{\Omega} H^{p(\cdot)}(\nabla \eta) \, dx + \int_{\Omega} \eta^2 \, dx \right).$$

Finally, since $H^{p(\cdot)}(\nabla u_{\varepsilon}) \geq a^{p^+} |\nabla u_{\varepsilon}|^{p(\cdot)}$, the above estimate implies in particular that

$$\int_{\Omega} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 |\nabla u_{\varepsilon}|^{p(\cdot)} \, dx \leq C_5.$$

Applying the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega') \hookrightarrow L^q(\Omega')$ with $q > 1$ sufficiently close to 1 (which is possible since $p^- > 1$ and Ω' is bounded), we obtain

$$\int_{\omega} (u_{\varepsilon} + \varepsilon)^{-\gamma} \, dx \leq C_6 \left(\int_{\Omega'} (u_{\varepsilon} + \varepsilon)^{-\gamma-1} \eta^2 |\nabla u_{\varepsilon}|^{p(\cdot)} \, dx + \int_{\Omega'} (u_{\varepsilon} + \varepsilon)^{-\gamma+1} \eta^2 \, dx \right)^{1/q} \leq C_{\omega}.$$

The constant C_{ω} is independent of ε . This proves Eq (4.3). The claim for the limit u follows directly from Fatou's lemma.

5. Convergence analysis

With uniform estimates in hand, we now study the limit as $\varepsilon \rightarrow 0$.

Lemma 5.1. *There exist functions $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$ and $b \in L^{\infty}(\Omega)$ such that, up to a subsequence $\varepsilon \rightarrow 0$:*

(C1) $u_\varepsilon \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)}(\Omega)$.

(C2) $u_\varepsilon \rightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$ and almost everywhere in Ω .

(C3) $\zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \xrightarrow{*} b$ weak-* in $L^\infty(\Omega)$.

(C4) $b(x) \in \zeta(u(x))$ for almost every $x \in \Omega$.

(C5) $u(x) \geq 0$ a.e., and for any $\omega \subset\subset \Omega$, $\int_\omega u^{-\gamma} dx < \infty$, hence $u > 0$ a.e.

Proof. By Lemma 4.1, $\{u_\varepsilon\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. By reflexivity, there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly. The compact embedding (Lemma 2.6(4)) gives $u_\varepsilon \rightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$ and, up to a subsequence, almost everywhere.

From Lemma 4.3, $\|u_\varepsilon\|_{L^\infty} \leq C_\infty$. Thus, $|T_{1/\varepsilon}(u_\varepsilon)| \leq \min\{C_\infty, 1/\varepsilon\}$. By Lemma 2.15(3), $|\zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))| \leq C_\zeta(1 + |T_{1/\varepsilon}(u_\varepsilon)|^{p^+-1}) \leq C_\zeta(1 + C_\infty^{p^+-1})$, independent of ε . Hence, $\{\zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\}$ is bounded in $L^\infty(\Omega)$. By the Banach-Alaoglu theorem, there exists $b \in L^\infty(\Omega)$ such that $\zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \xrightarrow{*} b$ in $L^\infty(\Omega)$.

Let $J_\varepsilon = (I + \varepsilon\zeta)^{-1}$ be the resolvent. Then $\zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \in \zeta(J_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))$. We have

$$|J_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - u| \leq |J_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - T_{1/\varepsilon}(u_\varepsilon)| + |T_{1/\varepsilon}(u_\varepsilon) - u|.$$

The first term tends to 0 by properties of resolvents and the second by a.e. convergence of u_ε . Thus, $J_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightarrow u$ in measure. By the closedness of maximal monotone graphs (see [7, Proposition 2.5]), we obtain $b(x) \in \zeta(u(x))$ a.e.

The nonnegativity of u follows from Lemma 4.2 and the strong convergence. The local integrability of $u^{-\gamma}$ follows from Lemma 4.4 and Fatou's lemma. Since $u^{-\gamma} \in L_{\text{loc}}^1$, we have $u > 0$ a.e. in Ω .

Lemma 5.2. For any $\varphi \in C_0^1(\Omega)$,

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega \frac{f\varphi}{(u_\varepsilon(x) + \varepsilon)^\gamma} dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx.$$

Proof. Let $\omega = \text{supp}(\varphi) \subset\subset \Omega$. By Lemma 4.4, the family

$$\left\{ \frac{f\varphi}{(u_\varepsilon + \varepsilon)^\gamma} \right\}$$

is uniformly integrable on ω . Indeed, it is bounded by $\|f\|_\infty \|\varphi\|_\infty (u_\varepsilon + \varepsilon)^{-\gamma}$, and the latter is uniformly bounded in $L^1(\omega)$ by Lemma 4.4. Moreover, we have the stronger equi-integrability estimate

$$\lim_{|E| \rightarrow 0} \sup_{\varepsilon > 0} \int_E \frac{f|\varphi|}{(u_\varepsilon + \varepsilon)^\gamma} dx = 0,$$

which follows from the uniform L^∞ bound on u_ε , the uniform L_{loc}^1 bound on $u_\varepsilon^{-\gamma}$ from Lemma 4.4, and the absolute continuity of the integral. Moreover, by Lemma 5.1(C2), $u_\varepsilon \rightarrow u$ a.e. in Ω , and hence

$$\frac{f\varphi}{(u_\varepsilon + \varepsilon)^\gamma} \rightarrow \frac{f\varphi}{u^\gamma} \quad \text{a.e. in } \Omega.$$

Vitali's convergence theorem then gives the desired L^1 -convergence.

Lemma 5.3. *Up to a subsequence, $\nabla u_\varepsilon \rightarrow \nabla u$ strongly in $(L^{p(\cdot)}(\Omega))^N$.*

Proof. Let u be the weak limit of $\{u_\varepsilon\}$ in $W_0^{1,p(\cdot)}(\Omega)$. By the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, we have $u_\varepsilon \rightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$ and almost everywhere.

Test the approximate equation (3.1) with $\varphi = u_\varepsilon - u$:

$$\begin{aligned} \int_{\Omega} \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))(u_\varepsilon - u)dx + \int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u)dx \\ + \int_{\Omega} G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(u_\varepsilon - u)dx = \int_{\Omega} \frac{f(u_\varepsilon - u)}{(u_\varepsilon + \varepsilon)^\gamma} dx. \end{aligned}$$

Since $\{\zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\}$ is bounded in $L^\infty(\Omega)$ and $u_\varepsilon - u \rightarrow 0$ strongly in $L^1(\Omega)$, the first term tends to 0.

For the natural growth term, note that $|G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \leq c(\|u_\varepsilon\|_\infty)(m(x) + |\nabla u_\varepsilon|^{p(x)})$ with $\|u_\varepsilon\|_\infty$ uniformly bounded by Lemma 4.3. Hence $\{G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\}$ is bounded in $L^1(\Omega)$. Since $u_\varepsilon - u \rightarrow 0$ strongly in $L^q(\Omega)$ for all $q < \infty$ (by interpolation between $L^{p(\cdot)}$ and L^∞), we have

$$\int_{\Omega} G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(u_\varepsilon - u)dx \rightarrow 0.$$

By Lemma 5.2, the right-hand side tends to $\int_{\Omega} \frac{f(u-u)}{u^\gamma} dx = 0$.

Thus,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u)dx = 0.$$

Now consider

$$I_\varepsilon = \int_{\Omega} [H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) - H^{p(x)-1}(\nabla u) \nabla H(\nabla u)] \cdot \nabla(u_\varepsilon - u)dx.$$

By Lemma 2.12, $I_\varepsilon \geq 0$. Moreover,

$$I_\varepsilon = \int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon) \nabla H(\nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u)dx - \int_{\Omega} H^{p(x)-1}(\nabla u) \nabla H(\nabla u) \cdot \nabla(u_\varepsilon - u)dx.$$

The first term tends to 0 by the previous limit. The second term tends to 0 because $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $(L^{p(\cdot)}(\Omega))^N$. Hence, $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$.

From Lemma 2.12, for $p(x) \geq 2$,

$$I_\varepsilon \geq C'_1 \int_{\Omega} |\nabla(u_\varepsilon - u)|^{p(x)} dx,$$

and for $1 < p(x) < 2$,

$$I_\varepsilon \geq C_1 \int_{\Omega} (|\nabla u_\varepsilon| + |\nabla u|)^{p(x)-2} |\nabla(u_\varepsilon - u)|^2 dx.$$

In both cases, $I_\varepsilon \rightarrow 0$ implies $\nabla u_\varepsilon \rightarrow \nabla u$ in measure (see [11, Lemma 5] for the anisotropic case).

Since $\{|\nabla u_\varepsilon|^{p(\cdot)}\}$ is uniformly integrable in $L^1(\Omega)$ (which follows from the uniform $W^{1,p(\cdot)}$ bound by the de la Vallée Poussin criterion), Vitali's convergence theorem (Lemma 2.7) gives $\nabla u_\varepsilon \rightarrow \nabla u$ strongly in $(L^{p(\cdot)}(\Omega))^N$.

Lemma 5.4. $G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \rightarrow G(x, u, \nabla u)$ strongly in $L^1(\Omega)$.

Proof. By Lemma 5.3, $\nabla u_\varepsilon \rightarrow \nabla u$ in $(L^{p(\cdot)}(\Omega))^N$. Hence, $|\nabla u_\varepsilon|^{p(\cdot)} \rightarrow |\nabla u|^{p(\cdot)}$ in $L^1(\Omega)$ by the continuous embedding of $L^{p(\cdot)}$ into L^1 and the norm-modular equivalence.

By Assumption 2.20 (G2) and Lemma 4.3,

$$|G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \leq c(\|u_\varepsilon\|_\infty)(m(x) + |\nabla u_\varepsilon|^{p(x)}) \leq C(m(x) + |\nabla u|^{p(x)}),$$

with C independent of ε . The right-hand side converges in $L^1(\Omega)$ to $C(m + |\nabla u|^{p(x)})$.

Moreover, by the continuity of G and the a.e. convergence of $(u_\varepsilon, \nabla u_\varepsilon)$, we have

$$G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \rightarrow G(x, u, \nabla u) \quad \text{a.e. in } \Omega.$$

Thus, by Vitali's convergence theorem (Lemma 2.7), $G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \rightarrow G(x, u, \nabla u)$ in $L^1(\Omega)$.

6. Passage to the limit and completion of the proof

We now have all the ingredients to pass to the limit in the approximate equation.

Proposition 6.1. *The functions (u, b) obtained in Lemma 5.1 satisfy, for every $\varphi \in C_0^1(\Omega)$,*

$$\int_{\Omega} b\varphi dx + \int_{\Omega} H^{p(x)-1}(\nabla u)\nabla H(\nabla u) \cdot \nabla \varphi dx + \int_{\Omega} G(x, u, \nabla u)\varphi dx = \int_{\Omega} \frac{f\varphi}{u^\gamma} dx. \quad (6.1)$$

Proof. Let $\varphi \in C_0^1(\Omega)$. From the weak formulation (3.1) of the approximate problem,

$$\int_{\Omega} \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\varphi dx + \int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon)\nabla H(\nabla u_\varepsilon) \cdot \nabla \varphi dx + \int_{\Omega} G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi dx = \int_{\Omega} \frac{f\varphi}{(u_\varepsilon + \varepsilon)^\gamma} dx. \quad (6.2)$$

We examine the limit of each term as $\varepsilon \rightarrow 0$.

By Lemma 5.1(C3) and (C4),

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \zeta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\varphi dx = \int_{\Omega} b\varphi dx,$$

with $b(x) \in \zeta(u(x))$ a.e.

By Lemma 5.3 and the continuity of the map $\xi \mapsto H^{p(x)-1}(\xi)\nabla H(\xi)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} H^{p(x)-1}(\nabla u_\varepsilon)\nabla H(\nabla u_\varepsilon) \cdot \nabla \varphi dx = \int_{\Omega} H^{p(x)-1}(\nabla u)\nabla H(\nabla u) \cdot \nabla \varphi dx.$$

By Lemma 5.4,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi dx = \int_{\Omega} G(x, u, \nabla u)\varphi dx.$$

By Lemma 5.2,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{f\varphi}{(u_\varepsilon + \varepsilon)^\gamma} dx = \int_{\Omega} \frac{f\varphi}{u^\gamma} dx.$$

Combining these limits, we obtain the desired identity Eq (6.1) for all $\varphi \in C_0^1(\Omega)$. By density, the identity holds for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Lemma 6.2. *The pair (u, b) obtained above satisfies all conditions of Definition 2.23.*

Proof of Theorem 2.24. We verify each condition.

By Lemma 5.1(C4), $b(x) \in \zeta(u(x))$ a.e., so (S1) holds.

By Lemma 5.1(C5), $u > 0$ a.e. and $\int_{\omega} u^{-\gamma} < \infty$ for all $\omega \subset\subset \Omega$, so (S2) holds.

By Lemma 5.4, $G(x, u, \nabla u) \in L^1(\Omega)$, so (S3) holds.

Equation (6.1) holds for all $\varphi \in C_0^1(\Omega)$ by Proposition 6.1, so (S4) holds.

To verify the boundary condition (S5), note that for any $\varepsilon > 0$, $(u_{\varepsilon} - \varepsilon)^+ \in W_0^{1,p(\cdot)}(\Omega)$. Since $u_{\varepsilon} \rightarrow u$ a.e. and in $L^{p(\cdot)}$, for any $\delta > 0$, $(u - \delta)^+$ is the limit in $W_0^{1,p(\cdot)}(\Omega)$ of $(u_{\varepsilon} - \delta)^+$ (by Lemma 5.3), thus, $(u - \delta)^+ \in W_0^{1,p(\cdot)}(\Omega)$ for all $\delta > 0$, which is precisely the definition of $u = 0$ on $\partial\Omega$.

Proof of Theorem 2.24. The existence of a weak solution (u, b) follows from Proposition 6.1 and Lemma 6.2. It remains to establish the additional regularity properties (R1)–(R3).

For (R1), by Lemma 4.3, $\|u_{\varepsilon}\|_{L^{\infty}} \leq C_{\infty}\|f\|_{\infty}$ for all ε . Since $u_{\varepsilon} \rightarrow u$ a.e., the same bound holds for u by Fatou's lemma.

For (R2), positivity and local integrability of $u^{-\gamma}$ are part of Lemma 5.1(C5).

For (R3), the estimate in Lemma 4.1 gives a uniform bound on $\|\nabla u_{\varepsilon}\|_{L^{p(\cdot)}}$. By weak lower semicontinuity,

$$\|\nabla u\|_{L^{p(\cdot)}} \leq \liminf_{\varepsilon \rightarrow 0} \|\nabla u_{\varepsilon}\|_{L^{p(\cdot)}} \leq C.$$

Hence $u \in W_0^{1,p(\cdot)}(\Omega)$.

7. Conclusions

We have established the existence of weak solutions for a comprehensive class of singular anisotropic elliptic equations in variable exponent spaces with $0 < \gamma \leq 1$. The solution enjoys natural regularity properties: L^{∞} bounds, positivity, and $W_0^{1,p(\cdot)}$ regularity.

The key innovations in our approach include the combination of Finsler anisotropy with variable exponents, requiring careful adaptation of all estimates to the non-homogeneous modular framework, the treatment of general maximal monotone operators via Yosida approximation, which adds flexibility for applications involving hysteresis or non-smooth phenomena, the handling of natural growth terms $G(x, u, \nabla u)$ with only L^1 integrability of the solution, and the delicate convergence analysis for the singular term.

Several questions remain open. Proving uniqueness for general maximal monotone ζ and general G under only the natural growth condition remains an open challenge. The optimal regularity of solutions in Besov or Lorentz spaces is unknown, particularly in the anisotropic variable exponent setting. The precise boundary behavior likely involves the Finsler distance function. Sign-changing f may cause nonexistence. Parabolic extensions and systems of singular equations also remain largely unexplored.

Author contributions

All authors contributed to the research, and approved the final version.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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