



Research article

Higher-order NIPG methods on Bakhvalov-type meshes for a singularly perturbed Volterra integro-differential equation

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Abstract: In the present article, a higher-order NIPG method on a Bakhvalov-type mesh is developed to solve a second-order singularly perturbed Volterra integro-differential equation. Drawing on the distinctive properties of Bakhvalov-type meshes, we construct a composite interpolation operator, which enables the derivation of parameter-uniform error estimates for the proposed NIPG method. It is shown that the accuracy of this method attains an order of k in the energy norm, where k denotes the degree of the piecewise polynomials employed in the finite element space. Numerical experiments are presented to confirm our theoretical results.

Keywords: singularly perturbed; Volterra integro-differential equation; NIPG method; Bakhvalov-type mesh; uniform convergence

1. Introduction

In the present article, we consider the nonsymmetric interior penalty Galerkin (NIPG) method for solving the second-order singularly perturbed Volterra integro-differential equation (SPVIDE) as follows:

$$\begin{cases} Lu := -\varepsilon u'' - b(x)u'(x) + c(x)u(x) + \int_0^x K(x,s)u(s)ds = f(x), & x \in \Omega = (0,1), \\ u(0) = u_0, \quad u(1) = u_1, \end{cases} \quad (1.1)$$

where ε denotes a small perturbation parameter satisfying $0 < \varepsilon \ll 1$. Here, $b(x)$, $c(x)$, $f(x)$, and $K(x,s)$ are sufficiently smooth functions. Furthermore, for two positive constants β and γ , we make

the following assumptions:

$$b(x) \geq \beta > 0, \quad M_K := \sup |K(x, s)|, \quad (1.2)$$

$$c(x) + \frac{1}{2}b'(x) - \frac{1}{2}M_K \geq \gamma^2 > 0, \quad x \in \bar{\Omega}. \quad (1.3)$$

Under the above assumptions, problem (1.1) admits a uniquely determined solution $u(x)$. As ε tends to 0, the exact solution $u(x)$ develops a boundary layer with a width of $O(\varepsilon \ln(1/\varepsilon))$ located at $x = 0$.

Volterra integro-differential equations (VIDEs) arise as models in various scientific and engineering fields, for instance, fluid dynamics, biomechanics, biological systems, electromagnetism, oscillation theory, and coupled heat and mass transfer (see [1, 2] for example). If the highest-order derivative term in VIDEs is accompanied by a small parameter ε , the corresponding problems are known as singularly perturbed Volterra integro-differential equations (SPVIDEs). As ε tends to zero, the solution may exhibit multiscale features, such as boundary layers, which can be difficult to resolve accurately using standard numerical methods on uniform meshes. Therefore, some conventional numerical methods, encompassing both finite difference and finite element approaches, when applied on a uniform mesh, are inadequate for accurately resolving boundary layer phenomena and oscillatory solution behaviors induced by the perturbation parameter.

Over the past two decades, there has been great interest in the use of layer-adapted mesh approaches for singularly perturbed Volterra integro-differential equations (see, e.g., [3–7]). However, a comprehensive review of the existing literature reveals that the majority of research efforts have been confined to the application of low-order finite difference schemes for solving first-order SPVIDEs. It is widely recognized that, in contrast to finite difference schemes, the finite element method exhibits greater versatility in handling complex geometries and achieving higher-order convergence rates. For this reason, Tao and Xie [8] developed a discontinuous Galerkin method on a Shishkin mesh to solve the first-order SPVIDEs, and established that the proposed method attains $p + 1$ order uniform convergence in the L^2 norm. Interior-penalty DG methods were developed for gradient-elastic beam problems exhibiting boundary-layer behavior, and stability and a priori error estimates were established in [9].

Among the family of discontinuous Galerkin (DG) methods, the NIPG method exhibits greater flexibility compared to the conventional continuous Galerkin method. This enhanced flexibility stems from the incorporation of an interior penalty term, which effectively enforces continuity constraints across element interfaces, thereby endowing it with advantages not shared by traditional finite element approaches. Over the last twenty years, some researchers [10–15] have investigated the NIPG method on Shishkin-type meshes for a class of singularly perturbed problems. Recently, there has been growing interest in developing the NIPG method on Bakhvalov-type meshes for the numerical solution of singularly perturbed problems (see [16–18]).

The key purpose of this article is to investigate the uniform convergence properties of the NIPG method in an energy norm for solving a second-order SPVIDE. More specifically, we first establish the regularity properties of the exact solution and present a suitable decomposition. Then, we define an NIPG method of any order and a suitable energy norm. Finally, we construct a composite interpolation operator combining Lagrange interpolation on the fine elements with Gauß-Radau projection on the remaining elements. The former eliminates interpolation error jumps within the fine mesh region, while the latter provides endpoint matching and moment orthogonality to control the convection terms, and an-optimal order parameter-uniform error estimate in the energy norm has been established.

The rest of this paper is structured as outlined below. Section 2 presents a priori estimates for the solution $u(x)$ of problem (1.1) and defines the Bakhvalov-type mesh. Section 3 introduces the NIPG method and the associated energy norm. Section 4 constructs a composite interpolation operator and establishes the corresponding interpolation error estimates. Section 5 establishes parameter-uniform convergence of the proposed method. Section 6 presents numerical experiments to illustrate the theoretical results.

Notation. Throughout this paper, the symbols C and C_i stand for a generic constant and a fixed constant, respectively, and these constants do not depend on either ε or N .

Assumption 1. Throughout our analysis, we impose the condition that N is an integer with $N \geq 16$ and that $\varepsilon \leq N^{-1}$, which is a technical restriction used to control ε -dependent terms in the error analysis.

2. Preliminary results

Prior to delving into the construction of Bakhvalov-type meshes, it is imperative to first outline certain key properties of the continuous exact solution $u(x)$ (see Lemma 2.1).

Lemma 2.1. *Let $b(x), c(x), f(x) \in C^{k+1}(\Omega)$ and $K(x, s) \in C^{k+1}(\Omega \times \Omega)$. Then the solution to Eq (1.1) satisfies the following estimates:*

$$|u^{(i)}(x)| \leq C \left(1 + \varepsilon^{-i} e^{-\beta x/\varepsilon}\right), \quad x \in \Omega, \quad i = 0, 1, \dots, k+1. \quad (2.1)$$

Furthermore, the solution $u(x)$ can be decomposed into

$$u = S + E, \quad (2.2)$$

with

$$|S^{(i)}(x)| \leq C, \quad |E^{(i)}(x)| \leq C\varepsilon^{-i} e^{-\beta x/\varepsilon}, \quad i = 0, 1, \dots, k+1. \quad (2.3)$$

Proof. The proof is similar to those in [19, Lemma 1] and Roos [20] (p.23–24).

To design an NIPG scheme for problem (1.1), we consider a Bakhvalov-type mesh $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ with $x_i = \psi(i/N)$, where N is the discretization parameter and $\psi(t)$ is a mesh-generating function (see [21]):

$$x = \psi(t) = \begin{cases} -\frac{\sigma\varepsilon}{\beta} \ln[1 - 2(1 - \varepsilon)t], & t \in [0, 1/2], \\ 1 - d(1 - t), & t \in [1/2, 1], \end{cases} \quad (2.4)$$

where σ satisfies $\sigma \geq k+1$, and d is a constant chosen appropriately to secure the continuity of $\psi(t)$ at $t = 1/2$. We set $\mathcal{T}_N = I_j, j = 1, 2, \dots, N$, with I_j referring to the subinterval $[x_{j-1}, x_j]$ of length $h_j = x_j - x_{j-1}$, and I denoting a generic subinterval.

Our next lemma establishes several key properties of Bakhvalov-type meshes.

Lemma 2.2. *For the Bakhvalov-type mesh given in Eq (2.4), one can derive*

$$h_1 \leq h_2 \leq \dots \leq h_{\frac{N}{2}-1}, \quad C_1\varepsilon \leq h_{\frac{N}{2}-1} \leq C_2\varepsilon, \quad (2.5)$$

$$C_3\varepsilon \leq h_{N/2} \leq C_4N^{-1}, \quad (2.6)$$

$$C_5 N^{-1} \leq h_j \leq C_6 N^{-1}, \quad N/2 + 1 \leq j \leq N. \quad (2.7)$$

Additionally, we have the following properties:

$$x_{N/2-1} \geq C\sigma\epsilon \ln N, \quad x_{N/2} \geq C\sigma\epsilon |\ln \epsilon|, \quad (2.8)$$

$$h_j^\nu e^{-\beta x_j/\epsilon} \leq C\epsilon^\nu N^{-\nu}, \quad 1 \leq j \leq N/2 - 1, \quad 0 \leq \nu \leq \sigma. \quad (2.9)$$

Proof. The proof of this statement is presented in Lemma 3 of [22].

3. The NIPG method

For each element I_j , $j = 1, 2, \dots, N$, and a nonnegative s , the broken Sobolev space of order s is defined by

$$H^s(\Omega, \mathcal{T}_N) = \{\vartheta \in L^2(\Omega) : \vartheta|_{I_j} \in H^s(I_j), \forall j = 1, 2, \dots, N\},$$

and its norm and seminorm are given by

$$\|\vartheta\|_{s, \mathcal{T}_N}^2 = \sum_{j=1}^N \|\vartheta\|_{s, I_j}^2, \quad |\vartheta|_{s, \mathcal{T}_N}^2 = \sum_{j=1}^N |\vartheta|_{s, I_j}^2,$$

where $\|\cdot\|_{s, I_j}$ and $|\cdot|_{s, I_j}$ are the standard Sobolev norm and seminorm associated with the space $H^s(I_j)$. It is important to observe that $\|\cdot\|_I$ and $(\cdot, \cdot)_I$ are customarily used to signify the $L^2(I)$ -norm and $L^2(I)$ -inner product. When I coincides with Ω , the subscript is removed from the notation for simplicity. Thereafter, on the Bakhvalov-type mesh considered above, the finite element space admits the representation

$$V_N^k = \{\vartheta \in L^2(\Omega) : \vartheta|_{I_j} \in \mathbb{P}_k(I_j), \forall j = 1, 2, \dots, N\},$$

where $\mathbb{P}_k(I_j)$ signifies the space of polynomials of order at most k defined on each element I_j . The functions belonging to V_N^k are fully discontinuous across the interfaces between neighboring elements.

For a function $l \in H^1(\Omega, \mathcal{T}_N)$, let $l(x_j^+)$ and $l(x_j^-)$ denote its right- and left-hand traces at the interior node x_j , respectively. For each $j = 0, 1, \dots, N$, we then define the jump $[l(x_j)]$ and the average $\{l(x_j)\}$ of l at the point x_j as

$$\begin{aligned} [l(x_j)] &= l(x_j^+) - l(x_j^-), \quad \{l(x_j)\} = \frac{1}{2} (l(x_j^+) + l(x_j^-)), \quad \forall j = 1, \dots, N-1, \\ [l(x_0)] &= l(x_0^+), \quad \{l(x_0)\} = l(x_0^+), \quad [l(x_N)] = -l(x_N^-), \quad \{l(x_N)\} = l(x_N^-). \end{aligned}$$

Consequently, the NIPG method for problem (1.1) is formulated as seeking $u_N \in V_N^k$ such that

$$a(u_N, v_N) = a_1(u_N, v_N) + a_2(u_N, v_N) = L(v_N), \quad \forall v_N \in V_N^k, \quad (3.1)$$

where

$$a_1(u, v) = \sum_{j=1}^N \int_{I_j} \epsilon u' v' dx + \sum_{j=0}^N \epsilon (\{u'(x_j)\} [v(x_j)] - \{v'(x_j)\} [u(x_j)]) + \sum_{j=0}^N \rho(x_j) [u(x_j)] [v(x_j)],$$

$$a_2(u, v) = \sum_{j=1}^N \int_{I_j} -bu'v dx - \sum_{j=1}^N b(x_j)[u(x_j)]v(x_j^-) + \sum_{j=1}^N \int_{I_j} cuv dx + \sum_{j=1}^N \int_{I_j} \int_0^x K(x, s)u(s)ds v(x) dx,$$

$$L(v) = \sum_{j=1}^N \int_{I_j} f v dx - \varepsilon u(x_0^+)v'(x_0^+) + \varepsilon u(x_N^-)v'(x_N^-) + \rho(x_0)u(x_0^+)v(x_0^+) + (\rho(x_N) + b(x_N))u(x_N^-)v(x_N^-).$$

Let $\rho(x_j)$ represent the penalty parameter attached to the grid point $x_j, j = 0, 1, \dots, N$. To attain an optimal order of uniform convergence, we choose $\rho(x_j) = \rho_j = N$.

Lemma 3.1. Denote by u the solution to problem (1.1). The bilinear form $a(\cdot, \cdot)$ introduced in Eq (3.1) possesses the Galerkin orthogonality property:

$$a(u - u_N, v) = 0, \quad \forall v \in V_N^k. \quad (3.2)$$

Proof. Since u is the exact solution, we have

$$[u(x_j)] = 0, \quad 1 \leq j \leq N-1, \quad [u'(x_j)] = 0, \quad 1 \leq j \leq N-1.$$

Then for all $v \in V_N^k$, one has

$$\begin{aligned} a(u, v) &= \sum_{j=1}^N \int_{I_j} \varepsilon u'v' dx + \sum_{j=0}^N \varepsilon \{u'(x_j)\} [v(x_j)] + \sum_{j=1}^N \int_{I_j} (-bu' + cu)v dx \\ &\quad + \sum_{j=1}^N \int_{I_j} \int_0^x K(x, s)u(s)ds v(x) dx - \varepsilon u(x_0^+)v'(x_0^+) + \varepsilon u(x_N^-)v'(x_N^-) \\ &\quad + \rho(x_0)u(x_0^+)v(x_0^+) + (\rho(x_N) + b(x_N))u(x_N^-)v(x_N^-). \end{aligned} \quad (3.3)$$

By a simple calculation, we conclude that

$$\begin{aligned} \sum_{j=1}^N \int_{I_j} \varepsilon u'v' dx &= \sum_{j=1}^N \left\{ \int_{I_j} -\varepsilon u''v dx + \varepsilon u'(x_j^-)v(x_j^-) - \varepsilon u'(x_{j-1}^+)v(x_{j-1}^+) \right\} \\ &= \sum_{j=1}^N \int_{I_j} -\varepsilon u''v dx - \sum_{j=0}^N \varepsilon [u'(x_j)v(x_j)] \\ &= \sum_{j=1}^N \int_{I_j} -\varepsilon u''v dx - \sum_{j=1}^{N-1} \varepsilon [u'(x_j)]\{v(x_j)\} - \sum_{j=0}^N \varepsilon \{u'(x_j)\}[v(x_j)] \\ &= \sum_{j=1}^N \int_{I_j} -\varepsilon u''v dx - \sum_{j=0}^N \varepsilon \{u'(x_j)\}[v(x_j)]. \end{aligned} \quad (3.4)$$

Inserting Eq (3.4) into Eq (3.3) and recalling $-\varepsilon u'' - bu' + cu + \int_0^x K(x, s)u(s)ds = f$, we have

$$\begin{aligned} a(u, v) &= \sum_{j=1}^N \int_{I_j} f v dx - \varepsilon u(x_0^+)v'(x_0^+) + \varepsilon u(x_N^-)v'(x_N^-) \\ &\quad + \rho(x_0)u(x_0^+)v(x_0^+) + (\rho(x_N) + b(x_N))u(x_N^-)v(x_N^-) \\ &= L(v). \end{aligned} \quad (3.5)$$

Using Eqs (3.1) and (3.5), we can prove this conclusion.

Corresponding to the bilinear form $a(\cdot, \cdot)$ given in Eq (3.1), the energy norm is formulated as follows:

$$\|v\|_{\varepsilon}^2 = \sum_{j=1}^N \left(\varepsilon \|v'\|_{I_j}^2 + \|\gamma v\|_{I_j}^2 \right) + \sum_{j=0}^N \left(\frac{1}{2} b(x_j) + \rho(x_j) \right) [v(x_j)]^2, \quad \forall v \in V_N^k. \quad (3.6)$$

Obviously, we can derive that

$$\|v_N\|_{\varepsilon}^2 \leq a(v_N, v_N), \quad \forall v_N \in V_N^k. \quad (3.7)$$

Proof. For any $v_N \in V_N^k$, the boundedness of $K(x, s)$ and the Cauchy–Schwarz inequality imply

$$\begin{aligned} & \left| \int_0^1 \left(\int_0^x K(x, s) v_N(s) ds \right) v_N(x) dx \right| \\ & \leq M_K \int_0^1 |v_N(x)| \left(\int_0^x |v_N(s)| ds \right) dx = \frac{M_K}{2} \left(\int_0^1 |v_N(x)| dx \right)^2 \leq \frac{M_K}{2} \|v_N\|_{L^2(0,1)}^2. \end{aligned}$$

Then by some simple calculations, we obtain

$$\begin{aligned} a(v_N, v_N) & \geq \varepsilon \sum_{j=1}^N \|v'_N\|_{I_j}^2 + \int_0^1 \left(c(x) + \frac{1}{2} b'(x) - \frac{1}{2} M_K \right) v_N(x)^2 dx \\ & \quad + \sum_{j=0}^N \left(\rho(x_j) + \frac{1}{2} b(x_j) \right) [v_N(x_j)]^2 \\ & \geq \|v_N\|_{\varepsilon}^2, \end{aligned}$$

where the last inequality follows from assumption (1.3) and the definition of the energy norm.

Then it follows from the Lax–Milgram lemma that Eq (3.1) has a unique u_N .

4. Interpolation and interpolation error

In the present section, we first formulate a newly constructed interpolation operator Π , given explicitly by the following definition:

$$(\Pi u)|_{I_j} = \begin{cases} (L_k u)|_{I_j}, & 1 \leq j \leq N/2 - 1, \\ (P_h u)|_{I_j}, & N/2 \leq j \leq N, \end{cases} \quad (4.1)$$

where $(L_k u)|_{I_j}$ denotes the degree- k Lagrange interpolation at $k + 1$ equidistant nodes including both endpoints of I_j , and $(P_h u)|_{I_j}$ denotes the Gauß–Radau projection on I_j . By standard interpolation theory in Sobolev spaces [23, Theorem 3.1.4], for all $v \in W^{k+1,m}(I_j)$, we have

$$\|v - L_k v\|_{W^{l,n}(I_j)} \leq C h_i^{k+1-l+1/n-1/m} |v|_{W^{k+1,m}(I_j)}, \quad (4.2)$$

where $1 \leq m, n \leq \infty$ and $l = 0, 1$. Furthermore, defining the Gauß–Radau projection $P_h u \in V_h^k$ (see [24]) by

$$\int_{I_j} (P_h u) v dx = \int_{I_j} u v dx, \quad \forall v \in \mathbb{P}_{k-1}(I_j), \quad (4.3)$$

$$(P_h u)(x_j^+) = u(x_j^+), \quad j = 0, 1, \dots, N-1 \quad (4.4)$$

and utilizing the projection results [24], for all $v \in H^{k+1}(I_j)$, we have

$$\|v - P_h v\|_{I_j} + h_j^{1/2} \|v - P_h v\|_{L^\infty(I_j)} \leq C h_j^{k+1} |v|_{k+1, I_j}, \quad (4.5)$$

$$\|P_h v\|_{L^\infty(I_j)} \leq C \|v\|_{L^\infty(I_j)}, \quad j = 1, 2, \dots, N. \quad (4.6)$$

Remark 1. By the definition of interpolation (4.1), we can easily obtain

$$[(u - \Pi u)(x_{N/2-1})] = 0.$$

Theorem 4.1. On the Bakhvalov-type mesh (2.4), one can conclude that

$$\|u - \Pi u\|_\varepsilon \leq C N^{-k}. \quad (4.7)$$

Proof. By virtue of the energy norm defined in Eq (3.6), one can conclude that

$$\begin{aligned} \|u - \Pi u\|_\varepsilon^2 &= \sum_{j=1}^N \varepsilon \|(u - \Pi u)'\|_{I_j}^2 + \sum_{j=1}^N \|\gamma(u - \Pi u)\|_{I_j}^2 \\ &\quad + \sum_{j=0}^N \rho(x_j) [(u - \Pi u)(x_j)]^2 + \frac{1}{2} \sum_{j=0}^N b(x_j) [(u - \Pi u)(x_j)]^2 \\ &:= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4. \end{aligned}$$

To begin with, we decompose Φ_1 into two subparts as outlined below:

$$\Phi_1 = \sum_{j=1}^{\frac{N}{2}-1} \varepsilon \|(u - L_k u)'\|_{I_j}^2 + \sum_{j=\frac{N}{2}}^N \varepsilon \|(u - P_h u)'\|_{I_j}^2.$$

For the first part, using Eqs (2.3), (2.9), (4.2), Lemma 2.2, and the triangle inequality, we can conclude that

$$\begin{aligned} \sum_{j=1}^{\frac{N}{2}-1} \varepsilon \|(u - L_k u)'\|_{I_j}^2 &\leq C \varepsilon \sum_{j=1}^{\frac{N}{2}-1} \|(S - L_k S)'\|_{I_j}^2 + C \varepsilon \sum_{j=1}^{\frac{N}{2}-1} \|(E - L_k E)'\|_{I_j}^2 \\ &\leq C \varepsilon \sum_{j=1}^{\frac{N}{2}-1} h_j^{2k+1} \|S^{(k+1)}\|_{L^\infty(I_j)}^2 + C \varepsilon \sum_{j=1}^{\frac{N}{2}-1} h_j^{2k+1} \|E^{(k+1)}\|_{L^\infty(I_j)}^2 \\ &\leq C N^{-(2k+1)} + C \varepsilon^{-(2k+1)} \sum_{j=1}^{\frac{N}{2}-1} \left(h_j^{k+\frac{1}{2}} e^{-\frac{\beta x_j}{\varepsilon}} \right)^2 \\ &\leq C N^{-2k}. \end{aligned} \quad (4.8)$$

For $\frac{N}{2} \leq j \leq N$, applying the triangle inequality, Eqs (4.5), (4.6), and the inverse inequality, we arrive at

$$\sum_{j=\frac{N}{2}}^N \varepsilon \|(u - P_h u)'\|_{I_j}^2 \leq C \varepsilon \sum_{j=\frac{N}{2}}^N \|(S - P_h S)'\|_{I_j}^2 + C \varepsilon \sum_{j=\frac{N}{2}}^N \|(E - P_h E)'\|_{I_j}^2$$

$$\begin{aligned}
&\leq C\varepsilon \sum_{j=\frac{N}{2}}^N \left(\|(S - L_k S)'\|_{I_j}^2 + \|(L_k S - P_h S)'\|_{I_j}^2 \right) + C\varepsilon \sum_{j=\frac{N}{2}}^N \left(\|E'\|_{I_j}^2 + \|(P_h E)'\|_{I_j}^2 \right) \\
&\leq C\varepsilon \sum_{j=\frac{N}{2}}^N h_j^{2k+1} \|S^{(k+1)}\|_{L^\infty(I_j)}^2 + C\varepsilon \sum_{j=\frac{N}{2}}^N h_j^{-2} \left(\|L_k S - S\|_{I_j}^2 + \|S - P_h S\|_{I_j}^2 \right) \\
&\quad + C\varepsilon \int_{x_{\frac{N}{2}-1}}^{x_N} (E')^2 dx + C\varepsilon \sum_{j=\frac{N}{2}}^N h_j^{-1} \|P_h E\|_{L^\infty(I_j)}^2 \\
&\leq CN^{-(2k+1)}.
\end{aligned} \tag{4.9}$$

A combination of Eqs (4.8) and (4.9) leads to

$$\Phi_1 \leq CN^{-2k}. \tag{4.10}$$

Following this, utilizing Eqs (2.3), (2.9), and the triangle inequality for Φ_2 , we derive

$$\begin{aligned}
\sum_{j=1}^{\frac{N}{2}-1} \|\gamma(u - \Pi u)\|_{I_j}^2 &\leq C \sum_{j=1}^{\frac{N}{2}-1} \|(S - L_k S)\|_{I_j}^2 + C \sum_{j=1}^{\frac{N}{2}-1} \|(E - L_k E)\|_{I_j}^2 \\
&\leq CN^{-2(k+1)} + C \sum_{j=1}^{\frac{N}{2}-1} h_j^{2(k+1)} \|E^{(k+1)}\|_{I_j}^2 \\
&\leq CN^{-2(k+1)} + C\varepsilon^{-2(k+1)} \sum_{j=1}^{\frac{N}{2}-1} h_j^{2(k+1)+1} e^{-\frac{2\beta x_{j-1}}{\varepsilon}} \\
&\leq CN^{-2(k+1)}.
\end{aligned} \tag{4.11}$$

In the case where $\frac{N}{2} \leq j \leq N$, it holds that

$$\begin{aligned}
\sum_{j=\frac{N}{2}}^N \|\gamma(u - \Pi u)\|_{I_j}^2 &\leq C \sum_{j=\frac{N}{2}}^N \|(S - P_h S)\|_{I_j}^2 + C \sum_{j=\frac{N}{2}}^N \|(E - P_h E)\|_{I_j}^2 \\
&\leq C \sum_{j=\frac{N}{2}}^N h_j \|(S - P_h S)\|_{L^\infty(I_j)}^2 + C \sum_{j=\frac{N}{2}}^N \|E\|_{I_j}^2 + C \sum_{j=\frac{N}{2}}^N \|P_h E\|_{I_j}^2 \\
&\leq CN^{-2(k+1)} + C \sum_{j=\frac{N}{2}}^N h_j \|E\|_{L^\infty(I_j)}^2 \\
&\leq CN^{-2(k+1)}.
\end{aligned} \tag{4.12}$$

From Eqs (4.11) and (4.12), we arrive at

$$\Phi_2 \leq CN^{-2(k+1)}. \tag{4.13}$$

To facilitate analysis of Φ_3 , by using Eq (4.1) and Remark 1, we first represent it as

$$\begin{aligned}\Phi_3 &= \sum_{j=\frac{N}{2}}^{N-1} \rho(x_j) [(u - \Pi u)(x_j)]^2 + \rho(x_N) [(u - \Pi u)(x_N)]^2 \\ &\leq C \sum_{j=\frac{N}{2}}^{N-1} \rho_j \| (u - P_h u) \|_{L^\infty(I_j \cup I_{j+1})}^2 + C \rho(x_N) \| (u - P_h u) \|_{L^\infty(I_N)}^2.\end{aligned}\quad (4.14)$$

Regarding the first term in Eq (4.14), it is straightforward to obtain

$$\begin{aligned}\sum_{j=\frac{N}{2}}^{N-1} \rho_j \| (u - P_h u) \|_{L^\infty(I_j \cup I_{j+1})}^2 &\leq C \sum_{j=\frac{N}{2}}^{N-1} \rho_j \| (S - P_h S) \|_{L^\infty(I_j \cup I_{j+1})}^2 + C \sum_{j=\frac{N}{2}}^{N-1} \rho_j \| (E - P_h E) \|_{L^\infty(I_j \cup I_{j+1})}^2 \\ &\leq C \sum_{j=\frac{N}{2}}^{N-1} \rho_j h_j^{2(k+1)} \| S^{(k+1)} \|_{L^\infty(I_j \cup I_{j+1})}^2 + C \sum_{j=\frac{N}{2}}^{N-1} \rho_j \| E \|_{L^\infty(I_j \cup I_{j+1})}^2 \\ &\leq CN^{-2k} + C \sum_{j=\frac{N}{2}}^{N-1} \rho_j \| E \|_{L^\infty(I_j \cup I_{j+1})}^2 \\ &\leq CN^{-2k}.\end{aligned}\quad (4.15)$$

Correspondingly, an estimate for the second term in Eq (4.14) can be derived as follows

$$\rho(x_N) \| (u - P_h u) \|_{L^\infty(I_N)}^2 \leq CN^{-(2k+1)}.\quad (4.16)$$

By virtue of Eqs (4.14)–(4.16), we arrive at

$$\Phi_3 \leq CN^{-2k}.\quad (4.17)$$

Likewise, for Φ_4 , it is straightforward to derive

$$\Phi_4 \leq CN^{-(2k+1)}.\quad (4.18)$$

Consequently, Theorem 4.1 can be deduced from Eqs (4.10), (4.13), (4.17), and (4.18).

Lemma 4.1. For all $\theta \in H^1(I_j)$, $j = 1, 2, \dots, N$, we have

$$|\theta(x_s)|^2 \leq 2 \left(h_j^{-1} \|\theta\|_{I_j}^2 + \|\theta\|_{I_j} \|\theta'\|_{I_j} \right).$$

Lemma 4.2. Let $\eta(x) = (\Pi u - u)(x)$. On the Bakhvalov-type mesh (2.4), the following estimate holds:

$$\{\eta'(x_j)\}^2 \leq \begin{cases} C\varepsilon^{-2}N^{-2k}, & 0 \leq j \leq N/2 - 1, \\ C\varepsilon^{-2}N^{-2(k+1)}, & N/2 \leq j \leq N. \end{cases}\quad (4.19)$$

Proof. By applying Lemma 4.1, we can prove this lemma.

5. Uniform convergence

Let $\lambda = \Pi u - u^N$. Then, using the Galerkin orthogonality and Eq (3.7), we obtain

$$\|\lambda\|_{\varepsilon}^2 \leq a(\lambda, \lambda) = a(\Pi u - u + u - u_N, \lambda) = a(\eta, \lambda) = \sum_{i=1}^8 \Gamma_i, \quad (5.1)$$

where

$$\begin{aligned} \Gamma_1 &= \sum_{j=1}^N \int_{I_j} \varepsilon \eta' \lambda' dx, \quad \Gamma_2 = \sum_{j=0}^N \varepsilon \{\eta'(x_j)\} [\lambda(x_j)], \quad \Gamma_3 = - \sum_{j=0}^N \varepsilon \{\lambda'(x_j)\} [\eta(x_j)], \\ \Gamma_4 &= \sum_{j=0}^N \rho(x_j) [\eta(x_j)] [\lambda(x_j)], \quad \Gamma_5 = \sum_{j=1}^N \int_{I_j} b \eta \lambda' dx, \quad \Gamma_6 = \sum_{j=0}^{N-1} b(x_j) [\lambda(x_j)] \eta(x_j^+), \\ \Gamma_7 &= \sum_{j=1}^N \int_{I_j} (b' + c) \eta \lambda dx, \quad \Gamma_8 = \sum_{j=1}^N \int_{I_j} \int_0^x K(x, s) \eta(s) ds \lambda(x) dx. \end{aligned}$$

Next, we will derive individual estimates for each Γ_i , $i = 1, \dots, 8$.

First, from the Hölder inequalities and Eq (4.7), it is easy to obtain

$$\begin{aligned} |\Gamma_1| &\leq C \left(\sum_{j=1}^N \varepsilon \|\eta'\|_{I_j}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N \varepsilon \|\lambda'\|_{I_j}^2 \right)^{\frac{1}{2}} \\ &\leq \|\eta\|_{\varepsilon} \|\lambda\|_{\varepsilon} \\ &\leq CN^{-k} \|\lambda\|_{\varepsilon}. \end{aligned} \quad (5.2)$$

For Γ_2 , using Eq (4.19) and $\rho(x_j) = N$, we obtain

$$\begin{aligned} |\Gamma_2| &\leq \left(\sum_{j=0}^N \frac{\varepsilon^2}{\rho(x_j)} \{\eta'(x_j)\}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^N \rho(x_j) [\lambda(x_j)]^2 \right)^{\frac{1}{2}} \\ &\leq CN^{-k} \|\lambda\|_{\varepsilon}. \end{aligned} \quad (5.3)$$

To establish the bound for Γ_3 , we first decompose it into two components:

$$\Gamma_3 = -\varepsilon \sum_{j=0}^{N/2-1} \{\lambda'(x_j)\} [\eta(x_j)] - \varepsilon \sum_{j=N/2}^N \{\lambda'(x_j)\} [\eta(x_j)].$$

By virtue of Eq (4.1), Remark 1, and the properties of Lagrange interpolation, it remains only to

estimate

$$\begin{aligned}
& \left| -\varepsilon \sum_{j=N/2}^N [\eta(x_j)] \{\lambda'(x_j)\} \right| \\
& \leq \left| \varepsilon \sum_{j=N/2}^{N-1} [\eta(x_j)] \{\lambda'(x_j)\} \right| + \left| \varepsilon [\eta(x_N)] \{\lambda'(x_N)\} \right| \\
& \leq C\varepsilon \sum_{j=N/2}^{N-1} \|\eta\|_{L^\infty(I_j \cup I_{j+1})} \|\lambda'\|_{L^\infty(I_j \cup I_{j+1})} + C\varepsilon \|\eta\|_{L^\infty(I_N)} \|\lambda'\|_{L^\infty(I_N)} \\
& \leq C\varepsilon \sum_{j=N/2}^{N-1} \|\eta\|_{L^\infty(I_j)} \max\{h_j^{-\frac{1}{2}}, h_{j+1}^{-\frac{1}{2}}\} \|\lambda'\|_{L^\infty(I_j \cup I_{j+1})} + C\varepsilon \|\eta\|_{L^\infty(I_N)} N^{\frac{1}{2}} \|\lambda'\|_{L^\infty(I_N)} \\
& \leq C \left(\sum_{j=N/2}^{N-1} \|\eta\|_{L^\infty(I_j)}^2 \right)^{\frac{1}{2}} \left(\sum_{j=N/2}^{N-1} \varepsilon \|\lambda'\|_{L^\infty(I_j \cup I_{j+1})}^2 \right)^{\frac{1}{2}} + CN^{-(k+1)} \|\lambda\|_\varepsilon \\
& \leq CN^{-(k+\frac{1}{2})} \|\lambda\|_\varepsilon.
\end{aligned} \tag{5.4}$$

For Γ_4 , applying an analogous approach to that for Γ_3 and using Eqs (4.15) and (4.16), we have

$$\begin{aligned}
|\Gamma_4| & \leq \left| \sum_{j=0}^{N/2-1} \rho(x_j) [\eta(x_j)] [\lambda(x_j)] \right| + \left| \sum_{j=N/2}^N \rho(x_j) [\eta(x_j)] [\lambda(x_j)] \right| \\
& \leq C \left(\sum_{j=N/2}^N \rho(x_j) [\eta(x_j)]^2 \right)^{\frac{1}{2}} \left(\sum_{j=N/2}^N \rho(x_j) [\lambda(x_j)]^2 \right)^{\frac{1}{2}} \\
& \leq C \left(\sum_{j=N/2}^{N-1} \rho(x_j) \|\eta\|_{L^\infty(I_j \cup I_{j+1})}^2 + \rho(x_N) \|\eta\|_{L^\infty(I_N)}^2 \right)^{\frac{1}{2}} \|\lambda\|_\varepsilon \\
& \leq CN^{-k} \|\lambda\|_\varepsilon.
\end{aligned} \tag{5.5}$$

Furthermore, in order to estimate $\Gamma_5 + \Gamma_6$, we first divide it into two portions:

$$\begin{aligned}
\Gamma_5 + \Gamma_6 & = \sum_{j=1}^{N/2-1} \int_{I_j} b(x) \eta \lambda' dx + \sum_{j=0}^{N/2-2} b(x_j) [\lambda(x_j)] \eta(x_j^+) \\
& \quad + \sum_{j=N/2}^N \int_{I_j} b(x) \eta \lambda' dx + \sum_{j=N/2-1}^{N-1} b(x_j) [\lambda(x_j)] \eta(x_j^+).
\end{aligned}$$

Then a simple calculation leads to

$$\left| \sum_{j=1}^{N/2-1} \int_{I_j} b(x) \eta \lambda' dx + \sum_{j=0}^{N/2-2} b(x_j) [\lambda(x_j)] \eta(x_j^+) \right| \leq CN^{-(k+\frac{1}{2})} \|\lambda\|_\varepsilon$$

and

$$\begin{aligned}
& \left| \sum_{j=N/2}^N \int_{I_j} b(x) \eta \lambda' dx + \sum_{j=N/2-1}^{N-1} b(x_j) [\lambda(x_j)] \eta(x_j^+) \right| \\
&= \left| \sum_{j=N/2}^N \int_{I_j} \left(b(x) - b(x_{j-\frac{1}{2}}) \right) \eta \lambda' dx \right| \\
&\leq C \sum_{j=N/2}^N h_j \|\eta\|_{L^\infty(I_j)} \|\lambda'\|_{L^1(I_j)} \\
&\leq C \sum_{j=N/2}^N h_j \|\eta\|_{L^\infty(I_j)} h_j^{-\frac{1}{2}} \|\lambda\|_{I_j} \\
&\leq CN^{-(k+1)} \|\lambda\|_\varepsilon.
\end{aligned}$$

Therefore, one has

$$|\Gamma_5 + \Gamma_6| \leq CN^{-(k+\frac{1}{2})} \|\lambda\|_\varepsilon. \quad (5.6)$$

For Γ_7 and Γ_8 , we observe that the kernel $K(x, s)$ is a sufficiently smooth function. By applying the Hölder inequalities and Eq (4.7), we obtain

$$|\Gamma_7| + |\Gamma_8| \leq C \|\eta\|_\varepsilon \|\lambda\|_\varepsilon \leq CN^{-k} \|\lambda\|_\varepsilon. \quad (5.7)$$

Ultimately, substituting Eqs (5.2)–(5.7) into Eq (5.1), we derive

$$\|\Pi u - u^N\|_\varepsilon \leq CN^{-k}. \quad (5.8)$$

Theorem 5.1. Denote by u and u^N the solutions to Eqs (1.1) and (3.1), respectively. On the Bakhvalov-type mesh (2.4), the following conclusion holds:

$$\|u - u^N\|_\varepsilon \leq CN^{-k}.$$

Proof. From Eqs (4.7), (5.8), and the triangle inequality, the theorem can be readily established.

6. Numerical experiments and discussion

To verify the efficiency of the NIPG method, we adopt the following typical example:

$$-\varepsilon u''(x) - u'(x) + u(x) + \int_0^x x u(s) ds = f(x), \quad u(0) = 1, \quad u(1) = e^{-1/\varepsilon}. \quad (6.1)$$

Here $f(x)$ is chosen such that $u(x) = e^{-x/\varepsilon}$ is the exact solution to Eq (6.1).

For fixed values of ε and N , we computed the energy-norm errors by $E_N = \|u - u^N\|_\varepsilon$, where u denotes the exact solution to problem (1.1) and u^N signifies the numerical approximation introduced in Eq (3.1). The corresponding rate of convergence is

$$r^N = \frac{\ln E_N - \ln E_{2N}}{\ln 2}.$$

For $\varepsilon = 10^{-5}, \dots, 10^{-10}$, $k = 1, 2, 3$, $N = 16, \dots, 1024$, and $\sigma = 2k + 1$, the corresponding error values and computed convergence orders are summarized in Tables 1–3. From these results, on the Bakhvalov-type mesh, it can be verified that the energy-norm errors $\|u - u^N\|_\varepsilon$ exhibit uniform convergence of order k , a finding fully consistent with the theoretical predictions of Theorem 5.1. For $k = 3$, convergence deteriorates on finer meshes. This deterioration may be related to round-off errors.

Table 1. Energy-norm errors E_N and convergence orders r^N for $k = 1$.

ε		N						
		16	32	64	128	256	512	1024
10^{-5}	E_N	9.448e-02	4.698e-02	2.345e-02	1.172e-02	5.860e-03	2.930e-03	1.465e-03
	r^N	1.00	1.00	1.00	1.00	1.00	1.00	–
10^{-6}	E_N	9.448e-02	4.698e-02	2.345e-02	1.172e-02	5.860e-03	2.930e-03	1.465e-03
	r^N	1.00	1.00	1.00	1.00	1.00	1.00	–
10^{-7}	E_N	9.448e-02	4.698e-02	2.345e-02	1.172e-02	5.860e-03	2.930e-03	1.465e-03
	r^N	1.00	1.00	1.00	1.00	1.00	1.00	–
10^{-8}	E_N	9.448e-02	4.698e-02	2.345e-02	1.172e-02	5.860e-03	2.930e-03	1.465e-03
	r^N	1.00	1.00	1.00	1.00	1.00	1.00	–
10^{-9}	E_N	9.448e-02	4.698e-02	2.345e-02	1.172e-02	5.860e-03	2.930e-03	1.465e-03
	r^N	1.00	1.00	1.00	1.00	1.00	1.00	–
10^{-10}	E_N	9.449e-02	4.698e-02	2.345e-02	1.172e-02	5.860e-03	2.930e-03	1.465e-03
	r^N	1.00	1.00	1.00	1.00	1.00	1.00	–

Table 2. Energy-norm errors E_N and convergence orders r^N for $k = 2$.

ε		N						
		16	32	64	128	256	512	1024
10^{-5}	E_N	1.585e-02	4.039e-03	1.023e-03	2.574e-04	6.460e-05	1.618e-05	4.049e-06
	r^N	1.97	1.98	1.99	1.99	2.00	2.00	–
10^{-6}	E_N	1.586e-02	4.040e-03	1.023e-03	2.575e-04	6.460e-05	1.618e-05	4.052e-06
	r^N	1.97	1.98	1.99	1.99	2.00	2.00	–
10^{-7}	E_N	1.587e-02	4.041e-03	1.023e-03	2.575e-04	6.460e-05	1.618e-05	4.058e-06
	r^N	1.97	1.98	1.99	1.99	2.00	2.00	–
10^{-8}	E_N	1.588e-02	4.041e-03	1.023e-03	2.575e-04	6.460e-05	1.618e-05	4.054e-06
	r^N	1.97	1.98	1.99	1.99	2.00	2.00	–
10^{-9}	E_N	1.589e-02	4.042e-03	1.023e-03	2.575e-04	6.460e-05	1.618e-05	4.050e-06
	r^N	1.97	1.98	1.99	1.99	2.00	2.00	–
10^{-10}	E_N	1.589e-02	4.042e-03	1.023e-03	2.575e-04	6.460e-05	1.618e-05	4.050e-06
	r^N	1.97	1.98	1.99	1.99	2.00	2.00	–

Table 3. Energy-norm errors E_N and convergence orders r^N for $k = 3$.

ε		N						
		16	32	64	128	256	512	1024
10^{-5}	E_N	2.028e-03	2.494e-04	3.094e-05	3.859e-06	4.825e-07	3.412e-07	2.760e-07
	r^N	3.02	3.01	3.00	3.00	0.50	0.30	–
10^{-6}	E_N	2.029e-03	2.494e-04	3.094e-05	3.859e-06	4.842e-07	1.717e-06	1.046e-05
	r^N	3.02	3.01	3.00	3.00	–1.83	–2.61	–
10^{-7}	E_N	2.030e-03	2.494e-04	3.094e-05	3.859e-06	4.855e-07	3.685e-07	3.784e-05
	r^N	3.03	3.01	3.00	3.00	0.40	–6.68	–
10^{-8}	E_N	2.030e-03	2.494e-04	3.094e-05	3.859e-06	4.821e-07	9.291e-07	2.398e-05
	r^N	3.03	3.01	3.00	3.00	–0.95	–4.69	–
10^{-9}	E_N	2.030e-03	2.494e-04	3.094e-05	3.859e-06	4.826e-07	1.824e-07	3.780e-06
	r^N	3.03	3.01	3.00	3.00	1.40	–4.37	–
10^{-10}	E_N	2.031e-03	2.494e-04	3.094e-05	3.859e-06	4.829e-07	6.383e-07	3.830e-05
	r^N	3.03	3.01	3.00	3.00	–0.40	–5.91	–

Author contributions

Weining Chen and Ke Liu conceived the study, carried out the theoretical proofs, and provided critical revisions to the manuscript. Lei Xu and Ying Liang performed the numerical experiments and wrote the manuscript. All authors reviewed and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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