



Research article

On the existence, uniqueness, and Ulam-Hyers stability of fractional Langevin equations with multi-point closed boundary conditions

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Abstract: This paper investigates the existence and uniqueness of mild solutions for a class of fractional Langevin equations subject to multi-point closed boundary conditions. By applying Krasnosel'skii fixed point theorem and the Banach contraction mapping principle, we establish sufficient conditions which ensure existence and uniqueness. Furthermore, we analyze the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the solutions within this framework. Illustrative examples are provided to demonstrate the applicability of the main results and to verify the stated hypotheses.

Keywords: fractional Langevin equation; multi-point closed boundary conditions; fixed point theorem; existence and uniqueness; Ulam-type stability

1. Introduction

Fractional differential equations (FDEs), owing to their capability to capture memory effects and nonlocal characteristics of dynamical systems, have emerged as powerful tools to model complex processes. Compared with classical integer-order models, fractional formulations exhibit greater flexibility and accuracy, and have found wide applications in engineering and applied mathematics, including the control theory, mechanical engineering, material science, behavioral science, heat conduction, and signal processing [1–6]. Consequently, the study of FDEs is not only of significant theoretical importance but also provides effective approaches to address practical problems.

As a fundamental stochastic dynamical model, the Langevin equation plays a crucial role in bridging deterministic mechanics with random perturbations. Recent studies have further underscored its theoretical significance in the analysis of complex systems and network dynamics. For instance, references [7,8] demonstrated how the Langevin framework characterizes the collective behaviors of fractional oscillators in random networks, capturing memory effects as well as the coupling between dissipation and stochastic fluctuations. Beyond theoretical exploration, the concepts derived from the

Langevin equation have also been translated into practical applications in mechanical engineering fault diagnoses. Works such as references [9,10] revealed that generalized stochastic resonance mechanisms, rooted in Langevin dynamics, can be effectively integrated with modern machine learning approaches to achieve weak signal enhancement and intelligent diagnoses under complex operating conditions. Overall, research on the Langevin equation not only advances the fundamental understanding of stochastic dynamical systems but also provides solid theoretical support and methodological innovation for engineering practice.

Fractional Langevin equations (FLEs), through the incorporation of nonlocal fractional derivatives, provide an effective framework for characterizing memory effects and long-range dependence in complex materials and media. Such a formulation plays a significant role in the dynamical analysis of viscoelastic structures, fractal media, and heterogeneous environments [11]. Studies have shown that Langevin models driven by fractional Brownian motion may exhibit spurious non-ergodicity and aging phenomena under far from equilibrium conditions. These findings revealed the intrinsic dynamical features of complex systems and challenged conventional assumptions of ergodicity [12]. Within this context, the systematic exploration of the existence and qualitative properties of solutions to FLEs with initial or boundary conditions is of fundamental importance. Such research establishes solid mathematical foundations for the analysis of complex dynamical phenomena and promotes the further development of FDEs in modeling non-equilibrium processes. Recent contributions have explored the qualitative properties of the (initial) boundary value problems (BVPs) associated with FLEs, and related results continue to emerge. These include investigations on initial value problem (IVP) [13], two-point BVP [14], BVPs for coupled systems [15–17], multipoint BVPs [18, 19], as well as integral BVPs [20–23]. For example, El-Sayed et al. [13] considered the following IVP of the nonlinear delayed FLE:

$$\begin{cases} \mathcal{D}^\beta(\mathcal{D}^\gamma + \lambda)x(t) = f(t, x(\phi(t))), & t \in (0, T), \\ x(0) = x_0, \end{cases}$$

where $\beta, \gamma \in (0, 1)$, $\mathcal{D}^\beta$ and $\mathcal{D}^\gamma$ denote the fractal fractional derivatives, $\lambda > 0$ is a parameter, and ϕ represents the delay function. By applying the Schauder fixed point theorem (FPT) together with the Banach contraction mapping principle (BCMP), the authors established the existence and uniqueness of solutions, analyzed the continuous dependence of solutions with respect to the initial data and parameters, and further verified the Ulam-Hyers (U-H) stability of the problem.

In [17], Almaghami et al. investigated a nonlinear coupled Langevin system under the Riemann-Stieltjes integral boundary conditions. The system was expressed in the following form:

$$\begin{cases} {}^H\mathcal{D}^{\alpha_i, \delta_i, \psi}({}^H\mathcal{D}^{\beta_i, \sigma_i, \psi} + \lambda_i)x_i(t) = f_i(t, x_1(t), x_2(t)), & t \in (0, 1), \quad i = 1, 2, \\ x_i(0) = 0, \quad x_i(1) = \int_0^1 x_j(s) d\theta_i(s), \end{cases}$$

where $i, j = 1, 2$ with $i \neq j$, and ${}^H\mathcal{D}^{\alpha_i, \delta_i, \psi}$, ${}^H\mathcal{D}^{\beta_i, \sigma_i, \psi}$ stand for the ψ -Hilfer fractional derivatives of orders $0 < \alpha_i, \beta_i \leq 1$ and types $0 \leq \delta_i, \sigma_i \leq 1$, respectively, with the constants $\lambda_i \in \mathbb{R}$. Applying the Schauder FPT, the Leray-Schauder nonlinear alternative, as well as the BCMP, the authors established the existence and uniqueness of solutions.

Closed boundary conditions (CBCs) hold an important position in the study of FDEs and related physical modeling. Such conditions are not only indispensable in fluid problems but also are widely

applied in gravitational effect modeling, radiation transfer under diffusion approximation, elastic wavefield analyses on closed free surfaces, and radiation heat transfer simulations [24]. In recent years, scholars have actively carried out qualitative analyses of FDEs with CBCs [24–28] and their nonlocal forms [29–32]. For example, Ahmad et al. [24] introduced a nonlocal variant of CBCs and investigated the following fractional integro-differential equation together with mixed nonlinearities:

$$\begin{cases} {}^C\mathcal{D}_{T-}^{\alpha}\phi(t) + \lambda\mathcal{J}_{T-}^{\beta}\mathcal{J}_{0+}^{\gamma}\chi_1(t, \phi(t)) = \chi_2(t, \phi(t)), & t \in (0, T), \quad T > 0, \\ \phi(T) = \sigma_1\phi(\xi) + \sigma_2T\phi'(\xi), \quad T\phi'(T) = \delta_1\phi(\xi) + \delta_2T\phi'(\xi), & \xi \in (0, T), \end{cases}$$

where ${}^C\mathcal{D}_{T-}^{\alpha}$ denotes the right Caputo fractional derivative (CFD) of order $\alpha \in (1, 2]$, the operators $\mathcal{J}_{T-}^{\beta}$ and $\mathcal{J}_{0+}^{\gamma}$ refer to the right and left Riemann-Liouville fractional integral operators of orders $\beta, \gamma > 0$, respectively, with the functions $\chi_1, \chi_2 \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and $\lambda, \sigma_1, \sigma_2, \delta_1, \delta_2 \in \mathbb{R}$. By employing Krasnosel'skii FPT, the Leray-Schauder nonlinear alternative, and the BCMP, the authors established existence and uniqueness results for the solutions of this problem.

More recently, Ahmad et al. [33] considered the nonlinear Caputo-type coupled fractional differential system with a new class of coupled multi-point CBCs, namely

$$\begin{cases} {}^C\mathcal{D}_{0+}^{q_1}\phi_1(t) = \chi_1(t, \phi_1(t), \phi_2(t)), & t \in (0, T), \quad T > 0, \\ {}^C\mathcal{D}_{0+}^{q_2}\phi_2(t) = \chi_2(t, \phi_1(t), \phi_2(t)), & t \in (0, T), \\ \phi_1(T) = \sum_{i=1}^m (\alpha_i\phi_2(\xi_i) + \beta_iT\phi_2'(\xi_i)), \quad T\phi_1'(T) = \sum_{i=1}^m (\gamma_i\phi_2(\xi_i) + \delta_iT\phi_2'(\xi_i)), \\ \phi_2(T) = \sum_{i=1}^m (\mu_i\phi_1(\xi_i) + \nu_iT\phi_1'(\xi_i)), \quad T\phi_2'(T) = \sum_{i=1}^m (\rho_i\phi_1(\xi_i) + \omega_iT\phi_1'(\xi_i)), \end{cases}$$

where $1 < q_1, q_2 \leq 2$, ${}^C\mathcal{D}_{0+}^{\kappa}$ denotes the CFD operator of order $\kappa \in \{q_1, q_2\}$, $\alpha_i, \beta_i, \gamma_i, \delta_i, \mu_i, \nu_i, \rho_i, \omega_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \dots, m$, $\chi_1, \chi_2 \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$. The authors employed the Leray-Schauder FPT together with the BCMP to establish the existence and uniqueness of solutions.

Motivated by previous studies, this paper investigates FLEs with multi-point CBCs and establishes existence and stability results. More precisely, we focus on the following BVP:

$$\begin{cases} {}^C\mathcal{D}_{0+}^{\beta}({}^C\mathcal{D}_{0+}^{\alpha} + \lambda)x(t) = f(t, x(t)), & t \in (0, T), \\ x(T) = \sum_{i=1}^m (a_i x(\xi_i) + Tb_i {}^C\mathcal{D}_{0+}^{\alpha} x(\xi_i)), \\ T {}^C\mathcal{D}_{0+}^{\alpha} x(T) = \sum_{i=1}^m (c_i x(\xi_i) + Td_i {}^C\mathcal{D}_{0+}^{\alpha} x(\xi_i)), \end{cases} \quad (1.1)$$

where $\lambda > 0$, $0 < \alpha, \beta < 1$, with $1 < \alpha + \beta < 2$, ${}^C\mathcal{D}_{0+}^{\alpha}$ and ${}^C\mathcal{D}_{0+}^{\beta}$ are CFD operators, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $a_i, b_i, c_i, d_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \dots, m$, with $\Lambda = \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21} \neq 0$, where

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{\Gamma(\alpha+1)} \left[-T^{\alpha} + \sum_{i=1}^m (a_i - \lambda Tb_i) \xi_i^{\alpha} \right] + T \sum_{i=1}^m b_i, \\ \varepsilon_{12} &= -1 + \sum_{i=1}^m (a_i - \lambda Tb_i), \quad \varepsilon_{22} = \lambda T + \sum_{i=1}^m (c_i - \lambda Td_i), \\ \varepsilon_{21} &= \frac{\lambda T^{\alpha+1}}{\Gamma(\alpha+1)} - T + \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^m (c_i - \lambda Td_i) \xi_i^{\alpha} + T \sum_{i=1}^m d_i. \end{aligned}$$

It should be emphasized that the multi-point CBCs studied in this work can be physically interpreted as multi-location constraints in complex media or layered structures, where the parameters may correspond to measurements or control conditions at different points. At the same time, they possess independent mathematical significance as a nonlocal constraint, providing a unified and flexible framework to analyze the existence, uniqueness, and stability of FLEs. The principal innovations and contributions of the present study are outlined below:

- Generality of the model: we investigate FLE subject to multi-point CBCs, which represent a more general and complex class of nonlocal BVPs.
- Analytical challenges: since Langevin equations are multi-term differential equations and the multi-point CBCs are inherently nonlocal, the analysis of such problems becomes considerably more difficult.
- Methodology and results: by utilizing Krasnosel'skii FPT and the BCMP, we establish the existence and uniqueness of mild solutions and analyze the U-H stability and Ulam-Hyers-Rassias (U-H-R) stability of the mild solutions.
- Extension of previous results: if the parameters in the multi-point CBCs take special values (e.g., $m = 1$), then the boundary conditions (BCs) of the BVP (1.1) reduce to

$$x(T) = \sum_{i=1}^1 (a_i x(\xi_i) + T b_i^C \mathcal{D}_{0+}^\alpha x(\xi_i)) = a_1 x(\xi_1) + T b_1^C \mathcal{D}_{0+}^\alpha x(\xi_1),$$

$$T^C \mathcal{D}_{0+}^\alpha x(T) = \sum_{i=1}^1 (c_i x(\xi_i) + T d_i^C \mathcal{D}_{0+}^\alpha x(\xi_i)) = c_1 x(\xi_1) + T d_1^C \mathcal{D}_{0+}^\alpha x(\xi_1),$$

which corresponds to the BCs studied in [32]. Moreover, the multi-point CBCs considered in this paper involve the fractional derivative ${}^C \mathcal{D}_{0+}^\alpha$, which are more general than the BCs discussed in [24, 33]. Furthermore, let $m = 1$. To consider the limiting case $\xi_i \rightarrow 0^+$, we interpret the endpoint fractional derivative, whenever the corresponding right trace exists, as follows:

$${}^C \mathcal{D}_{0+}^\alpha x(0^+) := \lim_{t \rightarrow 0^+} {}^C \mathcal{D}_{0+}^\alpha x(t).$$

Additionally, assume that the limiting boundary system remains nondegenerate, that is,

$$\Lambda_0 \neq 0,$$

where

$$\Lambda_0 = \varepsilon_{11}^0 \varepsilon_{22}^0 - \varepsilon_{12}^0 \varepsilon_{21}^0,$$

with

$$\varepsilon_{11}^0 = -\frac{T^\alpha}{\Gamma(\alpha + 1)} + T b_1, \quad \varepsilon_{12}^0 = -1 + a_1 - \lambda T b_1,$$

$$\varepsilon_{21}^0 = \frac{\lambda T^{\alpha+1}}{\Gamma(\alpha + 1)} - T + T d_1, \quad \varepsilon_{22}^0 = \lambda T + c_1 - \lambda T d_1.$$

Then, for $\xi_i \rightarrow 0^+$, the BCs of the BVP (1.1) reduce to the following:

$$x(T) = \lim_{\xi_i \rightarrow 0^+} \sum_{i=1}^1 (a_i x(\xi_i) + T b_i^C \mathcal{D}_{0+}^\alpha x(\xi_i)) = a_1 x(0) + T b_1^C \mathcal{D}_{0+}^\alpha x(0^+),$$

$$T^C \mathcal{D}_{0+}^\alpha x(T) = \lim_{\xi_i \rightarrow 0^+} \sum_{i=1}^1 (c_i x(\xi_i) + T d_i^C \mathcal{D}_{0+}^\alpha x(\xi_i)) = c_1 x(0) + T d_1^C \mathcal{D}_{0+}^\alpha x(0^+).$$

Hence, provided that the right trace exists and $\Lambda_0 \neq 0$, the above endpoint-type CBCs are recovered as a limiting special case of the multi-point CBCs considered in this paper, which correspond to the BCs studied in [25–28]. Therefore, under the above trace and nondegeneracy assumptions, the proposed multi-point CBCs include these endpoint-type conditions as limiting special cases.

The remaining sections of this paper are organized as follows: Section 2 recalls the fundamental definitions and properties of fractional calculus and introduces the theoretical tools required in this study, namely Krasnosel'skii FPT and the BCMP; Section 3 establishes the existence and uniqueness of mild solutions to the BVP (1.1) by applying the two FPTs presented in Section 2; Section 4 is devoted to the analysis of U-H stability and U-H-R stability of the mild solutions; Section 5 presents two concrete examples to indicate and validate the main results obtained in this work; and finally, Section 6 provides a conclusion together with some perspectives for future research.

2. Preliminaries

This section reviews the basic definitions as well as the key properties of fractional calculus, along with Krasnosel'skii FPT and the BCMP.

Definition 2.1. ([34]) For a function $\vartheta : [0, +\infty) \rightarrow \mathbb{R}$, the Riemann-Liouville fractional integral of order ι ($\iota > 0$) is defined as follows:

$$\mathcal{I}_{0+}^\iota \vartheta(t) = \frac{1}{\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} \vartheta(s) ds, \quad t > 0,$$

where the expression on the right is pointwise well-defined on $(0, +\infty)$.

Definition 2.2. ([34]) For $\vartheta(t) \in AC^n[0, +\infty)$, the CFD of order ι ($\iota > 0$) is defined by the following:

$${}^C \mathcal{D}_{0+}^\iota \vartheta(t) = \frac{1}{\Gamma(n-\iota)} \int_0^t (t-s)^{n-\iota-1} \vartheta^{(n)}(s) ds, \quad t > 0,$$

where $n = \lceil \iota \rceil$. In particular, ${}^C \mathcal{D}_{0+}^\iota \vartheta(t) = \vartheta^{(n)}(t)$, $\iota = n \in \mathbb{N}$.

Lemma 2.1. ([34]) Let $\iota > 0$ and $n = \lceil \iota \rceil$. If $\vartheta(t) \in AC^n[0, T]$, then

$$\mathcal{I}_{0+}^\iota {}^C \mathcal{D}_{0+}^\iota \vartheta(t) = \vartheta(t) - \sum_{k=0}^{n-1} \frac{\vartheta^{(k)}(0)}{k!} t^k, \quad 0 < t < T.$$

Lemma 2.2. ([34]) Let $\iota, \varrho > 0$; and $\vartheta(t) \in C[0, T]$; then,

$$(\mathcal{I}_{0+}^\iota \mathcal{I}_{0+}^\varrho \vartheta)(t) = (\mathcal{I}_{0+}^{\iota+\varrho} \vartheta)(t), \quad \mathcal{I}_{0+}^\iota t^{\varrho-1} = \frac{\Gamma(\varrho)}{\Gamma(\iota+\varrho)} t^{\iota+\varrho-1},$$

$${}^C \mathcal{D}_{0+}^{\varrho} t^{\varrho-1} = \frac{\Gamma(\varrho)}{\Gamma(\varrho - \iota)} t^{\varrho-\iota-1}, \quad \varrho > [\iota],$$

and

$${}^C \mathcal{D}_{0+}^{\iota} t^k = 0, \quad k = 0, 1, \dots, [\iota] - 1.$$

In particular, ${}^C \mathcal{D}_{0+}^{\iota} 1 = 0$.

For the sake of completeness, we now recall two fundamental FPTs that will be frequently used in the subsequent analysis: the Krasnosel'skii FPT and the BCMP.

Theorem 2.1. ([35]) Let χ denote a Banach space, and let Ω be a nonempty, bounded, convex, and closed subset. Let Υ and Ξ be two operators defined on Ω that satisfy the following three properties:

- (a) $\Upsilon x + \Xi y \in \Omega$, for all $x, y \in \Omega$;
- (b) Υ is a completely continuous operator; and
- (c) Ξ is a contraction mapping.

Then, there exists an element $\aleph \in \Omega$ verifying $\aleph = \Upsilon \aleph + \Xi \aleph$.

Theorem 2.2. ([35]) Let χ be a Banach space and $\Omega \subset \chi$ be a nonempty closed set, and let $\mathfrak{J} : \Omega \subseteq \chi \rightarrow \Omega$ be a contraction, meaning that for a fixed $\tau \in [0, 1)$,

$$\|\mathfrak{J}x - \mathfrak{J}y\|_{\chi} \leq \tau \|x - y\|_{\chi}, \quad \forall x, y \in \Omega.$$

Then, $\mathfrak{J}$ possesses a unique fixed point in the set Ω .

3. Main results

In the present section, we prove the existence and uniqueness of mild solutions to the BVP (1.1) via Theorems 2.1 and 2.2, respectively. We begin by introducing the Banach space $\chi = C[0, T]$, equipped with the norm

$$\|x\|_{\infty} = \max_{t \in [0, T]} |x(t)|.$$

Lemma 3.1. Let $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$. Consider the FLE

$${}^C \mathcal{D}_{0+}^{\beta} ({}^C \mathcal{D}_{0+}^{\alpha} + \lambda)x(t) = f(t, x(t)), \quad t \in (0, T), \quad (3.1)$$

subject to the multi-point closed BCs

$$\begin{aligned} x(T) &= \sum_{i=1}^m (a_i x(\xi_i) + b_i T^C \mathcal{D}_{0+}^{\alpha} x(\xi_i)), \\ T^C \mathcal{D}_{0+}^{\alpha} x(T) &= \sum_{i=1}^m (c_i x(\xi_i) + d_i T^C \mathcal{D}_{0+}^{\alpha} x(\xi_i)). \end{aligned} \quad (3.2)$$

If a function x satisfies the BVP (3.1) and (3.2), then x satisfies the following integral equation:

$$x(t) = -\frac{1}{\Lambda} \{ \varepsilon_{11} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{J}_2(x(t))] \}$$

$$\begin{aligned}
& -\varepsilon_{21} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t))] \\
& - \frac{t^\alpha}{\Gamma(\alpha + 1)} \frac{1}{\Lambda} \{\varepsilon_{22} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t))] \\
& - \varepsilon_{12} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{I}_2(x(t))]\} \\
& - \lambda \mathcal{J}_{0+}^\alpha x(t) + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)),
\end{aligned} \tag{3.3}$$

where $\Lambda = \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21} \neq 0$,

$$\begin{aligned}
\mathfrak{R}_1(f(t, x(t))) &= -\mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=T} + \sum_{i=1}^m (a_i - \lambda T b_i) \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=\xi_i} \\
&+ T \sum_{i=1}^m b_i \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=\xi_i}, \\
\mathfrak{R}_2(f(t, x(t))) &= \lambda T \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=T} + \sum_{i=1}^m (c_i - \lambda T d_i) \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=\xi_i} \\
&- T \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=T} + T \sum_{i=1}^m d_i \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=\xi_i}, \\
\mathfrak{I}_1(x(t)) &= \lambda \mathcal{J}_{0+}^\alpha x(t)|_{t=T} - \lambda \sum_{i=1}^m (a_i - \lambda T b_i) \mathcal{J}_{0+}^\alpha x(t)|_{t=\xi_i}, \\
\mathfrak{I}_2(x(t)) &= -\lambda^2 T \mathcal{J}_{0+}^\alpha x(t)|_{t=T} - \lambda \sum_{i=1}^m (c_i - \lambda T d_i) \mathcal{J}_{0+}^\alpha x(t)|_{t=\xi_i}.
\end{aligned}$$

Proof. Applying the operator $\mathcal{J}_{0+}^\beta$ to Eq (3.1), we deduce via Lemma 2.1 that

$${}^C \mathcal{D}_{0+}^\alpha x(t) + \lambda x(t) + \tilde{c}_0 = \mathcal{J}_{0+}^\beta f(t, x(t)). \tag{3.4}$$

By acting the operator $\mathcal{J}_{0+}^\alpha$ on each side of Eq (3.4) and invoking Lemmas 2.1 and 2.2, we arrive at

$$x(t) = -\tilde{c}_1 - \lambda \mathcal{J}_{0+}^\alpha x(t) - \frac{t^\alpha}{\Gamma(\alpha + 1)} \tilde{c}_0 + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)). \tag{3.5}$$

From Eqs (3.4) and (3.5), it is clear that

$$\begin{cases} x(\xi_i) = -\tilde{c}_1 - \lambda \mathcal{J}_{0+}^\alpha x(t)|_{t=\xi_i} - \frac{\xi_i^\alpha}{\Gamma(\alpha+1)} \tilde{c}_0 + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=\xi_i}, \\ x(T) = -\tilde{c}_1 - \lambda \mathcal{J}_{0+}^\alpha x(t)|_{t=T} - \frac{T^\alpha}{\Gamma(\alpha+1)} \tilde{c}_0 + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=T}, \\ {}^C \mathcal{D}_{0+}^\alpha x(\xi_i) = -\lambda x(\xi_i) - \tilde{c}_0 + \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=\xi_i}, \\ {}^C \mathcal{D}_{0+}^\alpha x(T) = -\lambda x(T) - \tilde{c}_0 + \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=T}. \end{cases} \tag{3.6}$$

Substituting Eq (3.6) into Eq (3.2) results in a system of linear equations involving the unknown coefficients $\tilde{c}_0$ and $\tilde{c}_1$,

$$\begin{cases} \varepsilon_{11} \tilde{c}_0 + \varepsilon_{12} \tilde{c}_1 = \mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t)), \\ \varepsilon_{21} \tilde{c}_0 + \varepsilon_{22} \tilde{c}_1 = \mathfrak{R}_2(f(t, x(t))) + \mathfrak{I}_2(x(t)). \end{cases} \tag{3.7}$$

By solving the system of above equations, we derive the following:

$$\begin{aligned}\tilde{c}_0 &= \frac{1}{\Lambda} \{ \varepsilon_{22} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t))] - \varepsilon_{12} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{I}_2(x(t))] \}, \\ \tilde{c}_1 &= \frac{1}{\Lambda} \{ \varepsilon_{11} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{I}_2(x(t))] - \varepsilon_{21} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t))] \}.\end{aligned}$$

Substituting $\tilde{c}_0$ and $\tilde{c}_1$ into Eq (3.5) directly results in Eq (3.3). Thus, the lemma is proven.

Definition 3.1. A function $x(t) \in \chi$ is called a mild solution of the BVP (3.1) and (3.2) if $x(t)$ satisfies the integral Eq (3.3) for every $t \in [0, T]$.

Based on the integral Eq (3.3), we introduce the operator $\mathfrak{J} : \chi \rightarrow \chi$ defined as follows:

$$\begin{aligned}\mathfrak{J}(x(t)) &= -\frac{1}{\Lambda} \{ \varepsilon_{11} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{I}_2(x(t))] \\ &\quad - \varepsilon_{21} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t))] \} \\ &\quad - \frac{t^\alpha}{\Gamma(\alpha + 1)} \frac{1}{\Lambda} \{ \varepsilon_{22} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{I}_1(x(t))] \\ &\quad - \varepsilon_{12} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{I}_2(x(t))] \} \\ &\quad - \lambda \mathcal{J}_{0+}^\alpha x(t) + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)).\end{aligned}$$

It is evident that establishing the existence of a mild solution to the BVP (1.1) is equivalent to showing that the operator $\mathfrak{J}$ admits a fixed point.

Theorem 3.1. Suppose that $\Lambda \neq 0$, together with the following hypotheses:

(C₁) $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

(C₂) There exist functions $p(t), q(t) \in C([0, T], \mathbb{R}^+)$ such that

$$|f(t, x(t))| \leq p(t) + q(t)|x(t)|, \quad (t, x) \in [0, T] \times \mathbb{R},$$

$q = \max_{t \in [0, T]} |q(t)|$, $p = \max_{t \in [0, T]} |p(t)|$. Then, the BVP (1.1) admits at least one mild solution on $[0, T]$, provided that

$$\tau + \varsigma q < 1, \quad (3.8)$$

where

$$\begin{aligned}\Delta_1 &= \frac{1}{\Gamma(\alpha + \beta + 1)} \sum_{i=1}^m |a_i - \lambda T b_i| \xi_i^{\alpha+\beta} + \frac{T}{\Gamma(\beta + 1)} \sum_{i=1}^m |b_i| \xi_i^\beta + \frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)}, \\ \Delta_2 &= \frac{1}{\Gamma(\alpha + \beta + 1)} \sum_{i=1}^m |c_i - \lambda T d_i| \xi_i^{\alpha+\beta} + \frac{T}{\Gamma(\beta + 1)} \sum_{i=1}^m |d_i| \xi_i^\beta \\ &\quad + \frac{\lambda T^{\alpha+\beta+1}}{\Gamma(\alpha + \beta + 1)} + \frac{T^{\beta+1}}{\Gamma(\beta + 1)}, \\ \Delta_3 &= \frac{\lambda}{\Gamma(\alpha + 1)} \sum_{i=1}^m |a_i - \lambda T b_i| \xi_i^\alpha + \frac{\lambda T^\alpha}{\Gamma(\alpha + 1)},\end{aligned}$$

$$\begin{aligned}\Delta_4 &= \frac{\lambda}{\Gamma(\alpha+1)} \sum_{i=1}^m |c_i - \lambda T d_i| \xi_i^\alpha + \frac{\lambda^2 T^{\alpha+1}}{\Gamma(\alpha+1)}, \\ \varsigma &= \frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_2 + |\varepsilon_{21}| \Delta_1) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha+1)} (|\varepsilon_{22}| \Delta_1 + |\varepsilon_{12}| \Delta_2) + \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \\ \tau &= \frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_4 + |\varepsilon_{21}| \Delta_3) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha+1)} (|\varepsilon_{22}| \Delta_3 + |\varepsilon_{12}| \Delta_4) + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Proof. Let $\varepsilon > 0$ be such that

$$\varepsilon \geq \frac{\varsigma p}{1 - \tau - \varsigma q}.$$

Define a closed, bounded subset as follows:

$$\Omega_\varepsilon = \{x \in \mathcal{X} : \|x\|_\infty \leq \varepsilon\}.$$

In turn, the operators $\Upsilon, \Xi : \Omega_\varepsilon \rightarrow \mathcal{X}$ are defined as follows:

$$\begin{aligned}\Upsilon(x(t)) &= -\frac{1}{\Lambda} [\varepsilon_{11} \mathfrak{R}_2(f(t, x(t))) - \varepsilon_{21} \mathfrak{R}_1(f(t, x(t)))] + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)) \\ &\quad - \frac{t^\alpha}{\Lambda \Gamma(\alpha+1)} [\varepsilon_{22} \mathfrak{R}_1(f(t, x(t))) - \varepsilon_{12} \mathfrak{R}_2(f(t, x(t)))], \\ \Xi(x(t)) &= -\frac{1}{\Lambda} [\varepsilon_{11} \mathfrak{I}_2(x(t)) - \varepsilon_{21} \mathfrak{I}_1(x(t))] - \lambda \mathcal{J}_{0+}^\alpha x(t) \\ &\quad - \frac{t^\alpha}{\Lambda \Gamma(\alpha+1)} [\varepsilon_{22} \mathfrak{I}_1(x(t)) - \varepsilon_{12} \mathfrak{I}_2(x(t))].\end{aligned}$$

To establish Theorem 3.1 by means of Theorem 2.1, the proof is decomposed into the following three steps.

Step 1. It is shown that $\Upsilon x + \Xi y \in \Omega_\varepsilon$ for all $x, y \in \Omega_\varepsilon$. In fact, given that $x, y \in \Omega_\varepsilon$, we can derive $\|x\|_\infty \leq \varepsilon, \|y\|_\infty \leq \varepsilon$, as well as $|f(t, x(t))| \leq p + q\varepsilon$. Then,

$$\begin{aligned}|\mathfrak{R}_1(f(t, x(t)))| &= \left| \sum_{i=1}^m (a_i - \lambda T b_i) \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=\xi_i} \right. \\ &\quad \left. + T \sum_{i=1}^m b_i \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=\xi_i} - \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=T} \right| \leq \Delta_1(q\varepsilon + p), \\ |\mathfrak{R}_2(f(t, x(t)))| &= \left| \sum_{i=1}^m (c_i - \lambda T d_i) \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=\xi_i} + T \sum_{i=1}^m d_i \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=\xi_i} \right. \\ &\quad \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t))|_{t=T} - T \mathcal{J}_{0+}^\beta f(t, x(t))|_{t=T} \right| \leq \Delta_2(q\varepsilon + p), \\ |\mathfrak{I}_1(x(t))| &= \left| -\lambda \sum_{i=1}^m (a_i - \lambda T b_i) \mathcal{J}_{0+}^\alpha x(t)|_{t=\xi_i} + \lambda \mathcal{J}_{0+}^\alpha x(t)|_{t=T} \right| \leq \Delta_3\varepsilon, \\ |\mathfrak{I}_2(x(t))| &= \left| -\lambda \sum_{i=1}^m (c_i - \lambda T d_i) \mathcal{J}_{0+}^\alpha x(t)|_{t=\xi_i} - \lambda^2 T \mathcal{J}_{0+}^\alpha x(t)|_{t=T} \right| \leq \Delta_4\varepsilon.\end{aligned}$$

Furthermore, for any $t \in [0, T]$, we have the following:

$$|\Upsilon x(t)| = \left| -\frac{1}{\Lambda} [\varepsilon_{11} \mathfrak{R}_2(f(t, x(t))) - \varepsilon_{21} \mathfrak{R}_1(f(t, x(t)))] - \frac{t^\alpha}{\Lambda \Gamma(\alpha + 1)} [\varepsilon_{22} \mathfrak{R}_1(f(t, x(t))) - \varepsilon_{12} \mathfrak{R}_2(f(t, x(t)))] + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)) \right| \quad (3.9)$$

$$\leq \left[\frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_2 + |\varepsilon_{21}| \Delta_1) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} (|\varepsilon_{22}| \Delta_1 + |\varepsilon_{12}| \Delta_2) + \frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} \right] (q\varepsilon + p),$$

$$|\Xi y(t)| = \left| -\frac{1}{\Lambda} [\varepsilon_{11} \mathfrak{J}_2(y(t)) - \varepsilon_{21} \mathfrak{J}_1(y(t))] - \frac{t^\alpha}{\Lambda \Gamma(\alpha + 1)} [\varepsilon_{22} \mathfrak{J}_1(y(t)) - \varepsilon_{12} \mathfrak{J}_2(y(t))] - \lambda \mathcal{J}_{0+}^\alpha y(t) \right|$$

$$\leq \left[\frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_4 + |\varepsilon_{21}| \Delta_3) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} (|\varepsilon_{22}| \Delta_3 + |\varepsilon_{12}| \Delta_4) + \frac{\lambda T^\alpha}{\Gamma(\alpha + 1)} \right] \varepsilon. \quad (3.10)$$

Combining Eq (3.9) with Eq (3.10), one can obtain the following:

$$|\Upsilon x(t) + \Xi y(t)| \leq \left[\frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_2 + |\varepsilon_{21}| \Delta_1) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} (|\varepsilon_{22}| \Delta_1 + |\varepsilon_{12}| \Delta_2) + \frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} \right] (q\varepsilon + p) + \left[\frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_4 + |\varepsilon_{21}| \Delta_3) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} (|\varepsilon_{22}| \Delta_3 + |\varepsilon_{12}| \Delta_4) + \frac{\lambda T^\alpha}{\Gamma(\alpha + 1)} \right] \varepsilon = \varsigma(q\varepsilon + p) + \tau\varepsilon.$$

Therefore,

$$\|\Upsilon x + \Xi y\|_\infty \leq \varsigma(p + q\varepsilon) + \tau\varepsilon \leq \varepsilon,$$

that is, $\Upsilon x + \Xi y \in \Omega_\varepsilon$, for any $x, y \in \Omega_\varepsilon$.

Step 2. We assert that Υ is a completely continuous operator on Ω_ε . To this end, let the sequence $\{x_n(t)\} \subset \Omega_\varepsilon$ satisfy $\|x_n - x\|_\infty \rightarrow 0, n \rightarrow \infty$. By this joint continuity of $f(t, x(t))$ on the compact strip $[0, T] \times [-\varepsilon, \varepsilon]$, it follows that $f(\cdot, x_n(\cdot)) \rightarrow f(\cdot, x(\cdot)), n \rightarrow \infty$. Since $f(t, x(t))$ satisfies condition (C_2) , then $|f(t, x_n(t))| \leq p + q\varepsilon$. Define the following:

$$\mathfrak{F}(s) = (p + q\varepsilon)(t - s)^\kappa, (t, s) \in [0, T] \times [0, t], (\kappa = \alpha + \beta - 1, \beta - 1).$$

Then, $\mathfrak{F}(s)$ is integrable on $[0, t]$. Hence, by the Lebesgue dominated convergence theorem, we obtain the following:

$$\lim_{n \rightarrow \infty} \Upsilon(x_n(t)) = -\frac{1}{\Lambda} \lim_{n \rightarrow \infty} [\varepsilon_{11} \mathfrak{R}_2(f(t, x_n(t))) - \varepsilon_{21} \mathfrak{R}_1(f(t, x_n(t)))]$$

$$\begin{aligned}
& - \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} \lim_{n \rightarrow \infty} [\varepsilon_{22}\mathfrak{R}_1(f(t, x_n(t))) - \varepsilon_{12}\mathfrak{R}_2(f(t, x_n(t)))] \\
& + \lim_{n \rightarrow \infty} \mathcal{J}_{0+}^{\alpha+\beta} f(t, x_n(t)) \\
& = - \frac{1}{\Lambda} \lim_{n \rightarrow \infty} [\varepsilon_{11}\mathfrak{R}_2(f(t, x(t))) - \varepsilon_{21}\mathfrak{R}_1(f(t, x(t)))] \\
& - \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} \lim_{n \rightarrow \infty} [\varepsilon_{22}\mathfrak{R}_1(f(t, x(t))) - \varepsilon_{12}\mathfrak{R}_2(f(t, x(t)))] \\
& + \lim_{n \rightarrow \infty} \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)) \\
& = \Upsilon(x(t)).
\end{aligned}$$

Therefore, the operator Υ is continuous on Ω_ε . On the other hand, it follows from Step 1 that Υ is uniformly bounded on Ω_ε for all $x(t) \in \Omega_\varepsilon$ and $t \in [0, T]$. By the Arzelà-Ascoli theorem, to establish that Υ is compact on Ω_ε , one must further verify that Υ is equicontinuous on Ω_ε . In fact, for any $x(t) \in \Omega_\varepsilon$ and $t_1, t_2 \in [0, T]$, without loss of generality, suppose that $0 \leq t_1 < t_2 \leq T$; then,

$$\begin{aligned}
& |\Upsilon x(t_2) - \Upsilon x(t_1)| \\
& = \left| - \frac{1}{\Lambda} [\varepsilon_{11}\mathfrak{R}_2(f(t, x(t))) - \varepsilon_{21}\mathfrak{R}_1(f(t, x(t)))] \right. \\
& \quad - \frac{t_2^\alpha}{\Lambda\Gamma(\alpha+1)} [\varepsilon_{22}\mathfrak{R}_1(f(t, x(t))) - \varepsilon_{12}\mathfrak{R}_2(f(t, x(t)))] \\
& \quad + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)) \Big|_{t=t_2} + \frac{1}{\Lambda} [\varepsilon_{11}\mathfrak{R}_2(f(t, x(t))) - \varepsilon_{21}\mathfrak{R}_1(f(t, x(t)))] \\
& \quad + \frac{t_1^\alpha}{\Lambda\Gamma(\alpha+1)} [\varepsilon_{22}\mathfrak{R}_1(f(t, x(t))) - \varepsilon_{12}\mathfrak{R}_2(f(t, x(t)))] - \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)) \Big|_{t=t_1} \Big| \\
& \leq \left[\frac{(t_2^\alpha - t_1^\alpha)}{|\Lambda|\Gamma(\alpha+1)} (|\varepsilon_{22}|\Delta_1 + |\varepsilon_{12}|\Delta_2) + \frac{t_2^{\alpha+\beta} - t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right] (q\varepsilon + p).
\end{aligned}$$

Given that t^α and $t^{\alpha+\beta}$ are uniformly continuous on $[0, T]$, it follows that

$$|\Upsilon x(t_2) - \Upsilon x(t_1)| \rightarrow 0, \quad \text{as } t_2 \rightarrow t_1.$$

Therefore, Υ is equicontinuous on Ω_ε .

Step 3. We demonstrate that Ξ is a contraction mapping on Ω_ε . In fact, given any $x, y \in \Omega_\varepsilon$, we can derive the following:

$$\begin{aligned}
& |\Xi x(t) - \Xi y(t)| \\
& = \left| - \frac{1}{\Lambda} [\varepsilon_{11}\mathfrak{J}_2(x(t)) - \varepsilon_{21}\mathfrak{J}_1(x(t))] - \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} [\varepsilon_{22}\mathfrak{J}_1(x(t)) - \varepsilon_{12}\mathfrak{J}_2(x(t))] \right. \\
& \quad - \lambda \mathcal{J}_{0+}^\alpha x(t) + \frac{1}{\Lambda} [\varepsilon_{11}\mathfrak{J}_2(y(t)) - \varepsilon_{21}\mathfrak{J}_1(y(t))] \\
& \quad + \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} [\varepsilon_{22}\mathfrak{J}_1(y(t)) - \varepsilon_{12}\mathfrak{J}_2(y(t))] + \lambda \mathcal{J}_{0+}^\alpha y(t) \Big| \\
& \leq \left[\frac{1}{|\Lambda|} (|\varepsilon_{11}|\Delta_4 + |\varepsilon_{21}|\Delta_3) + \frac{T^\alpha}{|\Lambda|\Gamma(\alpha+1)} (|\varepsilon_{22}|\Delta_3 + |\varepsilon_{12}|\Delta_4) + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)} \right] \|x - y\|_\infty.
\end{aligned}$$

Consequently,

$$\|\Xi x - \Xi y\|_\infty \leq \tau \|x - y\|_\infty.$$

Hence, Ξ is a contraction mapping. By virtue of Theorem 2.1, we deduce that the BVP (1.1) possesses at least one mild solution on $[0, T]$.

Theorem 3.2. Assume that $\Lambda \neq 0$, and suppose the following conditions are satisfied:

(C₁) $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

(C₃) There exists a constant $L > 0$, such that for all $x, y \in \mathbb{R}$ and $t \in [0, T]$,

$$|f(t, x) - f(t, y)| \leq L|x - y|.$$

If the inequality

$$\tau + \varsigma L < 1 \tag{3.11}$$

holds, then the BVP (1.1) possesses a unique mild solution.

Proof. Let $\rho > 0$ and

$$\rho \geq \frac{\varsigma \omega}{1 - \tau - \varsigma L},$$

where $\omega = \max_{t \in [0, T]} |f(t, 0)|$. Define the set

$$\Omega_\rho = \{x \in \mathcal{X} : \|x\|_\infty \leq \rho\},$$

where we can argue that $\mathfrak{J}\Omega_\rho \subset \Omega_\rho$. For $\forall x \in \Omega_\rho$ and $t \in [0, T]$, by condition (C₃), we obtain the following:

$$|f(t, x)| \leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq L\|x\|_\infty + \omega \leq L\rho + \omega.$$

Furthermore, we obtain the following:

$$\begin{aligned} & |\mathfrak{J}(x(t))| \\ &= \left| -\frac{1}{\Lambda} \{ \varepsilon_{11} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{J}_2(x(t))] - \varepsilon_{21} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{J}_1(x(t))] \} \right. \\ &\quad - \frac{t^\alpha}{\Lambda \Gamma(\alpha + 1)} \{ \varepsilon_{22} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{J}_1(x(t))] \\ &\quad \left. - \varepsilon_{12} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{J}_2(x(t))] \} - \lambda \mathcal{I}_{0+}^\alpha x(t) + \mathcal{I}_{0+}^{\alpha+\beta} f(t, x(t)) \right| \\ &\leq \frac{1}{|\Lambda|} \{ |\varepsilon_{11}| [\Delta_2(L\rho + \omega) + \Delta_4\rho] + |\varepsilon_{21}| [\Delta_1(L\rho + \omega) + \Delta_3\rho] \} \\ &\quad + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} \{ |\varepsilon_{22}| [\Delta_1(L\rho + \omega) + \Delta_3\rho] + |\varepsilon_{12}| [\Delta_2(L\rho + \omega) + \Delta_4\rho] \} \\ &\quad + \frac{\lambda T^\alpha}{\Gamma(\alpha + 1)} \rho + \frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} (L\rho + \omega) \end{aligned}$$

$$\leq \varsigma(L\rho + \omega) + \tau\rho \leq \rho.$$

Consequently, $\mathfrak{J}\Omega_\rho \subset \Omega_\rho$. Now, we prove that $\mathfrak{J}$ constitutes a contractive mapping on Ω_ρ . For $\forall x, y \in \Omega_\rho$, it follows that

$$\begin{aligned} & |\mathfrak{J}x(t) - \mathfrak{J}y(t)| \\ &= \left| -\frac{1}{\Lambda} \{ \varepsilon_{11} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{J}_2(x(t))] - \varepsilon_{21} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{J}_1(x(t))] \} \right. \\ &\quad - \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} \{ \varepsilon_{22} [\mathfrak{R}_1(f(t, x(t))) + \mathfrak{J}_1(x(t))] \\ &\quad - \varepsilon_{12} [\mathfrak{R}_2(f(t, x(t))) + \mathfrak{J}_2(x(t))] \} - \lambda \mathcal{J}_{0+}^\alpha x(t) + \mathcal{J}_{0+}^{\alpha+\beta} f(t, x(t)) \\ &\quad + \frac{1}{\Lambda} \{ \varepsilon_{11} [\mathfrak{R}_2(f(t, y(t))) + \mathfrak{J}_2(y(t))] - \varepsilon_{21} [\mathfrak{R}_1(f(t, y(t))) + \mathfrak{J}_1(y(t))] \} \\ &\quad + \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} \{ \varepsilon_{22} [\mathfrak{R}_1(f(t, y(t))) + \mathfrak{J}_1(y(t))] \\ &\quad - \varepsilon_{12} [\mathfrak{R}_2(f(t, y(t))) + \mathfrak{J}_2(y(t))] \} + \lambda \mathcal{J}_{0+}^\alpha y(t) - \mathcal{J}_{0+}^{\alpha+\beta} f(t, y(t)) \Big| \\ &\leq \frac{1}{|\Lambda|} [|\varepsilon_{11}|(\Delta_2 L \|y-x\|_\infty + \Delta_4 \|y-x\|_\infty) + |\varepsilon_{21}|(\Delta_1 L \|y-x\|_\infty + \Delta_3 \|y-x\|_\infty)] \\ &\quad + \frac{T^\alpha}{|\Lambda|\Gamma(\alpha+1)} [|\varepsilon_{22}|(\Delta_1 L \|y-x\|_\infty + \Delta_3 \|y-x\|_\infty) \\ &\quad + |\varepsilon_{12}|(\Delta_2 L \|y-x\|_\infty + \Delta_4 \|y-x\|_\infty)] + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)} \|y-x\|_\infty + \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} L \|y-x\|_\infty \\ &= \varsigma L \|y-x\|_\infty + \tau \|y-x\|_\infty. \end{aligned}$$

This implies that

$$\|\mathfrak{J}x(t) - \mathfrak{J}y(t)\|_\infty \leq (\varsigma L + \tau) \|y-x\|_\infty. \quad (3.12)$$

On the basis of Eq (3.11), it can be inferred that $\mathfrak{J}$ is a contraction operator. By using the BCMP, one guarantees the uniqueness of the fixed point $x \in \Omega_\rho$ associated with the operator $\mathfrak{J}$. Consequently, the BVP (1.1) exists for a unique mild solution on Ω_ρ .

4. Ulam stability analysis

This section is devoted to the investigation of the U-H and U-H-R stability of mild solutions to the BVP (1.1). Throughout this section, we assume that $\Lambda \neq 0$. Let $\mathfrak{J} : \chi \rightarrow \chi$ be the operator defined in Section 3, and let $\varphi(t) : [0, T] \rightarrow \mathbb{R}^+$ ($\mathbb{R}^+ = (0, +\infty)$) be a non-decreasing function. For convenience, we introduce the linear forcing operator $B : \chi \rightarrow \chi$ defined by the following:

$$\begin{aligned} (Bh)(t) &= -\frac{1}{\Lambda} [\varepsilon_{11} \mathfrak{R}_2(h) - \varepsilon_{21} \mathfrak{R}_1(h)] + \mathcal{J}_{0+}^{\alpha+\beta} h(t) \\ &\quad - \frac{t^\alpha}{\Lambda\Gamma(\alpha+1)} [\varepsilon_{22} \mathfrak{R}_1(h) - \varepsilon_{12} \mathfrak{R}_2(h)], \quad h \in \chi, \quad t \in [0, T], \end{aligned}$$

where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are the linear functions introduced in Lemma 3.1.

Definition 4.1. A function $\varpi \in \chi$ is called an ε -perturbed mild solution of the BVP (1.1) if there exists $u \in C([0, T], \mathbb{R})$ such that

$$\|u\|_\infty \leq \varepsilon$$

and

$$\varpi = \mathfrak{J}\varpi + Bu. \quad (4.1)$$

Definition 4.2. Let $\varepsilon > 0$, and let $\varphi(t) : [0, T] \rightarrow \mathbb{R}^+$ be non-decreasing continuous functions. A function $\varpi \in \chi$ is called an (ε, φ) -perturbed mild solution of the BVP (1.1) if there exists $v \in C([0, T], \mathbb{R})$ such that

$$|v(t)| \leq \varepsilon \varphi(t), \quad t \in [0, T],$$

and

$$\varpi = \mathfrak{J}\varpi + Bv. \quad (4.2)$$

Definition 4.3. The BVP (1.1) is said to be U-H stable if there exists a constant $c_f > 0$ such that, for every $\varepsilon > 0$ and every ε -perturbed mild solution $\varpi \in \chi$ of the BVP (1.1), there exists a mild solution $z \in \chi$ of the BVP (1.1) satisfying

$$\|z - \varpi\|_\infty \leq c_f \varepsilon.$$

Definition 4.4. Let $\varphi(t) : [0, T] \rightarrow \mathbb{R}^+$. The BVP (1.1) is said to be U-H-R stable with respect to φ if there exists a constant $c_{f,\varphi} > 0$ such that, for every $\varepsilon > 0$ and every (ε, φ) -perturbed mild solution $\varpi \in \chi$ of the BVP (1.1), there exists a mild solution $z \in \chi$ of the BVP (1.1) satisfying

$$\|z - \varpi\|_\infty \leq c_{f,\varphi} \varepsilon \varphi(t), \quad t \in [0, T].$$

Next, we establish sufficient conditions for the U-H and U-H-R stability of the BVP (1.1).

Theorem 4.1. Suppose that (C_1) , (C_3) , and Eq (3.11) are satisfied. Let $z \in \chi$ be a mild solution of the BVP (1.1), and let $\varpi \in \chi$ be an ε -perturbed mild solution of the BVP (1.1). Then, there exists a constant $c_f > 0$, such that for each $\varepsilon > 0$,

$$\|z - \varpi\|_\infty \leq c_f \varepsilon,$$

that is, the BVP (1.1) is U-H stable.

Proof. By Eq (4.1),

$$\varpi = \mathfrak{J}\varpi + Bu, \quad \|u\|_\infty \leq \varepsilon.$$

Since (C_1) , (C_3) , and Equation (3.11) hold and $z \in \chi$ is a mild solution of BVP (1.1), it follows from Theorem 3.2 that z is the unique mild solution of Problem (1.1), such that $\mathfrak{J}z = z$. Then, by Eq (3.12), we obtain

$$\|z - \varpi\|_\infty = \|\mathfrak{J}z - \mathfrak{J}\varpi - Bu\|_\infty \leq (\varsigma L + \tau)\|z - \varpi\|_\infty + \|Bu\|_\infty;$$

subsequently,

$$\|z - \varpi\|_\infty \leq \frac{\|Bu\|_\infty}{1 - (\varsigma L + \tau)}. \quad (4.3)$$

On the other hand, we have the following:

$$\begin{aligned}
& \|Bu\|_\infty \\
& \leq \frac{1}{|\Lambda|} \left\{ |\varepsilon_{11}| \left[\sum_{i=1}^m |c_i - \lambda T d_i| \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=\xi_i} + T \sum_{i=1}^m |d_i| \mathcal{J}_{0+}^\beta |u(t)| \Big|_{t=\xi_i} \right. \right. \\
& \quad \left. \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=T} + T \mathcal{J}_{0+}^\beta |u(t)| \Big|_{t=T} \right] \right. \\
& \quad \left. + |\varepsilon_{21}| \left[\sum_{i=1}^m |a_i - \lambda T b_i| \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=\xi_i} + T \sum_{i=1}^m |b_i| \mathcal{J}_{0+}^\beta |u(t)| \Big|_{t=\xi_i} \right. \right. \\
& \quad \left. \left. + \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=T} \right] \right\} + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha+1)} \left\{ |\varepsilon_{22}| \left[\sum_{i=1}^m |a_i - \lambda T b_i| \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=\xi_i} \right. \right. \\
& \quad \left. \left. + T \sum_{i=1}^m |b_i| \mathcal{J}_{0+}^\beta |u(t)| \Big|_{t=\xi_i} + \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=T} \right] \right. \\
& \quad \left. + |\varepsilon_{12}| \left[\sum_{i=1}^m |c_i - \lambda T d_i| \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=\xi_i} + T \sum_{i=1}^m |d_i| \mathcal{J}_{0+}^\beta |u(t)| \Big|_{t=\xi_i} \right. \right. \\
& \quad \left. \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=T} + T \mathcal{J}_{0+}^\beta |u(t)| \Big|_{t=T} \right] \right\} + \mathcal{J}_{0+}^{\alpha+\beta} |u(t)| \Big|_{t=T} \\
& \leq \left[\frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_2 + |\varepsilon_{21}| \Delta_1) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha+1)} (|\varepsilon_{22}| \Delta_1 + |\varepsilon_{12}| \Delta_2) + \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right] \varepsilon \\
& = \zeta \varepsilon.
\end{aligned}$$

It follows from Eq (4.3) that

$$\|z - \varpi\|_\infty \leq \frac{\zeta \varepsilon}{1 - (\zeta L + \tau)}.$$

Let

$$c_f = \frac{\zeta}{1 - (\zeta L + \tau)}.$$

Then, the BVP (1.1) is U-H stable. It has been demonstrated that Theorem 4.1 holds.

Theorem 4.2. Suppose that (C_1) , (C_3) , and Eq (3.11) are satisfied. Let $z \in \chi$ be a mild solution of the BVP (1.1), and let $\varpi \in \chi$ be an (ε, φ) -perturbed mild solution of the BVP (1.1), $\varphi \in C([0, T], \mathbb{R}^+)$, and there exists $\rho_\varphi > 0$ such that

$$\mathcal{J}_{0+}^\beta \varphi(t) \Big|_{t=T} \leq \rho_\varphi \varphi(t), \quad \mathcal{J}_{0+}^{\alpha+\beta} \varphi(t) \Big|_{t=T} \leq \rho_\varphi \varphi(t), \quad t \in [0, T]. \quad (4.4)$$

Then, there exists a constant $c_{f,\varphi} > 0$ such that for each $\varepsilon > 0$,

$$\|z - \varpi\|_\infty \leq c_{f,\varphi} \varepsilon \varphi(t), \quad t \in [0, T],$$

that is, the BVP (1.1) is U-H-R stable.

Proof. By Eq (4.2),

$$\varpi = \mathfrak{J}\varpi + Bv, \quad |v(t)| \leq \varepsilon\varphi(t), \quad t \in [0, T].$$

Note that $z \in \chi$ is a mild solution of the BVP (1.1), and (C_1) , (C_3) , and Eq (3.11) are satisfied; by Theorem 3.2, we derive that z is a unique mild solution of the BVP (1.1) and $\mathfrak{J}z = z$. This, combined with Eq (3.12), it follows

$$\|z - \varpi\|_\infty = \|\mathfrak{J}z - \mathfrak{J}\varpi - Bv\|_\infty \leq (\varsigma L + \tau)\|z - \varpi\|_\infty + \|Bv\|_\infty,$$

we can imply that

$$\|z - \varpi\|_\infty \leq \frac{\|Bv\|_\infty}{1 - (\varsigma L + \tau)}. \quad (4.5)$$

On the other hand, by using Eq (4.4), it can be obtained that

$$\begin{aligned} & \|Bv\|_\infty \\ & \leq \frac{1}{|\Lambda|} \left\{ |\varepsilon_{11}| \left[\sum_{i=1}^m |c_i - \lambda T d_i| \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=\xi_i} + T \sum_{i=1}^m |d_i| \mathcal{J}_{0+}^\beta |v(t)| \Big|_{t=\xi_i} \right. \right. \\ & \quad \left. \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=T} + T \mathcal{J}_{0+}^\beta |v(t)| \Big|_{t=T} \right] \right. \\ & \quad \left. + |\varepsilon_{21}| \left[\sum_{i=1}^m |a_i - \lambda T b_i| \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=\xi_i} + T \sum_{i=1}^m |b_i| \mathcal{J}_{0+}^\beta |v(t)| \Big|_{t=\xi_i} \right. \right. \\ & \quad \left. \left. + \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=T} \right] + \frac{T^\alpha}{|\Lambda|\Gamma(\alpha+1)} \left\{ |\varepsilon_{22}| \left[\sum_{i=1}^m |a_i - \lambda T b_i| \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=\xi_i} \right. \right. \right. \\ & \quad \left. \left. + T \sum_{i=1}^m |b_i| \mathcal{J}_{0+}^\beta |v(t)| \Big|_{t=\xi_i} + \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=T} \right] \right. \\ & \quad \left. \left. + |\varepsilon_{12}| \left[\sum_{i=1}^m |c_i - \lambda T d_i| \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=\xi_i} + T \sum_{i=1}^m |d_i| \mathcal{J}_{0+}^\beta |v(t)| \Big|_{t=\xi_i} \right. \right. \right. \\ & \quad \left. \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=T} + T \mathcal{J}_{0+}^\beta |v(t)| \Big|_{t=T} \right] + \mathcal{J}_{0+}^{\alpha+\beta} |v(t)| \Big|_{t=T} \right\} \\ & \leq \frac{1}{|\Lambda|} \left\{ |\varepsilon_{11}| \left[\sum_{i=1}^m |c_i - \lambda T d_i| \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=\xi_i} + T \sum_{i=1}^m |d_i| \mathcal{J}_{0+}^\beta |\varphi(t)\varepsilon| \Big|_{t=\xi_i} \right. \right. \\ & \quad \left. \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=T} + T \mathcal{J}_{0+}^\beta |\varphi(t)\varepsilon| \Big|_{t=T} \right] \right. \\ & \quad \left. + |\varepsilon_{21}| \left[\sum_{i=1}^m |a_i - \lambda T b_i| \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=\xi_i} + T \sum_{i=1}^m |b_i| \mathcal{J}_{0+}^\beta |\varphi(t)\varepsilon| \Big|_{t=\xi_i} \right. \right. \\ & \quad \left. \left. + \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=T} \right] + \frac{T^\alpha}{|\Lambda|\Gamma(\alpha+1)} \left\{ |\varepsilon_{22}| \left[\sum_{i=1}^m |a_i - \lambda T b_i| \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=\xi_i} \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
& +T \sum_{i=1}^m |b_i| \mathcal{J}_{0+}^\beta |\varphi(t)\varepsilon| \Big|_{t=\xi_i} + \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=T} \Big] \\
& + |\varepsilon_{12}| \left[\sum_{i=1}^m |c_i - \lambda T d_i| \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=\xi_i} + T \sum_{i=1}^m |d_i| \mathcal{J}_{0+}^\beta |\varphi(t)\varepsilon| \Big|_{t=\xi_i} \right. \\
& \left. + \lambda T \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=T} + T \mathcal{J}_{0+}^\beta |\varphi(t)\varepsilon| \Big|_{t=T} \right] + \mathcal{J}_{0+}^{\alpha+\beta} |\varphi(t)\varepsilon| \Big|_{t=T} \\
& \leq \left\{ \frac{1}{|\Lambda|} \left[|\varepsilon_{11}| \left(\sum_{i=1}^m |c_i - \lambda T d_i| + T \sum_{i=1}^m |d_i| + \lambda T + T \right) \right. \right. \\
& \quad \left. + |\varepsilon_{21}| \left(\sum_{i=1}^m |a_i - \lambda T b_i| + T \sum_{i=1}^m |b_i| + 1 \right) \right] \\
& \quad + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} \left[|\varepsilon_{22}| \left(\sum_{i=1}^m |a_i - \lambda T b_i| + T \sum_{i=1}^m |b_i| + 1 \right) \right. \\
& \quad \left. \left. + |\varepsilon_{12}| \left(\sum_{i=1}^m |c_i - \lambda T d_i| + T \sum_{i=1}^m |d_i| + \lambda T + T \right) \right] + 1 \right\} \rho_\varphi \varphi(t) \varepsilon, \quad t \in [0, T].
\end{aligned}$$

From Eq (4.5), we get the following:

$$\begin{aligned}
& \|z - \varpi\|_\infty \\
& \leq \frac{1}{1 - (\varsigma L + \tau)} \left\{ \frac{1}{|\Lambda|} \left[|\varepsilon_{11}| \left(\sum_{i=1}^m |c_i - \lambda T d_i| + T \sum_{i=1}^m |d_i| + \lambda T + T \right) \right. \right. \\
& \quad \left. + |\varepsilon_{21}| \left(\sum_{i=1}^m |a_i - \lambda T b_i| + T \sum_{i=1}^m |b_i| + 1 \right) \right] \\
& \quad + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} \left[|\varepsilon_{22}| \left(\sum_{i=1}^m |a_i - \lambda T b_i| + T \sum_{i=1}^m |b_i| + 1 \right) \right. \\
& \quad \left. \left. + |\varepsilon_{12}| \left(\sum_{i=1}^m |c_i - \lambda T d_i| + T \sum_{i=1}^m |d_i| + \lambda T + T \right) \right] + 1 \right\} \rho_\varphi \varphi(t) \varepsilon, \quad t \in [0, T].
\end{aligned}$$

Let

$$\begin{aligned}
c_{f,\varphi} = & \frac{1}{1 - (\varsigma L + \tau)} \left\{ \frac{1}{|\Lambda|} \left[|\varepsilon_{11}| \left(\sum_{i=1}^m |c_i - \lambda T d_i| + T \sum_{i=1}^m |d_i| + \lambda T + T \right) \right. \right. \\
& \left. + |\varepsilon_{21}| \left(\sum_{i=1}^m |a_i - \lambda T b_i| + T \sum_{i=1}^m |b_i| + 1 \right) \right] \\
& + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} \left[|\varepsilon_{22}| \left(\sum_{i=1}^m |a_i - \lambda T b_i| + T \sum_{i=1}^m |b_i| + 1 \right) \right. \\
& \left. \left. + |\varepsilon_{12}| \left(\sum_{i=1}^m |c_i - \lambda T d_i| + T \sum_{i=1}^m |d_i| + \lambda T + T \right) \right] + 1 \right\} \rho_\varphi.
\end{aligned}$$

Thus, the BVP (1.1) is U-H-R stable. The theorem has been proven.

5. Examples

Example 5.1. Let $m = 3, T = 1, \alpha = \frac{3}{5}, \beta = \frac{7}{10}, \lambda = 0.1, a_1 = 0.111, a_2 = 0.132, a_3 = 0.089, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \xi_3 = \frac{3}{4}, b_1 = 1, b_2 = 1.2, b_3 = 0.8, c_1 = 0.192, c_2 = 0.169, c_3 = 0.213, d_1 = 2, d_2 = 1.8,$ and $d_3 = 2.2$. Consider the following BVP:

$$\begin{cases} {}^C \mathcal{D}_{0+}^{0.7}({}^C \mathcal{D}_{0+}^{0.6} + 0.1)x(t) = f(t, x(t)), & t \in (0, 1), \\ x(1) = \sum_{i=1}^3 (a_i x(\xi_i) + b_i {}^C \mathcal{D}_{0+}^{0.6} x(\xi_i)), \\ {}^C \mathcal{D}_{0+}^{0.6} x(1) = \sum_{i=1}^3 (c_i x(\xi_i) + d_i {}^C \mathcal{D}_{0+}^{0.6} x(\xi_i)), \end{cases} \quad (5.1)$$

where

$$f(t, x) = 1 + t + 0.101 \frac{x}{1 + |x|}.$$

We choose the following:

$$p(t) = 1 + t, \quad q(t) \equiv 0.101.$$

The condition (C_1) is satisfied. Furthermore, for any $t \in [0, 1]$ and $x \in \mathbb{R}$,

$$|f(t, x)| \leq 1 + t + \frac{0.101|x|}{1 + |x|} \leq 1 + t + 0.101|x| = p(t) + q(t)|x(t)|.$$

Hence, condition (C_2) is satisfied. Besides, through calculation, we obtain the following:

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{\Gamma(\alpha + 1)} \left[-T^\alpha + \sum_{i=1}^m (a_i - \lambda T b_i) \xi_i^\alpha \right] + T \sum_{i=1}^m b_i \\ &= \frac{1}{\Gamma(1.6)} (-1 + 0.020278) + 3 \approx 1.903520, \\ \varepsilon_{21} &= \frac{\lambda T^{\alpha+1}}{\Gamma(\alpha + 1)} - T + \frac{1}{\Gamma(\alpha + 1)} \sum_{i=1}^m (c_i - \lambda T d_i) \xi_i^\alpha + T \sum_{i=1}^m d_i \\ &= \frac{0.1}{\Gamma(1.6)} - 1 + \frac{-0.016629}{\Gamma(1.6)} + 6 \approx 5.093306, \\ \varepsilon_{12} &= -1 + \sum_{i=1}^m (a_i - \lambda T b_i) = -1 + 0.032 = -0.968000, \\ \varepsilon_{22} &= \lambda T + \sum_{i=1}^m (c_i - \lambda T d_i) = 0.1 + (-0.026) = 0.074000, \\ \Delta_1 &= \frac{1}{\Gamma(\alpha + \beta + 1)} \sum_{i=1}^m |a_i - \lambda T b_i| \xi_i^{\alpha+\beta} + \frac{T}{\Gamma(\beta + 1)} \sum_{i=1}^m |b_i| \xi_i^\beta + \frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} \\ &= \frac{0.012882}{\Gamma(2.3)} + \frac{1.771682}{\Gamma(1.7)} + \frac{1}{\Gamma(2.3)} \approx 2.817987, \end{aligned}$$

$$\begin{aligned}
\Delta_2 &= \frac{1}{\Gamma(\alpha + \beta + 1)} \sum_{i=1}^m |c_i - \lambda T d_i| \xi_i^{\alpha+\beta} + \frac{T}{\Gamma(\beta + 1)} \sum_{i=1}^m |d_i| \xi_i^\beta \\
&\quad + \frac{\lambda T^{\alpha+\beta+1}}{\Gamma(\alpha + \beta + 1)} + \frac{T^{\beta+1}}{\Gamma(\beta + 1)} \\
&= \frac{0.010604}{\Gamma(2.3)} + \frac{3.664573}{\Gamma(1.7)} + \frac{0.1}{\Gamma(2.3)} + \frac{1}{\Gamma(1.7)} \approx 5.228430, \\
\Delta_3 &= \frac{\lambda}{\Gamma(\alpha+1)} \sum_{i=1}^m |a_i - \lambda T b_i| \xi_i^\alpha + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)} = \frac{0.002028}{\Gamma(1.6)} + \frac{0.1}{\Gamma(1.6)} \approx 0.114187, \\
\Delta_4 &= \frac{\lambda}{\Gamma(\alpha+1)} \sum_{i=1}^m |c_i - \lambda T d_i| \xi_i^\alpha + \frac{\lambda^2 T^{\alpha+1}}{\Gamma(\alpha+1)} = \frac{0.001663}{\Gamma(1.6)} + \frac{0.01}{\Gamma(1.6)} \approx 0.013053,
\end{aligned}$$

and

$$\begin{aligned}
\varsigma &= \frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_2 + |\varepsilon_{21}| \Delta_1) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} (|\varepsilon_{22}| \Delta_1 + |\varepsilon_{12}| \Delta_2) + \frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} \\
&= 4.792774 + \frac{5.269651}{|\Lambda| \Gamma(1.6)} + \frac{1}{\Gamma(2.3)} \approx 6.812913, \\
\tau &= \frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_4 + |\varepsilon_{21}| \Delta_3) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha + 1)} (|\varepsilon_{22}| \Delta_3 + |\varepsilon_{12}| \Delta_4) + \frac{\lambda T^\alpha}{\Gamma(\alpha + 1)} \\
&= 0.119586 + \frac{0.021085}{|\Lambda| \Gamma(1.6)} + \frac{0.1}{\Gamma(1.6)} \approx 0.236156, \\
p &= 2, \quad q = 0.101, \quad |\Lambda| = |\varepsilon_{11} \varepsilon_{22} - \varepsilon_{12} \varepsilon_{21}| = 5.071181.
\end{aligned}$$

Thus,

$$\tau + \varsigma q = 0.924260 < 1,$$

that is, condition (3.8) holds. According to Theorem 3.1, it can be concluded that the BVP (5.1) has at least one mild solution on $[0, 1]$.

Example 5.2. Let $m = 3, T = 1, \alpha = \frac{1}{2}, \beta = \frac{7}{10}, \lambda = 0.05, a_1 = 0.010, a_2 = 0.015, a_3 = 0.012, \xi_1 = 0.2, \xi_2 = 0.5, \xi_3 = 0.8, b_1 = 0.010, b_2 = 0.015, b_3 = 0.012, c_1 = 0.012, c_2 = 0.014, c_3 = 0.011, d_1 = 0.020, d_2 = 0.025$, and $d_3 = 0.022$. Consider the following BVP:

$$\begin{cases} {}^C \mathcal{D}_{0+}^{0.7} ({}^C \mathcal{D}_{0+}^{0.5} + 0.05)x(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(1) = \sum_{i=1}^3 (a_i x(\xi_i) + b_i {}^C \mathcal{D}_{0+}^{0.5} x(\xi_i)), \\ {}^C \mathcal{D}_{0+}^{0.5} x(1) = \sum_{i=1}^3 (c_i x(\xi_i) + d_i {}^C \mathcal{D}_{0+}^{0.5} x(\xi_i)), \end{cases} \quad (5.2)$$

where

$$f(t, x) = 0.082x + \cos t.$$

We take $L = 0.082$; then, conditions (C_1) and (C_3) are met. By calculation, we get the following:

$$\begin{aligned}
 \varepsilon_{11} &= \frac{1}{\Gamma(\alpha+1)} \left[-T^\alpha + \sum_{i=1}^m (a_i - \lambda T b_i) \xi_i^\alpha \right] + T \sum_{i=1}^m b_i \\
 &= \frac{1}{\Gamma(1.5)} (-1 + 0.024521) + 0.037 \approx -1.063710, \\
 \varepsilon_{21} &= \frac{\lambda T^{\alpha+1}}{\Gamma(\alpha+1)} - T + \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^m (c_i - \lambda T d_i) \xi_i^\alpha + T \sum_{i=1}^m d_i \\
 &= \frac{0.05}{\Gamma(1.5)} - 1 + \frac{0.022790}{\Gamma(1.5)} + 0.067 \approx -0.850866, \\
 \varepsilon_{12} &= -1 + \sum_{i=1}^m (a_i - \lambda T b_i) = -1 + 0.03515 = -0.964850, \\
 \varepsilon_{22} &= \lambda T + \sum_{i=1}^m (c_i - \lambda T d_i) = 0.05 + 0.03365 = 0.083650, \\
 |\Lambda| &= |\varepsilon_{11} \varepsilon_{22} - \varepsilon_{12} \varepsilon_{21}| = 0.909937, \\
 \Delta_1 &= \frac{1}{\Gamma(\alpha+\beta+1)} \sum_{i=1}^m |a_i - \lambda T b_i| \xi_i^{\alpha+\beta} + \frac{T}{\Gamma(\beta+1)} \sum_{i=1}^m |b_i| \xi_i^\beta + \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
 &= \frac{0.016297}{\Gamma(2.2)} + \frac{0.022599}{\Gamma(1.7)} + \frac{1}{\Gamma(2.2)} \approx 0.947425, \\
 \Delta_2 &= \frac{1}{\Gamma(\alpha+\beta+1)} \sum_{i=1}^m |c_i - \lambda T d_i| \xi_i^{\alpha+\beta} + \frac{T}{\Gamma(\beta+1)} \sum_{i=1}^m |d_i| \xi_i^\beta + \frac{\lambda T^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\beta+1}}{\Gamma(\beta+1)} \\
 &= \frac{0.014714}{\Gamma(2.2)} + \frac{0.040430}{\Gamma(1.7)} + \frac{0.05}{\Gamma(2.2)} + \frac{1}{\Gamma(1.7)} \approx 1.204068, \\
 \Delta_3 &= \frac{\lambda}{\Gamma(\alpha+1)} \sum_{i=1}^m |a_i - \lambda T b_i| \xi_i^\alpha + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)} = \frac{0.001226}{\Gamma(1.5)} + \frac{0.05}{\Gamma(1.5)} \approx 0.057802, \\
 \Delta_4 &= \frac{\lambda}{\Gamma(\alpha+1)} \sum_{i=1}^m |c_i - \lambda T d_i| \xi_i^\alpha + \frac{\lambda^2 T^{\alpha+1}}{\Gamma(\alpha+1)} = \frac{0.001140}{\Gamma(1.5)} + \frac{0.0025}{\Gamma(1.5)} \approx 0.004107,
 \end{aligned}$$

and

$$\begin{aligned}
 \varsigma &= \frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_2 + |\varepsilon_{21}| \Delta_1) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha+1)} (|\varepsilon_{22}| \Delta_1 + |\varepsilon_{12}| \Delta_2) + \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
 &= 2.294051 + \frac{1.240995}{|\Lambda| \Gamma(1.5)} + \frac{1}{\Gamma(2.2)} \approx 4.739985, \\
 \tau &= \frac{1}{|\Lambda|} (|\varepsilon_{11}| \Delta_4 + |\varepsilon_{21}| \Delta_3) + \frac{T^\alpha}{|\Lambda| \Gamma(\alpha+1)} (|\varepsilon_{22}| \Delta_3 + |\varepsilon_{12}| \Delta_4) + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)} \\
 &= 0.058855 + \frac{0.008798}{|\Lambda| \Gamma(1.5)} + \frac{0.05}{\Gamma(1.5)} \approx 0.126179.
 \end{aligned}$$

Hence,

$$\tau + \varsigma L = 0.126179 + 4.739985 \times 0.082 = 0.514858 < 1,$$

that is, condition (3.11) is satisfied. Therefore, the BVP (5.2) has a unique mild solution on $[0, 1]$. Besides, we can easily obtain that Eq (5.2) is U-H stable. Next, we will prove that Eq (5.2) is U-H-R stable. Let

$$\varphi(t) = t + 1, \quad t \in [0, 1],$$

and choose $\rho_\varphi = 1.75$; then, we obtain the following:

$$\begin{aligned} \mathcal{J}_{0+}^\beta \varphi(t)|_{t=1} &\leq 1.747929 \leq 1.75\varphi(t), \\ \mathcal{J}_{0+}^{\alpha+\beta} \varphi(t)|_{t=1} &\leq 1.32015 \leq 1.75\varphi(t). \end{aligned}$$

By Theorem 4.2, we deduce that the BVP (5.2) is U-H-R stable.

6. Conclusions

In the present paper, we have investigated the existence, uniqueness, and U-H and U-H-R stability of mild solutions to FLEs with multi-point CBCs. Illustrative examples were recomputed to verify the stated hypotheses and to demonstrate the applicability of our main results. These findings provide a useful reference for further studies on fractional Langevin systems. For future research, possible directions include the analysis of coupled fractional Langevin systems subject to nonlocal multi-point boundary conditions, and the study of higher-order systems such as triple fractional Langevin equations under cyclic multi-point conditions.

Author contributions

Wei Zhang: Conceptualization, Methodology, Formal analysis, Writing—original draft, Supervision, and Funding acquisition. Jianing Xu: Methodology, Validation, and Writing—original draft. Jinbo Ni: Investigation, Formal analysis. All authors have read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. O. Krejcar, H. Namazi, Review of the applications of different fractional models in engineering, *Appl. Math. Model.*, **153** (2026), 116623. <https://doi.org/10.1016/j.apm.2025.116623>
2. A. Dabiri, E. A. Butcher, M. Poursina, M. Nazari, Optimal periodic-gain fractional delayed state feedback control for linear fractional periodic time-delayed systems, *IEEE Trans. Autom. Control*, **63** (2018), 989–1002. <https://doi.org/10.1109/TAC.2017.2731522>
3. R. Zhang, K. Chen, H. Wang, The adaptive bearing fault diagnosis based on generalized stochastic resonance in a scale-transformed fractional oscillator driven by unilateral attenuated impulse signal, *Meas. Sci. Technol.*, **34** (2023), 015005. <https://doi.org/10.1088/1361-6501/ac93a2>
4. K. Zhao, A fractional differential equation model and dynamic analysis of animal avoidance learning, *Fractal Fract.*, **10** (2026), 327. <https://doi.org/10.3390/fractalfract10050327>
5. K. Zhao, X. Zhao, X. Lv, A general framework for the multiplicity of positive solutions to higher-order Caputo and Hadamard fractional functional differential coupled Laplacian systems, *Fractal Fract.*, **9** (2025), 701. <https://doi.org/10.3390/fractalfract9110701>
6. K. Zhao, Solvability, approximation and stability of periodic boundary value problem for a nonlinear Hadamard fractional differential equation with p -Laplacian, *Axioms*, **12** (2023), 733. <https://doi.org/10.3390/axioms12080733>
7. L. Meng, X. Zhang, L. Lin, S. Shi, H. Wang, Collective dynamics of fractional oscillators with fluctuating frequency in Erdős-Rényi random networks, *Commun. Nonlinear Sci. Numer. Simul.*, **159** (2026), 109919. <https://doi.org/10.1016/j.cnsns.2026.109919>
8. R. Zhang, X. Zhu, L. Lin, S. Shi, H. Wang, Collective dynamical behaviors of coupled Langevin oscillators with damping fluctuation in attractive-repulsive weighted random networks, *Physica A*, **689** (2026), 131431. <https://doi.org/10.1016/j.physa.2026.131431>
9. X. Wen, X. Zhang, L. Lin, S. Shi, H. Wang, Robust-GSR driven lightweight CNN framework for intelligent fault diagnosis under variable-speed conditions, *Nonlinear Dyn.*, **114** (2026), 662. <https://doi.org/10.1007/s11071-026-12544-2>
10. X. Wen, X. Zhang, L. Lin, H. Wang, GSR-enhanced sparse feature learning for bearing fault diagnosis under data scarcity, *Meas. Sci. Technol.*, **37** (2026), 356114. <https://doi.org/10.1088/1361-6501/ae9be0>
11. W. Chen, H. G. Sun, X. Li, *Fractional Derivative Modeling in Mechanics and Engineering*, Springer, Singapore, 2022. <https://doi.org/10.1007/978-981-16-8802-7>
12. Q. Wei, W. Wang, Y. Tang, R. Metzler, A. Chechkin, Fractional Langevin equation far from equilibrium: Riemann-Liouville fractional Brownian motion, spurious nonergodicity, and aging, *Phys. Rev. E*, **111** (2025), 014128. <https://doi.org/10.1103/PhysRevE.111.014128>
13. A. M. A. El-Sayed, S. I. Nasim, E. M. A. Hamdallah, On the qualitative properties of the solution of a non-linear Langevin equation of two fractal derivatives, *J. Appl. Anal. Comput.*, **16** (2026), 1475–1489. <https://doi.org/10.11948/20250229>

14. S. Zerbib, K. Hilal, A. Kajouni, On the Langevin fractional boundary value integro-differential equations involving ψ -Caputo derivative with two distinct variable-orders, *Comput. Appl. Math.*, **44** (2025), 212. <https://doi.org/10.1007/s40314-025-03178-y>
15. A. Alsaedi, H. A. Saeed, H. Alsulami, Existence and stability of solutions for a nonlocal multi-point and multi-strip coupled boundary value problem of nonlinear fractional Langevin equations, *Bull. Math. Sci.*, **15** (2025), 2450014. <https://doi.org/10.1142/S1664360724500140>
16. H. Bouzid, A. Benali, A. Salim, İ. M. Erhan, Existence and Ulam stability results of hybrid Langevin pantograph ψ -fractional coupled systems, *Filomat*, **39** (2025), 3401–3424. <https://doi.org/10.2298/FIL2510401B>
17. L. Almaghamisi, A. Alghamdi, A. Ghanmi, Existence of solution for a coupled system of Langevin equations involving the ψ -Hilfer fractional derivative, *Math. Methods Appl. Sci.*, **49** (2026), 10285–10300. <https://doi.org/10.1002/mma.70595>
18. A. Berhail, J. Alzabut, M. E. Samei, On multi order nonlinear Langevin type of FDE subject to multi-point boundary conditions, *Qual. Theory Dyn. Syst.*, **24** (2025), 58. <https://doi.org/10.1007/s12346-024-01211-9>
19. K. Zhao, J. Liu, X. Lv, A unified approach to solvability and stability of multipoint BVPs for Langevin and Sturm-Liouville equations with CH-fractional derivatives and impulses via coincidence theory, *Fractal Fract.*, **8** (2024), 111. <https://doi.org/10.3390/fractalfract8020111>
20. H. Lmou, K. Hilal, A. Kajouni, Existence and stability analysis for Φ -Caputo generalized proportional fractional differential Langevin equation, *Hacet. J. Math. Stat.*, **54** (2025), 1308–1328. <https://doi.org/10.15672/hujms.1487219>
21. L. Almaghamisi, Stability analysis of hybrid Langevin equation via two fractional operators, *Fractals*, **33** (2025), 2540109. <https://doi.org/10.1142/S0218348X25401097>
22. A. E. Allaoui, L. Mbarki, Y. Allaoui, J. V. C. Sousa, Solvability of Langevin fractional differential equation of higher-order with integral boundary conditions, *J. Appl. Anal. Comput.*, **15** (2025), 316–332. <https://doi.org/10.11948/20240092>
23. K. Zhao, Existence and UH-stability of integral boundary problem for a class of nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions, *Filomat*, **37** (2023), 1053–1063. <https://doi.org/10.2298/FIL2304053Z>
24. B. Ahmad, M. Alnahdi, S. K. Ntouyas, Existence results for a differential equation involving the right Caputo fractional derivative and mixed nonlinearities with nonlocal closed boundary conditions, *Fractal Fract.*, **7** (2023), 129. <https://doi.org/10.3390/fractalfract7020129>
25. A. Alsaedi, M. Alnahdi, B. Ahmad, S. K. Ntouyas, On a nonlinear coupled Caputo-type fractional differential system with coupled closed boundary conditions, *AIMS Math.*, **8** (2023), 17981–17995. <https://doi.org/10.3934/math.2023914>
26. A. Alsaedi, B. Ahmad, H. Al-Hutami, Nonlinear multi-term impulsive fractional q-difference equations with closed boundary conditions, *Qual. Theory Dyn. Syst.*, **23** (2024), 67. <https://doi.org/10.1007/s12346-023-00934-5>

27. W. Zhang, X. Fu, Existence, uniqueness, and stability analysis for a nonlinear multi-term tripled system of fractional differential equations with closed boundary conditions, *Filomat*, **39** (2025), 8175–8192. <https://doi.org/10.2298/FIL2523175Z>
28. W. Zhang, Z. Wang, Y. Zhang, J. Ni, Well-posedness of a triply coupled system of fractional Langevin equations with closed boundary conditions, *Electron. Res. Arch.*, **33** (2025), 7017–7033. <https://doi.org/10.3934/era.2025309>
29. B. Ahmad, M. Alnahdi, S. K. Ntouyas, A. Alsaedi, On a mixed nonlinear fractional boundary value problem with a new class of closed integral boundary conditions, *Qual. Theory Dyn. Syst.*, **22** (2023), 96. <https://doi.org/10.1007/s12346-023-00781-4>
30. B. Ahmad, M. Alnahdi, S. K. Ntouyas, A. Alsaedi, On a mixed nonlinear boundary value problem with the right Caputo fractional derivative and multipoint closed boundary conditions, *AIMS Math.*, **8** (2023), 11709–11726. <https://doi.org/10.3934/math.2023593>
31. B. Ahmad, M. Aldhuain, A. Alsaedi, Existence results for a right-Caputo type fractional differential equation with mixed nonlinearities and nonlocal multipoint substrips type closed boundary conditions, *Lobachevskii J. Math.*, **45** (2024), 6457–6469. <https://doi.org/10.1134/S1995080224606969>
32. A. E. Allaoui, Y. Allaoui, M. Lekbir, H. E. Khalfi, Analytical investigation of fractional differential equations with combined nonlinearities and nonlocal closed fractional boundary conditions, *SeMA J.*, **82** (2025), 481–495. <https://doi.org/10.1007/s40324-024-00368-2>
33. B. Ahmad, M. Aldhuain, A. Alsaedi, A nonlinear Caputo-type coupled fractional differential system with a new class of coupled multi-point closed boundary conditions, *Nonlinear Anal. Real World Appl.*, **88** (2026), 104469. <https://doi.org/10.1016/j.nonrwa.2025.104469>
34. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
35. Y. Zhou, *Basic Theory of Fractional Differential Equations*, Third edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2023. <https://doi.org/10.1142/13289>



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