



Research article

Unconditionally stable and high-order numerical method for variable-order Caputo-Fabrizio fractional diffusion equations

Xiao Chen¹, Shanshan Wang^{2,*} and Lingna Lu³

¹ School of General Education, Zhengzhou Business University, Zhengzhou 450046, China

² School of Statistics, Henan University of Economics and Law, Zhengzhou 450046, China

³ School of Basic Sciences, Zhengzhou University of Technology, Zhengzhou 450044, China

* **Correspondence:** Email: sswang2258@163.com.

Abstract: In this work, we present and analyze a high-precision numerical scheme to solve a class of variable-order fractional diffusion equations involving the Caputo-Fabrizio (CF) fractional derivative. The proposed scheme employs a finite difference method for temporal discretization and a local discontinuous Galerkin (LDG) method equipped with generalized alternating numerical fluxes for spatial discretization. For any variable-order function $\alpha(t) \in (0, 1)$, we demonstrate the stability and L^2 convergence of the proposed scheme. Moreover, we show that the method attains an accuracy order of $O(\tau^2 + h^{k+1})$, where τ , h , and k denote the temporal step size, spatial step size, and degree of the piecewise P^k polynomials, respectively. A series of numerical experiments are carried out to verify the theoretical findings and the effectiveness of the proposed algorithm.

Keywords: variable-order derivatives; finite difference method; local discontinuous Galerkin method; stability; error estimates

1. Introduction

Fractional partial differential equations (FPDEs), which extend classical partial differential equations (PDEs) by incorporating derivatives of non-integer order, have been widely applied in diverse fields including engineering, physics, and the applied sciences. These equations serve as effective tools to model complex phenomena characterized by memory effects and hereditary properties that traditional integer-order equations fail to accurately capture. In particular, FPDEs can describe nonlocal relationships inherent in systems with power-law memory kernels in both time and space. This capability facilitates the modeling of materials or processes where past states influence future behavior, a feature especially important to describe materials with memory, such as viscoelastic substances and certain biological systems [1]. The inclusion of fractional derivatives in FPDEs

enables the representation of long-range interactions, which play a key role in various physical systems, including diffusion processes, wave propagation, and heat conduction in heterogeneous media. Furthermore, fractional calculus has found utility in fluid dynamics, materials science, and biological systems, offering a more precise framework to characterize phenomena involving anomalous diffusion, fractal geometry, and complex rheological behaviors [2, 3].

In recent years, significant attention has been devoted to the theoretical analysis and development of numerical methods for solving FPDEs. Researchers have concentrated on enhancing the stability, convergence, and efficiency of numerical schemes, thereby proposing a variety of techniques to address the nonlocal nature of fractional derivatives. For a comprehensive review of theoretical foundations and numerical approaches for FPDEs, readers may consult the works of Kilbas et al. [4], Guo et al. [5], Wei et al. [6], Liu et al. [7], and others [8–10]. In addition to forward problems, fractional calculus has been widely applied in optimal control problems. For instance, the control parametrization technique has been developed for fractal-fractional optimal control problems with Caputo-Fabrizio (CF) derivatives [11], and numerical methods have been proposed for optimal control problems involving Atangana-Baleanu fractional derivatives [12] as well as for fractional time-delay systems with state constraints [13].

Over the past decade or more, it has become increasingly recognized that many dynamic processes can be described by fractional calculus, wherein the order of fractional operators varies with time or space. Accurately capturing such complex dynamics necessitates the development of variable-order fractional calculus [14, 15]. Numerous studies have investigated variable-order fractional operators. For instance, Samko [16] introduced variable-order Riemann-Liouville and Marchaud fractional derivatives, while Langlands and Henry [17] further elaborated on these definitions. Wang and Zheng [18] examined the well-posedness of variable-order fractional diffusion equations and the regularity properties of their solutions. Lee and Kim [19] proposed a normalized variable-order time-fractional diffusion equation, thereby introducing a normalization factor to ensure consistent scaling of the fractional operator across varying orders, thus enabling a fair comparison of the effects of different fractional-order profiles on the solution dynamics. Recently, various numerical methods based on the Caputo fractional derivative have been applied to variable-order fractional problems, including finite difference methods [20, 21], dual reciprocity boundary element methods [22], meshless methods [23], Galerkin methods [24], and spectral collocation methods [25].

In 2015, Caputo and Fabrizio [26] proposed a novel fractional derivative free of singular kernels. This new formulation, commonly known as the CF fractional derivative, offers a more flexible framework to describe different scales and configurations of matter that lie beyond the reach of conventional local theories [27]. More recently, increasing attention has been paid to FPDEs involving the CF fractional derivative [28–30].

Among the various existing definitions of variable-order fractional derivatives, the CF definition [26] is particularly attractive due to its non-singular exponential kernel, which is well-suited for modeling physical phenomena with exponentially decaying memory effects, such as rapid stress relaxation in viscoelastic materials and anomalous heat transfer with a finite propagation speed. In this work, we focus on the CF formulation for two main reasons. From the modeling perspective, the exponential kernel naturally captures memory mechanisms that decay faster than the power-law kernel of the classical Caputo derivative, thus making it appropriate for a distinct class of practical problems. From the numerical analysis perspective, the exponential structure enables a clean and

efficient temporal discretization with explicit coefficient computation, as will be demonstrated in Section 3, and the non-singular nature of the kernel facilitates the stability and convergence analysis within the local discontinuous Galerkin (LDG) framework. While we recognize that other variable-order definitions exist [16], the CF definition is particularly well-suited for the class of problems considered in this work. Nevertheless, relatively few efforts have been devoted to the numerical treatment of variable-order fractional equations based on the CF derivative, especially using the LDG method.

The present work focuses on the development and analysis of a discontinuous Galerkin method incorporating generalized alternating numerical fluxes to solve the following problem governed by a variable-order time-fractional diffusion equation:

$$\begin{aligned} \rho(t)_0^{CF} \partial_t^{1-\alpha(t)} u + \delta u - u_{xx} &= f(x, t), & (x, t) \in (a, b) \times (0, T], \\ u(x, 0) &= u_0(x), & x \in [a, b], \end{aligned} \quad (1.1)$$

where $0 < \alpha(t) \leq \bar{\alpha} < 1$, $\alpha(t) \in C^1[0, T]$, $\delta > 0$, and $\rho(t)$ are continuous functions on the interval $[0, T]$, while f and u_0 are given smooth functions. Throughout this paper, we consider solutions that are either periodic or compactly supported.

The CF fractional derivative is defined as follows [26, 31]:

$${}_0^{CF} \partial_t^{1-\alpha(t)} z(t) = \left(\frac{1}{\alpha(t)} \int_0^\zeta \frac{\partial z(\xi)}{\partial \xi} \exp \left[\frac{\alpha(t)-1}{\alpha(t)} (\zeta - \xi) \right] d\xi \right)_{\zeta=t}.$$

The remainder of this paper is organized as follows: Section 2 presents the necessary preliminaries that will be employed throughout the work; in Section 3, we describe the LDG scheme for solving the above problem, together with its stability analysis and optimal error estimate; numerical experiments are reported in Section 4; and finally, Section 5 provides concluding remarks.

2. Preliminaries

Let $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N+\frac{1}{2}} = b$ be a partition of the domain $\Omega = [a, b]$. The intervals are denoted by

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \quad j = 1, \dots, N,$$

and we set $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $1 \leq j \leq N$, with

$$h = \max_{1 \leq j \leq N} h_j.$$

The left and right limits of the function $u(x, t)$ at $x_{j+\frac{1}{2}}$ are defined as follows:

$$u_{j+\frac{1}{2}}^+ = \lim_{t \rightarrow 0^+} u(x_{j+\frac{1}{2}} + t), \quad u_{j+\frac{1}{2}}^- = \lim_{t \rightarrow 0^+} u(x_{j+\frac{1}{2}} - t).$$

Let V_h^k denote the corresponding discontinuous Galerkin space:

$$V_h^k = \left\{ v : v \in P^k(I_j), x \in I_j, j = 1, 2, \dots, N \right\},$$

where $P^k(I_j)$ denotes the space of polynomials of degree at most k on each interval I_j .

Now, we introduce the generalized Gauss-Radau projection, which will play an essential role in the subsequent convergence analysis.

For a periodic function ϖ defined on the interval $[a, b]$, its generalized Gauss-Radau projection, denoted by $Q_\theta \varpi$, is uniquely defined following [32]. The corresponding projection error is given by $\varpi^e = Q_\theta \varpi - \varpi$. For $\delta \neq \frac{1}{2}$, the projection satisfies the following conditions on each element I_j for $j = 1, 2, \dots, N$:

$$\int_{I_j} \varpi^e v dx = 0, \quad \forall v \in P^{k-1}(I_j), \quad \text{and} \quad (\varpi^e)^{(\theta)}_{j+\frac{1}{2}} = 0. \quad (2.1)$$

From these defining properties, the following approximation result can be derived [33, 34].

Lemma 2.1. Assume $\delta \neq \frac{1}{2}$ and let $\varpi \in H^{s+1}[a, b]$. Then, the projection error ϖ^e obeys the estimate

$$\|\varpi^e\| + h^{\frac{1}{2}} \|\varpi^e\|_{L^2(\Gamma_h)} \leq Ch^{\min(k+1, s+1)} \|\varpi\|_{s+1}, \quad (2.2)$$

where the constant $C > 0$ is independent of the mesh size h and the function ϖ .

Throughout this article, the symbol C denotes a generic constant, which may assume different values at different occurrences. We shall employ the standard notation commonly used in Sobolev spaces. Let the scalar inner product on $L^2(E)$ be denoted by $(\cdot, \cdot)_E$, and the associated norm by $\|\cdot\|_E$. When $E = \Omega$, the subscript E will be omitted for simplicity.

3. Fully discrete LDG scheme

In this section, we first describe the numerical scheme for the model problem (1.1), and then demonstrate that the method is unconditionally stable and convergent with an accuracy of $O(\tau^2 + h^{k+1})$.

We start by rewriting model (1.1) into the following equivalent form:

$$p = u_x, \quad \rho(t) {}^{CF}_0 \partial_t^{1-\alpha(t)} u + \delta u - p_x = f(x, t). \quad (3.1)$$

Let $t_n = \frac{n}{M}T$, with $\tau = t_n - t_{n-1}$. Then, the temporal derivatives u_t and ${}^{CF}_0 \partial_t^{1-\alpha(t)} u$ at t_n are approximated as follows:

$$\begin{aligned} {}^{CF}_0 \partial_t^{1-\alpha(t)} u(x, t_n) &= \frac{1}{\alpha(t_n)} \int_0^{t_n} \frac{\partial u(x, \varsigma)}{\partial \varsigma} \exp \left[\frac{\alpha(t_n) - 1}{\alpha(t_n)} (t_n - \varsigma) \right] d\varsigma \\ &= \frac{1}{(1 - \alpha(t_n))\tau} \sum_{k=1}^n (u(x, t_k) - u(x, t_{k-1})) \\ &\quad \times \left(\exp \left[\frac{(\alpha(t_n) - 1)\tau}{\alpha(t_n)} (n - k) \right] - \exp \left[\frac{(\alpha(t_n) - 1)\tau}{\alpha(t_n)} (n - k + 1) \right] \right) \\ &\quad + \frac{1}{\alpha(t_n)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (\varsigma - t_{k-\frac{1}{2}}) \frac{\partial^2 u(x, c_k)}{\partial \varsigma^2} \exp \left[\frac{\alpha(t_n) - 1}{\alpha(t_n)} (t_n - \varsigma) \right] d\varsigma \\ &= \frac{1}{(1 - \alpha(t_n))\tau} \sum_{k=1}^n (u(x, t_k) - u(x, t_{k-1})) \Phi_k^n(x) + \gamma^n(x), \end{aligned} \quad (3.2)$$

where $c_k \in (t_{k-1}, t_k)$, and

$$\Phi_k^n = \exp\left[\frac{(\alpha(t_n) - 1)\tau}{\alpha(t_n)}(n - k)\right] - \exp\left[\frac{(\alpha(t_n) - 1)\tau}{\alpha(t_n)}(n - k + 1)\right].$$

Here, $\gamma^n(x)$ denotes the temporal truncation error.

By employing a technique similar to that used in the proof of Lemma 3.1 in [28], we derive the following bound for the truncation error:

$$\|\gamma^n(x)\| \leq C\tau^2. \quad (3.3)$$

Here, the constant C depends on the regularity of u , specifically on the bound of $\|u_{tt}\|_{L^\infty(0,T;L^2)}$.

Furthermore, after some algebraic manipulations, we arrive at the following expression for the fractional derivative:

$${}_0^{CF}\partial_t^{1-\alpha(t)}u(x, t_n) = \frac{1}{(1 - \alpha(t_n))\tau} \left(\Phi_n^n u(x, t_n) + \sum_{k=1}^{n-1} (\Phi_k^n - \Phi_{k+1}^n) u(x, t_k) - \Phi_1^n u(x, t_0) \right) + \gamma^n(x). \quad (3.4)$$

To derive the properties of the coefficients Φ_k^n , let

$$\lambda_n = \frac{(1 - \alpha(t_n))\tau}{\alpha(t_n)} > 0,$$

since $\alpha(t_n) \in (0, 1)$ and $\tau > 0$. Then,

$$\Phi_k^n = e^{-\lambda_n(n-k)} - e^{-\lambda_n(n-k+1)} = (1 - e^{-\lambda_n})e^{-\lambda_n(n-k)}.$$

This explicit representation immediately yields the following properties:

$$\begin{aligned} 0 < \Phi_1^n < \Phi_2^n < \dots < \Phi_n^n = \Phi_{n-1}^{n-1}, \\ \sum_{k=2}^n \Phi_k^n \leq 1, \quad |\Phi_{k+1}^n - \Phi_k^n| < 1, \quad \forall k \leq n-1. \end{aligned} \quad (3.5)$$

Indeed, the monotonicity immediately follows from the fact that the sequence $e^{-\lambda_n(n-k)}$ is increasing in k . Moreover,

$$\sum_{k=2}^n \Phi_k^n = (1 - e^{-\lambda_n}) \sum_{k=2}^n e^{-\lambda_n(n-k)} = (1 - e^{-\lambda_n}) \frac{1 - e^{-\lambda_n(n-1)}}{1 - e^{-\lambda_n}} = 1 - e^{-\lambda_n(n-1)} < 1.$$

Now, we present the fully discrete LDG method. Let u_h^n and $p_h^n \in V_h^k$ be the approximations of $u(\cdot, t_n)$ and $p(\cdot, t_n)$, respectively, and set $f^n(x) = f(x, t_n)$. We seek u_h^n and $p_h^n \in V_h^k$ such that for all test functions $v, w \in V_h^k$, the following system holds:

$$\begin{aligned}
& \left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \int_{\Omega} u_h^n v \, dx + \left(\int_{\Omega} p_h^n v_x \, dx - \sum_{j=1}^N \left((\widehat{p_h^n v^-})_{j+\frac{1}{2}} - (\widehat{p_h^n v^+})_{j-\frac{1}{2}} \right) \right) \\
& = \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_{\Omega} u_h^k v \, dx + \Phi_1^n \int_{\Omega} u_h^0 v \, dx \right) \\
& \quad + \int_{\Omega} f^n v \, dx, \\
& \int_{\Omega} p_h^n w \, dx + \int_{\Omega} u_h^n w_x \, dx - \sum_{j=1}^N \left((\widehat{u_h^n w^-})_{j+\frac{1}{2}} - (\widehat{u_h^n w^+})_{j-\frac{1}{2}} \right) = 0.
\end{aligned} \tag{3.6}$$

The terms indicated by a “hat” ($\widehat{u_h^n}$, $\widehat{p_h^n}$) in the interface sums arise from integration by parts and are known as *numerical fluxes*. The choice of these fluxes is essential for both the stability analysis and the practical performance of the LDG method. In this work, we adopt the generalized alternating numerical fluxes, which provide greater flexibility and broader applicability relative to traditional choices. These fluxes are defined as follows:

$$\widehat{u_h^n} = \theta(u_h^n)^- + (1-\theta)(u_h^n)^+, \quad \widehat{p_h^n} = (1-\theta)(p_h^n)^- + \theta(p_h^n)^+, \tag{3.7}$$

where the parameter θ belongs to $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. The case $\theta = \frac{1}{2}$ is excluded, as it results in a symmetric flux that complicates the analysis of uniqueness and approximation properties associated with the underlying generalized Gauss-Radau projection.

To simplify the notation, we introduce the following bilinear form:

$$\begin{aligned}
\Psi_{\Omega}(u_h^n, p_h^n; w, v) &= \int_{\Omega} u_h^n w_x \, dx - \sum_{j=1}^N \left((\widehat{u_h^n w^-})_{j+\frac{1}{2}} - (\widehat{u_h^n w^+})_{j-\frac{1}{2}} \right) \\
& \quad + \int_{\Omega} p_h^n v_x \, dx - \sum_{j=1}^N \left((\widehat{p_h^n v^-})_{j+\frac{1}{2}} - (\widehat{p_h^n v^+})_{j-\frac{1}{2}} \right).
\end{aligned} \tag{3.8}$$

Now, we proceed to analyze the stability of the numerical method (3.6).

3.1. Stability analysis

To simplify the notation and without loss of generality, we set $f = 0$ in the subsequent numerical analysis. Under this assumption, the following stability result for scheme (3.6) can be established.

Theorem 3.1. *For periodic or compactly supported boundary conditions, the fully discrete LDG scheme (3.6) is unconditionally stable. Moreover, the numerical solution u_h^n satisfies the following inequality:*

$$\|u_h^n\| \leq \|u_h^0\|, \quad n = 1, 2, \dots, M. \tag{3.9}$$

Proof. By choosing the test functions $v = u_h^n$ and $w = p_h^n$ in scheme (3.6) and employing the fluxes defined in Eq (3.7), we obtain the following relation:

$$\begin{aligned} & \left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|u_h^n\|^2 + \|p_h^n\|^2 + \Psi_\Omega(u_h^n, p_h^n; p_h^n, u_h^n) \\ &= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_\Omega u_h^k u_h^n dx + \Phi_1^n \int_\Omega u_h^0 u_h^n dx \right). \end{aligned} \quad (3.10)$$

On each cell $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, the expression for Ψ_{I_j} becomes the following:

$$\begin{aligned} \Psi_{I_j}(u_h^n, p_h^n; p_h^n, u_h^n) &= \int_{I_j} u_h^n (p_h^n)_x dx - ((u_h^n)^- (p_h^n)^-)_{j+\frac{1}{2}} + ((u_h^n)^- (p_h^n)^+)_{j-\frac{1}{2}} \\ &\quad + \int_{I_j} p_h^n (u_h^n)_x dx - ((p_h^n)^+ (u_h^n)^-)_{j+\frac{1}{2}} + ((p_h^n)^+ (u_h^n)^+)_{j-\frac{1}{2}} \\ &= ((u_h^n)^- (p_h^n)^-)_{j+\frac{1}{2}} - ((u_h^n)^+ (p_h^n)^+)_{j-\frac{1}{2}} \\ &\quad - ((u_h^n)^- (p_h^n)^-)_{j+\frac{1}{2}} + ((u_h^n)^- (p_h^n)^+)_{j-\frac{1}{2}} \\ &\quad - ((p_h^n)^+ (u_h^n)^-)_{j+\frac{1}{2}} + ((p_h^n)^+ (u_h^n)^+)_{j-\frac{1}{2}}. \end{aligned} \quad (3.11)$$

After straightforward algebraic manipulation and summing Eq (3.11) over $j = 1$ to N , we readily obtain the following result:

$$\Psi_\Omega(u_h^n, p_h^n; p_h^n, u_h^n) = 0. \quad (3.12)$$

By combining Eqs (3.5) and (3.12) together with the Cauchy-Schwarz inequality, Eq (3.10) reduces to:

$$\left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|u_h^n\| \leq \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \|u_h^k\| + \Phi_1^n \|u_h^0\| \right). \quad (3.13)$$

Then, Theorem 3.1 can be proven by induction. First, setting $n = 1$ in Eq (3.13) yields the following:

$$\left(\delta + \frac{\rho(t_1)\Phi_1^1}{(1-\alpha(t_1))\tau} \right) \|u_h^1\| \leq \frac{\rho(t_1)}{(1-\alpha(t_1))\tau} \Phi_1^1 \|u_h^0\|. \quad (3.14)$$

Hence, for $n = 1$, we have the following:

$$\|u_h^1\| \leq \|u_h^0\|. \quad (3.15)$$

Now, suppose that the following inequalities hold:

$$\|u_h^j\| \leq \|u_h^0\|, \quad j = 1, 2, 3, \dots, n-1. \quad (3.16)$$

We aim to show the following:

$$\|u_h^n\| \leq \|u_h^0\|.$$

From Eq (3.13), we obtain the following:

$$\begin{aligned}
 \left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|u_h^n\| &\leq \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \|u_h^k\| + \Phi_1^n \|u_h^0\| \right) \\
 &\leq \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) + \Phi_1^n \right) \|u_h^0\| \\
 &= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \Phi_n^n \|u_h^0\|.
 \end{aligned} \tag{3.17}$$

Thus, we conclude the following:

$$\|u_h^n\| \leq \|u_h^0\|.$$

This completes the proof of Theorem 3.1. $\square$

Remark 3.1. For the inhomogeneous case with $f \neq 0$, following the same procedure as in the proof of Theorem 3.1 and applying the Cauchy-Schwarz inequality to the source term $\int_{\Omega} f^n u_h^n dx$, we obtain the stability estimate

$$\|u_h^n\| \leq \|u_h^0\| + C \max_{1 \leq k \leq n} \|f^k\|.$$

Thus, the unconditional stability still holds for the general inhomogeneous case.

3.2. Error estimate

Theorem 3.2. Let $u(x, t_n)$ be the exact solution of problem (1.1), assumed to be sufficiently smooth with bounded derivatives, and let u_h^n denote the numerical solution obtained from the fully discrete LDG scheme (3.6). Then, the following error estimate holds:

$$\|u(x, t_n) - u_h^n\| \leq C(\tau^2 + h^{k+1}), \tag{3.18}$$

where C is a constant that depends on u and T .

Proof.

$$\begin{aligned}
 e_u^n &= u(x, t_n) - u_h^n = \xi_u^n - \eta_u^n, & \xi_u^n &= \mathcal{Q}_{\theta} e_u^n, & \eta_u^n &= \mathcal{Q}_{\theta} u(x, t_n) - u(x, t_n), \\
 e_p^n &= p(x, t_n) - p_h^n = \xi_p^n - \eta_p^n, & \xi_p^n &= \mathcal{Q}_{1-\theta} e_p^n, & \eta_p^n &= \mathcal{Q}_{1-\theta} p(x, t_n) - p(x, t_n).
 \end{aligned} \tag{3.19}$$

With the fluxes defined in Eq (3.7), we obtain the following error equation:

$$\begin{aligned}
 \left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \int_{\Omega} e_u^n v dx + \int_{\Omega} e_p^n w dx + \Psi_{\Omega}(e_u^n, e_p^n; w, v) + \int_{\Omega} \gamma^n(x) v dx \\
 - \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_{\Omega} e_u^k v dx \right. \\
 \left. - \Phi_1^n \int_{\Omega} e_u^0 v dx \right) = 0.
 \end{aligned} \tag{3.20}$$

By employing notation (3.19) and selecting the test functions $v = \xi_u^n$ and $w = \xi_p^n$, the error equation (3.20) can be reformulated as follows:

$$\begin{aligned}
& \left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|\xi_u^n\|^2 + \|\xi_p^n\|^2 + \Psi_\Omega(\xi_u^n, \xi_p^n, \xi_p^n, \xi_u^n) \\
&= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_\Omega \xi_u^k \xi_u^n dx \right. \\
&\quad \left. + \Phi_1^n \int_\Omega \xi_u^0 \xi_u^n dx \right) \\
&\quad - \int_\Omega \gamma^n(x) \xi_u^n dx + \left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \int_\Omega \eta_u^n \xi_u^n dx \\
&\quad + \Psi_\Omega(\eta_u^n, \eta_p^n, \xi_p^n, \xi_u^n) + \int_\Omega \eta_p^n \xi_p^n dx \\
&\quad - \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_\Omega \eta_u^k \xi_u^n dx \right. \\
&\quad \left. + \Phi_1^n \int_\Omega \eta_u^0 \xi_u^n dx \right). \tag{3.21}
\end{aligned}$$

Using property (2.1), we obtain the following:

$$\Psi_\Omega(\eta_u^n, \eta_p^n; \xi_p^n, \xi_u^n) = 0.$$

From Eq (3.12), and noticing that $\xi_u^0 = 0$, we can derive the following equality:

$$\begin{aligned}
\left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \|\xi_u^n\|^2 + \|\xi_p^n\|^2 \right) &= \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_\Omega \xi_u^k \xi_u^n dx \\
&\quad - \int_\Omega \gamma^n(x) \xi_u^n dx + \int_\Omega \eta_p^n \xi_p^n dx + \mathfrak{F}_g, \tag{3.22}
\end{aligned}$$

where

$$\mathfrak{F}_g = - \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\Phi_n^n \int_\Omega \eta_u^n \xi_u^n dx - \sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \int_\Omega \eta_u^k \xi_u^n dx + \Phi_1^n \int_\Omega \eta_u^0 \xi_u^n dx \right).$$

With the aid of the estimate

$$\begin{aligned}
\left\| \frac{\eta_u^i - \eta_u^{i-1}}{\tau} \right\| &\leq \left\| \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \frac{\partial}{\partial t} (Q_\theta u(x, t) - u(x, t)) dt \right\| \\
&\leq Ch^{k+1} \|u_t\|_{L^\infty(H^2(\Omega))},
\end{aligned}$$

we arrive at the following inequality:

$$\mathfrak{F}_g \leq Ch^{k+1} \|\xi_u^n\|.$$

Then, based on the previously derived bounds for $\mathfrak{F}_g$ and applying the Cauchy-Schwarz inequality to Eq (3.22), we obtain the following:

$$\left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|\xi_u^n\| \leq \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \left(\sum_{k=1}^{n-1} (\Phi_{k+1}^n - \Phi_k^n) \|\xi_u^k\| + \Phi_1^n \|\xi_u^0\| \right) + C\tau(h^{k+1} + \tau^2). \quad (3.23)$$

Now, we establish the error estimates using mathematical induction. First, taking $n = 1$ in Eq (3.23) yields the following:

$$\|\xi_u^1\| \leq C\tau(h^{k+1} + \tau^2).$$

Next, suppose that the following inequality holds for all $m = 1, 2, \dots, n-1$:

$$\|\xi_u^m\|_\Omega \leq Cm\tau(h^{k+1} + \tau^2), \quad m = 1, 2, \dots, n-1. \quad (3.24)$$

By substituting the induction hypothesis (3.24) into Eq (3.23), we have the following:

$$\left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|\xi_u^n\| \leq C\tau(h^{k+1} + \tau^2) \left[\frac{\rho(t_n)}{(1-\alpha(t_n))\tau} \sum_{k=1}^{n-1} k(\Phi_{k+1}^n - \Phi_k^n) + 1 \right].$$

Using summation by parts,

$$\sum_{k=1}^{n-1} k(\Phi_{k+1}^n - \Phi_k^n) = (n-1)\Phi_n^n - \sum_{k=1}^{n-1} \Phi_k^n \leq (n-1)\Phi_n^n,$$

where the inequality follows from the monotonicity of Φ_k^n given in Eq (3.5). Consequently,

$$\left(\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau} \right) \|\xi_u^n\| \leq \frac{\rho(t_n)}{(1-\alpha(t_n))\tau} (n-1)\Phi_n^n \cdot C\tau(h^{k+1} + \tau^2) + C\tau(h^{k+1} + \tau^2).$$

Dividing both sides by $\delta + \frac{\rho(t_n)\Phi_n^n}{(1-\alpha(t_n))\tau}$ and absorbing the lower-order term into the constant C , we obtain the following:

$$\|\xi_u^n\| \leq Cn\tau(h^{k+1} + \tau^2).$$

Noting that $m\tau \leq n\tau = T$, and applying the interpolation property from Eq (2.1) together with the triangle inequality, the proof of Theorem 3.2 is concluded. $\square$

4. Numerical experiment

In this section, we carry out a series of numerical experiments to validate the theoretical analysis and assess the efficiency of the proposed method (3.6) to solve the variable-order fractional diffusion equation. The equation involves the newly introduced fractional derivative in the form of Eq (1.1).

We consider the following model problem associated with the fractional diffusion equation:

$$\rho(t) {}_0^{CF} \partial_t^{1-\alpha(t)} u + \delta u - p_x = f(x, t),$$

where the parameters are chosen as $\rho(t) = 1.0$, $\delta = 0.5$, and the exact solution is prescribed as

$$u(x, t) = e^t \sin(2\pi x).$$

Table 1. Spatial accuracy test using piecewise P^k polynomials when $\alpha(t) = \frac{2+3\sin t}{10}$, $M = 2000$, $T = 1$.

θ	P^k	N	L^2 -error	order	L^∞ -error	order
$\theta = 0.3$	P^0	5	1.815564546443454	-	1.726645566447588	-
		10	0.920454441518513	0.98	0.887594090299581	0.96
		20	0.476457098649061	0.95	0.453122164450959	0.97
		40	0.239885559924296	0.99	0.228136939173803	0.99
	P^1	5	0.532619821961024	-	0.681212312319272	-
		10	0.139774978285209	1.93	0.183795890627358	1.89
		20	0.036176033062342	1.95	0.047240774109138	1.96
		40	0.009106914317791	1.99	0.012226679102576	1.95
	P^2	5	0.052212315246546	-	0.0923415214632625	-
		10	0.006946653938161	2.91	0.012200831759944	2.92
		20	0.000911502316781	2.93	0.001578886645345	2.95
		40	0.000121271985902	2.91	0.000201507801251	2.97
$\theta = 0.6$	P^0	5	1.823783914538455	-	1.763215364685745	-
		10	0.937528729684956	0.96	0.912697510118668	0.95
		20	0.492069774463119	0.93	0.479036901736399	0.93
		40	0.251204603210424	0.97	0.247965050796671	0.95
	P^1	5	0.492314624517427	-	0.664315345365475	-
		10	0.128305284905193	1.94	0.183003119403389	1.86
		20	0.033438466612157	1.94	0.047693621526288	1.94
		40	0.008535269969017	1.97	0.012516197554637	1.93
	P^2	5	0.053243675896455	-	0.089253412536555	-
		10	0.007034940746045	2.92	0.011874833873265	2.91
		20	0.000923086833677	2.93	0.001568989204089	2.92
		40	0.000121964929525	2.92	0.000207306236771	2.92

We emphasize that the choice of the solution is solely motivated by the purpose of convergence verification. The exponentially growing case is adopted here as it provides a particularly stringent test for the temporal discretization due to its larger time derivatives, while the conclusions drawn from this test equally apply to decaying solutions.

For numerical implementation, the proposed scheme is applied using piecewise P^k polynomials with various choices of $\alpha(t)$, and uniform discretization is adopted for both the spatial and temporal domains. The performance and accuracy of the method are tested for different $\alpha(t)$ functions, and the numerical results are summarized in the tables below.

To isolate the effect of spatial accuracy and minimize temporal errors, a fixed small time step $\tau = \frac{1}{2000}$ is employed. This ensures that the temporal discretization does not interfere with the assessment

of the spatial accuracy. In the numerical tests, we utilize several mesh sizes $N = 5, 10, 20$ and 40 , and report the corresponding L^2 and L^∞ errors for different polynomial degrees P^k .

For the cases $\alpha(t) = \frac{2+3\sin(t)}{10}$ and $\alpha(t) = \frac{2+e^t}{7}$, Tables 1 and 2 present the spatial accuracy test results with $M = 2000$, $T = 1$, and different values of θ . The tables demonstrate that the method achieves uniform $(k + 1)$ -th order convergence in both the L^2 and L^∞ norms for various polynomial degrees P^k . In particular, for P^0 polynomials, the convergence rate is approximately 1.0 for both norms, thus suggesting that the scheme effectively captures the solution behavior.

Table 2. Spatial accuracy test using piecewise P^k polynomials when $\alpha(t) = \frac{2+e^t}{7}$, $M = 2000$, $T = 1$.

θ	P^k	N	L^2 -error	order	L^∞ -error	order
$\theta = 0.4$	P^0	5	1.759187219785498	-	1.722123987492347	-
		10	0.936213211046355	0.91	0.910158061431456	0.92
		20	0.484614349316598	0.95	0.467873043700081	0.96
		40	0.247398563471974	0.97	0.247275123941674	0.92
	P^1	5	0.523243255348754	-	0.692121524165447	-
		10	0.137314293683208	1.93	0.186739272992958	1.89
		20	0.035786362409707	1.94	0.049689928516681	1.91
		40	0.009262091438661	1.95	0.013130787719084	1.92
	P^2	5	0.062332543265456	-	0.092928764784599	-
		10	0.008066336957069	2.95	0.012363825791119	2.91
		20	0.001073196067341	2.91	0.001599979395754	2.95
		40	0.000140818977085	2.93	0.000211400885729	2.92
$\theta = 0.7$	P^0	5	1.763257982455679	-	1.785725644323475	-
		10	0.931897748809757	0.92	0.943772110771504	0.92
		20	0.482380525974742	0.95	0.488527081190739	0.95
		40	0.244557153589095	0.98	0.256407513763799	0.93
	P^1	5	0.552346525685743	-	0.661456789235567	-
		10	0.145960060572808	1.92	0.180956999529158	1.87
		20	0.038570604306722	1.92	0.047818703267044	1.92
		40	0.009982698432529	1.95	0.012636308007379	1.92
	P^2	5	0.063947538645482	-	0.096221465552357	-
		10	0.008390849774099	2.93	0.012713478122262	2.92
		20	0.001108659948358	2.92	0.001668193761477	2.93
		40	0.000145472355531	2.93	0.000220413862629	2.92

The temporal accuracy is assessed in Figure 1 corresponding to $\alpha(t) = 0.5 + 0.3 \cos t$. A fixed spatial resolution ($N = 40$) together with piecewise P^1 elements is employed. The observed temporal convergence rate approaches unity, which agrees well with the theoretical expectation for the time discretization used.

The numerical results summarized in both tables and the figure indicate that the proposed method yields highly accurate solutions to the variable-order fractional diffusion equation. These findings

confirm the efficiency and effectiveness of the proposed scheme, even when $\alpha(t)$ takes different functional forms.

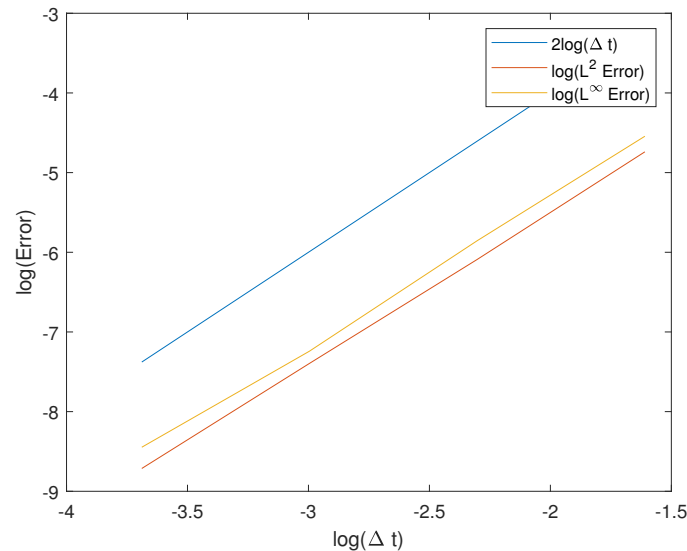


Figure 1. Error in L^2 and L^∞ norm versus Δt , order for $\alpha(t) = 0.5 + 0.3 \cos t$, $N = 40$, $k = 1$.

5. Conclusions

In this paper, we developed an efficient and accurate numerical scheme to solve a class of variable-order (VO) fractional diffusion equations involving the CF fractional derivative. By combining a finite difference method for temporal discretization with a LDG method equipped with generalized alternating numerical fluxes for spatial discretization, we constructed the fully discrete scheme. We rigorously analyzed the stability and convergence properties of the method, demonstrating that it is unconditionally stable and convergent with an error bound of $O(h^{k+1} + \tau^2)$. Moreover, the proposed method was validated through several numerical experiments, which confirm its effectiveness and high accuracy. The results indicate that the method attains the expected convergence order and robustly performs across a range of test cases, underscoring its potential to solve fractional diffusion problems with variable-order derivatives.

It is worth noting that, although the present analysis focused on time-fractional derivatives, the proposed methodology can, in principle, be extended to problems that involve spatial fractional derivatives. Such an extension would require appropriate modifications to handle the nonlocal character of spatial fractional operators, which constitutes an interesting direction for future research. This generalization could potentially open new avenues to address complex fractional models in various scientific and engineering fields.

Acknowledgments

This work is supported by the Henan Provincial Philosophy and Social Sciences Planning Project(Grant No. 2024CJJ181).

Authors contribution

Xiao Chen performed the coding and contributed to the writing and original draft preparation. Shanshan Wang and Lingna Lu developed the methodology, conducted the stability analysis, and performed the error estimation. All authors have read and approved the final manuscript.

Data Availability Statement

The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare no conflicts of interest.

References

1. C. P. Li, F. H. Zeng, *Numerical Methods for Fractional Calculus*, CRC Press, Boca Raton, FL, 2015.
2. L. Wei, L. Feng, I. Turner, Z. Mao, F. Liu, Numerical investigation of the 2D unsteady natural convection heat transfer equation with tempered fractional constitutive relationship, *Commun. Nonlinear Sci. Numer. Simul.*, **161** (2026), 110071. <https://doi.org/10.1016/j.cnsns.2026.110071>
3. Z. Li, Y. Nam, J. Ma, Y. Hwang, J. Kim, An efficient numerical scheme on nonuniform grids for a normalized time-fractional Black-Scholes equation, *Comput. Appl. Math.*, **45** (2026), 325. <https://doi.org/10.1007/s40314-026-03711-7>
4. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Netherlands, 2006.
5. L. Guo, Z. Wang, S. Vong, Fully discrete local discontinuous Galerkin methods for some time-fractional fourth-order problems, *Int. J. Comput. Math.*, **93** (2016), 1665–1682. <https://doi.org/10.1080/00207160.2015.1070840>
6. L. Wei, Y. Chen, F. Qin, J. Huang, P. Li, The discontinuous Galerkin method for solving the two-dimensional transport equation with distributed-order memory kernel on triangular meshes, *Appl. Numer. Math.*, **228** (2026), 57–70. <https://doi.org/10.1016/j.apnum.2026.05.009>
7. T. Liu, H. Zhang, S. Wang, A new high-order compact CN-ADI scheme on graded meshes for three-dimensional nonlinear PIDEs with multiple weakly singular kernels, *Appl. Math. Lett.*, **171** (2025), 109697. <https://doi.org/10.1016/j.aml.2025.109697>
8. Z. Chen, H. Zhang, H. Chen, ADI compact difference scheme for the two-dimensional integro-differential equation with two fractional Riemann-Liouville integral kernels, *Fractal Fract.*, **8** (2024), 707. <https://doi.org/10.3390/fractalfract8120707>

9. X. Yang, Y. Shi, H. Zhang, Z. Zhang, Analysis of a new time-space two-grid method for the two-dimensional nonlinear nonlocal mobile/immobile transport model, *J. Sci. Comput.*, **107** (2026), 96. <https://doi.org/10.1007/s10915-026-03314-8>
10. C. Li, Z. Wang, Non-uniform L1/discontinuous Galerkin approximation for the time-fractional convection equation with weak regular solution, *Math. Comput. Simulation.*, **182** (2021), 838–857. <https://doi.org/10.1016/j.matcom.2020.12.007>
11. C. Liu, X. Yi, Z. Gong, M. Han, The control parametrization technique for numerically solving fractal-fractional optimal control problems involving Caputo-Fabrizio derivatives, *J. Comput. Appl. Math.*, **472** (2026), 116814. <https://doi.org/10.1016/j.cam.2025.116814>
12. C. Liu, C. Yu, Z. Gong, H. T. Cheong, K. L. Teo, Numerical computation of optimal control problems with Atangana-Baleanu fractional derivatives, *J. Optim. Theory Appl.*, **197** (2023), 798–816. <https://doi.org/10.1007/s10957-023-02212-5>
13. C. Liu, Z. Gong, C. Yu, S. Wang, K. L. Teo, Optimal control computation for nonlinear fractional time-delay systems with state inequality constraints, *J. Optim. Theory Appl.*, **191** (2021), 83–117. <https://doi.org/10.1007/s10957-021-01926-8>
14. H. Sun, W. Chen, Y. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, *Physica A*, **388** (2009), 4586–4592. <https://doi.org/10.1016/j.physa.2009.07.024>
15. C. Lorenzo, T. Hartley, Initialization, conceptualization, and application in the generalized (fractional) calculus, *Crit. Rev. Biomed. Eng.*, **35** (2007), 447–553. <https://doi.org/10.1615/CritRevBiomedEng.v35.i6.10>
16. S. G. Samko, Fractional integration and differentiation of variable order, *Anal. Math.*, **21** (1995), 213–236. <https://doi.org/10.1007/BF01911126>
17. T. A. M. Langlands, B. I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, *J. Comput. Phys.*, **205** (2005), 719–736. <https://doi.org/10.1016/j.jcp.2004.11.025>
18. H. Wang, X. Zheng, Wellposedness and regularity of the variable-order time-fractional diffusion equations, *J. Math. Anal. Appl.*, **475** (2019), 1778–1802. <https://doi.org/10.1016/j.jmaa.2019.03.052>
19. C. Lee, J. Kim, A normalized variable-order time-fractional diffusion equation, *Physica A*, **680** (2025), 131052. <https://doi.org/10.1016/j.physa.2025.131052>
20. R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, *Appl. Math. Comput.*, **212** (2009), 435–445. <https://doi.org/10.1016/j.amc.2009.02.047>
21. S. Haq, A. Ghafoor, M. Hussain, Numerical solutions of variable order time fractional (1+1)- and (1+2)-dimensional advection dispersion and diffusion models, *Appl. Math. Comput.*, **360** (2019), 107–121. <https://doi.org/10.1016/j.amc.2019.04.085>
22. M. Dehghan, M. Safarpour, The dual reciprocity boundary elements method for the linear and nonlinear two-dimensional time-fractional partial differential equations, *Math. Meth. Appl. Sci.*, **39** (2016), 3979–3995. <https://doi.org/10.1002/mma.3839>

23. A. Tayebia, Y. Shekaria, M. H. Heydarib, A meshless method for solving two-dimensional variable-order time fractional advection-diffusion equation, *J. Comput. Phys.*, **340** (2017), 655–669. <https://doi.org/10.1016/j.jcp.2017.03.061>
24. D. Li, H. Liao, W. Sun, J. Wang, J. Zhang, Analysis of L1-Galerkin FEMs for time-fractional nonlinear parabolic problems, *Commun. Comput. Phys.*, **24** (2018), 86–103. <https://doi.org/10.4208/cicp.OA-2017-0080>
25. F. Zeng, Z. Zhang, G. Karniadakis, A generalized spectral collocation method with tunable accuracy for variable-order fractional differential equations, *SIAM J. Sci. Comput.*, **37** (2015), A2710–A2732. <https://doi.org/10.1137/141001299>
26. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. <https://dx.doi.org/10.12785/pfda/010201>
27. M. Zhang, Y. Liu, H. Li, High-order local discontinuous Galerkin method for a fractal mobile/immobile transport equation with the Caputo-Fabrizio fractional derivative, *Numer. Meth. Part. Differ. Equ.*, **35** (2019), 1588–1612. <https://doi.org/10.1002/num.22366>
28. Z. G. Liu, A. J. Cheng, X. L. Li, A second order finite difference scheme for quasilinear time fractional parabolic equation based on new fractional derivative, *Int. J. Comput. Math.*, **95** (2018), 396–411. <https://doi.org/10.1080/00207160.2017.1290434>
29. A. Atangana, On the new fractional derivative and application to nonlinear fishers reaction-diffusion equation, *Appl. Math. Comput.*, **273** (2016), 948–956. <https://doi.org/10.1016/j.amc.2015.10.021>
30. D. Baleanu, A. Mousalou, S. Rezapour, A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative, *Adv. Differ. Equ.*, **2017** (2017), 51. <https://doi.org/10.1186/s13662-017-1088-3>
31. E. F. Doungmo Goufo, S. Kumar, Shallow water wave models with and without singular kernel: Existence, uniqueness, and similarities, *Math. Probl. Eng.*, **2017** (2017), 1–9. <https://doi.org/10.1155/2017/4609834>
32. Y. Cheng, X. Meng, Q. Zhang, Application of generalized Gauss-Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations, *Math. Comput.*, **86** (2017), 1233–1267.
33. L. Wei, Y. He, A fully discrete local discontinuous Galerkin method with the generalized numerical flux to solve the tempered fractional reaction-diffusion equation, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 4907–4926. <https://doi.org/10.3934/dcdsb.2020319>
34. L. Wei, Y. Yang, Optimal order finite difference/local discontinuous Galerkin method for variable-order time-fractional diffusion equation, *J. Comput. Appl. Math.*, **383** (2021), 113129. <https://doi.org/10.1016/j.cam.2020.113129>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)