



Research article

Consensus of discrete-time opinion dynamics with stubbornness under fixed and switching topologies

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Abstract: We consider a discrete-time consensus dynamics of an opinion model with stubbornness under fixed and switching Topologies. Regarding the underlying network structures, both static and dynamically switching interaction topologies are considered. For the fixed topology, we prove that as long as the system parameters and initial states meet certain conditions, the presence of a rooted directed graph suffices for consensus. For the dynamically switching topology, under similar parametric conditions, consensus is guaranteed provided that the union of switched directed graphs over a bounded time interval contains a directed spanning tree. Numerical simulations are established to show the consensus and non-consensus phenomenon.

Keywords: discrete opinion model; consensus behavior; Rayleigh friction; static topology; dynamically switching topology

1. Introduction

From browsing the internet or news media, exchanging opinions on social networks, to interacting with others offline, human beings are constantly immersed in diverse information streams, forming and updating their views on various topics. Opinion dynamics has therefore become a pivotal field spanning sociology, economics, public policy, and collective decision-making. Foundational models include the DeGroot model [1], the Friedkin–Johnsen (FJ) model [2], and the Hegselmann–Krause (HK) model [3].

The HK model captures opinion exchange via local averaging among agents with bounded confidence, and the DeGroot model prescribes updates as convex combinations of own and others' views. In contrast, the FJ model introduces a linear stubbornness term representing an individual's inclination to resist external influence. Such linear stubbornness already accounts for persistent disagreement [4], but modeling an agent's desire to maintain convictions often requires nonlinear mechanisms. Hence, nonlinear stubbornness terms have been introduced. Reference [5] adds a nonlinear term to the continuous-time DeGroot model; reference [6] uses bias parameters to model

confirmation bias; and recently [7–9] employ a nonlinear term inspired by Rayleigh’s self-propulsion and friction force. Rayleigh friction was first introduced in [10] and later incorporated into the Cucker–Smale (C–S) flocking model [11, 12], where dynamics arise from competition between flocking and nonlinear dissipation.

Although Rayleigh friction has been extensively studied within C–S flocking, its direct use as a conviction mechanism in opinion dynamics is novel and largely unexplored. In particular, reference [8] considers adaptively changing personal traits, while reference [9] establishes exponential convergence to a Dirac measure. Inspired by these works, we investigate an opinion model with Rayleigh friction, where views are shaped by both external influence and internal predispositions.

Beyond stubbornness mechanisms, the interaction network plays a crucial role. The study of collective behavior often begins with all-to-all or undirected connected topologies. Generalizing to arbitrary network structures is vital for realism. A significant line of research explores leadership hierarchies. Shen [13] studies the C–S model under hierarchical leadership. Li et al. [14–16] systematically investigate the discrete-time C–S model under hierarchical, rooted, and jointly rooted leadership, progressively relaxing constraints. A consistent necessary condition emerges: information from at least one leader must reach every agent within a finite time window. As shown in [16], it suffices that the union of communication graphs over a finite interval contains a directed path from every agent to some leader. Further studies include [17], and Dong et al. [18, 19] consider the C–S model with Lipschitz nonlinearities over general graphs. More recently, reference [20] studies predefined-time optimal fault-tolerant consensus tracking control for nonlinear multi-agent systems over an undirected connected graph.

Motivated by the above, this paper investigates opinion dynamics for the proposed Rayleigh-friction opinion model on two classes of directed graphs. We model the social network as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with N vertices and directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A directed edge from j to i means i receives information from j . We consider both static and dynamically switching interaction graphs. For the static topology, we assume the directed graph contains at least one spanning tree. For switching topologies defined by a finite set of digraphs, we require that the union of graphs over any bounded successive time interval is rooted (contains a spanning tree). This ensures all individuals can communicate at least once within a finite window. The graph condition adopted in this paper is motivated by the works in [15, 16, 18] and the continuous-time model [21], as we aim to address general social network scenarios.

Unlike the unique Nash equilibrium studied in [7], the main objective of this paper is to determine the parameter conditions under which consensus can still be achieved, rather than to address Nash equilibria or polarization.

We adopt a discrete-time formulation because most real-world data is inherently discrete. This framework is widely employed in other collective behavior models. For example, references [19, 22, 23] study the discrete-time C–S model [24] under various topologies, while reference [25] introduces an improved communication function. Similarly, reference [26] analyzes consensus in the time-discrete infinite Kuramoto model. References [27–29] investigate consensus and synchronization issues in bounded-confidence opinion dynamics, focusing respectively on noise-induced quasi-consensus, noise-induced synchronization, and practical consensus. Reference [30] studies discrete-time leader–follower consensus with communication delays. The present model can be seen as a discrete-time counterpart of the continuous model studied in [21].

The rest of this paper is organized as follows. Section 2 establishes the mathematical models under the two network topologies. Section 3 presents basic properties, i.e., directed graphs, scrambling matrices and state transition matrices. The main results on consensus emergence and proof for the static topology case is in Section 4, and that for the switching topology case in Section 5. Several numerical simulations are performed in Section 6.

2. Mathematical formulation of opinion models

In this section, we first state several concepts on the directed graphs, and formally derive two discrete-time models, where one with a static topology and the other with a dynamically switching topology.

2.1. A directed graph

The interaction topology among individuals is modeled using a digraph. Formally, a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, \dots, N\}$ of vertices (nodes) with $|\mathcal{G}| = N$, and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of arcs.

If $\{j, i\} \in \mathcal{E}$, we say that j is a neighbor of i , i.e. there is a directed edge from j to i , and we denote the neighbor set of the vertex i by $\mathcal{N}_i := \{j : \{j, i\} \in \mathcal{E}\}$. For the case when $\{i, i\} \in \mathcal{E}$, we understand that $\mathcal{G}$ contains a self-loop at vertex i , if $\mathcal{G}$ does not have any self-loop at any vertices, then $\mathcal{G}$ is said to be simple. We say that a non-negative matrix $A = (a_{ij})$ is an adjacency matrix of a given digraph $\mathcal{G}$ when $a_{ij} > 0$ if and only if $\{j, i\} \in \mathcal{E}$. In this case, $\mathcal{G}(A)$ is induced by A , we use $\mathcal{G}(A)$ instead of $\mathcal{G}$ to emphasize this dependence. A path in a digraph $\mathcal{G}$ from i_0 to i_l is a sequence of distinct vertices $i_0, i_1, \dots, i_l$ where $\{i_{m-1}, i_m\} \in \mathcal{E}$ for all $1 \leq m \leq l$. If there exists a path from i to j , then the vertex j is said to be reachable from i , and we define the distance from i to j , $\text{dist}(i, j)$, as the length of shortest path from i to j . We say that a digraph $\mathcal{G}$ is rooted or $\mathcal{G}$ has a spanning tree if we can find a vertex $r \in \mathcal{V}$ (called a root) such that any other vertex of $\mathcal{G}$ is reachable from it, for such a root r , we define the depth of the spanning tree as $\max_{j \in \mathcal{V}} \text{dist}(r, j)$, and the smallest depth of $\mathcal{G}$ denoted by γ_g is

$$\gamma_g := \min_{r \text{ is a root}} \max_{j \in \mathcal{V}} \text{dist}(r, j).$$

Let n_r be the cardinality of the set $\{r : \max_{j \in \mathcal{V}} \text{dist}(r, j) = \gamma_g\}$.

2.2. Dynamical systems in different network topologies

In this section, we briefly introduce our opinion dynamics model with Rayleigh conviction under both static and dynamically switching digraph topologies. Considering a group of N individuals, given time-step $h > 0$ and let $t \in \mathbb{N}$ be the discrete time mark, we use $x_i[t]$ and $\theta_i[t]$ represent the opinion and the conviction value of individual i at time th , respectively, i.e., $x_i[t] := x_i(th)$ and $\theta_i[t] := \theta_i(th)$.

In static interaction topology setting, we denote by $\mathcal{N}_i$ the neighbor set of individual i , which consists of individuals that influence i directly. We assume that this fixed interaction topology has a spanning tree. Then the dynamics of $(x_i[t], \theta_i[t])$ are governed by the following discrete-time system:

$$\begin{cases} x_i[t+1] = x_i[t] + h \sum_{j \in \mathcal{N}_i} \phi(|x_i[t] - x_j[t]|)(x_j[t] - x_i[t]) + h\sigma(\theta_i[t] - x_i^2[t])x_i[t], \\ \theta_i[t+1] = \theta_i[t] + h \sum_{j \in \mathcal{N}_i} \psi(|x_i[t] - x_j[t]|)(\theta_j[t] - \theta_i[t]), \end{cases} \quad t \in \mathbb{N}, \quad i = 1, 2, \dots, N. \quad (2.1)$$

Here $\mathbb{N}$ denotes the set of all natural numbers (including zero). We assume $x_i[0], \theta_i[0] \in \mathbb{R}^+ \cup \{0\}$ nonnegative for simplicity. The communication weights ϕ, ψ which quantify the manner in which individuals influence one another, are assumed to be bounded, Lipschitz continuous, and positive non-increasing functions defined on the non-negative real numbers:

$$\begin{aligned} 0 < \phi(r) \leq \phi(0) &:= \kappa_1, \quad 0 < \psi(r) \leq \psi(0) := \kappa_2, \quad r \geq 0, \\ (\phi(r) - \phi(s))(r - s) &\leq 0, \quad (\psi(r) - \psi(s))(r - s) \leq 0, \quad r, s \geq 0. \end{aligned} \quad (2.2)$$

The second term on the right-hand side of system (2.1)₁ represents the alignment force, with weights to indicate that agent are influenced more by those whose opinions are closer to their own.

The third term, which we refer as the self-adjustment term, is inspired by Rayleigh's self-propulsion and friction force. Analogous to physical systems where propulsion drives the state toward a target and friction prevents overshooting, here it depends on the opinion, and pulls opinions toward the agent's conviction value $\theta_i^{1/2}[t]$. The positive constant parameter σ is the strength coefficient, measuring the degree of stubbornness, i.e., the extent to which individuals adhere to their own convictions. A larger σ indicates that individuals are more stubborn in their convictions or beliefs, while a smaller σ signifies a higher degree of openness or a greater tendency to follow the crowd.

In dynamically switching digraphs scenario, we consider the dynamically changing interaction topology of the group, where individuals can switch among a finite set of modes $\mathcal{P} := \{1, 2, \dots, p\}$. The switching law is governed by a right-continuous, piecewise constant signal $\chi(t) : [0, \infty) \rightarrow \mathcal{P}$. We denote $\mathcal{G}_{\chi(t)} := (\mathcal{V}, \mathcal{E}_{\chi(t)})$ to be the interaction topology at time t and $\mathcal{N}_i^{\chi(t)}$ to be the set of neighbors of individual i at time t , i.e., the set of individuals that can influence i directly at moment t . The set of discontinuities of $\chi(t)$ is known as the sequence of switching instants, denoted by $\{t_l\}_{l \in \mathbb{N}}$ with $t_0 = 0$. We have $\mathcal{G}_{\chi(s)} = \mathcal{G}_{\chi(t_l)}$, for all $s \in [t_l, t_{l+1})$, $l \in \mathbb{N}$. Since the state is updated at integer t , we only consider $\chi[t] : \mathbb{N} \rightarrow \mathcal{P}$. In this paper, we work initially with the situation that each individual can communicate with others at least once within a certain period of time. Therefore, we propose the following

Assumption 2.1. For the switching topology $\mathcal{G}_{\chi[t]}$, we assume

- There exists a positive integer $\tau_1 > 0$ such that $t_{l+1} - t_l \geq \tau_1$ for all $l \in \mathbb{N}$;
- There exists a positive integer $\tau_2 > 0$ and an increasing time sequence $\{t_l^*\}_{l \in \mathbb{N}}$ with $t_0^* = 0$ such that $t_{l+1}^* - t_l^* \leq \tau_2$ and the union digraph $\mathcal{G}([t_l^*, t_{l+1}^*)) := (\mathcal{V}, \cup_{s \in [t_l^*, t_{l+1}^*)} \mathcal{E}_{\chi[s]})$ is rooted for all $l \in \mathbb{N}$.

Under this setup, $(x_i[t], \theta_i[t])$ evolves according to the following system:

$$\begin{cases} x_i[t+1] = x_i[t] + h \sum_{j \in \mathcal{N}_i^{\chi[t]}} \phi(|x_i[t] - x_j[t]|)(x_j[t] - x_i[t]) + h\sigma(\theta_i[t] - x_i^2[t])x_i[t], \\ \theta_i[t+1] = \theta_i[t] + h \sum_{j \in \mathcal{N}_i^{\chi[t]}} \psi(|x_i[t] - x_j[t]|)(\theta_j[t] - \theta_i[t]), \end{cases} \quad t \in \mathbb{N}, \quad i = 1, 2, \dots, N. \quad (2.3)$$

We assume that each vertex always has a self-loop in both fixed and switching topologies context. This is natural because each individual persistently takes their own current position into account.

Before discussing our main contributions in this paper, we introduce the definition of consensus as follows. We first define $X[t]$ and $\Theta[t]$ and their diameters:

$$X[t] := (x_1[t], \dots, x_N[t])^T, \quad \Theta[t] := (\theta_1[t], \dots, \theta_N[t])^T, \\ D(X[t]) := \max_{1 \leq i, j \leq N} |x_i[t] - x_j[t]|, \quad D(\Theta[t]) := \max_{1 \leq i, j \leq N} |\theta_i[t] - \theta_j[t]|.$$

Definition 2.1 (Consensus). *We say that the group of individuals reaches consensus if and only if the following condition holds:*

$$\lim_{t \rightarrow \infty} D(X[t]) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} D(\Theta[t]) = 0.$$

Notation. For $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the greatest integer not exceeding s . For $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \geq B$ means that $a_{ij} \geq b_{ij}$ for all i, j . A matrix $A = (a_{ij})$ is said to be non-negative if all the entries are non-negative.

3. Preliminaries

In this section, we first announce the Lipschitz continuity property of the Rayleigh friction term and prove the uniformly boundness property of the solutions for two systems.

Secondly, we review the concept of scrambling matrices, state transition matrices and their properties, which are crucial for the later proof of the main theorems.

3.1. Basic properties for the opinion dynamic systems

Set

$$\bar{\theta} := \max_{1 \leq i \leq N} \theta_i[0], \quad \underline{\theta} := \min_{1 \leq i \leq N} \theta_i[0], \\ \bar{x} := \max \left\{ \max_{1 \leq i \leq N} x_i[0], \bar{\theta}^{\frac{1}{2}} \right\}, \quad \underline{x} := \min \left\{ \min_{1 \leq i \leq N} x_i[0], \underline{\theta}^{\frac{1}{2}} \right\} \geq 0, \\ \mathcal{B} := \left\{ (x, \theta) \in (\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\}) : x \in [\underline{x}, \bar{x}], \theta \in [\underline{\theta}, \bar{\theta}] \right\}.$$

For the functional $f(x, \theta) := \sigma(\theta - x^2)x$ with $(x, \theta) \in \mathcal{B}$, the following property holds.

Proposition 3.1. *There exists a constant $L > 0$ such that for any $(x_i, \theta_i), (x_j, \theta_j) \in \mathcal{B}$, $i \neq j$, we have*

$$|f(x_i, \theta_i) - f(x_j, \theta_j)| \leq L(|\theta_i - \theta_j| + |x_i - x_j|),$$

where $L := \max \left\{ \sigma \bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2) \right\}$.

Proof. On the convex set $\mathcal{B}$ we have

$$\begin{aligned} & |f(x_i, \theta_i) - f(x_j, \theta_j)| \\ &= \sigma \left| (\theta_i - \theta_j)x_i + \theta_j(x_i - x_j) + x_j^3 - x_i^3 \right| \\ &= \sigma |(\theta_i - \theta_j)x_i + \theta_j(x_i - x_j) + (x_j - x_i)(x_j^2 + x_i x_j + x_i^2)| \\ &\leq \sigma \bar{x} |\theta_i - \theta_j| + \sigma(\bar{\theta} + 3\bar{x}^2) |x_i - x_j| \\ &\leq \max \left\{ \sigma \bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2) \right\} (|\theta_i - \theta_j| + |x_i - x_j|) \\ &= L(|\theta_i - \theta_j| + |x_i - x_j|), \end{aligned}$$

which completes our proof. $\square$

Next, we show that the solutions of systems (2.1) and (2.3) lie in set $\mathcal{B}$ for all times. The proof is very different from the continuous model studied in [21], since the continuity property is invalid.

Proposition 3.2. *Let $(x_i, \theta_i)_{i=1}^N$ be the solution to system (2.1) or system (2.3), suppose that $\min_{1 \leq i \leq N} \{x_i[0], \theta_i[0]\} \geq 0$, $h < \min\{\frac{1}{N\kappa_1}, \frac{1}{N\kappa_2}\}$ and $\sigma < \min\{\frac{1-hN\kappa_1}{2h\bar{x}^2}, \frac{1-hN\kappa_2}{2h\bar{x}^2}\}$, then for any $i = 1, 2, \dots, N$ and $t \in \mathbb{N}$, we have*

1. *The values of individuals' convictions are uniformly bounded:*

$$\underline{\theta} \leq \theta_i[t] \leq \bar{\theta},$$

and $D(\Theta[t])$ is non-increasing in t .

2. *The values of individuals' opinions are uniformly bounded:*

$$\underline{x} \leq x_i[t] \leq \bar{x}.$$

Proof. We only provide the proof for the case of the static topology since the proof for the other case is analogous.

1. We first define the maximal and minimal conviction values at time t as follows:

$$\theta_{m_t}[t] := \min_{1 \leq i \leq N} \theta_i[t], \quad \theta_{M_t}[t] := \max_{1 \leq i \leq N} \theta_i[t], \quad t \in \mathbb{N}.$$

Clearly, $\underline{\theta} = \theta_{m_0}[0]$, $\bar{\theta} = \theta_{M_0}[0]$. Next we prove the following relation:

$$\theta_{m_t}[t] \leq \theta_{m_{t+1}}[t+1] \leq \theta_{M_{t+1}}[t+1] \leq \theta_{M_t}[t], \quad t \in \mathbb{N}. \quad (3.1)$$

By direct calculation, $\forall t \in \mathbb{N}$, we have

$$\begin{aligned} & \theta_{m_{t+1}}[t+1] - \theta_{m_t}[t] \\ &= \theta_{m_{t+1}}[t] + h \sum_{j \in \mathcal{N}_{m_{t+1}}} \psi(|x_{m_{t+1}}[t] - x_j[t]|)(\theta_j[t] - \theta_{m_{t+1}}[t]) - \theta_{m_t}[t] \\ &\geq (1 - h \sum_{j \in \mathcal{N}_{m_{t+1}}} \psi(|x_{m_{t+1}}[t] - x_j[t]|))(\theta_{m_{t+1}}[t] - \theta_{m_t}[t]) \\ &\geq (1 - hN\kappa_2)(\theta_{m_{t+1}}[t] - \theta_{m_t}[t]) \\ &\geq 0, \end{aligned}$$

where $\theta_{m_{t+1}}[t]$ denotes the conviction value at time t of the agent m_{t+1} who achieves the minimum conviction at time $t+1$. Similarly, $x_{m_{t+1}}[t]$ denotes the opinion value at time t of the same agent. And the third inequality follows from Eq (2.2) and the definition of $\theta_{m_t}[t]$,

$$\theta_i[t] - \theta_{m_t}[t] \geq 0, \quad i = 1, \dots, N.$$

Similarly, one has

$$\begin{aligned}
 & \theta_{M_{t+1}}[t+1] - \theta_{M_t}[t] \\
 &= \theta_{M_{t+1}}[t] + h \sum_{j \in \mathcal{N}_{M_{t+1}}} \psi(|x_{M_{t+1}}[t] - x_j[t]|)(\theta_j[t] - \theta_{M_{t+1}}[t]) - \theta_{M_t}[t] \\
 &\leq (1 - h \sum_{j \in \mathcal{N}_{M_{t+1}}} \psi(|x_{M_{t+1}}[t] - x_j[t]|))(\theta_{M_{t+1}}[t] - \theta_{M_t}[t]) \\
 &\leq (1 - hN\kappa_2)(\theta_{M_{t+1}}[t] - \theta_{M_t}[t]) \\
 &\leq 0,
 \end{aligned}$$

where the third inequality follows from Eq (2.2) and the definition of $\theta_{M_t}[t]$, i.e.,

$$\theta_i[t] - \theta_{M_t}[t] \leq 0, \quad i = 1, \dots, N.$$

Combining the two results above yields the desired relation in Eq (3.1), and

$$D(\Theta[t+1]) = \theta_{M_{t+1}}[t+1] - \theta_{m_{t+1}}[t+1] \leq \theta_{M_t}[t] - \theta_{m_t}[t] = D(\Theta[t]).$$

Therefore, $D(\Theta[t])$ is non-increasing and we obtain our desired result.

2. Similarly, we let

$$x_{m_t}[t] := \min_{1 \leq i \leq N} x_i[t], \quad x_{M_t}[t] := \max_{1 \leq i \leq N} x_i[t], \quad t \in \mathbb{N}.$$

Note that $\underline{x}$ is determined by the values of $x_{m_0}[0]$ and $\theta^{\frac{1}{2}}$. Our aim is to show that, whatever the relation between them, the lower bound of $x_i[t]$ is not less than $\underline{x}$, and similarly for the upper bound. We prove this claim by induction in two steps.

- Step A (Bound estimate at $t = 1$): We begin by establishing that $x_i[t]$ remains within its initial bounds $\underline{x}$ and $\bar{x}$ at $t = 1$.

- ◊ Step A.1 (Lower bound estimate at $t = 1$): For the different choice of $\underline{x}$, let us observe how the minimum values of $x_i[t]$ changes at $t = 1$. We split the proof in two cases.

- Case 1 ($\underline{x} = x_{m_0}[0]$): At the initial time, we have $x_i[0] \leq \bar{x}$, $1 \leq i \leq N$. Therefore,

$$\begin{aligned}
 & x_{m_1}[1] - x_{m_0}[0] \\
 &= x_{m_1}[0] + h \sum_{j \in \mathcal{N}_{m_1}} \phi(|x_{m_1}[0] - x_j[0]|)(x_j[0] - x_{m_1}[0]) \\
 &\quad + h\sigma(\theta_{m_1}[0] - x_{m_1}^2[0])x_{m_1}[0] - x_{m_0}[0] \\
 &\geq (1 - hN\kappa_1)(x_{m_1}[0] - x_{m_0}[0]) + h\sigma(\underline{\theta} - x_{m_1}^2[0])x_{m_1}[0].
 \end{aligned}$$

Note that $(x_{m_1}[0] - x_{m_0}[0]) \geq 0$, $x_{m_1}[0] \geq x_{m_0}[0] \geq 0$.

If $\underline{\theta} \geq x_{m_1}^2[0]$, then $x_{m_1}[1] \geq x_{m_0}[0]$. If $\underline{\theta} < x_{m_1}^2[0]$, then $x_{m_0}[0] \leq \underline{\theta}^{\frac{1}{2}} < x_{m_1}[0] \leq \bar{x}$. We incorporate this observation to refine our estimates on $x_{m_1}[1] - x_{m_0}[0]$:

$$\begin{aligned}
 & x_{m_1}[1] - x_{m_0}[0] \\
 & \geq (1 - hN\kappa_1)(x_{m_1}[0] - x_{m_0}[0]) + h\sigma(\underline{\theta} - x_{m_1}^2[0])x_{m_1}[0] \\
 & \geq (1 - hN\kappa_1)(x_{m_1}[0] - \underline{\theta}^{\frac{1}{2}}) + h\sigma(\underline{\theta} - x_{m_1}^2[0])x_{m_1}[0] \\
 & = (1 - hN\kappa_1)(x_{m_1}[0] - \underline{\theta}^{\frac{1}{2}}) + h\sigma(\underline{\theta}^{\frac{1}{2}} - x_{m_1}[0])(\underline{\theta}^{\frac{1}{2}} + x_{m_1}[0])x_{m_1}[0] \\
 & \geq (1 - hN\kappa_1 - 2h\sigma\bar{x}^2)(x_{m_1}[0] - \underline{\theta}^{\frac{1}{2}}) \\
 & > 0,
 \end{aligned}$$

which gives $x_{m_1}[1] > x_{m_0}[0]$.

• Case 2 ($\underline{x} = \underline{\theta}^{\frac{1}{2}}$): Through similar calculations, we can get $x_{m_1}[1] - \underline{\theta}^{\frac{1}{2}} \geq 0$.

Thus $x_{m_1}[1] \geq \underline{x} := \min\{x_{m_0}[0], \underline{\theta}^{\frac{1}{2}}\} \geq 0$.

♦ Step A.2 (Upper bound estimate at $t = 1$): We still consider two cases for $\bar{x}$.

• Case 3 ($\bar{x} = x_{M_0}[0]$): We have

$$\begin{aligned}
 & x_{M_1}[1] - x_{M_0}[0] \\
 & = x_{M_1}[0] + h \sum_{j \in \mathcal{N}_{M_1}} \phi(|x_{M_1}[0] - x_j[0]|)(x_j[0] - x_{M_1}[0]) \\
 & \quad + h\sigma(\theta_{M_1}[0] - x_{M_1}^2[0])x_{M_1}[0] - x_{M_0}[0] \\
 & \leq (1 - hN\kappa_1)(x_{M_1}[0] - x_{M_0}[0]) + h\sigma(\bar{\theta} - x_{M_1}^2[0])x_{M_1}[0].
 \end{aligned}$$

Note that $(x_{M_1}[0] - x_{M_0}[0]) \leq 0$, $x_{M_1}[0] \geq x_{m_0}[0] \geq 0$.

If $\bar{\theta} \leq x_{M_1}^2[0]$, then $x_{M_1}[1] \leq x_{M_0}[0]$. If $\bar{\theta} > x_{M_1}^2[0]$, then $x_{M_1}[0] < \bar{\theta}^{\frac{1}{2}} \leq x_{M_0}[0] \leq \bar{x}$, and we conclude that

$$\begin{aligned}
 & x_{M_1}[1] - x_{M_0}[0] \\
 & \leq (1 - hN\kappa_1)(x_{M_1}[0] - x_{M_0}[0]) + h\sigma(\bar{\theta} - x_{M_1}^2[0])x_{M_1}[0] \\
 & \leq (1 - hN\kappa_1)(x_{M_1}[0] - \bar{\theta}^{\frac{1}{2}}) + h\sigma(\bar{\theta}^{\frac{1}{2}} - x_{M_1}[0])(\bar{\theta}^{\frac{1}{2}} + x_{M_1}[0])x_{M_1}[0] \\
 & \leq (1 - hN\kappa_1 - 2h\sigma\bar{x}^2)(x_{M_1}[0] - \bar{\theta}^{\frac{1}{2}}) \\
 & < 0,
 \end{aligned}$$

i.e., $x_{M_1}[1] < x_{M_0}[0]$.

• Case 4 ($\bar{x} = \bar{\theta}^{\frac{1}{2}}$): It is easy to get $x_{M_1}[1] - \bar{\theta}^{\frac{1}{2}} \leq 0$.

Thus $x_{M_1}[1] \leq \bar{x} := \max\{x_{M_0}[0], \bar{\theta}^{\frac{1}{2}}\}$. We combine all four cases to get

$$0 \leq \underline{x} \leq x_i[1] \leq \bar{x}, \quad 1 \leq i \leq N.$$

• Step B (Maintenance of the bounds): The procedure is repeated starting from $t = 2$, the following four cases will arise:

If $x_{m_1}[1] \leq \underline{\theta}^{\frac{1}{2}}$, we return to Case 1 and obtain $x_{m_2}[2] - x_{m_1}[1] \geq 0$;

If $x_{m_1}[1] > \underline{\theta}^{\frac{1}{2}}$, we return to Case 2 and obtain $x_{m_2}[2] - \underline{\theta}^{\frac{1}{2}} \geq 0$;

If $x_{M_1}[1] \geq \bar{\theta}^{\frac{1}{2}}$, we return to Case 3 and obtain $x_{M_2}[2] - x_{M_1}[1] \leq 0$;

If $x_{M_1}[1] < \bar{\theta}^{\frac{1}{2}}$, we return to Case 4 and obtain $x_{M_2}[2] - \bar{\theta}^{\frac{1}{2}} \leq 0$.

This indicates that $x_{m_2}[2] \geq \min\{x_{m_1}[1], \underline{\theta}^{\frac{1}{2}}\} \geq \underline{x}$, $x_{M_2}[2] \leq \max\{x_{M_1}[1], \bar{\theta}^{\frac{1}{2}}\} \leq \bar{x}$. Applying the argument inductively on $t \in \mathbb{N}$, the claim follows. $\square$

Remark 3.1. Following the same arguments as in the proof of Proposition 3.2, if $x_i[0] = \theta_i[0] = 0$, for all $i = 1, \dots, N$, we obtain $\bar{x} = 0$ and $x_i[t] = \theta_i[t] = 0$ for all $t \in \mathbb{N}$. Consequently, consensus is trivially achieved without any constraint on σ . Accordingly, in Sections 4 and 5, we exclude this case and focus only on the nontrivial scenario where $\bar{x} > 0$.

3.2. Scrambling matrices

In this subsection, we review scrambling matrix and its properties, which are essential for establishing our consensus results.

Definition 3.1. Let $A = (a_{ij})$ be a non-negative $N \times N$ matrix.

1. A is a (row) stochastic matrix if each of its row-sums equal to 1:

$$\sum_{j=1}^N a_{ij} = 1, \quad 1 \leq i \leq N.$$

2. A is a scrambling matrix if for each pair of indexes i and j , there exists an index k such that:

$$a_{ik} > 0 \quad \text{and} \quad a_{jk} > 0.$$

Remark 3.2. Define the ergodicity coefficient of A as follows:

$$\mu(A) := \min_{i,j} \sum_{k=1}^N \min\{a_{ik}, a_{jk}\}.$$

Then it's easy to see that:

1. A is scrambling if and only if $\mu(A) > 0$, and $\mu(A) \leq 1$ if A is a stochastic matrix.
2. For non-negative matrices A and B ,

$$A \geq B \quad \Rightarrow \quad \mu(A) \geq \mu(B).$$

Proposition 3.3 ([31], Proposition 1). Let $A = (a_{ij})$ be a non-negative matrix whose diagonal entries are all positive. Suppose that $\mathcal{G}(A)$ is rooted with the smallest depth γ_g , then we have

$$\mu(A^{\gamma_g}) \geq n_r \underline{a}^{\gamma_g},$$

where $\underline{a} := \min\{a_{ij} : a_{ij} > 0\}$, and n_r is the cardinality of the set $\{r : \max_{j \in V} \text{dist}(r, j) = \gamma_g\}$.

Lemma 3.1 ([18], Lemma 2.2). Suppose that $A = (a_{ij})$ is stochastic. Let $W = (w_1, w_2, \dots, w_N)^T$, $Z = (z_1, z_2, \dots, z_N)^T$ and $B = (b_1, b_2, \dots, b_N)^T \in (\mathbb{R}^d)^N$ be such that

$$W = AZ + B,$$

then

$$\max_{i,j} \|w_i - w_j\| \leq (1 - \mu(A)) \max_{i,j} \|z_i - z_j\| + \max_{i,j} \|b_i - b_j\|.$$

$\|\cdot\|$ is the norm of l^2 .

Lemma 3.2 ([32], Theorem 5.1). If A_i are non-negative $N \times N$ matrices with positive diagonal elements such that $\mathcal{G}(A_i)$ has a spanning tree for all $1 \leq i \leq N-1$, the $A_1 A_2 \dots A_{N-1}$ is scrambling.

4. Consensus behavior in a static topology

In this section, we prove the consensus result for static interaction topology. The proof proceeds by first establishing the convergence of the conviction diameter depending solely on initial convictions, then that of the opinion diameter, in contrast, whose dynamics depend on both initial opinions and convictions.

4.1. Main result

For the convenience of later calculations, given a constant $n \in \mathbb{N}$ with $n \geq \gamma_g > 0$, we define

$$c_1 := \binom{n}{\gamma_g} (1 - hN\kappa_2)^{n-\gamma_g} h^{\gamma_g} n_r, \quad c_2 := \binom{n}{\gamma_g} (1 - hN\kappa_1)^{n-\gamma_g} h^{\gamma_g} n_r, \quad c_3 := c_2 \phi(2\bar{x})^{\gamma_g} - Lhn. \quad (4.1)$$

Here $L := \max \{ \sigma\bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2) \}$.

First, we summarize the consensus result for the system (2.1) with static topology.

Theorem 4.1. Consider the system (2.1) with communicate functions satisfying Eq (2.2). Assume that $\min_{1 \leq i \leq N} \{x_i[0], \theta_i[0]\} \geq 0$, $h < \min \{ \frac{1}{N\kappa_1}, \frac{1}{N\kappa_2} \}$, $\sigma < \min \{ \frac{1-hN\kappa_1}{2h\bar{x}^2}, \frac{1-hN\kappa_2}{2h\bar{x}^2} \}$ and there exists a constant $n \in \mathbb{N}$ with $n \geq \gamma_g > 0$ such that $c_3 > 0$, i.e.,

$$\binom{n}{\gamma_g} (1 - hN\kappa_1)^{n-\gamma_g} h^{\gamma_g-1} n^{-1} n_r \phi(2\bar{x})^{\gamma_g} > \max \{ \sigma\bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2) \}.$$

Then the system (2.1) reaches consensus.

Remark 4.1. Compared with the continuous-time model studied in [21], our discrete-time framework naturally introduces the time step h into the conditions. While our conditions are only sufficient, γ_g is not artificial, it emerges directly from the graph topology and it reflects the maximum information propagation distance over the graph, as also observed in [21].

4.2. Convergence of conviction diameter

In this subsection, we need to rewrite Eq (2.1)₂ in a matrix form to obtain the solution formula of Θ . Hence, we define an $N \times N$ matrix $L[t]$ by $L[t] := D[t] - A[t]$, where $A[t] = (a_{ij}[t])$ and $D[t] = \text{diag}(d_1[t], \dots, d_N[t])$ are given by

$$a_{ij}[t] := \begin{cases} \psi(|x_i[t] - x_j[t]|), & \text{if } j \in \mathcal{N}_i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_i[t] := \sum_{j=1}^N a_{ij}[t].$$

Then Eq (2.1)₂ can be rewritten as

$$\Theta[t+1] = \Theta[t] - hL[t]\Theta[t]. \quad (4.2)$$

For $t_1, t_2 \in \mathbb{N}$ with $t_2 > t_1$, let $\Psi[t_2, t_1]$ be the transition matrix of $-L[t]$ over time interval $[t_1, t_2]$. Then we derive the solution formula for Θ :

$$\Theta[t_2] = \Psi[t_2, t_1]\Theta[t_1]. \quad (4.3)$$

We can discover that

$$\Psi[t_1 + 1, t_1] = I - hL[t_1],$$

then we have

$$\Psi[t_2, t_1] = (I - hL[t_2 - 1])(I - hL[t_2 - 2]) \cdots (I - hL[t_1]), \quad (4.4)$$

with the convention $\Psi[t_1, t_1] = I$.

Before providing the formal proof, we first present two technical lemmas which will assist in the subsequent proof. The first lemma is cited from Lemma A.2 in [18].

Lemma 4.1 ([18], Lemma A.2). *Let $f[t] : \mathbb{N} \rightarrow [0, \infty)$ be any function. Assume that there exist two constants $0 < \rho < 1$ and $n \in \mathbb{N}$ with $n \geq 1$ such that*

$$f[t] \leq \rho \max\{f[t-n], f[t-n+1], \dots, f[t-1], f[t]\}, \quad \forall t \geq n.$$

Then we have

$$f[t] \leq \rho^{\lfloor \frac{t}{n} \rfloor} \max\{f[0], f[1], \dots, f[n]\}, \quad \forall t \geq 0.$$

The second lemma is an extension version.

Lemma 4.2. *Let $f[t], g[t] : \mathbb{N} \rightarrow [0, \infty)$ be two functions. Assume that there exist four constants $0 < \rho_1, \rho_2 < 1$, $K > 0$ and $n \in \mathbb{N}$ with $n \geq 1$ such that*

$$\begin{aligned} f[t] &\leq \rho_1 \max\{f[t-n], f[t-n+1], \dots, f[t]\} + K \max\{g[t-n], g[t-n+1], \dots, g[t]\}, \\ g[t] &\leq \rho_2 \max\{g[t-n], g[t-n+1], \dots, g[t]\}, \quad \forall t \geq n. \end{aligned} \quad (4.5)$$

Then for any $t \geq 0$, we have

$$f[t] \leq \begin{cases} \rho_1^{\lfloor \frac{t}{n} \rfloor} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C \bar{g}[n] \} + C \bar{g}[n] \lfloor \frac{t}{n} \rfloor \rho_1^{\lfloor \frac{t}{n} \rfloor - 1}, & \text{if } \rho_1 = \rho_2, \\ \rho_1^{\lfloor \frac{t}{n} \rfloor} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C \bar{g}[n] \} + C \bar{g}[n] \frac{\rho_2^{\lfloor \frac{t}{n} \rfloor} - \rho_1^{\lfloor \frac{t}{n} \rfloor}}{\rho_2 - \rho_1}, & \text{otherwise.} \end{cases}$$

Where we denote by $\bar{f}[n] := \max\{f[0], f[1], \dots, f[n]\}$, $\bar{g}[n] := \max\{g[0], g[1], \dots, g[n]\}$ and $C := \frac{K}{1-\rho_1}$.

Proof. We consider three cases.

- Case $t \geq 2n$.

For the simplicity, we denote $\bar{f}[t] := \max\{f[t-n], f[t-n+1], \dots, f[t]\}$ and $\bar{g}[t] := \max\{g[t-n], g[t-n+1], \dots, g[t]\}$. Using Lemma 4.1, we can derive

$$\bar{g}[t] \leq \rho_2 \bar{g}[t-n] \leq \bar{g}[t-n], \quad \forall t \geq 2n, \quad (4.6)$$

and

$$\bar{g}[t] \leq \rho_2^{\lfloor \frac{t}{n} \rfloor - 1} \bar{g}[n], \quad \forall t \geq 0. \quad (4.7)$$

Then for any $t \geq 2n$, we can get the following estimate from Eqs (4.5) and (4.6)

$$\begin{aligned} \bar{f}[t] &\leq \rho_1 \max\{f[t-2n], f[t-2n+1], \dots, f[t]\} + K \max\{g[t-2n], g[t-2n+1], \dots, g[t]\} \\ &= \rho_1 \max\{\bar{f}[t-n], \bar{f}[t]\} + K \max\{\bar{g}[t-n], \bar{g}[t]\} \\ &= \rho_1 \max\{\bar{f}[t-n], \bar{f}[t]\} + K \bar{g}[t-n]. \end{aligned}$$

If $\bar{f}[t-n] \geq \bar{f}[t]$, then

$$\bar{f}[t] \leq \rho_1 \bar{f}[t-n] + K \bar{g}[t-n]. \quad (4.8)$$

Otherwise, if $\bar{f}[t-n] < \bar{f}[t]$, we have

$$\bar{f}[t] \leq C \bar{g}[t-n], \quad (4.9)$$

where $C := \frac{K}{1-\rho_1}$. Combining Eqs (4.8) and (4.9), we can get the estimate for $\bar{f}[t]$:

$$\bar{f}[t] \leq \rho_1 \bar{f}[t-n] + C \bar{g}[t-n]. \quad (4.10)$$

We use Eq (4.10) again to deduce

$$\begin{aligned} \bar{f}[t] &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \bar{f}[t - (\lfloor \frac{t}{n} \rfloor - 1)n] + C \sum_{m=1}^{\lfloor \frac{t}{n} \rfloor - 2} \rho_1^m \bar{g}[t - (m+1)n] + C \bar{g}[t-n] \\ &= \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \bar{f}[t - (\lfloor \frac{t}{n} \rfloor - 1)n] + C \sum_{m=1}^{\lfloor \frac{t}{n} \rfloor - 1} \rho_1^{m-1} \bar{g}[t - mn] \\ &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \max\{\bar{f}[n], \bar{f}[2n]\} + C \sum_{m=1}^{\lfloor \frac{t}{n} \rfloor - 1} \rho_1^{m-1} \rho_2^{\lfloor \frac{t}{n} \rfloor - m - 1} \bar{g}[n]. \end{aligned} \quad (4.11)$$

Here, the last inequality in Eq (4.11) holds due to Eq (4.7).

Next, we discuss two cases: $\rho_1 = \rho_2$ and $\rho_1 \neq \rho_2$, respectively. First, if $\rho_1 = \rho_2$, we have

$$\begin{aligned}\bar{f}[t] &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \max \{ \bar{f}[n], \bar{f}[2n] \} + C\bar{g}[n] \left(\lfloor \frac{t}{n} \rfloor - 1 \right) \rho_1^{\lfloor \frac{t}{n} \rfloor - 2} \\ &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + C\bar{g}[n] \left(\lfloor \frac{t}{n} \rfloor - 1 \right) \rho_1^{\lfloor \frac{t}{n} \rfloor - 2},\end{aligned}$$

where the inequality $\bar{f}[2n] \leq \rho_1 \bar{f}[n] + C\bar{g}[n]$ is used. Then we can obtain

$$\begin{aligned}f[t] &\leq \rho_1 \bar{f}[t] + K\bar{g}[t] \\ &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + C\bar{g}[n] \left(\lfloor \frac{t}{n} \rfloor - 1 \right) \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} + C\rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \bar{g}[n] \\ &= \rho_1^{\lfloor \frac{t}{n} \rfloor} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + C\bar{g}[n] \lfloor \frac{t}{n} \rfloor \rho_1^{\lfloor \frac{t}{n} \rfloor - 1}.\end{aligned}$$

Second, when $\rho_1 \neq \rho_2$, we can derive

$$\begin{aligned}\bar{f}[t] &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor - 1} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + C\bar{g}[n] \frac{\rho_2^{\lfloor \frac{t}{n} \rfloor - 1} - \rho_1^{\lfloor \frac{t}{n} \rfloor - 1}}{\rho_2 - \rho_1}, \\ f[t] &\leq \rho_1^{\lfloor \frac{t}{n} \rfloor} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + \rho_1 C\bar{g}[n] \frac{\rho_2^{\lfloor \frac{t}{n} \rfloor - 1} - \rho_1^{\lfloor \frac{t}{n} \rfloor - 1}}{\rho_2 - \rho_1} + C\rho_2^{\lfloor \frac{t}{n} \rfloor - 1} \bar{g}[n] \\ &= \rho_1^{\lfloor \frac{t}{n} \rfloor} \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + C\bar{g}[n] \frac{\rho_2^{\lfloor \frac{t}{n} \rfloor} - \rho_1^{\lfloor \frac{t}{n} \rfloor}}{\rho_2 - \rho_1}.\end{aligned}$$

- Case $n \leq t < 2n$. For the case of $n \leq t \leq 2n - 1$, we use Eqs (4.5), (4.6) and (4.10) to see that

$$\begin{aligned}f[t] &\leq \rho_1 \max \{ f[0], f[1], \dots, f[2n - 1] \} + K \max \{ g[0], g[1], \dots, g[2n - 1] \} \\ &\leq \rho_1 \max \{ \bar{f}[n], \bar{f}[2n] \} + K\bar{g}[n] \\ &\leq \rho_1 \max \{ \bar{f}[n], \rho_1 \bar{f}[n] + C\bar{g}[n] \} + C\bar{g}[n].\end{aligned}$$

- Case $0 \leq t < n$. In this case, the inequality obviously holds. □

Lemma 4.3. Let $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.1) with communication functions satisfying Eq (2.2), then for any $t, n \in \mathbb{N}$ with $t \geq n \geq \gamma_g$, the following assertions hold under the assumptions of Proposition 3.2:

1. The ergodicity coefficient $\mu(\Psi[t, t - n])$ satisfies

$$\mu(\Psi[t, t - n]) \geq c_1 \psi(2\bar{x})^{\gamma_g},$$

where the constant c_1 is defined in Eq (4.1).

2. The state transition matrix $\Psi[t, t - n]$ is stochastic.

Proof. 1. We have

$$-L[t] = A[t] - D[t] \geq A - N\kappa_2 I, \quad (4.12)$$

where $A = (a_{ij})$ is given by

$$a_{ij} := \begin{cases} \psi(2\bar{x}), & \text{if } j \in \mathcal{N}_i, \\ 0, & \text{otherwise.} \end{cases}$$

According to Eqs (4.4) and (4.12), we can calculate that

$$\begin{aligned} \Psi[t, t-n] &= (I - hL[t-1])(I - hL[t-2]) \dots (I - hL[t-n]) \\ &= (I - hD[t-1] + hA[t-1]) \dots (I - hD[t-n] + hA[t-n]) \\ &\geq ((1 - hN\kappa_2)I + hA)^n \\ &\geq \binom{n}{\gamma_g} (1 - hN\kappa_2)^{n-\gamma_g} h^{\gamma_g} (A)^{\gamma_g}. \end{aligned} \quad (4.13)$$

Since each vertex has a self-loop in both fixed and switching topologies, A is a non-negative matrix whose diagonal entries are all positive. Consequently, we can apply Proposition 3.3 to obtain

$$\begin{aligned} \mu(\Psi[t, t-n]) &\geq \binom{n}{\gamma_g} (1 - hN\kappa_2)^{n-\gamma_g} h^{\gamma_g} \mu(A^{\gamma_g}) \\ &\geq \binom{n}{\gamma_g} (1 - hN\kappa_2)^{n-\gamma_g} h^{\gamma_g} n_r \psi(2\bar{x})^{\gamma_g} \\ &= c_1 \psi(2\bar{x})^{\gamma_g}. \end{aligned} \quad (4.14)$$

2. Let $(1, \dots, 1)^T = \mathbf{1}$. Equation (4.13) gives that $\Psi[t, t-n]$ is non-negative and $L[t]\mathbf{1} = \mathbf{0}$. Then, $\Theta[t] = \mathbf{1}$ is a solution to Eq (4.2), and we have

$$(1, \dots, 1)^T = \Psi[t, t-n](1, \dots, 1)^T,$$

which gives that $\Psi[t, t-n]$ is stochastic and $0 < \mu(\Psi[t, t-n]) \leq 1$.

□

Proposition 4.1. *Given that the hypotheses of Proposition 3.2 are satisfied, let $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.1) with the communication functions satisfying Eq (2.2), then for any $n \in \mathbb{N}$ with $n \geq \gamma_g$ we have the following conviction diameter shrinkage property:*

$$D(\Theta[t]) \leq (1 - c_1 \psi(2\bar{x})^{\gamma_g})^{\lfloor \frac{t}{n} \rfloor} D(\Theta[0]), \quad t \geq 0,$$

where $0 < c_1 \psi(2\bar{x})^{\gamma_g} \leq 1$ is a lower bound of the ergodicity coefficient for state transition matrix $\Psi[t, t-n]$ associated with Eq (4.2) on time interval $[t-n, t]$, which is estimated from Eq (4.14).

Proof. We first discuss the case where $t \geq n \geq \gamma_g$. For any $t \geq n$, we use Eq (4.3), Lemmas 3.1 and 4.3 to obtain

$$\begin{aligned} D(\Theta[t]) &\leq (1 - \mu(\Psi[t, t - n])) D(\Theta[t - n]) \\ &\leq (1 - c_1 \psi(2\bar{x})^{\gamma_g}) D(\Theta[t - n]) \\ &\leq (1 - c_1 \psi(2\bar{x})^{\gamma_g})^{\lfloor \frac{t}{n} \rfloor} \max\{D(\Theta[0]), \dots, D(\Theta[n])\}. \end{aligned}$$

Here the last inequality is derived by Lemma 4.1. According to Eq (4.2)

$$\Theta[t + 1] = (I - hL[t])\Theta[t],$$

we combine $I - hL[t] \geq hA + (1 - hN\kappa_2)I$ and $(I - hL[t])\mathbf{1} = \mathbf{1}$ to see that $I - hL[t]$ is scrambling. Re-applying Lemma 3.1 to get

$$D(\Theta[t + 1]) \leq D(\Theta[t]), \quad \forall t \geq 0. \quad (4.15)$$

Therefore, we have for $t \geq n$,

$$D(\Theta[t]) \leq (1 - c_1 \psi(2\bar{x})^{\gamma_g})^{\lfloor \frac{t}{n} \rfloor} D(\Theta[0]).$$

For the case $0 \leq t < n$, it follows from Eq (4.15) that the conclusion of Proposition 4.1 clearly holds. $\square$

4.3. Convergence of opinions diameter

In Section 4.2, we have proven the shrinkage property of $D(\Theta[t])$, that is, there exists a constant θ^* such that for all $i = 1, 2, \dots, N$, we have $\lim_{t \rightarrow \infty} \theta_i[t] = \theta^*$. However, the opinions in system (2.1) cannot reach consensus unconditionally. We provide a sufficient condition to conclude the shrinkage property of opinion diameter in system (2.1), which is proposed in Theorem 4.1. In order to rewrite Eq (2.1)₁ in a matrix form, we define an $N \times N$ matrix $\tilde{L}[t]$ by $\tilde{L}[t] := \tilde{D}[t] - \tilde{A}[t]$, where $\tilde{A}[t] = (\tilde{a}_{ij}[t])$ and $\tilde{D}[t] = \text{diag}(\tilde{d}_1[t], \dots, \tilde{d}_N[t])$ is given by

$$\tilde{a}_{ij}[t] := \begin{cases} \phi(|x_i[t] - x_j[t]|), & \text{if } j \in \mathcal{N}_i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\tilde{d}_i[t] := \sum_{j=1}^N \tilde{a}_{ij}[t].$$

Besides, we define an $N \times 1$ vector

$$F(X[t], \Theta[t]) := (f(x_1[t], \theta_1[t]), \dots, f(x_N[t], \theta_N[t]))^T,$$

where $f(x_i[t], \theta_i[t]) := \sigma(\theta_i[t] - x_i^2[t])x_i[t]$, $i = 1, 2, \dots, N$ is the self-adjustment term of individual i . Then Eq (2.1)₁ can be rewritten as

$$X[t + 1] = X[t] - h\tilde{L}[t]X[t] + hF(X[t], \Theta[t]). \quad (4.16)$$

Let $\Phi[t_2, t_1]$ be the state transition matrix of $-\tilde{L}[t]$ over $[t_1, t_2]$, then the solution of Eq (4.16) can be represented as

$$X[t_2] = \Phi[t_2, t_1]X[t_1] + h \sum_{k=0}^{t_2-t_1-1} \Phi[t_2, t_1 + k + 1]F(X[t_1 + k], \Theta[t_1 + k]). \quad (4.17)$$

We also have

$$\Phi[t_2, t_1] = (I - h\tilde{L}[t_2 - 1])(I - h\tilde{L}[t_2 - 2]) \cdots (I - h\tilde{L}[t_1]). \quad (4.18)$$

Lemma 4.4. *Let the assumptions in Proposition 3.2 hold and $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.1) with communication functions satisfying Eq (2.2), then for any $n \in \mathbb{N}$ with $t \geq n \geq \gamma_g$, the following assertions hold:*

1. *The ergodicity coefficient $\mu(\Phi[t, t - n])$ satisfies*

$$\mu(\Phi[t, t - n]) \geq c_2 \phi(2\bar{x})^{\gamma_g},$$

where the constant c_2 is defined in Eq (4.1).

2. *The state transition matrix $\Phi[t, t - n]$ is stochastic.*

Proof. 1. We have

$$-\tilde{L}[t] = \tilde{A}[t] - \tilde{D}[t] \geq \tilde{A} - N\kappa_1 I, \quad (4.19)$$

where $\tilde{A} = (a_{ij})$ is given by

$$a_{ij} := \begin{cases} \phi(2\bar{x}), & \text{if } j \in \mathcal{N}_i, \\ 0, & \text{otherwise.} \end{cases}$$

According to Eqs (4.18) and (4.19), we can calculate that

$$\begin{aligned} \Phi[t, t - n] &= (I - h\tilde{L}[t - 1])(I - h\tilde{L}[t - 2]) \cdots (I - h\tilde{L}[t - n]) \\ &= (I - h\tilde{D}[t - 1] + h\tilde{A}[t - 1]) \cdots (I - h\tilde{D}[t - n] + h\tilde{A}[t - n]) \\ &\geq ((1 - hN\kappa_1)I + h\tilde{A})^n \\ &\geq \binom{n}{\gamma_g} (1 - hN\kappa_1)^{n-\gamma_g} h^{\gamma_g} (\tilde{A})^{\gamma_g}. \end{aligned} \quad (4.20)$$

Since each vertex has a self-loop in both fixed and switching topologies, $\tilde{A}$ is a non-negative matrix whose diagonal entries are all positive. Consequently, we can apply Proposition 3.3 to obtain

$$\begin{aligned} \mu(\Phi[t, t - n]) &\geq \binom{n}{\gamma_g} (1 - hN\kappa_1)^{n-\gamma_g} h^{\gamma_g} \mu(\tilde{A}^{\gamma_g}) \\ &\geq \binom{n}{\gamma_g} (1 - hN\kappa_1)^{n-\gamma_g} h^{\gamma_g} n_r \phi(2\bar{x})^{\gamma_g} \\ &= c_2 \phi(2\bar{x})^{\gamma_g}. \end{aligned} \quad (4.21)$$

2. Let $(1, \dots, 1)^T = \mathbf{1}$. Equation (4.20) gives that $\Phi[t, t-n]$ is non-negative and $\tilde{L}[t]\mathbf{1} = \mathbf{0}$. According to Eq (4.18), we have

$$(1, \dots, 1)^T = \Phi[t, t-n](1, \dots, 1)^T,$$

which gives that $\Phi[t, t-n]$ is stochastic and $0 < \mu(\Phi[t, t-n]) \leq 1$.

□

From the form of the solution, the results of Lemmas 3.1 and 4.4, we can give a rough estimate of $D(X[t])$.

Lemma 4.5. *Let the hypotheses of Theorem 4.1 be satisfied and $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.1) with communication functions satisfying Eq (2.2), then for any $t \geq n \geq \gamma_g$, we have the following inequality estimate for $D(X[t])$:*

$$D(X[t]) \leq (1 - c_3) \max\{D(X[t-n]), \dots, D(X[t-1])\} + hnL \max\{D(\Theta[t-n]), \dots, D(\Theta[t-1])\},$$

where the constants c_3 , L are defined in Eq (4.1) and Proposition 3.1 respectively.

Proof. For simplicity, we denote $B[t, k] := \Phi[t, t-n+k+1]F(X[t-n+k], \Theta[t-n+k]) = (B_1[t, k], \dots, B_N[t, k]) \in \mathbb{R}^{dN}$. With the help of Lemma 3.1, we have

$$D(B[t, k]) \leq D(F(X[t-n+k], \Theta[t-n+k])). \quad (4.22)$$

Using Lemmas 3.1 and 4.4, Eqs (4.17) and (4.22), one can have

$$\begin{aligned} D(X[t]) &\leq (1 - \mu(\Phi[t, t-n])) D(X[t-n]) + h \sum_{k=0}^{n-1} \max_{i,j} \|(B_i[t, k] - B_j[t, k])\| \\ &\leq (1 - c_2 \phi(2\bar{x})^{\gamma_g}) D(X[t-n]) + h \sum_{k=0}^{n-1} D(F(X[t-n+k], \Theta[t-n+k])) \\ &\leq (1 - c_2 \phi(2\bar{x})^{\gamma_g}) D(X[t-n]) + hnL \sum_{k=0}^{n-1} \max\{D(\Theta[t-n+k]) + D(X[t-n+k])\} \\ &\leq (1 - (c_2 \phi(2\bar{x})^{\gamma_g} - hnL)) \max\{D(X[t-n]), \dots, D(X[t-1])\} \\ &\quad + hnL \max\{D(\Theta[t-n]), \dots, D(\Theta[t-1])\} \\ &= (1 - c_3) \max\{D(X[t-n]), \dots, D(X[t-1])\} \\ &\quad + hnL \max\{D(\Theta[t-n]), \dots, D(\Theta[t-1])\}. \end{aligned}$$

Here the second term of the third inequality is derived from Proposition 3.1. Therefore, we finish the proof of this Lemma. □

Proposition 4.2. *Let the hypotheses of Theorem 4.1 be satisfied and $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.1) with communication functions satisfying Eq (2.2), then we have the shrinkage property of*

$D(X[t])$: for any $t \in \mathbb{N}$ with $t \geq 0$, we have

$$D(X[t]) \leq \begin{cases} (1-c_3)^{\lfloor \frac{t}{n} \rfloor} \max \left\{ G(X[0], \Theta[0]), (1-c_3)G(X[0], \Theta[0]) + \frac{hnL}{c_3}D(\Theta[0]) \right\} \\ + \frac{hnL}{c_3}D(\Theta[0]) \lfloor \frac{t}{n} \rfloor (1-c_3)^{\lfloor \frac{t}{n} \rfloor - 1}, \text{ if } c_1\psi(2\bar{x})^{\gamma_g} = c_3, \\ (1-c_3)^{\lfloor \frac{t}{n} \rfloor} \max \left\{ G(X[0], \Theta[0]), (1-c_3)G(X[0], \Theta[0]) + \frac{hnL}{c_3}D(\Theta[0]) \right\} \\ + \frac{hnL}{c_3}D(\Theta[0]) \frac{(1-c_1\psi(2\bar{x})^{\gamma_g})^{\lfloor \frac{t}{n} \rfloor} - (1-c_3)^{\lfloor \frac{t}{n} \rfloor}}{c_3 - c_1\psi(2\bar{x})^{\gamma_g}}, \text{ if } c_1\psi(2\bar{x})^{\gamma_g} \neq c_3, \end{cases}$$

where $G(X[0], \Theta[0]) := (1 + hL)^n D(X[0]) + \sum_{m=1}^n (1 + hL)^{m-1} hLD(\Theta[0])$.

Proof. For any $t \geq n \geq \gamma_g$, from Lemma 4.2, we know that if $c_1\psi(2\bar{x})^{\gamma_g} = c_3$, we have

$$D(X[t]) \leq (1-c_3)^{\lfloor \frac{t}{n} \rfloor} \max \left\{ \bar{D}(X[n]), (1-c_3)\bar{D}(X[n]) + \frac{hnL}{c_3}\bar{D}(\Theta[n]) \right\} \\ + \frac{hnL}{c_3}\bar{D}(\Theta[n]) \lfloor \frac{t}{n} \rfloor (1-c_3)^{\lfloor \frac{t}{n} \rfloor - 1},$$

if $c_1\psi(2\bar{x})^{\gamma_g} \neq c_3$, we have

$$D(X[t]) \leq (1-c_3)^{\lfloor \frac{t}{n} \rfloor} \max \left\{ \bar{D}(X[n]), (1-c_3)\bar{D}(X[n]) + \frac{hnL}{c_3}\bar{D}(\Theta[n]) \right\} \\ + \frac{hnL}{c_3}\bar{D}(\Theta[n]) \frac{(1-c_1\psi(2\bar{x})^{\gamma_g})^{\lfloor \frac{t}{n} \rfloor} - (1-c_3)^{\lfloor \frac{t}{n} \rfloor}}{c_3 - c_1\psi(2\bar{x})^{\gamma_g}}.$$

Here $\bar{D}(\Theta[n]) = \max\{D(\Theta[0]), \dots, D(\Theta[n])\}$. From Eq (4.15), we have $\bar{D}(\Theta[n]) = D(\Theta[0])$. Following the same idea with $X[t+1] = X[t] - h\tilde{L}[t]X[t] + hF(X[t], \Theta[t])$, we have

$$\begin{aligned} D(X[t+1]) &\leq D(X[t]) + hD(F(X[t], \Theta[t])) \\ &\leq (1+hL)D(X[t]) + hLD(\Theta[t]) \\ &\leq (1+hL)D(X[t]) + hLD(\Theta[0]). \end{aligned}$$

This shows that for $t \in [0, N]$,

$$D(X[t]) \leq (1+hL)^t D(X[0]) + \sum_{m=1}^t (1+hL)^{m-1} hLD(\Theta[0]).$$

For convenience of calculation, denote $G(X[0], \Theta[0]) := (1 + hL)^n D(X[0]) + \sum_{m=1}^n (1 + hL)^{m-1} hLD(\Theta[0])$, we can obtain

$$\bar{D}(X[n]) \leq G(X[0], \Theta[0]).$$

Substituting the upper bound estimate of $\bar{D}(\Theta[n])$ and $\bar{D}(X[n])$ to get the result. □

4.4. Proof of Theorem 4.1

By Propositions 4.1 and 4.2, one can see that if the constant c_3 defined in Eq (4.1) satisfies $c_3 > 0$, which is equivalent to

$$\binom{n}{\gamma_g}(1 - hN\kappa_1)^{n-\gamma_g}h^{\gamma_g-1}n^{-1}n_r\phi(2\bar{x})^{\gamma_g} > \max\{\sigma\bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2)\},$$

the consensus is reached as $t \rightarrow \infty$. This concludes the proof of Theorem 4.1.

5. Consensus behavior in dynamically switching topology

This section presents the consensus proof for the dynamically switching topology case. We begin with the convergence of the conviction diameter and then move on to the convergence of the opinion diameter.

5.1. Main result

Now we state the main result for the system (2.3) with the dynamically switching topology. Define

$$\begin{aligned} \lambda &:= 2(N-1)\tau_2, \quad \tilde{c}_1 := h^{N-1}(1 - hN\kappa_2)^{(2\tau_2-1)(N-1)}, \\ \tilde{c}_2 &:= h^{N-1}(1 - hN\kappa_1)^{(2\tau_2-1)(N-1)} \quad \tilde{c}_3 := \tilde{c}_2\phi(2\bar{x})^{N-1} - hL\lambda. \end{aligned} \quad (5.1)$$

Theorem 5.1. *Consider the system (2.3) with communication functions satisfying Eq (2.2). Assume that $\min_{1 \leq i \leq N} \{x_i[0], \theta_i[0]\} \geq 0$, $h < \min\{\frac{1}{N\kappa_1}, \frac{1}{N\kappa_2}\}$, $\sigma < \min\{\frac{1-hN\kappa_1}{2h\bar{x}^2}, \frac{1-hN\kappa_2}{2h\bar{x}^2}\}$ and Assumption 2.1 are being satisfied. If the following holds*

$$(h\phi(2\bar{x}))^{N-1}(1 - hN\kappa_1)^{(2\tau_2-1)(N-1)} \frac{1}{h\lambda} > \max\{\sigma\bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2)\},$$

then the system (2.3) reaches consensus.

Remark 5.1. *As in the static topology case, the conditions for the switching topology likewise include the time step h . The exponent $N-1$ comes directly from Lemma 3.2, which states that the product of $N-1$ rooted matrices is scrambling. Therefore, $N-1$ is a natural parameter, even though our conditions are only sufficient.*

5.2. Convergence of conviction diameter

Define an $N \times N$ matrix $L_{\chi[t]}[t] := D_{\chi[t]}[t] - A_{\chi[t]}[t]$, where $A_{\chi[t]}[t] = (a_{ij}^{\chi[t]}[t])$ and $D_{\chi[t]}[t] = \text{diag}(d_1^{\chi[t]}[t], \dots, d_N^{\chi[t]}[t])$. Here $d_i^{\chi[t]}[t] := \sum_{j=1}^N a_{ij}^{\chi[t]}[t]$,

$$a_{ij}^{\chi[t]}[t] := \begin{cases} \psi(|x_i[t] - x_j[t]|), & \text{if } j \in \mathcal{N}_i^{\chi[t]}, \\ 0, & \text{otherwise.} \end{cases}$$

Then Eq (2.3)₂ can be expressed as

$$\Theta[t+1] = \Theta[t] - hL_{\chi[t]}[t]\Theta[t]. \quad (5.2)$$

The solution of Eq (5.2) is given by

$$\Theta[t_2] = \Psi_{\chi[t]}[t_2, t_1]\Theta[t_1],$$

where $\Psi_{\chi[t]}[t_2, t_1]$ is the transition matrix of $-L_{\chi[t]}[t]$ over time interval $[t_1, t_2]$. Similarly, we have

$$\Psi_{\chi[t]}[t_2, t_1] = (I - hL_{\chi[t_2-1]}[t_2 - 1]) \dots (I - hL_{\chi[t_1]}[t_1]).$$

Lemma 5.1. *Let $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.3) with communication function satisfying Eq (2.2) and Assumption 2.1 holds, then for any $t \in \mathbb{N}$ with $t \geq \lambda > 0$, the following assertions hold under the assumptions of Proposition 3.2:*

1. *The ergodicity coefficient $\mu(\Psi_{\chi[t]}[t, t - \lambda])$ satisfies*

$$\mu(\Psi_{\chi[t]}[t, t - \lambda]) \geq \tilde{c}_1 \psi(2\bar{x})^{N-1}.$$

2. *The state transition matrix $\Psi_{\chi[t]}[t, t - \lambda]$ is stochastic.*

Proof. 1. We first consider the matrix $\Psi_{\chi[t]}[t, t - 2\tau_2]$. Note that for all $t \in \mathbb{N}$, we have

$$-L_{\chi[t]}(t) \geq A_{\chi[t]} - N\kappa_2 I,$$

where $A_{\chi[t]} = (a_{ij}^{\chi[t]})$ is given by

$$a_{ij}^{\chi[t]} := \begin{cases} \psi(2\bar{x}), & \text{if } j \in \mathcal{N}_i^{\chi[t]}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \Psi_{\chi[t]}[t, t - 2\tau_2] &= (I - hL_{\chi[t-1]}[t - 1]) \dots (I - hL_{\chi[t-2\tau_2]}[t - 2\tau_2]) \\ &\geq ((1 - hN\kappa_2)I + hA_{\chi[t-1]}) \dots ((1 - hN\kappa_2)I + hA_{\chi[t-2\tau_2]}) \\ &\geq (1 - hN\kappa_2)^{2\tau_2-1} h(A_{\chi[t-1]} + \dots + A_{\chi[t-2\tau_2]}) \\ &\geq (1 - hN\kappa_2)^{2\tau_2-1} h\psi(2\bar{x})F_1, \end{aligned}$$

where F_1 is the $(0, 1)$ -adjacency matrix of the union digraph $\mathcal{G}([t - 2\tau_2, t])$. By Assumption 2.1, the interval $[t - 2\tau_2, t)$ contains at least one subinterval $[t_l, t_{l+1})$ for some $l \in \mathbb{N}$. This implies that $\mathcal{G}(F_1) = \mathcal{G}([t - 2\tau_2, t])$ has a spanning tree. Let F_i , $i = 1, 2, \dots, N - 1$ be the $(0, 1)$ -adjacency matrix of the union digraph $\mathcal{G}([t - 2i\tau_2, t - 2(i - 1)\tau_2])$, then for each $1 \leq i \leq N - 1$, $\mathcal{G}(F_i) = \mathcal{G}([t - 2i\tau_2, t - 2(i - 1)\tau_2])$ has a spanning tree, and the following inequality holds:

$$\Psi_{\chi[t]}[t - 2(i - 1)\tau_2, t - 2i\tau_2] \geq (1 - hN\kappa_2)^{2\tau_2-1} h\psi(2\bar{x})F_i. \quad (5.3)$$

With the help of Lemma 3.2, we know that $F_1 F_2 \dots F_{N-1}$ is scrambling, which indicates that $\mu(F_1 F_2 \dots F_{N-1}) \geq 1$. Finally, we use Eq (5.3) to derive

$$\begin{aligned} \mu(\Psi_{\chi[t]}[t, t - \lambda]) &= \mu\left(\prod_{i=1}^{N-1} \Psi_{\chi[t]}[t - 2(i - 1)\tau_2, t - 2i\tau_2]\right) \\ &\geq (1 - hN\kappa_2)^{(2\tau_2-1)(N-1)} h^{N-1} \psi(2\bar{x})^{N-1} \mu\left(\prod_{i=1}^{N-1} F_i\right) \\ &\geq (1 - hN\kappa_2)^{(2\tau_2-1)(N-1)} h^{N-1} \psi(2\bar{x})^{N-1} \\ &= \tilde{c}_1 \psi(2\bar{x})^{N-1}. \end{aligned}$$

Then the proof is complete.

2. Let $(1, \dots, 1)^T = \mathbf{1}$. Equation (5.3) gives that $\Psi_{\chi[t]}[t, t-n]$ is non-negative and $L[t]_{\chi[t]} \mathbf{1} = \mathbf{0}$. Then, $\Theta[t] = \mathbf{1}$ is a solution to Eq (5.2), and we have

$$(1, \dots, 1)^T = \Psi_{\chi[t]}[t, t-n](1, \dots, 1)^T,$$

which gives that $\Psi_{\chi[t]}[t, t-n]$ is stochastic and $0 < \mu(\Psi_{\chi[t]}[t, t-n]) \leq 1$.

□

Proposition 5.1. *Let $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.3) with communication functions satisfying Eq (2.2) and Assumption 2.1, we have the following shrinkage property for conviction diameter under the assumptions of Proposition 3.2:*

$$D(\Theta[t]) \leq \left(1 - \tilde{c}_1 \psi(2\bar{x})^{N-1}\right)^{\lfloor \frac{t}{\lambda} \rfloor} D(\Theta[0]), \quad \forall t \in \mathbb{N}.$$

Proof. We use Lemmas 3.1, 4.1, and 5.1 to complete our proof. The details are similar to the proof of Proposition 4.1. □

5.3. Convergence of opinions diameter

Similar to Eqs (4.2), (4.16) and (5.2), we define an $N \times N$ matrix $\tilde{L}_{\chi[t]} := \tilde{D}_{\chi[t]}(t) - \tilde{A}_{\chi[t]}(t)$, where $\tilde{A}_{\chi[t]}[t] = (\tilde{a}_{ij}^{\chi[t]}[t])$, $\tilde{D}_{\chi[t]}[t] = \text{diag}(\tilde{d}_1^{\chi[t]}, \dots, \tilde{d}_N^{\chi[t]})$,

$$\tilde{a}_{ij}^{\chi[t]}[t] := \begin{cases} \phi(|x_i[t] - x_j[t]|), & \text{if } j \in \mathcal{N}_i^{\chi[t]}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\tilde{d}_i^{\chi[t]}[t] := \sum_{j=1}^N \tilde{a}_{ij}^{\chi[t]}[t].$$

Then the matrix form of Eq (2.3)₁ is

$$X[t+1] = X[t] - h\tilde{L}_{\chi[t]}[t]X[t] + hF(X[t], \Theta[t]). \quad (5.4)$$

The solution of Eq (5.4) can be represented as

$$X[t_2] = \Phi_{\chi[t]}[t_2, t_1]X[t_1] + g \sum_{k=0}^{t_2-t_1-1} \Phi_{\chi[t]}[t_2, t_1+k+1]F(X[t_1+k], \Theta[t_1+k]), \quad (5.5)$$

in which $\Phi_{\chi[t]}[t_2, t_1]$ is the transition matrix of $-\tilde{L}_{\chi[t]}[t]$ on time interval $[t_1, t_2]$, i.e.,

$$\Phi_{\chi[t]}[t_2, t_1] = (I - h\tilde{L}_{\chi[t_2-1]}[t_2-1]) \dots (I - h\tilde{L}_{\chi[t_1]}[t_1]). \quad (5.6)$$

Lemma 5.2. *Let the assumptions of Proposition 3.2 hold and $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.3) with communication functions satisfying Eq (2.2) and Assumption 2.1 holds, then for any $t \in \mathbb{N}$ with $t \geq \lambda$, the following assertions hold:*

1. The ergodicity coefficient $\mu(\Phi_{\chi[t]}[t, t - \lambda])$ satisfies

$$\mu(\Phi_{\chi[t]}[t, t - \lambda]) \geq \tilde{c}_2 \phi(2\bar{x})^{N-1},$$

where $\tilde{c}_2$ is defined in Eq (5.1).

2. The state transition matrix $\Phi_{\chi[t]}[t, t - \lambda]$ is stochastic.

Proof. 1. We first consider the matrix $\Phi_{\chi[t]}[t, t - 2\tau_2]$. Note that for all $t \in \mathbb{N}$, we have

$$-\tilde{L}_{\chi[t]}(t) \geq \tilde{A}_{\chi[t]} - N\kappa_1 I,$$

where $\tilde{A}_{\chi[t]} = (a_{ij}^{\chi[t]})$ is given by

$$a_{ij}^{\chi[t]} := \begin{cases} \phi(2\bar{x}), & \text{if } j \in \mathcal{N}_i^{\chi[t]}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \Phi_{\chi[t]}[t, t - 2\tau_2] &= (I - h\tilde{L}_{\chi[t-1]}[t-1]) \dots (I - h\tilde{L}_{\chi[t-2\tau_2]}[t-2\tau_2]) \\ &\geq ((1 - hN\kappa_1)I + h\tilde{A}_{\chi[t-1]}) \dots ((1 - hN\kappa_1)I + h\tilde{A}_{\chi[t-2\tau_2]}) \\ &\geq (1 - hN\kappa_1)^{2\tau_2-1} h(\tilde{A}_{\chi[t-1]} + \dots + \tilde{A}_{\chi[t-2\tau_2]}) \\ &\geq (1 - hN\kappa_1)^{2\tau_2-1} h\phi(2\bar{x})\tilde{F}_1, \end{aligned}$$

where $\tilde{F}_1$ is the $(0, 1)$ -adjacency matrix of the union digraph $\mathcal{G}([t - 2\tau_2, t])$. By Assumption 2.1, the interval $[t - 2\tau_2, t)$ contains at least one subinterval $[t_l, t_{l+1})$ for some $l \in \mathbb{N}$. This implies that $\mathcal{G}(\tilde{F}_1) = \mathcal{G}([t - 2\tau_2, t])$ has a spanning tree. Let $\tilde{F}_i$, $i = 1, 2, \dots, N - 1$ be the $(0, 1)$ -adjacency matrix of the union digraph $\mathcal{G}([t - 2i\tau_2, t - 2(i - 1)\tau_2])$, then for each $1 \leq i \leq N - 1$, $\mathcal{G}(\tilde{F}_i) = \mathcal{G}([t - 2i\tau_2, t - 2(i - 1)\tau_2])$ has a spanning tree, and the following inequality holds:

$$\Phi_{\chi[t]}[t - 2(i - 1)\tau_2, t - 2i\tau_2] \geq (1 - hN\kappa_1)^{2\tau_2-1} h\phi(2\bar{x})\tilde{F}_i. \quad (5.7)$$

With the help of Lemma 3.2, we know that $\tilde{F}_1 \tilde{F}_2 \dots \tilde{F}_{N-1}$ is scrambling, which indicates that $\mu(\tilde{F}_1 \tilde{F}_2 \dots \tilde{F}_{N-1}) \geq 1$. Finally, we use Eq (5.7) to derive

$$\begin{aligned} \mu(\Phi_{\chi[t]}[t, t - \lambda]) &= \mu\left(\prod_{i=1}^{N-1} \Phi_{\chi[t]}[t - 2(i - 1)\tau_2, t - 2i\tau_2]\right) \\ &\geq (1 - hN\kappa_1)^{(2\tau_2-1)(N-1)} h^{N-1} \phi(2\bar{x})^{N-1} \mu\left(\prod_{i=1}^{N-1} \tilde{F}_i\right) \\ &\geq (1 - hN\kappa_1)^{(2\tau_2-1)(N-1)} h^{N-1} \phi(2\bar{x})^{N-1} \\ &= \tilde{c}_2 \phi(2\bar{x})^{N-1}. \end{aligned}$$

2. Let $(1, \dots, 1)^T = \mathbf{1}$. Equation (5.7) gives that $\Phi_{\chi[t]}[t, t - n]$ is non-negative and $\tilde{L}[t]_{\chi[t]} \mathbf{1} = \mathbf{0}$. According to Eq (5.6), we have

$$(1, \dots, 1)^T = \Phi_{\chi[t]}[t, t - n](1, \dots, 1)^T,$$

which gives that $\Phi_{\chi[t]}[t, t - n]$ is stochastic and $0 < \mu(\Phi_{\chi[t]}[t, t - n]) \leq 1$.

□

Lemma 5.3. *Let the hypotheses of Theorem 5.1 be satisfied and $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.3) with communication functions satisfying Eq (2.2), then for any $t \in \mathbb{N}$ with $t \geq \lambda$, we have the following inequality estimate for $D(X(t))$:*

$$D(X[t]) \leq (1 - \tilde{c}_3) \max\{D(X[t - \lambda]), \dots, D(X[t - 1])\} + hL\lambda \max\{D(\Theta[t - \lambda]), \dots, D(\Theta[t - 1])\},$$

where $\tilde{c}_3$ is defined in Eq (5.1).

Proof. We use Eq (5.5), Lemmas 3.1, 5.2, and Proposition 3.1 to complete our proof, which is similar to that of Lemma 4.5. $\square$

Proposition 5.2. *Let the hypotheses of Theorem 5.1 be satisfied and $(x_i[t], \theta_i[t])_{i=1}^N$ be the solution to system (2.3) with communication functions satisfying Eq (2.2), then we have the shrinkage property of $D(X[t])$: for any $t \in \mathbb{N}$ with $t \geq 0$, we have*

$$D(X[t]) \leq \begin{cases} (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor} \max \left\{ \tilde{G}(X[0], \Theta[0]), (1 - \tilde{c}_3) \tilde{G}(X[0], \Theta[0]) + \frac{hL\lambda}{\tilde{c}_3} D(\Theta[0]) \right\} \\ + \frac{hL\lambda}{\tilde{c}_3} D(\Theta[0]) \lfloor \frac{t}{\lambda} \rfloor (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor - 1}, & \text{if } \tilde{c}_1 \psi(2\bar{x})^{N-1} = \tilde{c}_3, \\ (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor} \max \left\{ \tilde{G}(X[0], \Theta[0]), (1 - \tilde{c}_3) \tilde{G}(X[0], \Theta[0]) + \frac{hL\lambda}{\tilde{c}_3} D(\Theta[0]) \right\} \\ + \frac{hL\lambda}{\tilde{c}_3} D(\Theta[0]) \frac{(1 - \tilde{c}_1 \psi(2\bar{x})^{N-1})^{\lfloor \frac{t}{\lambda} \rfloor} - (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor}}{\tilde{c}_3 - \tilde{c}_1 \psi(2\bar{x})^{N-1}}, & \text{if } \tilde{c}_1 \psi(2\bar{x})^{N-1} \neq \tilde{c}_3, \end{cases}$$

here $\tilde{G}(X[0], \Theta[0]) := (1 + hL)^\lambda D(X[0]) + \sum_{m=1}^\lambda (1 + hL)^{m-1} hL D(\Theta[0])$.

Proof. First, from Lemma 4.2, if $\tilde{c}_1 \psi(2\bar{x})^{N-1} = \tilde{c}_3$, we have

$$D(X[t]) \leq (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor} \max \left\{ \bar{D}(X[\lambda]), (1 - \tilde{c}_3) \bar{D}(X[\lambda]) + \frac{hL\lambda}{\tilde{c}_3} \bar{D}(\Theta[\lambda]) \right\} \\ + \frac{hL\lambda}{\tilde{c}_3} \bar{D}(\Theta[\lambda]) \lfloor \frac{t}{\lambda} \rfloor (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor - 1},$$

if $\tilde{c}_1 \psi(2\bar{x})^{N-1} \neq \tilde{c}_3$, we have

$$D(X[t]) \leq (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor} \max \left\{ \bar{D}(X[\lambda]), (1 - \tilde{c}_3) \bar{D}(X[\lambda]) + \frac{hL\lambda}{\tilde{c}_3} \bar{D}(\Theta[\lambda]) \right\} \\ + \frac{hL\lambda}{\tilde{c}_3} \bar{D}(\Theta[\lambda]) \frac{(1 - \tilde{c}_1 \psi(2\bar{x})^{N-1})^{\lfloor \frac{t}{\lambda} \rfloor} - (1 - \tilde{c}_3)^{\lfloor \frac{t}{\lambda} \rfloor}}{\tilde{c}_3 - \tilde{c}_1 \psi(2\bar{x})^{N-1}}.$$

Next, we estimate $\bar{D}(\Theta[\lambda])$ and $\bar{D}(X[\lambda])$. From Eq (4.15), we have

$$\bar{D}(\Theta[\lambda]) = \max\{D(\Theta[0]), \dots, D(\Theta[\lambda])\} = D(\Theta[0]).$$

Similar to the proof of Proposition 4.2, we have

$$D(X[t]) \leq (1 + hL)^t D(X[0]) + \sum_{m=1}^t (1 + hL)^{m-1} hL D(\Theta[0]), \quad t \in [0, \lambda].$$

For convenience of calculation, we denote

$$\tilde{G}(X[0], \Theta[0]) := (1 + hL)^\lambda D(X[0]) + \sum_{m=1}^{\lambda} (1 + hL)^{m-1} hLD(\Theta[0]),$$

then we can obtain

$$\bar{D}(X[\lambda]) = \max\{D(X[0]), \dots, D(X[\lambda])\} \leq \tilde{G}(X[0], \Theta[0]).$$

Substituting the upper bound estimate of $\bar{D}(\Theta[\lambda])$ and $\bar{D}(X[\lambda])$ to get the result. $\square$

5.4. Proof of Theorem 5.1

We use Propositions 5.1 and 5.2 to know that as $t \rightarrow \infty$, the sufficient condition for consensus is that the constant $\tilde{c}_3$ defined in Eq (5.1) should satisfy $\tilde{c}_3 > 0$, which is equivalent to

$$(h\phi(2\bar{x}))^{N-1}(1 - hN\kappa_1)^{(2\tau_2-1)(N-1)} \frac{1}{h\lambda} > \max\{\sigma\bar{x}, \sigma(\bar{\theta} + 3\bar{x}^2)\},$$

hence we complete the proof.

6. Numerical simulations

In this section, we provide several numerical simulations to illustrate the consensus behavior of our proposed discrete opinion model under both fixed and switching topologies. In the settings of both fixed and switching topologies, we consider a group of five agents, i.e., $N = 5$, and employ the same communication functions, given by

$$\phi(r) := \frac{\kappa_1}{\ln(r^2 + e)}, \quad \psi(r) := \frac{\kappa_2}{\sqrt{1 + r^2}}, \quad r \in [0, \infty).$$

In the fixed topology context, the interaction topology with self-loop at each vertex is shown in Figure 1.

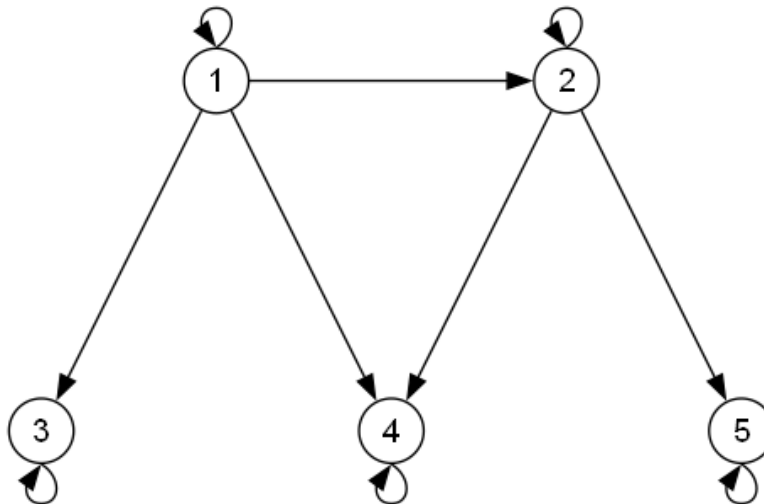


Figure 1. Fixed interaction topology.

We can see that vertex 1 is the only root, and $\gamma_g = 2$, $n_r = 1$. First, we present a scenario where consensus is achieved under the fixed topology context. We set $\kappa_1 = 0.1$, $\kappa_2 = 0.01$, $n = 2$, the initial opinions $X[0] = (0.1, 0.8, 0.2, 0.5, 0.6)^T$ and the initial convictions $\Theta[0] = (0.2, 0.5, 0.1, 0.7, 0.3)^T$, respectively. This yields $\bar{x} = \sqrt{0.7}$, $\bar{\theta} = 0.7$, and $h < 2$. Choosing $h = 0.2$ yields the calculated bound $\sigma < \frac{0.9}{0.28} \approx 3.2143$ in Proposition 3.2. Under this constraint, we set $\sigma = 3$. The consensus result is shown in Figure 2. It is worth noting that although the chosen parameters do not satisfy the inequality in Theorem 4.1, consensus still occurs. This indicates that the proposed condition in Theorem 4.1 is merely a sufficient condition for consensus.

However, the parameter σ cannot be arbitrarily large. Keeping all other parameters unchanged, we set the strength coefficient σ to $\sigma = 18$, which violates our computed bound on σ . As illustrated in Figure 3, consensus is not reached in this case.

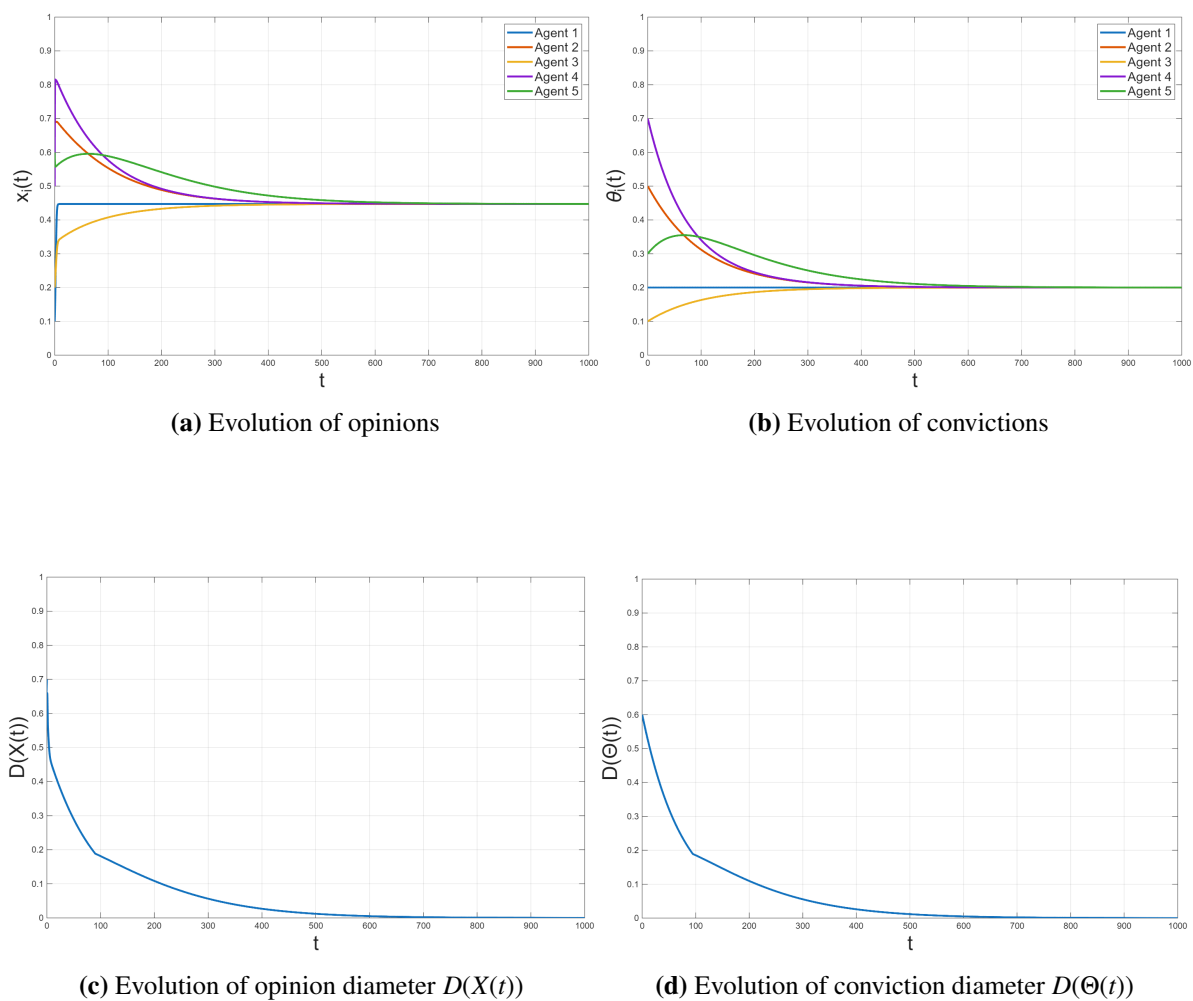


Figure 2. Consensus in fixed topology context.

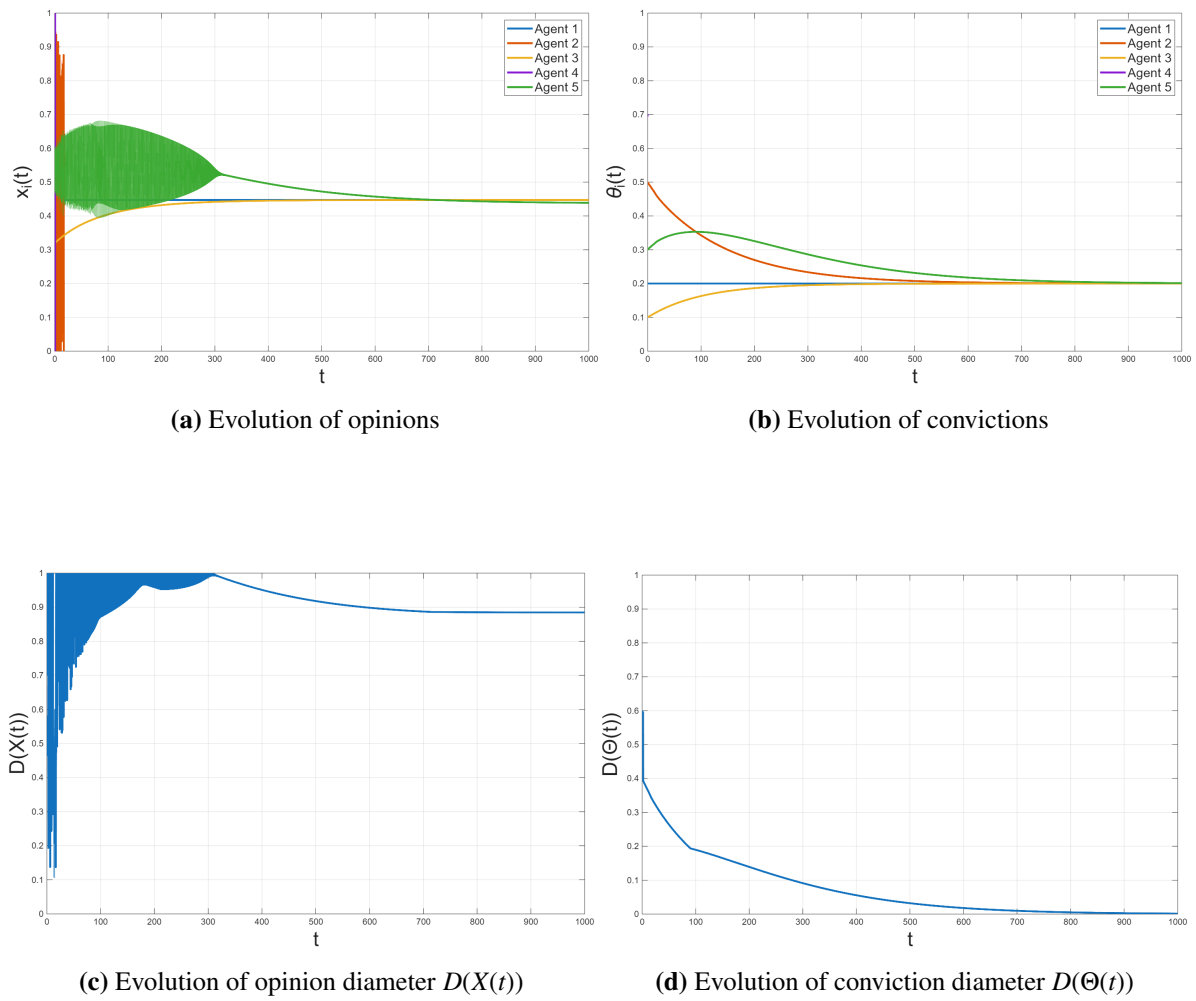


Figure 3. Non-consensus behavior in fixed topology context.

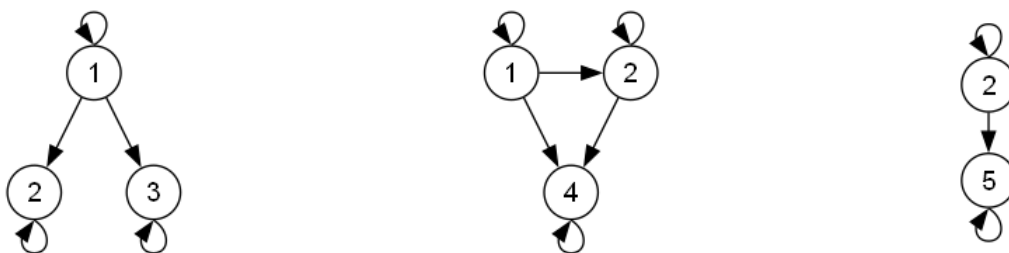


Figure 4. Set of switching topologies.

In the switching context, we split Figure 1 into three subfigures shown in Figure 4, where each vertex in every subfigure is still assumed to have a self-loop. We can see that the union of these three subfigures in Figure 4 has a spanning tree. We set $\tau_1 = 2$ and $\tau_2 = 6$. Starting from $t = 0$, the system cyclically switches among the topologies shown in Figure 4, with each topology remaining active for a dwell time of $\tau_1 h$. Under this setup, the union of graphs over any time interval of length $\tau_2 h$ contains a spanning tree. As in the fixed topology case, we maintain $\kappa_1 = 0.1$ and $\kappa_2 = 0.01$, which again yields the step size bound $h < 2$. By choosing $h = 0.2$, the condition in Theorem 5.1 further gives the following sufficient constraint on σ :

$$\sigma < 6.78 \times 10^{-12}.$$

This result is extremely conservative. We emphasize this condition is only a sufficient condition for achieving consensus, rather than a necessary one. In practice, consensus can still be achieved even when σ does not satisfy this theoretical bound. Setting $\sigma = 3$, the consensus result under switching topologies is shown in Figure 5.

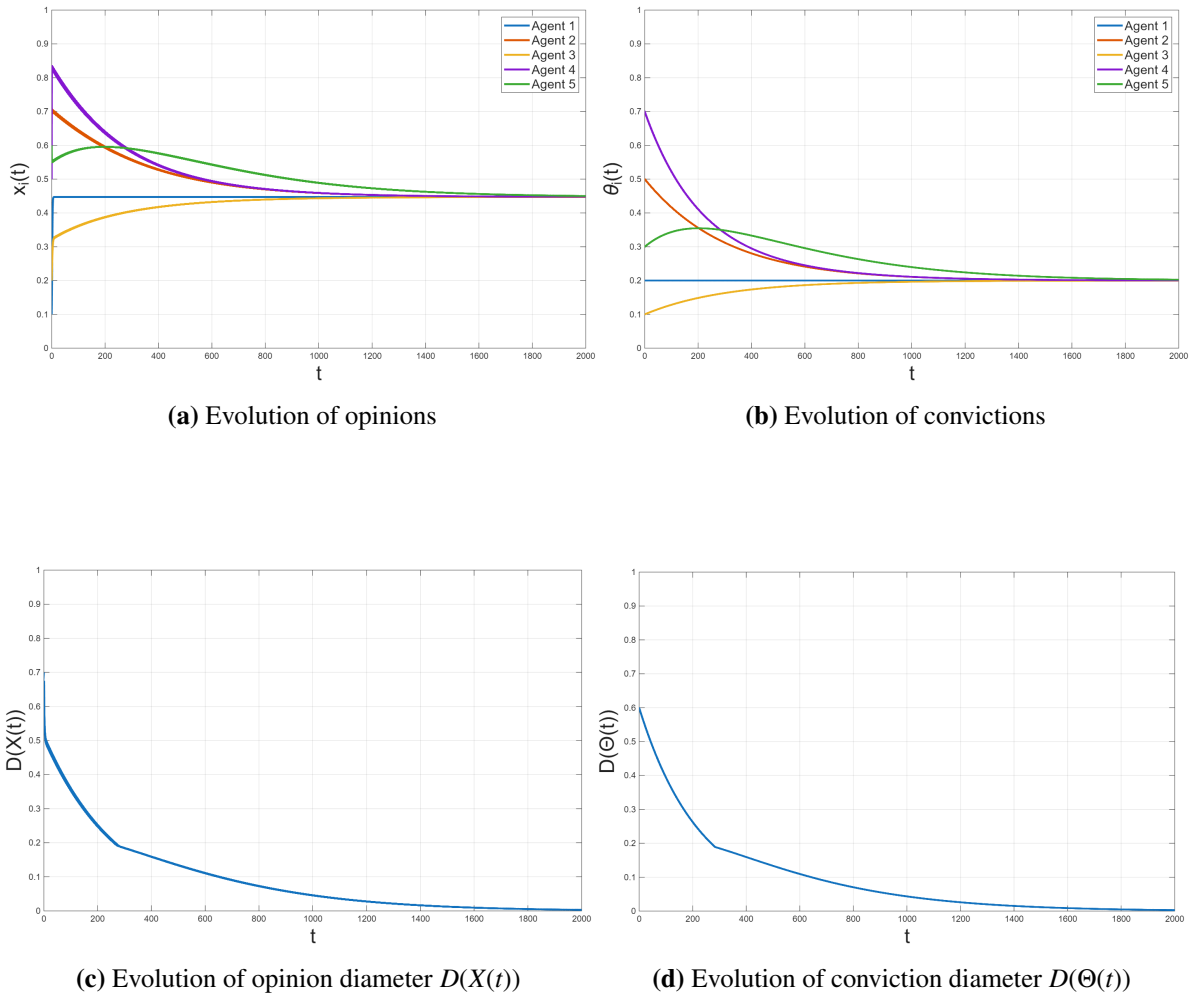


Figure 5. Consensus in switching context.

Next, keeping all other parameters unchanged, we set $\sigma = 26$. In this case, consensus is not reached, as illustrated in Figure 6, which also indicates that the parameter σ cannot be arbitrarily large. Therefore, relaxing the sufficient conditions for consensus and characterizing the conditions under which consensus fails to occur are interesting directions for future discussion.

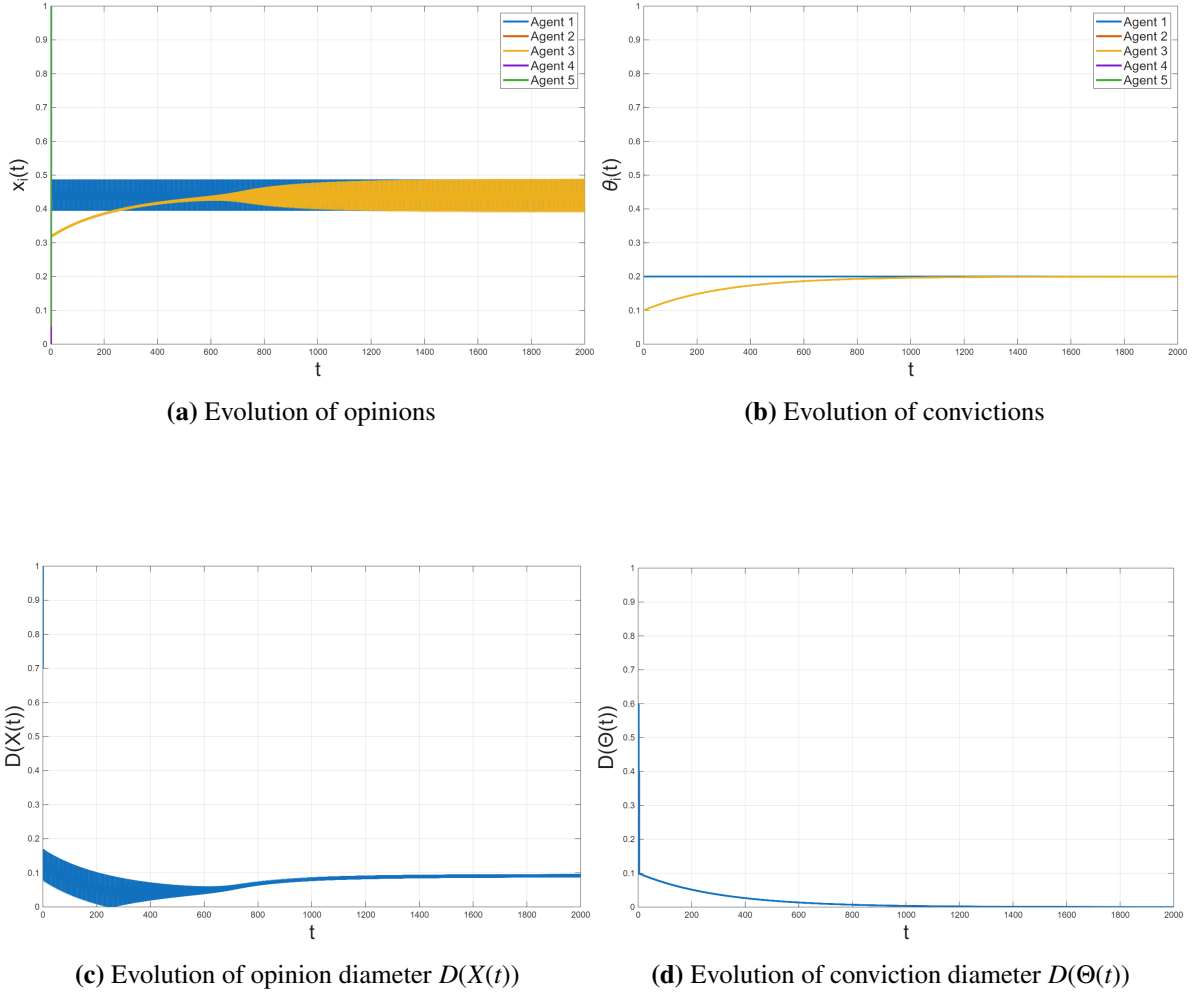


Figure 6. Non-consensus behavior in switching context.

7. Conclusions

This paper investigates the discrete-time consensus dynamics of an opinion model with stubbornness under fixed and switching topologies.

For fixed topology, we establish sufficient consensus conditions when the underlying digraph contains a rooted spanning tree. For switching topologies, consensus is achieved if the union of interaction graphs over every bounded time interval possesses a directed spanning tree. We prove uniform boundedness for agent opinions and conviction values, and analyse the diameter decay of opinion and conviction states via scrambling matrices and stochastic state transition matrices.

Numerical simulations illustrate the theoretical results. We observe that the derived theoretical

criteria are only sufficient rather than necessary: Consensus can still occur even if the sufficient inequalities are not satisfied. Meanwhile, large stubbornness strength will prevent the group from reaching consensus.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

Dianxingzi Chen: Investigation, Study design, Manuscript writing. Linglong Du: Conceptualization, Methodology, Supervision, Manuscript writing, review. Yu Huang: Conceptualization, Software, Manuscript writing.

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