



Research article

Asymptotic behavior for a time-dependent semilinear reaction–diffusion equation with a mixed local–nonlocal operator

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Abstract: In this paper, we consider a semilinear reaction–diffusion equation with a mixed local–nonlocal spatial operator and a time integral operator. The equation involves the mixed local–nonlocal Laplacian operator $\mathcal{L} = -\Delta + (-\Delta)^{\alpha/2}$ and the Riemann–Liouville fractional integral operator $I_{0t}^{1-\gamma}$. Our main goal is to characterize the large–time asymptotic behavior of global mild solutions.

Keywords: global mild solution; anomalous diffusion; reaction–diffusion; mixed local–nonlocal Laplacian operator; nonlinear memory

1. Introduction

Anomalous diffusion processes in heterogeneous media have been widely studied in mathematics and physics [1–3]. In this paper, we study the Cauchy problem for a semilinear delay reaction–diffusion equation with a mixed local–nonlocal spatial operator and nonlinear memory:

$$\begin{cases} \partial_t u(x, t) + t^\beta \mathcal{L}u(x, t) = -h(t)I_{0t}^{1-\gamma}(|u|^{p-1})u(x, t), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $p > 1$, $\beta \geq 0$, $0 < h \in L^\infty[0, +\infty)$, and $u_0 \in (L^1 \cap C_0^\infty)(\mathbb{R}^N)$, $C_0^\infty(\mathbb{R}^N)$ is the space of continuous functions on $\mathbb{R}^N$ vanishing at infinity. The mixed local–nonlocal Laplacian operator is given by $\mathcal{L} = -\Delta + (-\Delta)^{\alpha/2}$. This operator appears naturally in applied sciences and probability theory. It describes the superposition of different types of stochastic processes, such as classical random walks and Lévy flights. Here, $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian of order $0 < \alpha < 2$, and $I_{0t}^{1-\gamma}$ is the Riemann–Liouville fractional integral operator defined by

$$I_{0t}^{1-\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} f(\tau) d\tau, \quad \forall f \in L^q(0, T) \ (1 \leq q \leq \infty), \quad (1.2)$$

the nonlinear memory term

$$I_{0|t}^{1-\gamma}(|u|^{p-1}) = g_{1-\gamma}(t) \star |u|^{p-1}(t) = \int_0^t g_{1-\gamma}(t-\tau)|u|^{p-1}(\tau, x) d\tau \quad (1.3)$$

is a term with convolution kernel $g_{1-\gamma}(t)$. Here, the Gelfand–Shilov function $g_\gamma(t) = t^{\gamma-1}/\Gamma(\gamma) \in L_{Loc}^1(0, +\infty)$ ($\gamma > 0$) is nonincreasing and satisfies the convolution semigroup property $g_\alpha * g_\beta = g_{\alpha+\beta}$, where $*$ denotes convolution in time. The gamma function is defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$. Because the limit $\lim_{\gamma \rightarrow 1} s_+^{-\gamma} = \Gamma(1-\gamma)\delta(s)$ exists in the distributional sense [4], $I_{0|t}^{1-\gamma}(|u|^{p-1})u$ reduces to the local term $|u|^{p-1}u$ as $\gamma \rightarrow 1$. Thus, the convolution kernel $g_{1-\gamma}$ is weakly singular.

In a recent paper, Kirane et al. [5] considered the Cauchy problem of the reaction–diffusion equation driven by both the classical and anomalous diffusion

$$\begin{cases} \partial_t u(x, t) + t^\beta \mathcal{L}u(x, t) = -h(t)u^p, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \quad (1.4)$$

They showed that the nonlinear term determines the large–time asymptotics when $1 < p \leq 1 + \alpha/(N(\beta + 1))$, whereas the diffusion effects governed by Δ and $(-\Delta)^{\alpha/2}$ dominate when $p > 1 + \alpha/(N(\beta + 1))$. The critical exponent is $p_c = 1 + \alpha/(N(\beta + 1))$. The cases $\alpha = 2$ and $0 < \alpha < 2$ correspond to local diffusion and nonlocal diffusion, respectively. Compared with Brownian motion, which corresponds to $\alpha = 2$, symmetric α –stable processes with $0 < \alpha < 2$ exhibit jumps in any time interval. Motivated by applications such as semiconductor growth [6], the asymptotic properties of solutions involving local and nonlocal operators have attracted considerable attention.

For $\beta = 0$ and $h(t) \equiv 1$, Fino and Karch [7] proved that the critical Fujita exponent for the large time behavior of the fractional operator $(-\Delta)^{\alpha/2}$ is $p_F = 1 + \alpha/N$.

Fino and Kirane [8] investigated the spatiotemporally nonlinear parabolic equation

$$\begin{cases} \partial_t u(x, t) + (-\Delta)^{\alpha/2} u(x, t) = I_{0|t}^{1-\gamma}(|u|^{p-1}u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.5)$$

where $p > 1$, $0 < \alpha \leq 2$, $0 < \gamma < 1$, and $u_0 \in C_0(\mathbb{R}^N)$. The nonlocal operator $(-\Delta)^{\alpha/2}$ is defined by

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u)(\xi))(x), & \forall u \in \mathcal{D}((-\Delta)^{\alpha/2}), \\ \mathcal{D}((-\Delta)^{\alpha/2}) = H^\alpha(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \mid |\xi|^\alpha \mathcal{F}(u) \in L^2(\mathbb{R}^N)\}. \end{cases} \quad (1.6)$$

Here, $\mathcal{F}$ denotes the Fourier transform, and $\mathcal{F}^{-1}$ denotes its inverse.

For the initial value problem (IVP) of the semilinear reaction–diffusion equation with mixed local–nonlocal spatial operator $\mathcal{L} = -\Delta + (-\Delta)^{\alpha/2}$

$$\begin{cases} \partial_t u(x, t) + \mathcal{L}u(x, t) = u^p, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.7)$$

many results have been given.

For the purely local spatial differential operator $\mathcal{L} = -\Delta$, Fujita [9], Hayakawa [10], and Kobayashi et al. [11] established that Eq (1.7) has the Fujita phenomenon: (a) If $1 < p \leq 1 + 2/N$, every

solution of Eq (1.7) blows up in finite time with $u_0 \neq 0$; (b) if $p > 1 + 2/N$, there exists a global-in-time solution of Eq (1.7) for sufficiently small u_0 , and the critical Fujita exponent is $p = 1 + 2/N$.

For the nonlocal spatial operator $\mathcal{L} = (-\Delta)^{\alpha/2}$, Sugitani [12] and Fino et al. [13] proved that the critical Fujita exponent of Eq (1.7) is $p = 1 + \alpha/N$. This indicates that the IVP with $\mathcal{L} = -\Delta + (-\Delta)^{\alpha/2}$ has the same critical behavior as the IVP with $\mathcal{L} = (-\Delta)^{\alpha/2}$.

In a recent and closely related article, Kumar and Torebek [14] discussed the Fujita critical exponent for equations $u_t(x, t) + \mathcal{L}_{a,b}u(x, t) = |u|^p + f(x)$, where with the mixed local-nonlocal operator $\mathcal{L}_{a,b} = -a\Delta + b(-\Delta)^{\alpha/2}$ with $a, b \in \mathbb{R}_+$. They obtained that the critical exponent is determined by the nonlocal component of the operator and, as a result, coincides with that of the fractional Laplacian.

The operator $(-\Delta)^{\alpha/2}$ is a positive definite self-adjoint operator on $L^2(\mathbb{R}^N)$. It generates a strongly continuous semigroup $T(t) = e^{-t(-\Delta)^{\alpha/2}}$ on $L^2(\mathbb{R}^N)$; see [15] or [16, Theorem 4.9]:

$$T(t)u_0 = P_\alpha(t) * u_0 = \int_{\mathbb{R}^N} P_\alpha(x - y, t)u_0(y)dy, \quad (1.8)$$

where $*$ denotes convolution in $\mathbb{R}^N$. Here, $P_\alpha(t)$ is the heat kernel associated with $\partial_t + (-\Delta)^{\alpha/2}$, that is, the fundamental solution of the fractional diffusion equation

$$\partial_t u + (-\Delta)^{\alpha/2} u(x, t) = 0.$$

It is defined via the Fourier transform by

$$P_\alpha(t)(x) = P_\alpha(x, t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi - t|\xi|^\alpha} d\xi. \quad (1.9)$$

Lemma 1. [16] *The fractional heat kernel $P_\alpha(t)$ ($0 < \alpha \leq 2$) has the following properties.*

- (i) $P_\alpha \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$, $P_\alpha(t) \geq 0$, $\forall x \in \mathbb{R}^N$, $\forall t > 0$.
- (ii) $P_\alpha(x, 1) \in L^1 \cap L^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} P_\alpha(x, t)dx = 1$, $\forall t > 0$.
- (iii) *Similarity:* $P_\alpha(x, t) = t^{-N/\alpha} P_\alpha(xt^{-1/\alpha}, 1)$.
- (iv) $\forall v \in L^r(\mathbb{R}^N)$, $1 \leq r \leq q \leq \infty$, $\forall t > 0$,

$$\|P_\alpha(t) * v\|_q \leq Ct^{-\frac{N}{\alpha}(\frac{1}{r} - \frac{1}{q})} \|v\|_r. \quad (1.10)$$

- (v) $\forall 1 \leq q \leq \infty$, $\forall t > 0$,

$$\|\nabla P_\alpha(t)\|_q \leq Ct^{-\frac{N}{\alpha}(1 - \frac{1}{q}) - \frac{1}{\alpha}}. \quad (1.11)$$

The mixed local-nonlocal operator $\mathcal{L} = -\Delta + (-\Delta)^{\alpha/2}$ generates a strongly continuous contractive semigroup $S(t) = e^{-t\mathcal{L}}$ on $L^2(\mathbb{R}^N)$, and

$$S(t)u_0 = E_\alpha(t) * u_0 = \int_{\mathbb{R}^N} E_\alpha(x - y, t)u_0(y)dy, \quad (1.12)$$

where $E_\alpha(t) = P_\alpha(t) * P_2(t)$ is the heat kernel associated with $\partial_t + \mathcal{L}$; that is, the fundamental solution of $\partial_t u + \mathcal{L}u = 0$, and $P_2(x, t)$ is the Gaussian heat kernel of the operator $\partial_t - \Delta$, that is:

$$P_2(t)(x) = P_2(x, t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}. \quad (1.13)$$

Using the semigroup $S(t) = e^{-t\mathcal{L}}$ or equivalently the fundamental solution $E_\alpha(t)$, Kirane et al. [5, Lemma 7] gave the solution of the Cauchy problem

$$\begin{cases} u_t(x, t) + t^\beta \mathcal{L}u(x, t) = 0, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \quad (1.14)$$

as

$$u(t) = E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * u(s), \quad \forall t \geq s \geq 0. \quad (1.15)$$

Lemma 2. [17] *The mixed local–nonlocal heat kernel $E_\alpha(t)$ has the following properties.*

(i) $E_\alpha \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$, $E_\alpha(t) \geq 0$, $\forall x \in \mathbb{R}^N$, $\forall t > 0$.

(ii) $\int_{\mathbb{R}^N} E_\alpha(x, t) dx = 1$, $\forall t > 0$.

(iii) $\forall t > 0, \tau > 0$, $E_\alpha(t) * E_\alpha(\tau) = E_\alpha(t + \tau)$.

(iv) $\forall v \in L^r(\mathbb{R}^N)$, $1 \leq r \leq q \leq \infty$, $\forall t > 0$,

$$\|E_\alpha(t) * v\|_q \leq C \min \left\{ t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})}, t^{-\frac{N}{\alpha}(\frac{1}{r}-\frac{1}{q})} \right\} \|v\|_r. \quad (1.16)$$

(v) $\forall 1 \leq q \leq \infty$, $\forall t > 0$,

$$\|\nabla E_\alpha(t)\|_q \leq C \min \left\{ t^{-\frac{N}{2}(1-\frac{1}{q})-\frac{1}{2}}, t^{-\frac{N}{\alpha}(1-\frac{1}{q})-\frac{1}{\alpha}} \right\}. \quad (1.17)$$

(vi) Let $g \in L^1(\mathbb{R}^N)$, and $M_g = \int_{\mathbb{R}^N} g(x) dx$. Then,

$$\lim_{t \rightarrow +\infty} \|E_\alpha(t) * g - M_g E_\alpha(t)\|_1 = 0. \quad (1.18)$$

(vii) Let $g, xg \in L^1(\mathbb{R}^N)$, and $M_g = \int_{\mathbb{R}^N} g(x) dx$. Then,

$$\|E_\alpha(t) * g - M_g E_\alpha(t)\|_1 \leq C \min \left\{ t^{-\frac{1}{2}}, t^{-\frac{1}{\alpha}} \right\} \|xg\|_1, \quad \forall t > 0. \quad (1.19)$$

The kernels in Eqs (1.2) and (1.3) are singular in time. Consequently, the nonlinear memory term in Eq (1.1) is singular, and the weak singularity makes the mild–solution framework more convenient. We therefore consider mild solutions of the IVP in Eq (1.1).

Definition 1. Let $u_0 \in C_0(\mathbb{R}^N)$, $\alpha \in (0, 2)$, $\beta \geq 0$, $0 < \gamma < 1$, $p > 1$, $h : [0, +\infty) \rightarrow (0, +\infty)$, $h \in L^1_{loc}[0, +\infty)$, and $T > 0$. We say that $u \in C([0, T], C_0(\mathbb{R}^N))$ is a mild solution of the IVP in Eq (1.1) if u satisfies the following integral equation for $t \in [0, T]$:

$$u(t) = E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0 - \int_0^t h(s) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * [I^{1-\gamma}_{0|s}(|u|^{p-1})u(s)] ds. \quad (1.20)$$

The rest of this paper is organized as follows. Section 2 presents the main results. Section 3 proves the global existence result. Section 4 is devoted to the proof of the asymptotic behavior. Throughout this paper, $\|\cdot\|_q$ ($1 \leq q \leq +\infty$) denotes the usual $L^q(\mathbb{R}^N)$ -norm, and $a \lesssim b$ means that there exists a constant $C > 0$ such that $a \leq Cb$.

2. Main results

In this section, we state the main results concerning the existence, uniqueness, and asymptotic behavior of solutions to the problem in Eq (1.1), where $0 < \alpha < 2$, $\beta \geq 0$, and $0 < \gamma < 1$. First, following the arguments in [8, 18], we obtain the global existence, uniqueness, and regularity of mild solutions.

Theorem 1. (*Global existence*) Let $0 \leq u_0 \in (L^1 \cap C_0^\infty)(\mathbb{R}^N)$, $\alpha \in (0, 2)$, $\beta \geq 0$, $0 < \gamma < 1$, and $0 < h \in L^\infty[0, +\infty)$. Assume that $p > \max\{1 + \alpha(2 - \gamma)/(N(\beta + 1)), 1/\gamma\}$. Denote $p_0 := N(\beta + 1)(p - 1)/(\alpha(2 - \gamma))$. If $\|u_0\|_{p_0}$ is sufficiently small, then the Cauchy problem in Eq (1.1) has a unique non-negative global mild solution $u \in C([0, \infty), (L^1 \cap C_0)(\mathbb{R}^N))$. Moreover, for every $0 < a < T < \infty$, the mild solution satisfies $u \in H^1(a, T; L^2(\mathbb{R}^N)) \cap L^2(a, T; H^2(\mathbb{R}^N))$. In particular, we have $u \in H_{\text{loc}}^1((0, \infty); L^2(\mathbb{R}^N)) \cap L_{\text{loc}}^2((0, \infty); H^2(\mathbb{R}^N))$. The equation holds in $L^2(\mathbb{R}^N)$ for almost everywhere (a.e.) $t > 0$.

For a non-negative nontrivial global mild solution $u(x, t)$ of the Cauchy problem in Eq (1.1), we denote by

$$M(t) = \int_{\mathbb{R}^N} u(x, t) dx$$

the mass associated with Eq (1.1), and we set $M_\infty = \lim_{t \rightarrow +\infty} M(t)$, provided that the limit exists. The following result is conditional on the existence of a non-negative global mild solution.

Theorem 2. (*Asymptotic behavior*) Assume that the Cauchy problem in Eq (1.1) has a nonnegative nontrivial global mild solution $u(x, t)$. Let $0 < h(t) \in L^\infty[0, +\infty)$, $0 \leq u_0 \in (L^1 \cap C_0^\infty)(\mathbb{R}^N)$. If

$$p > \max\left\{1 + \frac{\alpha(2 - \gamma)}{N(\beta + 1)}, \frac{1}{\gamma}\right\}, \quad \int_1^{+\infty} h(t) dt < +\infty, \quad (2.1)$$

then $M_\infty > 0$. Moreover, $\forall q \in [1, +\infty)$, we have

$$t^{\frac{N}{\alpha}(1 - \frac{1}{q})(\beta + 1)} \left\| u(t) - M_\infty E_\alpha\left(\frac{t^{\beta + 1}}{\beta + 1}\right) \right\|_q \rightarrow 0 \quad (t \rightarrow +\infty). \quad (2.2)$$

3. Global existence

Theorem 1 establishes the global existence of mild solutions to the IVP in Eq (1.1). In this section, we prove Theorem 1.

Proof of Theorem 1. We divide the proof into several parts.

I. Local existence.

Let $u_0 \in C_0^\infty(\mathbb{R}^N)$, and $T > 0$. Define the Banach space

$$X_T = \left\{ u \in L^\infty((0, T), C_0^\infty(\mathbb{R}^N)) \mid \|u\| \leq 2\|u_0\|_\infty \right\},$$

where $\forall u \in X_T$,

$$\|\cdot\|_\infty = \|\cdot\|_{C_0^\infty(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \cdot(x), \quad \|\cdot\| = \|\cdot\|_{L^\infty((0, T), C_0^\infty(\mathbb{R}^N))} = \text{ess sup}_{t \in (0, T)} \sup_{x \in \mathbb{R}^N} \cdot(x, t).$$

For $u \in X_T$ and $t \in [0, T]$, define the map

$$\Phi(u)(t) = E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0 - \int_0^t h(s) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * \left[I_{0|s}^{1-\gamma}(|u|^{p-1})u\right] ds.$$

A. $\Phi : X_T \rightarrow X_T$. Let $u \in X_T$. Using the properties of fundamental solution $E_\alpha(x, t)$ associated with the operator $\partial_t + \mathcal{L}$ given in Lemma 2, we have

$$\begin{aligned} \|\Phi(u)(t)\|_\infty &\leq \|E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0\|_\infty \\ &\quad + \|h\|_\infty \int_0^t \|E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * \left[I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)\right]\|_\infty ds \\ &\leq \|E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0\|_\infty + \|h\|_\infty \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)\|_\infty ds \\ &\leq \|u_0\|_\infty + C\|u\| \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})\|_\infty ds \\ &\leq \|u_0\|_\infty + C\|u\| \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|u(\tau)\|_\infty^{p-1} d\tau ds \\ &\leq \|u_0\|_\infty + \frac{C}{\Gamma(1-\gamma)} \|u\|^p \int_0^t s^{1-\gamma} ds. \end{aligned}$$

Because $\|u\| \leq 2\|u_0\|_\infty$, we have

$$\|\Phi(u)(t)\|_\infty \leq \|u_0\|_\infty + \frac{C}{\Gamma(3-\gamma)} T^{2-\gamma} 2^p \|u_0\|_\infty^p.$$

Choose $T > 0$ sufficiently small so that $CT^{2-\gamma}2^p\|u_0\|_\infty^{p-1}/\Gamma(3-\gamma) \leq 1$, so $\|\Phi(u)(t)\|_\infty \leq 2\|u_0\|_\infty$, and $\|\Phi(u)\| \leq 2\|u_0\|_\infty$. Because $S(t)$ is a strongly continuous semigroup on $C_0^\infty(\mathbb{R}^N)$, hence we have $\Phi(u) \in X_T$.

B. Φ is contractive for $p \geq 2$ and has Hölder continuity for $1 < p < 2$.

$\forall u, v \in X_T$, we have

$$\begin{aligned} &\|\Phi(u)(t) - \Phi(v)(t)\|_\infty \\ &\leq \int_0^t h(s) \|E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * \left[I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v\right]\|_\infty ds \\ &\leq C\|h\|_\infty \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v\|_\infty ds, \end{aligned} \tag{3.1}$$

as

$$\begin{aligned} &I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v \\ &= I_{0|s}^{1-\gamma}(|u|^{p-1})(u - v) + [I_{0|s}^{1-\gamma}(|u|^{p-1}) - I_{0|s}^{1-\gamma}(|v|^{p-1})]v, \\ &\|I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v\|_\infty \\ &\leq \|I_{0|s}^{1-\gamma}(|u|^{p-1})\|_\infty \|u - v\|_\infty + \|I_{0|s}^{1-\gamma}(|u|^{p-1}) - I_{0|s}^{1-\gamma}(|v|^{p-1})\|_\infty \|v\|_\infty. \end{aligned}$$

We then have the estimate of the integral

$$\begin{aligned} & \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v\|_\infty ds \\ & \leq \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})\|_\infty \|(u-v)(s)\|_\infty ds + \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1} - |v|^{p-1})\|_\infty \|v(s)\|_\infty ds, \end{aligned} \quad (3.2)$$

and furthermore,

$$\begin{aligned} & \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v\|_\infty ds \\ & \leq \|u-v\| \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})\|_\infty ds + \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1} - |v|^{p-1})\|_\infty \|v(s)\|_\infty ds \\ & \leq \|u-v\| \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|u(s)\|_\infty^{p-1} d\tau ds \\ & \quad + \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|(|u|^{p-1} - |v|^{p-1})(\tau)\|_\infty \|v(s)\|_\infty d\tau ds. \end{aligned} \quad (3.3)$$

For $p \geq 2$, $\left| |a|^{p-1} - |b|^{p-1} \right| \leq C_p(|a|^{p-2} + |b|^{p-2})|a-b|$, and thus,

$$\begin{aligned} & \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|(|u|^{p-1} - |v|^{p-1})(\tau)\|_\infty \|v(s)\|_\infty d\tau ds \\ & \leq C_p(\|u\|^{p-2} + \|v\|^{p-2})\|v\| \|u-v\| \int_0^T \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} d\tau ds. \end{aligned}$$

Using the Hölder inequality, and putting Eqs (3.2) and (3.3) into Eq (3.1),

$$\begin{aligned} \| \Phi(u) - \Phi(v) \| & \leq C(\|u\|^{p-1} + \|v\|^{p-1})\|u-v\| \int_0^T \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} d\tau ds \\ & \leq \frac{C(p)}{\Gamma(3-\gamma)} T^{2-\gamma} 2^p \|u_0\|_\infty^{p-1} \|u-v\|. \end{aligned}$$

Choosing $T > 0$ small enough such that $C(p)T^{2-\gamma}2^p\|u_0\|_\infty^{p-1}/\Gamma(3-\gamma) \leq 1/2$, we have

$$\| \Phi(u) - \Phi(v) \| \leq \frac{1}{2} \|u-v\|,$$

so Φ is contractive. By the Banach fixed-point theorem, there exists a mild solution $u \in L^\infty((0, T), L^\infty(\mathbb{R}^N))$ such that Eq (1.20) holds true.

For $1 < p < 2$, $\left| |a|^{p-1} - |b|^{p-1} \right| \leq C_p|a-b|^{p-1}$; thus,

$$\begin{aligned} & \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|(|u|^{p-1} - |v|^{p-1})(\tau)\|_\infty \|v(s)\|_\infty d\tau ds \\ & \leq C_p \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|(u-v)(\tau)\|_\infty^{p-1} \|v(s)\|_\infty d\tau ds \end{aligned}$$

$$\begin{aligned}
&\leq C \|v\| \|u - v\|^{p-1} \int_0^T \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} d\tau ds \\
&\leq \frac{C(p)}{\Gamma(3-\gamma)} T^{2-\gamma} 2 \|u_0\| \|u - v\|^{p-1}.
\end{aligned}$$

Equations (3.1) and (3.3) imply that

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\| &\leq \frac{C(p)T^{2-\gamma}}{\Gamma(3-\gamma)} \left[2^p \|u_0\|_\infty^{p-1} \|u - v\| + 2 \|u_0\| \|u - v\|^{p-1} \right] \\
&\leq \frac{C(p, \|u_0\|)}{\Gamma(3-\gamma)} T^{2-\gamma} \|u - v\|^{p-1}.
\end{aligned}$$

The nonlinear term only yields the Hölder continuity with exponent $p - 1 \in (0, 1)$ rather than Lipschitz continuity. This sublinear estimate fails to satisfy the linear contraction condition, which implies that the Banach fixed-point theorem cannot be directly applied here. To overcome this difficulty, we adopt a classical framework: the Schauder fixed-point theorem for solution existence and the Bihari nonlinear Grönwall inequality for solution uniqueness.

II. Uniqueness.

If $u, v \in X_T$ are two mild solutions of Eq (1.1) for some $T > 0$, then from Eq (1.20) we have

$$\begin{aligned}
\|u(t) - v(t)\|_\infty &\leq \int_0^t h(s) \|E_\alpha(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}) * [I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v]\|_\infty ds \\
&\leq C \|h\|_\infty \int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})u - I_{0|s}^{1-\gamma}(|v|^{p-1})v\|_\infty ds.
\end{aligned} \tag{3.4}$$

To estimate the last integral, we divide it into two parts similar to Eq (3.2). For $u, v \in X_T$, we consider the first integral in Eq (3.2),

$$\begin{aligned}
\int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1})\|_\infty \|(u-v)(s)\|_\infty ds &\leq \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|u(\tau)\|_\infty^{p-1} \|u(s) - v(s)\|_\infty d\tau ds \\
&\leq \frac{\|u\|^{p-1}}{\Gamma(1-\gamma)} \int_0^t \int_0^s (s-\tau)^{-\gamma} \|u(s) - v(s)\|_\infty d\tau ds \\
&\leq \frac{2^{p-1} \|u_0\|^{p-1}}{\Gamma(2-\gamma)} \int_0^t s^{1-\gamma} \|u(s) - v(s)\|_\infty ds.
\end{aligned} \tag{3.5}$$

For the second integral in Eq (3.2), we also study two cases: $p \geq 2$ and $1 < p < 2$.

Now, for $p \geq 2$, $\left| |a|^{p-1} - |b|^{p-1} \right| \leq C_p (|a|^{p-2} + |b|^{p-2}) |a - b|$, and thus,

$$\begin{aligned}
&\int_0^t \|I_{0|s}^{1-\gamma}(|u|^{p-1} - |v|^{p-1})\|_\infty \|v(s)\|_\infty ds \\
&\leq \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|(|u|^{p-1} - |v|^{p-1})(\tau)\|_\infty \|v(s)\|_\infty d\tau ds \\
&\leq C_p (\|u\|^{p-2} + \|v\|^{p-2}) \|v\| \int_0^t \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \|u(\tau) - v(\tau)\|_\infty d\tau ds \\
&\leq \frac{C_p 2^p \|u_0\|_\infty^{p-1}}{\Gamma(1-\gamma)} \int_0^t \int_0^s (s-\tau)^{-\gamma} \|u(\tau) - v(\tau)\|_\infty d\tau ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_p 2^p \|u_0\|_\infty^{p-1}}{\Gamma(1-\gamma)} \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|u(\tau) - v(\tau)\|_\infty ds d\tau \\
&= \frac{C_p 2^p \|u_0\|_\infty^{p-1}}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} \|u(s) - v(s)\|_\infty ds.
\end{aligned} \tag{3.6}$$

Substituting Eqs (3.5) and (3.6) into Eq (3.4), we have

$$\begin{aligned}
\|u(t) - v(t)\|_\infty &\leq C \|h\|_\infty \frac{2^{p-1} \|u_0\|_\infty^{p-1}}{\Gamma(2-\gamma)} \int_0^t s^{1-\gamma} \|u(s) - v(s)\|_\infty ds \\
&\quad + C \|h\|_\infty \frac{C_p 2^p \|u_0\|_\infty^{p-1}}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} \|u(s) - v(s)\|_\infty ds \\
&\leq C \int_0^t \max\{s^{1-\gamma}, (t-s)^{1-\gamma}\} \|u(s) - v(s)\|_\infty ds.
\end{aligned} \tag{3.7}$$

Uniqueness then follows from a weighted Grönwall inequality.

Similarly, for $1 < p < 2$, $\left| |a|^{p-1} - |b|^{p-1} \right| \leq C_p |a - b|^{p-1}$, thus

$$\int_0^t \|I_{0|s}^{1-\gamma} (|u|^{p-1} - |v|^{p-1})\|_\infty \|v(s)\|_\infty ds \leq C \int_0^t (t-s)^{1-\gamma} \|u(s) - v(s)\|_\infty^{p-1} ds. \tag{3.8}$$

Substituting Eqs (3.5) and (3.8) into Eq (3.4), we have

$$\begin{aligned}
\|u(t) - v(t)\|_\infty &\leq C \|h\|_\infty \frac{2^{p-1} \|u_0\|_\infty^{p-1}}{\Gamma(2-\gamma)} \int_0^t s^{1-\gamma} \|u(s) - v(s)\|_\infty ds \\
&\quad + C \int_0^t (t-s)^{1-\gamma} \|u(s) - v(s)\|_\infty^{p-1} ds \\
&\leq C \|h\|_\infty \frac{4 \|u_0\|_\infty}{\Gamma(2-\gamma)} \int_0^t s^{1-\gamma} \|u(s) - v(s)\|_\infty^{p-1} ds \\
&\quad + C \int_0^t (t-s)^{1-\gamma} \|u(s) - v(s)\|_\infty^{p-1} ds \\
&\leq C \int_0^t \max\{s^{1-\gamma}, (t-s)^{1-\gamma}\} \|u(s) - v(s)\|_\infty^{p-1} ds.
\end{aligned} \tag{3.9}$$

Uniqueness then follows from a weighted Bihari–type nonlinear Grönwall inequality with $p - 1 \in (0, 1)$.

III. The maximal time interval of solution.

By uniqueness of the local solution, the maximal interval of existence is of the form $[0, T_{\max})$, where

$$T_{\max} = \sup \left\{ T > 0 \mid \text{The mild solution } u \in L^\infty((0, T), L^\infty(\mathbb{R}^N)) \right\} \leq +\infty.$$

From the continuity of the semigroup $S(t) = e^{-t\mathcal{L}}$, we have

$$u \in C([0, T_{\max}), L^\infty(\mathbb{R}^N)). \tag{3.10}$$

IV. Regularity.

For $0 \leq u_0 \in (L^1 \cap C_0^\infty)(\mathbb{R}^N)$, we repeat the fixed-point argument in the space

$$X_{T,1} = \left\{ u \in L^\infty((0, T), L^1 \cap C_0^\infty(\mathbb{R}^N)) \mid \|u\| \leq 2\|u_0\|_\infty, \|u\|_{\infty,1} \leq 2\|u_0\|_1 \right\},$$

instead of X_T , where $\|\cdot\|_{\infty,1} = \|\cdot\|_{L^\infty((0,T),L^1(\mathbb{R}^N))}$. Using the estimate

$$\|u^p(t)\|_1 \leq \|u(t)\|_\infty^{p-1} \|u(t)\|_1$$

in the contraction mapping argument, we obtain a unique solution in $X_{T,1}$. Therefore,

$$u \in C([0, T_{\max}), (L^1 \cap C_0)(\mathbb{R}^N)). \quad (3.11)$$

Let the forcing term in Eq (1.1) be denoted by $F(t) = -h(t)I_{0|t}^{1-\gamma}(|u|^{p-1})u(t)$. For any $T > 0$, we obtain

$$\|I_{0|t}^{1-\gamma}(|u|^{p-1})\|_\infty \leq C \int_0^t (t-s)^{-\gamma} \|u(s)\|_\infty^{p-1} ds \leq C_T,$$

and $\|F(t)\|_2 \leq C_T \|u(t)\|_2$. Because $u(t) \in (L^1 \cap C_0)(\mathbb{R}^N)$, $\|u(t)\|_2^2 \leq \|u(t)\|_1 \|u(t)\|_\infty$. Then, we have

$$F(t) \in L^\infty(0, T; L^2(\mathbb{R}^N)) \subset L^2(0, T; L^2(\mathbb{R}^N)).$$

We consider the equation $\partial_t u + t^\beta \mathcal{L}u = F(t)$. For any $a > 0$, t^β is bounded from above and below on the interval $[a, T]$. Because $\mathcal{L}$ is a non-negative, self-adjoint operator and forms a sectorial operator family, using L^2 -maximal regularity or energy estimate, we obtain

$$u(t) \in H^1(a, T; L^2(\mathbb{R}^N)) \cap L^2(a, T; D(\mathcal{L})).$$

Because $0 < \alpha < 2$ with equivalent norms, $H^2 \subset H^\alpha$, so we obtain $D(\mathcal{L}) = H^2$. This implies that $u(t) \in H^1(a, T; L^2(\mathbb{R}^N)) \cap L^2(a, T; H^2(\mathbb{R}^N))$. By the arbitrariness of a and T , it follows that

$$u(t) \in H_{loc}^1(0, \infty; L^2(\mathbb{R}^N)) \cap L_{loc}^2(0, \infty; H^2(\mathbb{R}^N)).$$

V. Global existence.

By definition, u is global if $T_{\max} = +\infty$. If $T_{\max} < +\infty$, then u blows up in finite time; in this case, one usually has $\|u(t)\|_{C_0(\mathbb{R}^N)} \rightarrow \infty$ as $t \rightarrow T_{\max}$.

Set $p_0 = N(\beta + 1)(p - 1)/(\alpha(2 - \gamma))$. The condition $p > \max\{1/\gamma, 1 + \alpha(2 - \gamma)/(N(\beta + 1))\}$ implies $p_0 > 1$. Choose $q \geq p$ such that

$$\frac{2 - \gamma}{p - 1} - \frac{1}{p} < \frac{N(\beta + 1)}{\alpha q} < \frac{1}{p - 1}. \quad (3.12)$$

Then,

$$q > \frac{N}{\alpha}(\beta + 1)(p - 1). \quad (3.13)$$

Define

$$b := \frac{N}{\alpha} \left(\frac{1}{p_0} - \frac{1}{q} \right) (\beta + 1) = \frac{2 - \gamma}{p - 1} - \frac{N}{\alpha q} (\beta + 1). \quad (3.14)$$

From Eqs (3.12) and (3.14), we observe that

$$pb = p \left[\frac{2-\gamma}{p-1} - \frac{N}{\alpha q}(\beta+1) \right] < 1, \quad 2-\gamma-(p-1)b - \frac{N}{\alpha q}(\beta+1)(p-1) = 0. \quad (3.15)$$

Using the properties of fundamental solution $E_\alpha(x, t)$ to the operator $\partial_t + \mathcal{L}$ given in Lemma 2, Eq (3.14), and $p_0 > 1$, one obtains

$$\begin{aligned} \sup_{t>0} t^b \|E_\alpha(\frac{t^{\beta+1}}{\beta+1}) * u_0\|_q &\leq C \sup_{t>0} t^b \min \left\{ t^{-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q})(\beta+1)}, t^{-\frac{N}{\alpha}(\frac{1}{p_0}-\frac{1}{q})(\beta+1)} \right\} \|u_0\|_{p_0} \\ &= C \sup_{t>0} t^{b-\frac{N}{\alpha}(\frac{1}{p_0}-\frac{1}{q})(\beta+1)} \min \left\{ t^{\frac{N}{\alpha}-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q})(\beta+1)}, 1 \right\} \|u_0\|_{p_0} \\ &\leq C \max \left\{ \sup_{t \geq 1}, \sup_{t \in (0,1)} \right\} \min \left\{ t^{\frac{N}{\alpha}-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{q})(\beta+1)}, 1 \right\} \|u_0\|_{p_0} \\ &= C \|u_0\|_{p_0} := \eta < +\infty. \end{aligned} \quad (3.16)$$

Set

$$\mathcal{X} := \left\{ u \in L_{\text{loc}}^\infty((0, \infty); L^q(\mathbb{R}^N)) : \|u\|_{\mathcal{X}} := \text{ess sup}_{t>0} t^b \|u(t)\|_{L^q} < \infty \right\}.$$

Then, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space. For a fixed $\delta > 0$, define

$$\mathcal{X}_\delta := \{u \in \mathcal{X} : \|u\|_{\mathcal{X}} \leq \delta\}.$$

We equip $\mathcal{X}_\delta$ with the metric

$$d_{\mathcal{X}}(u, v) := \|u - v\|_{\mathcal{X}}.$$

Because $\mathcal{X}_\delta$ is a closed subset of the Banach space $\mathcal{X}$, $(\mathcal{X}_\delta, d_{\mathcal{X}})$ is complete. For the map

$$\Phi(u)(t) = E_\alpha(\frac{t^{\beta+1}}{\beta+1}) * u_0 - \int_0^t h(s) E_\alpha(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}) * [I_{0|s}^{1-\gamma}(|u|^{p-1})u] ds, \quad \forall t > 0,$$

from Eq (3.16), we derive

$$\begin{aligned} t^b \|\Phi(u)(t)\|_q &\leq t^b \|E_\alpha(\frac{t^{\beta+1}}{\beta+1}) * u_0\|_q \\ &\quad + t^b \left\| \int_0^t h(s) E_\alpha(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}) * [I_{0|s}^{1-\gamma}(|u|^{p-1})u] ds \right\|_q \\ &\leq \eta + \|h\|_\infty t^b \int_0^t \|E_\alpha(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}) * [I_{0|s}^{1-\gamma}(|u|^{p-1})u]\|_q ds. \end{aligned} \quad (3.17)$$

Using the estimate (1.16) of $E_\alpha(x, t)$ given in Lemma 2(iv), we obtain

$$\begin{aligned} &\|E_\alpha(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}) * [I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)]\|_q \\ &\leq C \min \left\{ (t^{\beta+1} - s^{\beta+1})^{-\frac{N(p-1)}{2q}}, (t^{\beta+1} - s^{\beta+1})^{-\frac{N(p-1)}{\alpha q}} \right\} \|I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)\|_{q/p}. \end{aligned} \quad (3.18)$$

By Minkowski's integral inequality and the positivity of the kernel, we have

$$\|I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)\|_{\frac{q}{p}} \leq I_{0|s}^{1-\gamma}(\|u^{p-1}\|_{\frac{q}{p-1}}) \|u(s)\|_q = I_{0|s}^{1-\gamma}(\|u(\tau)\|_q^{p-1}) \|u(s)\|_q$$

$$= I_{0|s}^{1-\gamma}(\tau^{-(p-1)b} \cdot (\tau^b \|u(\tau)\|_q)^{p-1}) s^{-b} [s^b \|u(s)\|_q],$$

and $u \in X_\delta$ implies that

$$\|I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)\|_{\frac{q}{p}} \leq \delta^p I_{0|s}^{1-\gamma}(\tau^{-(p-1)b}) s^{-b} = \frac{\delta^p s^{-b}}{\Gamma(1-\gamma)} \int_0^s (s-\tau)^{-\gamma} \tau^{-(p-1)b} d\tau.$$

Because $0 < \gamma < 1$, and $pb < 1$ (see Eq (3.15)), the improper integral

$$\int_0^s (s-\tau)^{-\gamma} \tau^{-(p-1)b} d\tau = B(1-\gamma, 1-(p-1)b) s^{1-\gamma-(p-1)b}$$

converges, and $B(1-\gamma, 1-(p-1)b)$ is a beta function; thus,

$$\begin{aligned} \|I_{0|s}^{1-\gamma}(|u|^{p-1})u(s)\|_{q/p} &\leq \frac{\delta^p}{\Gamma(1-\gamma)} B(1-\gamma, 1-(p-1)b) s^{1-\gamma-(p-1)b-b} \\ &= \frac{\delta^p}{\Gamma(1-\gamma)} B(1-\gamma, 1-(p-1)b) s^{1-\gamma-pb}. \end{aligned} \quad (3.19)$$

Putting Eqs (3.18) and (3.19) into Eq (3.17), we have

$$\begin{aligned} &t^b \|\Phi(u)(t)\|_q - \eta \\ &\leq C\delta^p t^b \int_0^t \min \left\{ (t^{\beta+1} - s^{\beta+1})^{-\frac{N}{2q}(p-1)}, (t^{\beta+1} - s^{\beta+1})^{-\frac{N}{\alpha q}(p-1)} \right\} s^{1-\gamma-pb} ds \\ &\leq C\delta^p t^b \min \left\{ \int_0^t (t^{\beta+1} - s^{\beta+1})^{-\frac{N}{2q}(p-1)} s^{1-\gamma-pb} ds, \int_0^t (t^{\beta+1} - s^{\beta+1})^{-\frac{N}{\alpha q}(p-1)} s^{1-\gamma-pb} ds \right\}, \end{aligned} \quad (3.20)$$

as,

$$\begin{aligned} &\int_0^t (t^{\beta+1} - s^{\beta+1})^{-\frac{N}{2q}(p-1)} s^{1-\gamma-pb} ds \\ &= t^{2-\gamma-pb-\frac{N}{2q}(p-1)(\beta+1)} \int_0^1 (1-y^{\beta+1})^{-\frac{N}{2q}(p-1)} y^{1-\gamma-pb} dy \\ &= \frac{1}{\beta+1} t^{2-\gamma-pb-\frac{N}{2q}(p-1)(\beta+1)} B(1-N(p-1)/(2q), (2-\gamma-pb)/(\beta+1)), \\ &\int_0^t (t^{\beta+1} - s^{\beta+1})^{-\frac{N}{\alpha q}(p-1)} s^{1-\gamma-pb} ds \\ &= t^{2-\gamma-pb-\frac{N}{\alpha q}(p-1)(\beta+1)} \int_0^1 (1-y^{\beta+1})^{-\frac{N}{\alpha q}(p-1)} y^{1-\gamma-pb} dy \\ &= \frac{1}{\beta+1} t^{2-\gamma-pb-\frac{N}{\alpha q}(p-1)(\beta+1)} B(1-N(p-1)/(\alpha q), (2-\gamma-pb)/(\beta+1)). \end{aligned}$$

Because $0 < \gamma < 1$, and $pb < 1$ (see Eq (3.15)), for $q > \max\{N(p-1)/2, N(\beta+1)(p-1)/\alpha\}$, beta functions $B(1-N(p-1)/(2q), (2-\gamma-pb)/(\beta+1))$ and $B(1-N(p-1)/(\alpha q), (2-\gamma-pb)/(\beta+1))$ converge; using the second identity in Eq (3.15), Eq (3.20) becomes

$$t^b \|\Phi(u)(t)\|_q \leq \eta + C\delta^p t^b \min \left\{ t^{2-\gamma-pb-\frac{N}{2q}(p-1)(\beta+1)}, t^{2-\gamma-pb-\frac{N}{\alpha q}(p-1)(\beta+1)} \right\}$$

$$\begin{aligned}
&= \eta + C\delta^p t^{2-\gamma-(p-1)b-\frac{N}{\alpha q}(p-1)(\beta+1)} \min \left\{ t^{(\frac{1}{\alpha}-\frac{1}{2})\frac{N}{q}(p-1)(\beta+1)}, 1 \right\} \\
&= \eta + C\delta^p \min \left\{ t^{(\frac{1}{\alpha}-\frac{1}{2})\frac{N}{q}(p-1)(\beta+1)}, 1 \right\} \leq \eta + C\delta^p,
\end{aligned} \tag{3.21}$$

this yields

$$t^b \|\Phi(u)(t)\|_q \leq \eta + C\delta^p.$$

Choosing $\delta > 0$ sufficiently small and then taking $\|u_0\|_{p_0}$ sufficiently small so that $\eta + C\delta^p \leq \delta$, we obtain $\Phi(X_\delta) \subset X_\delta$.

Similarly, for $p \geq 2$, using

$$||a|^{p-1}a - |b|^{p-1}b| \leq C_p(|a|^{p-1} + |b|^{p-1})|a - b|,$$

one can prove, in the same way, that Φ is a strict contraction on X_δ , provided that δ is sufficiently small. Hence, by the Banach fixed point theorem, Φ has a unique fixed-point $u \in X_\delta$, which is a mild solution of Eq (1.1).

For $1 < p < 2$, using

$$||a|^{p-1} - |b|^{p-1}| \leq C_p|a - b|^{p-1},$$

the Schauder fixed-point theorem for solution existence and the Bihari nonlinear Grönwall inequality for solution uniqueness holds.

We now prove that $u \in C([0, \infty), (L^1 \cap C_0(\mathbb{R}^N)))$. First, for $T > 0$ sufficiently small, we have $u \in C([0, T], (L^1 \cap C_0(\mathbb{R}^N)))$. Indeed, for any $T > 0$, there exists a unique solution in

$$U_T := \left\{ u \in L^\infty((0, T), L^q(\mathbb{R}^N)) \mid \sup_{t \in (0, T)} t^b \|u(t)\|_q \leq \delta \right\}.$$

Let $\tilde{u}$ be the local solution of Eq (1.20). Because $u_0 \in (L^1 \cap C_0)(\mathbb{R}^N)$, and thus, $u_0 \in L^q(\mathbb{R}^N)$ for $q > \max\{N(p-1)/2, N(\beta+1)(p-1)/\alpha\}$, we obtain $\tilde{u} \in C([0, T_{\max}), L^q(\mathbb{R}^N))$. For $T > 0$ sufficiently small, it follows that $\sup_{t \in (0, T)} t^b \|\tilde{u}(t)\|_q \leq \delta$, and uniqueness of the local solution implies that $u = \tilde{u}$ on $[0, T]$ so that $u \in C([0, T], (L^1 \cap C_0)(\mathbb{R}^N))$.

Now, by a bootstrap method, iterating a finite number of times (refer to [8] for details), we can also prove that $u \in C([T, \infty), (L^1 \cap C_0)(\mathbb{R}^N))$. Combining the above two steps, we obtain $u \in C([0, \infty), (L^1 \cap C_0)(\mathbb{R}^N))$.

VI. Non-negativity of mild solutions.

The following energy argument is justified on every interval $[a, T]$, $a > 0$ by the regularity obtained in Step IV:

$$u(t) \in H^1(a, T; L^2(\mathbb{R}^N)) \cap L^2(a, T; H^2(\mathbb{R}^N)). \tag{3.22}$$

Passing to the limit as $a \rightarrow 0$ gives the result on $[0, T]$.

Let $T > 0$. If $u(x, t)$ is a mild solution of Eq (1.1) on $[0, T]$ with the regularity in Eq (3.22), and $u_0 \geq 0$, then $u(x, t) \geq 0$ for a.e. $x \in \mathbb{R}^N$ and for all $t \in [0, T]$.

We now prove the non-negativity of the mild solution. Let $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. Then, $u = u^+ - u^-$, $u^+ u^- = (\Delta u^+) u^- = 0$, and

$$\int_{\mathbb{R}^N} [(-\Delta)^{\alpha/2} u^+] u^- dx \leq 0. \tag{3.23}$$

Because $u(t) \in H^1(a, T; L^2(\mathbb{R}^N))$, and the map $t \mapsto u^-(t)$ also belongs to $H^1(a, T; L^2(\mathbb{R}^N))$, we have

$$\frac{1}{2} \frac{d}{dt} \|u^-(t)\|^2 = -(u_t(t), u^-(t)), \text{ for a.e. } t \in [0, T].$$

$u(t) \in H^2(\mathbb{R}^N)$ implies that $(\mathcal{L}u(t), u^-(t))$ makes sense, and

$$\begin{aligned} \int_{\mathbb{R}^N} u_t u^- dx &= \int_{\mathbb{R}^N} t^\beta (\Delta u) u^- dx - \int_{\mathbb{R}^N} t^\beta [(-\Delta)^{\alpha/2} u] u^- dx \\ &\quad - \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma} (|u|^{p-1}) u u^- dx. \end{aligned} \quad (3.24)$$

Using Eq (3.23) and substituting $u_t = u_t^+ - u_t^-$ and $\Delta u = \Delta u^+ - \Delta u^-$ into Eq (3.24), we obtain

$$\begin{aligned} - \int_{\mathbb{R}^N} u_t^- u^- dx &\geq - \int_{\mathbb{R}^N} t^\beta (\Delta u^-) u^- dx + \int_{\mathbb{R}^N} t^\beta [(-\Delta)^{\alpha/2} u^-] u^- dx \\ &\quad + \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma} (|u|^{p-1}) (u^-)^2 dx, \\ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (u^-)^2 dx &\leq t^\beta \int_{\mathbb{R}^N} (\Delta u^-) u^- dx - t^\beta \int_{\mathbb{R}^N} [(-\Delta)^{\alpha/2} u^-] u^- dx \\ &\quad - h(t) \int_{\mathbb{R}^N} I_{0|t}^{1-\gamma} (|u|^{p-1}) (u^-)^2 dx, \end{aligned} \quad (3.25)$$

as,

$$\int_{\mathbb{R}^N} (\Delta u^-) u^- dx = - \int_{\mathbb{R}^N} |\nabla u^-|^2 dx \leq 0, \quad (3.26)$$

$$\int_{\mathbb{R}^N} I_{0|t}^{1-\gamma} (|u|^{p-1}) (u^-)^2 dx \geq 0. \quad (3.27)$$

Using the self-adjoint property of $(-\Delta)^{\alpha/2}$, we have

$$\int_{\mathbb{R}^N} [(-\Delta)^{\alpha/2} u^-] u^- dx = \int_{\mathbb{R}^N} [(-\Delta)^{\alpha/4} u^-]^2 dx \geq 0. \quad (3.28)$$

Putting Eqs (3.26)–(3.28) into Eq (3.25), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} [u^-(x, t)]^2 dx \leq 0, \quad (3.29)$$

as $u_0 \geq 0$, Eq (3.29) implies that

$$\int_{\mathbb{R}^N} [u^-(x, t)]^2 dx \leq \int_{\mathbb{R}^N} [u_0^-(x)]^2 dx = 0, \quad \forall t \geq 0.$$

We therefore have $u^-(x, t) = 0$; that is, $u = u^+ \geq 0$ for a.e. $x \in \mathbb{R}^N$, $\forall t \in [0, T]$.

This completes the proof of Theorem 1. $\square$

4. Asymptotic behavior of mild solutions

In this section, we prove Theorem 2.

Lemma 3. (*Comparison principle for mild solutions*) Let $T > 0$ and $u(x, t), v(x, t)$ be mild solutions of Eq (1.1) with initial data u_0 and v_0 , respectively. If $0 \leq u_0 \leq v_0$, then $0 \leq u(x, t) \leq v(x, t)$ for a.e. $x \in \mathbb{R}^N$ and for all $t \in [0, T)$.

Proof. Let $w(x, t) = v(x, t) - u(x, t)$. By the non-negativity of mild solutions established in Theorem 1, we have $u(x, t) \geq 0, v(x, t) \geq 0$ for a.e. $x \in \mathbb{R}^N, \forall t \in [0, T)$. Because u and v are mild solutions on $[0, T)$, Definition 1 implies that $\|u\|_{L^\infty([0, T) \times \mathbb{R}^N)} < +\infty, \|v\|_{L^\infty([0, T) \times \mathbb{R}^N)} < +\infty$; that is, $Q := \|u\|_{L^\infty([0, T) \times \mathbb{R}^N)}^{p-1} + \|v\|_{L^\infty([0, T) \times \mathbb{R}^N)}^{p-1} < +\infty$. Thus, $u(x, t) \geq 0$ and $v(x, t) \geq 0$ imply that $|v|^{p-1} - |u|^{p-1} = v^{p-1} - u^{p-1}$. Moreover, because $u(x, t) \geq 0$, and $v(x, t) \geq 0$, we have

$$\begin{cases} \partial_t w(x, t) + t^\beta \mathcal{L}w(x, t) + h(t)I_{0|t}^{1-\gamma}(v^{p-1})w = -h(t)u(t)I_{0|t}^{1-\gamma}(v^{p-1} - u^{p-1}), \\ w(x, 0) = v_0(x) - u_0(x) \geq 0, \quad x \in \mathbb{R}^N. \end{cases} \quad (4.1)$$

Because $w(x, 0) = v_0(x) - u_0(x) \geq 0$, it remains to show that $w \geq 0$. Define

$$b(x, t) = \begin{cases} \frac{v^{p-1}(x, t) - u^{p-1}(x, t)}{v(x, t) - u(x, t)}, & v(x, t) \neq u(x, t), \\ (p-1)u^{p-2}(x, t), & v(x, t) = u(x, t). \end{cases} \quad (4.2)$$

Thus, $b \geq 0$, and $v^{p-1} - u^{p-1} = bw$. Hence

$$\partial_t w + t^\beta \mathcal{L}w + a(x, t)w + h(t)uI_{0|t}^{1-\gamma}(bw) = 0,$$

where $a(x, t) = h(t)I_{0|t}^{1-\gamma}(v^{p-1})(x, t) \geq 0$.

Let $w^+ = \max\{w, 0\}$, and $w^- = -\min\{w, 0\}$. Then, $w = w^+ - w^-$, and

$$w^+w^- = (\Delta w^+)w^- = 0, \quad ((-\Delta)^{\alpha/2}w^+)w^- \leq 0 \quad \text{a.e. } x \in \mathbb{R}^N.$$

Arguing as in the proof of Theorem 1, we obtain

$$\int_{\mathbb{R}^N} w_t w^- dx = -\frac{1}{2} \frac{d}{dt} \|w^-(t)\|_2^2, \quad \int_{\mathbb{R}^N} \mathcal{L}w \cdot w^- dx \leq 0.$$

Moreover, the local zero-order term satisfies

$$\int_{\mathbb{R}^N} a(x, t)ww^- dx = -\int_{\mathbb{R}^N} a(x, t)(w^-)^2 dx \leq 0.$$

For the memory term $\int_{\mathbb{R}^N} h(t)u(x, t)I_{0|t}^{1-\gamma}(bw)(x, t)w^-(x, t)dx$, because $w = w^+ - w^-$, we have

$$I_{0|t}^{1-\gamma}(bw) = I_{0|t}^{1-\gamma}(bw^+) - I_{0|t}^{1-\gamma}(bw^-),$$

and because $I_{0|t}^{1-\gamma}(bw^+) \geq 0, I_{0|t}^{1-\gamma}(bw^-) \geq 0$, thus we have $I_{0|t}^{1-\gamma}(bw) \geq -I_{0|t}^{1-\gamma}(bw^-)$, and

$$\int_{\mathbb{R}^N} h(t)u(x, t)I_{0|t}^{1-\gamma}(bw)(x, t)w^-(x, t)dx \geq -\int_{\mathbb{R}^N} h(t)u(x, t)I_{0|t}^{1-\gamma}(bw^-)(x, t)w^-(x, t)dx.$$

Then we have

$$\frac{1}{2} \frac{d}{dt} \|w^-(t)\|_2^2 \leq \int_{\mathbb{R}^N} h(t) u(x, t) I_{0|t}^{1-\gamma} (bw^-)(x, t) w^-(x, t) dx.$$

On the finite time interval $[0, T]$, the mild solutions u, v are bounded; that is,

$$\mathcal{Q} := \|u\|_{L^\infty([0, T] \times \mathbb{R}^N)}^{p-1} + \|v\|_{L^\infty([0, T] \times \mathbb{R}^N)}^{p-1} < +\infty.$$

Case 1. $p \geq 2$.

The function $b(x, t)$ given in Eq (4.2) is bounded; that is, $0 \leq b(x, t) \leq C_T$ for some constant $C_T > 0$. Hence,

$$\begin{aligned} I_{0|t}^{1-\gamma} (bw^-)(x, t) &\leq C_T \int_0^t (t-s)^{-\gamma} w^-(x, s) ds, \\ \frac{1}{2} \frac{d}{dt} \|w^-(t)\|_2^2 &\leq C_T \int_{\mathbb{R}^N} w^-(x, t) \int_0^t (t-s)^{-\gamma} w^-(x, s) ds dx, \end{aligned}$$

and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} w^-(x, t) w^-(x, s) dx &\leq \|w^-(t)\|_2 \|w^-(s)\|_2, \\ \frac{d}{dt} \|w^-(t)\|_2^2 &\leq C_T \|w^-(t)\|_2 \int_0^t (t-s)^{-\gamma} \|w^-(s)\|_2 ds. \end{aligned}$$

Setting $Y(t) = \sup_{0 \leq s \leq t} \|w^-(s)\|_2$, one gets

$$\frac{d}{dt} \|w^-(t)\|_2^2 \leq C_T Y^2(t) \int_0^t (t-s)^{-\gamma} ds = C_T t^{1-\gamma} Y^2(t).$$

Integrating with respect to t , $\|w^-(t)\|_2^2 \leq C_T \int_0^t s^{1-\gamma} Y^2(s) ds$. Taking the supremum with respect to t ,

$$Y^2(t) \leq C_T \int_0^t s^{1-\gamma} Y^2(s) ds,$$

as $w^-(0) = 0$, and Grönwall's inequality yields $Y^2(t) = 0$. We therefore have $w^-(t) = 0, 0 \leq t \leq T$, which implies that $v - u = w = w^+ \geq 0$; that is, $u \leq v$ for a.e. $x \in \mathbb{R}^N, \forall t \in [0, T]$.

Case 2. $1 < p < 2$.

Because the derivative of the map $r \mapsto r^{p-1}$ is singular near $r = 0$, namely, $(p-1)r^{p-2} \rightarrow +\infty$ as $r \rightarrow 0^+$, the function $b(x, t)$ may be unbounded. We therefore use a regularization argument. Define $f_\epsilon(r) = (r + \epsilon)^{p-1}$. We consider the regularized equation

$$\partial_t u_\epsilon + t^\beta \mathcal{L} u_\epsilon + a(x, t) u_\epsilon = -h(t) u_\epsilon I_{0|t}^{1-\gamma} (f_\epsilon(u_\epsilon)).$$

By solving the corresponding regularized problems with the same initial data and using the continuous dependence of mild solutions on the regularized nonlinearities, we obtain $u_\epsilon \rightarrow u$ and $v_\epsilon \rightarrow v$ uniformly strongly in $C([0, T]; L_{\text{loc}}^2(\mathbb{R}^N))$. Because $f'_\epsilon(r) = (p-1)(r + \epsilon)^{p-2}$ is bounded on every finite interval, the argument in Case 1 applies and gives $u_\epsilon \leq v_\epsilon$. Passing to the limit $\epsilon \rightarrow 0$, we obtain $u \leq v$ for a.e. $x \in \mathbb{R}^N, \forall t \in [0, T]$. $\square$

Lemma 4. (Decay of the mass) Let $u(x, t)$ be a non-negative, nontrivial, global, mild solution of the Cauchy problem in Eq (1.1). Then, the mass $M(t) := \int_{\mathbb{R}^N} u(x, t) dx$ of Eq (1.1) is nonincreasing and the limit $M_\infty := \lim_{t \rightarrow +\infty} M(t)$ exists.

Proof. Because $u(x, t)$ is a non-negative nontrivial global mild solution of the Cauchy problem in Eq (1.1), it satisfies the integral equation (1.20). Hence,

$$\begin{aligned} M(t) &= \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0 dx \\ &\quad - \int_{\mathbb{R}^N} \int_0^t h(s) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * [I_{0|s}^{1-\gamma}(u^{p-1})u(s)] ds dx. \end{aligned} \quad (4.3)$$

By Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0 dx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right)(x-y) u_0(y) dy dx \\ &= \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right)(x-y) u_0(y) dx \\ &= \int_{\mathbb{R}^N} u_0(y) dy \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right)(x) dx. \end{aligned}$$

Using Lemma 2(ii), namely $\int_{\mathbb{R}^N} E_\alpha(x, t) dx = 1$ for all $t > 0$, we obtain

$$\int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right)(x) dx = 1, \quad \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0 dx = \int_{\mathbb{R}^N} u_0(x) dx. \quad (4.4)$$

By Fubini's theorem again,

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_0^t h(s) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * I_{0|s}^{1-\gamma}(u^{p-1})u(s) dx \\ &= \int_0^t \int_{\mathbb{R}^N} h(s) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * I_{0|s}^{1-\gamma}(u^{p-1})u(s) dx ds \\ &= \int_0^t h(s) ds \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right)(x-y) I_{0|s}^{1-\gamma}(u^{p-1})u(y) dy \\ &= \int_0^t h(s) ds \int_{\mathbb{R}^N} I_{0|s}^{1-\gamma}(u^{p-1})u(y) dy \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right)(x-y) dx. \end{aligned}$$

Because $\int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right)(x-y) dx = 1, \forall y \in \mathbb{R}^N$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_0^t h(s) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1}\right) * I_{0|s}^{1-\gamma}(u^{p-1})u(s) dx \\ &= \int_0^t h(s) ds \int_{\mathbb{R}^N} I_{0|s}^{1-\gamma}(u^{p-1})u(y) dy \\ &= \int_0^t \int_{\mathbb{R}^N} h(s) I_{0|s}^{1-\gamma}(u^{p-1})u(x, s) dx ds. \end{aligned} \quad (4.5)$$

Substituting Eqs (4.4) and (4.5) into Eq (4.3) we have

$$M(t) = \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^t \int_{\mathbb{R}^N} h(s) I_{0|s}^{1-\gamma}(u^{p-1}) u(x, s) dx ds, \quad (4.6)$$

and,

$$M'(t) = \frac{d}{dt} \int_{\mathbb{R}^N} u(x, t) dx = - \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(u^{p-1}) u dx \leq 0.$$

Thus, $M'(t) \leq 0$, and hence, $M(t)$ is nonincreasing. Because $M(t) \geq 0$, the limit $\lim_{t \rightarrow +\infty} M(t)$ exists. We denote it by $M_\infty = \lim_{t \rightarrow +\infty} M(t) \geq 0$. Under additional assumptions, we will show that $M_\infty > 0$. $\square$

Proof of Theorem 2. Because $u(x, t) \geq 0$ is a global, mild solution of the Cauchy problem in Eq (1.1), from Eq (4.6), we have

$$0 \leq \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^t \int_{\mathbb{R}^N} h(s) I_{0|s}^{1-\gamma}(u^{p-1}) u(x, s) dx ds. \quad (4.7)$$

For $0 \leq u_0 \in L^1(\mathbb{R}^N)$, Eq (4.7) implies that

$$0 \leq \int_{\mathbb{R}^N} u(x, t) dx \leq \int_{\mathbb{R}^N} u_0(x) dx = \|u_0\|_{L^1(\mathbb{R}^N)} < +\infty, \forall t \geq 0,$$

and thus, $u(x, t) \in L^\infty([0, +\infty), L^1(\mathbb{R}^N))$, and

$$\int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(u^{p-1}) u(x, t) dx dt \leq \int_{\mathbb{R}^N} u_0(x) dx = \|u_0\|_{L^1(\mathbb{R}^N)} < +\infty.$$

On the other hand, because $u(x, t) \geq 0$ is a global mild solution, Eq (1.20) implies that

$$0 \leq u(t) \leq E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0, \forall t \geq 0.$$

Choose $p_1 = 1 + \alpha(1 - \gamma)/(N(\beta + 1))$. Since $p > 1 + \alpha(2 - \gamma)/(N(\beta + 1))$, we have $p_1 < p$ and $p_1 < 1 + \alpha/(N(\beta + 1)) < 1 + 2/(N(\beta + 1))$. By Lemma 2(iv) with $q = p$ and $r = p/p_1$, we obtain

$$\begin{aligned} \|u(t)\|_p^p &\leq \|E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) * u_0\|_p^p \\ &\leq C \min \left\{ t^{-\frac{N}{2}(p_1-1)(\beta+1)} \|u_0\|_{p/p_1}^p, t^{-\frac{N}{\alpha}(p_1-1)(\beta+1)} \|u_0\|_{p/p_1}^p, \|u_0\|_p^p \right\} \\ &:= H(t, p, \alpha, \beta, u_0). \end{aligned} \quad (4.8)$$

For a fixed $\varepsilon \in (0, 1]$, let $u^\varepsilon = u^\varepsilon(x, t)$ denote the solution of the IVP in Eq (1.1) with initial datum εu_0 . The comparison principle implies that $0 \leq u^\varepsilon(x, t) \leq u(x, t)$, $\forall x \in \mathbb{R}^N, t > 0$, and hence,

$$M_\infty \geq M_\infty^\varepsilon = \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^N} u^\varepsilon(x, t) dx,$$

and

$$M_\infty^\varepsilon = \varepsilon \int_{\mathbb{R}^N} u_0(x) dx - \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt$$

$$= \varepsilon \left[\int_{\mathbb{R}^N} u_0(x) dx - \frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma} (|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt \right]. \quad (4.9)$$

Applying Eq (4.8) to u^ε , we obtain

$$\|u^\varepsilon(t)\|_p^p \leq H(t, p, \alpha, \beta, \varepsilon u_0) = \varepsilon^p H(t, p, \alpha, \beta, u_0). \quad (4.10)$$

A direct calculation gives

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma} (|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt \\ &= \frac{1}{\varepsilon} \int_0^{+\infty} h(t) \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \int_{\mathbb{R}^N} [u^\varepsilon(x, s)]^{p-1} u^\varepsilon(x, t) dx ds dt \\ &\leq \frac{1}{\varepsilon} \int_0^{+\infty} h(t) \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \|u^\varepsilon(s)\|_p^{p-1} \|u^\varepsilon(t)\|_p ds dt, \end{aligned}$$

and using Eq (4.10), we obtain

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma} (|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt \\ &\leq \varepsilon^{p-1} \int_0^{+\infty} h(t) H^{1/p}(t, p, \alpha, \beta, u_0) \int_0^t (t-s)^{-\gamma} H^{\frac{p-1}{p}}(s, p, \alpha, \beta, u_0) ds dt. \end{aligned} \quad (4.11)$$

Because the function $H(t, p, \alpha, \beta, u_0)$ is defined in Eq (4.8), we compute the following integrals directly: $\int_0^t (t-s)^{-\gamma} H(s, p, \alpha, \beta, u_0) ds$. First, we use the beta function $B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$ to calculate the following two integrals:

$$\begin{aligned} & \int_0^t (t-s)^{-\gamma} s^{-\frac{N}{2}(p_1-1)(\beta+1)} ds \\ &= t^{1-\gamma-\frac{N}{2}(p_1-1)(\beta+1)} \int_0^1 (1-\tau)^{-\gamma} \tau^{-\frac{N}{2}(p_1-1)(\beta+1)} d\tau \\ &= t^{1-\gamma-\frac{N}{2}(p_1-1)(\beta+1)} B(1-\gamma, 1-\frac{N}{2}(p_1-1)(\beta+1)), \end{aligned}$$

$$\begin{aligned} & \int_0^t (t-s)^{-\gamma} s^{-\frac{N}{\alpha}(p_1-1)(\beta+1)} ds \\ &= t^{1-\gamma-\frac{N}{\alpha}(p_1-1)(\beta+1)} \int_0^1 (1-\tau)^{-\gamma} \tau^{-\frac{N}{\alpha}(p_1-1)(\beta+1)} d\tau \\ &= t^{1-\gamma-\frac{N}{\alpha}(p_1-1)(\beta+1)} B(1-\gamma, 1-\frac{N}{\alpha}(p_1-1)(\beta+1)). \end{aligned}$$

Both $B(1-\gamma, 1-N(p_1-1)(\beta+1)/2)$ and $B(1-\gamma, 1-N(p_1-1)(\beta+1)/\alpha)$ converge. Thus,

$$\int_0^t (t-s)^{-\gamma} H^{\frac{p-1}{p}}(s, p, \alpha, \beta, u_0) ds$$

$$\begin{aligned}
&= C \min \left\{ \int_0^t (t-s)^{-\gamma} s^{-\frac{N}{2}(p_1-1)(\beta+1)\frac{p-1}{p}} ds \|u_0\|_{p/p_1}^{p-1}, \right. \\
&\quad \left. \int_0^t (t-s)^{-\gamma} s^{-\frac{N}{\alpha}(p_1-1)(\beta+1)\frac{p-1}{p}} ds \|u_0\|_{p/p_1}^{p-1}, \int_0^t (t-s)^{-\gamma} ds \|u_0\|_p^{p-1} \right\} \\
&= C \min \left\{ t^{1-\gamma-\frac{N}{2}(p_1-1)(\beta+1)\frac{p-1}{p}} B(1-\gamma, 1-\frac{N}{2}(p_1-1)(\beta+1)\frac{p-1}{p}) \|u_0\|_{p/p_1}^{p-1}, \right. \\
&\quad \left. t^{1-\gamma-\frac{N}{\alpha}(p_1-1)(\beta+1)\frac{p-1}{p}} B(1-\gamma, 1-\frac{N}{\alpha}(p_1-1)(\beta+1)\frac{p-1}{p}) \|u_0\|_{p/p_1}^{p-1}, t^{1-\gamma} \|u_0\|_p^{p-1} \right\} \\
&= C t^{1-\gamma} \min \left\{ t^{-\frac{N}{2}(p_1-1)(\beta+1)\frac{p-1}{p}} \|u_0\|_{p/p_1}^{p-1}, t^{-\frac{N}{\alpha}(p_1-1)(\beta+1)\frac{p-1}{p}} \|u_0\|_{p/p_1}^{p-1}, \|u_0\|_p^{p-1} \right\},
\end{aligned}$$

and therefore,

$$\int_0^t (t-s)^{-\gamma} H^{\frac{p-1}{p}}(s, p, \alpha, \beta, u_0) ds \leq C t^{1-\gamma} H^{\frac{p-1}{p}}(t, p, \alpha, \beta, u_0). \quad (4.12)$$

Putting Eq (4.12) into Eq (4.11), we obtain

$$\frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt \leq C \varepsilon^{p-1} \int_0^{+\infty} h(t) t^{1-\gamma} H(t, p, \alpha, \beta, u_0) dt. \quad (4.13)$$

For $0 \leq h(t) \in L^\infty[0, +\infty)$, we have

$$\int_0^1 t^{1-\gamma} h(t) H(t, p, \alpha, \beta, u_0) dt \leq \|u_0\|_p^p \|h\|_\infty \int_0^1 t^{1-\gamma} dt \leq C \|h\|_\infty \|u_0\|_p^p < +\infty,$$

and Condition (2.1) implies that

$$\begin{aligned}
&\int_1^{+\infty} t^{1-\gamma} h(t) H(t, p, \alpha, \beta, u_0) dt \\
&\leq C \|u_0\|_{p/p_1}^p \int_1^{+\infty} t^{1-\gamma} h(t) \min \left\{ t^{-\frac{N}{2}(p_1-1)(\beta+1)}, t^{-\frac{N}{\alpha}(p_1-1)(\beta+1)} \right\} dt \\
&= C \|u_0\|_{p/p_1}^p \int_1^{+\infty} h(t) \min \left\{ t^{(1-\frac{\alpha}{2})(1-\gamma)}, 1 \right\} dt \\
&= C \|u_0\|_{p/p_1}^p \int_1^{+\infty} h(t) dt < +\infty.
\end{aligned}$$

Combining the above two estimates, we obtain

$$\int_0^{+\infty} t^{1-\gamma} h(t) H(t, p, \alpha, \beta, u_0) dt < +\infty.$$

Because $p > 1$, Eq (4.13) implies that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt = 0. \quad (4.14)$$

Equation (4.14) implies that, for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u^\varepsilon|^{p-1}) u^\varepsilon(x, t) dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} u_0(x) dx.$$

Hence, we can choose a constant $\varepsilon_0 \in (0, 1)$ such that

$$\frac{1}{\varepsilon_0} \int_0^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma} (|u^{\varepsilon_0}|^{p-1}) u^{\varepsilon_0}(x, t) dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} u_0(x) dx.$$

Equation (4.9) implies that

$$M_{\infty}^{\varepsilon_0} \geq \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} u_0(x) dx > 0,$$

and thus, we have $M_{\infty} \geq M_{\infty}^{\varepsilon_0} > 0$. This completes the proof of the first part of Theorem 2.

Next, we prove that Eq (2.2) holds true for $q \in [1, +\infty)$.

Case I: $q = 1$. By Minkowski's inequality, we obtain, $\forall t \geq \tau \geq 0$,

$$\begin{aligned} & \left\| u(t) - M_{\infty} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1 \\ & \leq \left\| u(t) - E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) * u(\tau) \right\|_1 \\ & \quad + \left\| E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) * u(\tau) - M(\tau) E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) \right\|_1 \\ & \quad + \left\| M(\tau) E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) - M(\tau) E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1 \\ & \quad + \left\| M(\tau) E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) - M_{\infty} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1. \end{aligned} \quad (4.15)$$

Because u is a mild solution of Eq (1.1), Eq (1.20) gives

$$\begin{aligned} & \left\| u(t) - E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) * u(\tau) \right\|_1 \\ & = \left\| \int_{\tau}^t h(s) E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1} \right) * [I_{0|s}^{1-\gamma} (|u|^{p-1}) u] ds \right\|_1 \\ & = \int_{\tau}^t h(s) ds \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1} \right) (x-y) I_{0|s}^{1-\gamma} (|u|^{p-1}) u dy \Big\|_1 \\ & = \int_{\tau}^t h(s) ds \int_{\mathbb{R}^N} I_{0|s}^{1-\gamma} (|u|^{p-1}) u(y) dy \int_{\mathbb{R}^N} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1} \right) (x-y) dx \Big\|_1 \\ & = \int_{\tau}^t \int_{\mathbb{R}^N} h(s) I_{0|s}^{1-\gamma} (|u|^{p-1}) u(y) dy ds. \end{aligned}$$

By direct calculation or using [17, Lemma 6] with $g = u(\tau)$, we have

$$\lim_{t \rightarrow +\infty} \left\| E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) * u(\tau) - M(\tau) E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) \right\|_1 = 0.$$

From the property given in Lemma 2(iii) of the fundamental solution $E_{\alpha}(x, t)$ to the operator $\partial_t + \mathcal{L}$, that is, $E_{\alpha}(t) * E_{\alpha}(\tau) = E_{\alpha}(t + \tau)$, $\forall t, \tau > 0$, we obtain

$$E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) - E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) = E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) - E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1} \right) * E_{\alpha} \left(\frac{\tau^{\beta+1}}{\beta+1} \right),$$

as

$$\left\| E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 = \int_{\mathbb{R}^N} E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right)(x) dx = 1,$$

by direct calculation or using [17, Lemma 6] with $g = E_\alpha(\tau^{\beta+1}/(\beta+1))$, we have

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \left\| E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1}\right) - E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 \\ &= \lim_{t \rightarrow +\infty} \left\| E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1}\right) * E_\alpha\left(\frac{\tau^{\beta+1}}{\beta+1}\right) - 1 \cdot E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1}\right) \right\|_1 = 0. \end{aligned}$$

Together with the monotonicity of $M(t)$, this gives

$$\begin{aligned} & \left\| M(\tau) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1}\right) - M(\tau) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 \\ & \leq M(0) \left\| E_\alpha\left(\frac{t^{\beta+1}}{\beta+1} - \frac{\tau^{\beta+1}}{\beta+1}\right) - E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 \rightarrow 0 \text{ as } t \rightarrow +\infty. \end{aligned}$$

For the last term, by the formula (4.6), we have

$$\begin{aligned} & \left\| M(\tau) E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) - M_\infty E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 \leq |M(\tau) - M_\infty| \left\| E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 \\ &= |M(\tau) - M_\infty| \leq \int_\tau^{+\infty} \int_{\mathbb{R}^N} h(s) I_{0|s}^{1-\gamma}(|u|^{p-1}) u(y, s) dy ds. \end{aligned}$$

Substituting the above estimates into Eq (4.15) we obtain

$$\limsup_{t \rightarrow +\infty} \left\| u(t) - M_\infty E_\alpha\left(\frac{t^{\beta+1}}{\beta+1}\right) \right\|_1 \leq 2 \int_\tau^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u|^{p-1}) u(y, t) dy dt. \quad (4.16)$$

By Fubini's theorem,

$$\begin{aligned} & \int_\tau^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u|^{p-1}) u(y, t) dy dt \\ &= \int_\tau^{+\infty} h(t) \int_{\mathbb{R}^N} \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} |u|^{p-1}(y, s) u(y, t) ds dy dt \\ &= \int_\tau^{+\infty} h(t) \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \int_{\mathbb{R}^N} |u|^{p-1}(y, s) u(y, t) dy ds dt \\ &\leq \int_\tau^{+\infty} h(t) \|u(t)\|_p \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \|u(s)\|_p^{p-1} ds dt. \end{aligned}$$

Using the estimate of $\|u\|_p$ given in Eq (4.8) and $u_0 \in (L^1 \cap C_0)(\mathbb{R}^N)$, $\forall \tau > 1$, we have

$$\begin{aligned} & \int_\tau^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u|^{p-1}) u(y, t) dy dt \\ &\leq C \|u_0\|_{p/p_1}^{p-1} \int_\tau^{+\infty} h(t) \|u(t)\|_p \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \min \left\{ s^{-\frac{N}{2}(p_1-1)(\beta+1)\frac{p-1}{p}}, s^{-\frac{N}{\alpha}(p_1-1)(\beta+1)\frac{p-1}{p}} \right\} ds dt. \end{aligned}$$

Substituting the estimate of $\|u\|_p$ again,

$$\begin{aligned}
 & \int_{\tau}^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u|^{p_1-1}) u(y, t) dy dt \\
 & \leq C \|u_0\|_{p/p_1}^p \int_{\tau}^{+\infty} h(t) t^{1-\gamma} \min \left\{ t^{-\frac{N}{2}(p_1-1)(\beta+1)}, t^{-\frac{N}{\alpha}(p_1-1)(\beta+1)} \right\} dt \\
 & = C \|u_0\|_{p/p_1}^p \int_{\tau}^{+\infty} h(t) \min \left\{ t^{(1-\frac{\alpha}{2})(1-\gamma)}, 1 \right\} dt \\
 & \leq C \|u_0\|_{p/p_1}^p \int_{\tau}^{+\infty} h(t) dt.
 \end{aligned}$$

By Condition (2.1), $\int_1^{+\infty} h(t) dt$ converges. Hence, $\int_1^{+\infty} h(t) dt$ converges, so we have

$$\lim_{\tau \rightarrow +\infty} \int_{\tau}^{+\infty} h(t) dt = 0,$$

and thus,

$$\lim_{\tau \rightarrow +\infty} \int_{\tau}^{+\infty} \int_{\mathbb{R}^N} h(t) I_{0|t}^{1-\gamma}(|u|^{p_1-1}) u(y, t) dy dt = 0.$$

Returning to Eq (4.16), we obtain

$$\lim_{t \rightarrow +\infty} \left\| u(t) - M_{\infty} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1 = 0. \quad (4.17)$$

Thus, Eq (2.2) holds for $q = 1$.

Case II: $q > 1$.

For any $m \in [1, +\infty]$, using $\|P_2(t)\|_1 = 1$ for all $t > 0$ and Lemmas 1 and 2, we have

$$\|u(t)\|_m \leq \left\| E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) * u_0 \right\|_m \leq C t^{-\frac{N}{\alpha}(1-\frac{1}{m})(\beta+1)} \|u_0\|_1 \quad (4.18)$$

and

$$\begin{aligned}
 \left\| E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_m &= \left\| P_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) * P_2 \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_m \\
 &\leq C t^{-\frac{N}{\alpha}(1-\frac{1}{m})(\beta+1)} \left\| P_2 \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1 \\
 &= C t^{-\frac{N}{\alpha}(1-\frac{1}{m})(\beta+1)}.
 \end{aligned} \quad (4.19)$$

Fix $m \in [1, +\infty]$. For any $q \in [1, m)$, Hölder's inequality and Minkowski's inequality yield

$$\begin{aligned}
 & \left\| u(t) - M_{\infty} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_q \\
 & \leq \left\| u(t) - M_{\infty} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1^{1-\delta} \left\| u(t) - M_{\infty} E_{\alpha} \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_m^{\delta}
 \end{aligned}$$

$$\leq \left\| u(t) - M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1^{1-\delta} \left(\|u(t)\|_m^\delta + \left\| M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_m^\delta \right), \quad (4.20)$$

where $1/q = 1 - \delta + \delta/m$, i.e. $\delta = (1 - 1/q)/(1 - 1/m)$.

For $q < m$, putting Eqs (4.18) and (4.19) into Eq (4.20), one obtains the estimate

$$\begin{aligned} t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1)} \left\| u(t) - M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_q &\leq \left\| u(t) - M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_1^{1-\delta} \\ &\cdot \left(t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1)} \|u(t)\|_m^\delta + t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1)} \left\| M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_m^\delta \right), \end{aligned}$$

and for any $t > 0$,

$$\begin{aligned} t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1)} \|u(t)\|_m^\delta &\leq C t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1) - \frac{N}{\alpha}\delta(1-\frac{1}{m})(\beta+1)} \|u_0\|_1^\delta = C \|u_0\|_1^\delta, \\ t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1)} \left\| M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_m^\delta &\leq C t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1) - \frac{N}{\alpha}\delta(1-\frac{1}{m})(\beta+1)} = C. \end{aligned}$$

Thus, Eq (4.17) implies that

$$\lim_{t \rightarrow +\infty} t^{\frac{N}{\alpha}(1-\frac{1}{q})(\beta+1)} \left\| u(t) - M_\infty E_\alpha \left(\frac{t^{\beta+1}}{\beta+1} \right) \right\|_q = 0.$$

Therefore, Eq (2.2) holds for $q > 1$.

This completes the proof of Theorem 2. $\square$

Remark 1. Let $0 < h(t) \in L^\infty[0, +\infty)$, $0 \leq u_0 \in (L^1 \cap C_0)(\mathbb{R}^N)$, and let $u(x, t)$ be a non-negative, nontrivial, global, mild solution of the Cauchy problem (1.1). It is natural to expect that

$$1 < p \leq 1 + \frac{\alpha(2-\gamma)}{N(\beta+1)}, \quad \inf_{t \geq 0} h(t) > 0, \quad (4.21)$$

is a sufficient condition for $M_\infty = 0$. In contrast to the reaction–diffusion equation without memory term, it needs to deal with the nonlinear term $\int_0^T \int_{\mathbb{R}^N} h(t) u^p(x, t) \varphi(x, t) dx dt$. It is a local term, so we can use the test function $\varphi(x, t)$ to help us. However, because the nonlinear memory term $I_{0t}^{1-\gamma}(|u|^{p-1})u$ is singular in time, the present method does not apply; we now face a nonlocal term

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} h(t) I_{0t}^{1-\gamma}(|u|^{p-1}) u(x, t) \varphi(x, t) dx dt \\ &= \int_0^T \int_{\mathbb{R}^N} h(t) \int_0^t (t-s)^{-\gamma} |u(x, s)|^{p-1} ds u(x, t) \varphi(x, t) dx dt \\ &= \int_0^T \int_{\mathbb{R}^N} |u(x, s)|^{p-1} \left[\int_0^t h(t)(t-s)^{-\gamma} u(x, t) \varphi(x, t) dt \right] dx ds. \end{aligned} \quad (4.22)$$

The nonlocality leads the difficulty in proving $M_\infty = 0$ by the rescaled test function method. New techniques are needed to overcome this difficulty. One possible approach is to adapt the method of Fino and Kirane [8] for the purely nonlocal operator $\mathcal{L} = (-\Delta)^{\alpha/2}$.

5. Conclusions

Without the memory term $-h(t)I_{0t}^{1-\gamma}(|u|^{p-1})u(x, t)$, under the assumption $\inf_{t \geq 0} h(t) > 0$, we can get inequality $h(t)u^p \varphi \geq Cu^p \varphi$ for a non-negative solution $u(x, t)$ and the test function φ . However, for the equation with memory term $\partial_t u(x, t) + t^\beta \mathcal{L}u(x, t) = -h(t)I_{0t}^{1-\gamma}(|u|^{p-1})u(x, t)$, we need an estimate of the non-local term $\int_s^T h(t)(t-s)^{-\gamma} \varphi(t) dt$, even if $h(t) \geq C > 0$, we will need the precise lower bound of $\int_s^T (t-s)^{-\gamma} \varphi(t) dt$, it is very difficult to choose a suitable test function. We overcome this main difficulty to characterize the large-time asymptotic behavior of the global mild solutions. However, the nonlocality also leads the difficulties in proving $M_\infty = 0$, this is a difficult problem, which merits further investigation in future work.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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Guoqiang Wang: Writing—original draft, Formal analysis; Yanbin Tang: Writing—review & editing, Formal analysis.

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