



Research article

Novel average-valued condition for multi-agent systems via a quantized adaptive control protocol

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Abstract: A novel average-valued condition for consensus of multi-agent systems (MASs) is developed via a quantized adaptive control protocol, a framework that simultaneously addresses multiple self-delays, time-varying coupling, and logarithmic quantizers. A adaptive controller is designed, where each follower applies a quantized error signal and a shared adaptive gain that is updated by the quantized sum of all followers' squared errors with a fixed adaptation rate. By constructing a Lyapunov functional and applying a generalized Halanay inequality that exploits the time-average property of the coupling strength, we prove that the adaptive gain is nondecreasing, remains bounded, and converges to a finite limit. A gain threshold, depending explicitly on the average coupling strength and the delay-related Lipschitz constants, is established: If the gain exceeds this threshold, exponential consensus is achieved; otherwise, asymptotic consensus still holds. Numerical simulations on a leader-follower network with Lorenz-type node dynamics and two self-delays validate the theoretical results, including the effect of different adaptation rates, quantization densities, and delay configurations.

Keywords: multi-agent system; consensus; time-varying coupling; logarithmic quantizer; adaptive control; time-delay

1. Introduction

Cooperative control of multi-agent systems (MASs) has become a vibrant research area owing to its broad applications in unmanned aerial vehicle formations [1, 2], robotic swarms [3, 4], smart grids [5, 6], and distributed sensor networks [7]. A fundamental problem is consensus, where all agents asymptotically reach a common state.

In practical MASs, communication among agents is often hampered by limited bandwidth, which necessitates signal quantization before transmission. Among various quantization schemes, the logarithmic quantizer has attracted considerable attention because of its sector-bounded

property [8–10]. This property facilitates the analysis and design of quantized consensus protocols. For instance, recent advances include quantized iterative learning control under limited communication [11], delay-dependent quantized consensus with logarithmic quantization [12], and a unified approach that combines a time-varying coupling gain with logarithmic quantization [13]. However, most existing consensus protocols either treat quantization alone or assume constant coupling. The simultaneous presence of multiple self-delays, time-varying coupling gains, and vector logarithmic quantizers remains largely unexplored, especially when explicit exponential convergence thresholds are considered.

Time delays are another realistic challenge. Many results have been derived for MASs with single delays [14, 15] or constant delays [16–18]. The analysis becomes even more involved when multiple independent self-delays are present [19]. Moreover, coupling strength may vary over time due to environmental changes, link fading, or mobility. A common assumption is that the time-average of the coupling strength is positive [14]. Consensus protocols with time-varying coupling have been studied in [14, 20], but this makes Lyapunov-based analysis challenging because the derivative of the Lyapunov function depends on a time-varying coefficient.

Various adaptive designs have been developed to address different challenges: neural network-based approaches for nonlinear unknown dynamics [21], fully distributed consensus protocols for linear MASs [22, 23], pinning control for complex networks [24], and event-triggered schemes to reduce communication load [25–27]. When quantization is further imposed on the feedback loop, adaptive control becomes more challenging; relevant results include quantized adaptive feedback for uncertain switched linear systems [28] and passification-based adaptive control for linear systems with quantized measurements and bounded disturbances [29]. However, many existing adaptive consensus results assume constant coupling or neglect the boundedness of the adaptive gain. In practice, an unbounded gain is infeasible, yet few works provide rigorous proof that the gain remains bounded when quantization and time delays are present. The recent work [15] addressed synchronization of complex networks with time-varying delay but did not consider quantization. Conversely, the authors in [22, 23] investigated fully distributed consensus for linear MASs without accounting for delayed dynamics or time-varying coupling.

Therefore, it remains an open problem to develop a unified framework that simultaneously addresses five issues: (i) multiple self-delays and nonlinear node dynamics; (ii) time-varying coupling strength with a positive average; (iii) logarithmic quantization of control inputs; (iv) a provably bounded adaptive gain; and (v) both exponential and asymptotic consensus guarantees.

Motivated by the above investigation, this paper proposes a novel average-valued condition for MASs via a quantized adaptive control protocol to fill this gap. The main contributions are listed as follows: (1) A unified MAS model is formulated that incorporates multiple self-delays, time-varying coupling, and vector logarithmic quantizers. (2) A control protocol with a shared adaptive gain is designed, and it is rigorously proved that the adaptive gain remains bounded and converges to a finite limit. (3) By constructing a novel Lyapunov functional and applying the generalized Halanay inequality [14], sufficient conditions for exponential consensus are derived, and an explicit gain threshold characterizing the average-valued condition is given. It is shown that once the gain exceeds this threshold, exponential convergence is guaranteed; otherwise, asymptotic consensus is still achieved.

The remainder of this paper is organized as follows. Section 2 introduces graph theory,

agent dynamics, and necessary assumptions. Section 3 presents the logarithmic quantizer and its sector-bound property. Section 4 states and proves the main results, including auxiliary lemmas, the adaptive quantized controller design, and convergence theorems. Section 5 provides numerical simulations. Section 6 concludes the paper.

2. Preliminaries and problem formulation

2.1. Preliminaries

Throughout this paper, let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and positive integers, respectively; $\mathbb{R}^n$ denotes the n -dimensional Euclidean space, $\mathbb{R}^{n \times m}$ the space of $n \times m$ real matrices, and I_N the N -dimensional identity matrix. $\text{Diag}\{d_1, \dots, d_N\}$ indicates the diagonal matrix with diagonal entries d_1 through d_N . We use $\mathbf{1}_N$ ($\mathbf{0}_N$) to represent the N -dimensional column vector with all entries equal to 1 (0). The symbols $\otimes$ and $\|\cdot\|$ denote the Kronecker product and the Euclidean norm. For a given constant $\Delta > 0$, we partition $\mathbb{R}$ into intervals $\mathcal{I}_n := [n\Delta, (n+1)\Delta)$, $n \in \mathbb{N}$. For any locally integrable function $f(t)$, denote its average over $\mathcal{I}_n$ by $\text{avg}[f]_n := \frac{1}{\Delta} \int_{\mathcal{I}_n} f(s) ds$.

Let $\mathcal{G}$ be a directed graph and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \dots, N\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $A = (a_{ij})_{N \times N}$ is the adjacency matrix. If $(j, i) \in \mathcal{E}$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. The graph is assumed to have no self-loops, i.e., $a_{ii} = 0$. The in-degree of node i is defined as $d_i = \sum_{j=1}^N a_{ij}$, and the Laplacian matrix is $L = D - A$, where $D = \text{diag}(d_1, \dots, d_N)$.

Definition 1. A matrix $\bar{A} = [\bar{a}_{ij}] \in \mathbb{R}^{n \times n}$ is referred to as a *nonsingular M-matrix* if $\bar{a}_{ij} \leq 0$ holds for all $i \neq j$ ($i, j = 1, 2, \dots, n$) and every leading principal minor of $\bar{A}$ is positive.

Lemma 1. [30] Assume the matrix $\bar{A} = [\bar{a}_{ij}] \in \mathbb{R}^{n \times n}$ satisfies $\bar{a}_{ij} \leq 0$ for all $i \neq j$ ($i, j = 1, 2, \dots, n$). Then the following statements are equivalent:

1. $\bar{A}$ is a nonsingular M-matrix;
2. There exists a positive definite diagonal matrix $\Phi = \text{diag}\{\phi_1, \dots, \phi_n\} \in \mathbb{R}^{n \times n}$ such that $\bar{A}^T \Phi + \Phi \bar{A} > 0$;
3. All eigenvalues of $\bar{A}$ have positive real parts.

Lemma 2. [30] Let A, B, C, D be matrices of compatible dimensions. The following properties of the Kronecker product hold:

1. $(A \otimes B)^T = A^T \otimes B^T$;
2. $A \otimes (B + C) = A \otimes B + A \otimes C$;
3. $(A \otimes B)(C \otimes D) = AC \otimes BD$;
4. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, provided that A and B are invertible.

2.2. Problem formulation

Consider a MAS with N agents, whose dynamics are described by

$$\dot{x}_i(t) = f(t, x_i(t), \hat{x}_i(t)) + c(t) \sum_{j=1}^N a_{ij} (x_j(t) - x_i(t)) + u_i(t), \quad i = 1, 2, \dots, N, \quad (2.1)$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of agent i . $\hat{x}_i(t) = (x_i(t - \tau_1)^T, x_i(t - \tau_2)^T, \dots, x_i(t - \tau_{\mathcal{R}})^T)^T \in \mathbb{R}^{n \times \mathcal{R}}$ is its delayed version with $\tau_r > 0$, $r = 1, 2, \dots, \mathcal{R}$, denoting the delay constants and $\tau := \max\{\tau_r\}$. The function $f(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times \mathcal{R}} \rightarrow \mathbb{R}^n$ is a continuous nonlinear function. $c(t)$ denotes the nonnegative time-varying coupling strength with $c(t) \leq c_M$. The term $u_i(t) \in \mathbb{R}^n$, $i = 1, 2, \dots, N$ is the control to be designed.

Assumption 1. *The communication topology $\mathcal{G}$ has a directed spanning tree with agent 1 as the root node.*

The leader (agent 1) is not influenced by other agents, which means $a_{1j} = 0$ for $j = 1, \dots, N$. However, agent 1 can send its state to some other agents, i.e., $a_{i1} > 0$ for some $i = 2, \dots, N$. Thus

$$\dot{x}_1(t) = f(t, x_1(t), \hat{x}_1(t)).$$

Define the error $e_i(t) = x_i(t) - x_1(t)$, $i = 2, \dots, N$ and the global error vector $e(t) = (e_2^T(t), \dots, e_N^T(t))^T \in \mathbb{R}^{n(N-1)}$. Then the error dynamics can be derived:

$$\dot{e}_i(t) = f(t, x_i(t), \hat{x}_i(t)) - f(t, x_1(t), \hat{x}_1(t)) + c(t) \sum_{j=1}^N a_{ij} (x_j(t) - x_i(t)) + u_i(t), \quad i = 2, \dots, N. \quad (2.2)$$

Under Assumption 1, the Laplacian matrix can be written as

$$L = \begin{pmatrix} 0 & 0_{N-1}^T \\ z & \widehat{L} \end{pmatrix}, \quad (2.3)$$

where $z \in \mathbb{R}^{N-1}$ is a column vector, $0_{N-1} \in \mathbb{R}^{N-1}$ is a column vector of all zeros, and $\widehat{L} \in \mathbb{R}^{(N-1) \times (N-1)}$ is a nonsingular M-matrix [31, Theorem 2.3, Chapter 6].

For $i = 2, \dots, N$,

$$\sum_{j=1}^N a_{ij} (x_j(t) - x_i(t)) = - \sum_{j=2}^N \widehat{l}_{ij} e_j(t),$$

where $\widehat{L} = [\widehat{l}_{ij}]_{(N-1) \times (N-1)}$ (with $\widehat{l}_{ii} = \sum_{j=2}^N a_{ij}$ for $i \geq 2$). Then the system (2.2) reduces to

$$\dot{e}_i(t) = f(t, x_i(t), \hat{x}_i(t)) - f(t, x_1(t), \hat{x}_1(t)) - c(t) \sum_{j=2}^N \widehat{l}_{ij} e_j(t) + u_i(t), \quad i = 2, \dots, N. \quad (2.4)$$

Lemma 3. [24] *Under Assumption 1, there exists a positive diagonal matrix $\Theta = \text{diag}(\theta_1, \dots, \theta_{N-1}) \in \mathbb{R}^{(N-1) \times (N-1)}$ such that $\Theta \widehat{L} + \widehat{L}^T \Theta$ is positive definite, where*

$$\theta \triangleq (\theta_1, \theta_2, \dots, \theta_{N-1})^T = (\widehat{L}^{-1})^T \mathbf{1}_{N-1}.$$

For brevity, we define

$$\theta_{\max} = \max_{1 \leq i \leq N-1} \theta_i, \quad \theta_{\min} = \min_{1 \leq i \leq N-1} \theta_i,$$

and let $\lambda_{\min}$ denote the minimum eigenvalue of $\Theta \widehat{L} + \widehat{L}^T \Theta$.

Assumption 2. *There exist time-varying bounded positive functions $\rho_0(t), \rho_1(t), \dots, \rho_{\mathcal{R}}(t)$ such that*

$$\|f(t, x(t), \hat{x}(t)) - f(t, y(t), \hat{y}(t))\| \leq \rho_0(t)\|x(t) - y(t)\| + \sum_{r=1}^{\mathcal{R}} \rho_r(t)\|x(t - \tau_r) - y(t - \tau_r)\|, \quad (2.5)$$

where $\hat{x}(t) = (x(t - \tau_1)^T, x(t - \tau_2)^T, \dots, x(t - \tau_{\mathcal{R}})^T)^T$ and $\hat{y}(t) = (y(t - \tau_1)^T, y(t - \tau_2)^T, \dots, y(t - \tau_{\mathcal{R}})^T)^T$.

Remark 1. *Unlike the traditional global Lipschitz condition, Assumption 2 allows the Lipschitz coefficients to be time-varying and bounded. Many practical systems, such as chaotic systems (e.g., Lorenz system, Chua's circuit), do not satisfy a constant Lipschitz condition but can satisfy the time-varying one. Similar conditions have been adopted in [32, 33] for nonautonomous complex networks.*

3. Logarithmic quantizer

Definition 2. Given $\pi_0 > 0$ and $0 < \rho < 1$, the scalar logarithmic quantizer has its quantization level set defined as

$$\Omega = \{\pm\pi_i : \pi_i = \rho^i \pi_0, i = 0, \pm 1, \pm 2, \dots\} \cup \{0\}.$$

Let $\delta = \frac{1-\rho}{1+\rho}$. The quantizer $q : \mathbb{R} \rightarrow \Omega$ is defined by

$$q(\iota) = \begin{cases} \rho^l \pi_0, & \text{if } \frac{1}{1+\delta} \rho^l \pi_0 < \iota \leq \frac{1}{1-\delta} \rho^l \pi_0, \\ 0, & \text{if } \iota = 0, \\ -q(-\iota), & \text{if } \iota < 0, \end{cases} \quad (3.1)$$

where $l \in \mathbb{Z}$. This quantizer satisfies the sector-bounded property [34]: For any $\iota \in \mathbb{R}$, there exists $\alpha \in [-\delta, \delta]$ such that $q(\iota) = (1 + \alpha)\iota$, and consequently

$$(1 - \delta)|\iota| \leq |q(\iota)| \leq (1 + \delta)|\iota|. \quad (3.2)$$

For the vector case, the scalar quantizer is extended component-wise. For a vector $e_i(t) = (e_{i1}(t), \dots, e_{in}(t))^T$,

$$q(e_i(t)) = (q(e_{i1}(t)), \dots, q(e_{in}(t)))^T.$$

Thus there exists a diagonal matrix $\Lambda_i(t) = \text{diag}(\alpha_{i1}(t), \dots, \alpha_{in}(t))$ with $|\alpha_{im}(t)| \leq \delta$ for all $m = 1, 2, \dots, n$ such that

$$q(e_i(t)) = (I_n + \Lambda_i(t)) e_i(t)$$

and

$$(1 - \delta)\|e_i(t)\| \leq \|q(e_i(t))\| \leq (1 + \delta)\|e_i(t)\|. \quad (3.3)$$

Remark 2. *The parameter $\delta = (1 - \rho)/(1 + \rho)$ directly determines the quantization density; a smaller ρ (coarser quantization) yields a larger δ . The sector-bound representation $q(\iota) = (1 + \alpha)\iota$ with $|\alpha| \leq \delta$ is crucial for Lyapunov analysis because it converts the quantization error into a multiplicative uncertainty.*

Assumption 3. *The quantizer parameter $\delta \in (0, 1)$ is known, and the vector quantizer satisfies the sector property.*

4. Main results

We first give the definition of consensus for the MAS.

Definition 3. The MAS (2.1) is said to achieve consensus if

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_1(t)\| = 0, \quad i = 2, \dots, N. \quad (4.1)$$

Lemma 4. [14] Suppose $\tau > 0$, $m_1(t) : [0, +\infty) \rightarrow \mathbb{R}$ and $m_2(t) : [0, +\infty) \rightarrow \mathbb{R}^+$ are bounded and integrable on each $\mathcal{I}_n$. Let $w(t) \geq 0$ be a scalar function satisfying

$$\dot{w}(t) \leq -m_1(t)w(t) + m_2(t) \left(\sup_{-\tau \leq s < 0} w(t+s) \right).$$

Let $\epsilon(t) = m_1(t) - m_2(t)$ and $\epsilon^* = \inf_n \text{avg}[\epsilon]_n$. If

$$\epsilon^* - (e^{\tau\mu} - 1)m_2^\times > 0, \quad (4.2)$$

where $\mu = \max_n \sup_{t \in \mathcal{I}_n} |\text{avg}[\epsilon]_n - \epsilon(t)|$ and $m_2^\times = \sup_{t \geq 0} |m_2(t)|$, then $w(t)$ is exponentially stable for any delay $\tau < (1/\mu) \ln(1 + \epsilon^*/m_2^\times)$.

Remark 3. The extended Halanay inequality in Lemma 4 does not require $m_1(t) > m_2(t)$ pointwise; it only needs the time-averaged condition $\epsilon^* > (e^{\tau\mu} - 1)m_2^\times$. This makes it especially suitable for systems with time-varying coefficients like $c(t)$ in our paper. Unlike most existing works (see, e.g., [35, 36]) where the pointwise condition $m_1(t) - m_2(t) > 0$ is required to hold for all $t \geq 0$, Lemma 4 only demands a time-averaged condition for stability analysis, which is significantly less restrictive. Specifically, in the special case where $m_1(t)$ is periodic with period T and $m_2(t) \equiv m_2 > 0$ is constant, condition (4.2) reduces to

$$\bar{m}_1 > e^{\tau\mu} m_2,$$

where $\bar{m}_1 = \frac{1}{T} \int_t^{t+T} m_1(s) ds$ and $\mu = \sup_{t \in [0, T]} |m_1(t) - \bar{m}_1|$ for any delay $\tau < \frac{1}{\mu} \ln\left(\frac{\bar{m}_1}{m_2}\right)$.

Lemma 5. [37, Lemma 8.2] Let $h : [0, +\infty) \rightarrow \mathbb{R}$ be a uniformly continuous function. If the integral $\int_0^{+\infty} h(\omega) d\omega$ is convergent, then $\lim_{t \rightarrow +\infty} h(t) = 0$.

4.1. Adaptive controller design

For the followers $i = 2, \dots, N$, the control input is designed as

$$u_i(t) = -\gamma(t)q(e_i(t)), \quad (4.3)$$

where $\gamma(t) \in \mathbb{R}$ is the adaptive gain shared by all followers.

The update law for the adaptive gain $\gamma(t)$ is given by

$$\dot{\gamma}(t) = \eta q \left(\sum_{i=2}^N \|e_i(t)\|^2 \right), \quad \gamma(0) = \gamma_0 > 0, \quad (4.4)$$

where $\eta > 0$ is the adaptive rate. Here, $q(\cdot)$ acts on scalars and also satisfies the sector property:

$$q(s) = (1 + \alpha_s(t))s, \quad \alpha_s(t) \in [-\delta, \delta],$$

which implies

$$(1 - \delta) \sum_{i=2}^N \|e_i(t)\|^2 \leq q \left(\sum_{i=2}^N \|e_i(t)\|^2 \right) \leq (1 + \delta) \sum_{i=2}^N \|e_i(t)\|^2. \quad (4.5)$$

4.2. Convergence analysis

Theorem 1. Consider the MAS (2.1) under Assumptions 1–3. Suppose the quantized control input is given by $u_i(t) = -\gamma q(e_i(t))$ with a constant gain $\gamma > 0$, and there exists $\Delta > 0$ such that

$$\inf_n \frac{1}{\Delta} \int_{n\Delta}^{(n+1)\Delta} \left(\frac{\lambda_{\min}}{\theta_{\max}} c(s) - 2 \sum_{r=0}^{\mathcal{R}} \rho_r(s) \right) ds > (e^{\tau\mu} - 1) \rho^\zeta - 2\gamma(1 - \delta), \quad (4.6)$$

where

$$\rho^\zeta := \sup_{t \geq 0} \sum_{r=1}^{\mathcal{R}} \rho_r(t), \quad \mu = \frac{\lambda_{\min}}{\theta_{\max}} \max_n \sup_{t \in \mathcal{I}_n} |\text{avg}[\epsilon]_n - \epsilon(t)|$$

and

$$\epsilon(t) = \frac{\lambda_{\min}}{\theta_{\max}} c(t) + 2\gamma(1 - \delta) - 2 \sum_{r=0}^{\mathcal{R}} \rho_r(t).$$

Define the gain threshold

$$\gamma^* = \frac{1}{2(1 - \delta)} \left[(e^{\tau\mu} - 1) \rho^\zeta - \inf_n \left(\frac{\lambda_{\min}}{\theta_{\max}} \text{avg}[c]_n - 2 \text{avg} \left[\sum_{r=0}^{\mathcal{R}} \rho_r \right]_n \right) \right].$$

If $\gamma > \gamma^*$, then the error $e(t)$ converges to zero exponentially for any $\tau < (1/\mu) \ln \left(1 + \frac{\inf \text{avg}[\epsilon]_n}{\rho^\zeta} \right)$.

Proof. Consider the Lyapunov function candidate

$$V(t) = \frac{1}{2} e^T(t) (\Theta \otimes I_n) e(t),$$

where Θ is given by Lemma 3 and satisfies

$$\Theta \widehat{L} + \widehat{L}^T \Theta \geq \lambda_{\min} I_{N-1}, \quad \theta_{\min} I_{N-1} \leq \Theta \leq \theta_{\max} I_{N-1}.$$

Under Assumption 2, it follows from the error system (2.4) that

$$\begin{aligned} \dot{V}(t) &= \sum_{i=2}^N \theta_{i-1} e_i^T(t) [f(t, x_i(t), \hat{x}_i(t)) - f(t, x_1(t), \hat{x}_1(t))] - c(t) e^T(t) ((\Theta \widehat{L}) \otimes I_n) e(t) \\ &\quad - \gamma \sum_{i=2}^N \theta_{i-1} e_i^T(t) q(e_i(t)) \\ &\leq \rho_0(t) \sum_{i=2}^N \theta_{i-1} \|e_i(t)\|^2 + \sum_{r=1}^{\mathcal{R}} \rho_r(t) \sum_{i=2}^N \theta_{i-1} \|e_i(t)\| \|e_i(t - \tau_r)\| - c(t) e^T(t) ((\Theta \widehat{L}) \otimes I_n) e(t) \\ &\quad - \gamma \sum_{i=2}^N \theta_{i-1} e_i^T(t) q(e_i(t)). \end{aligned} \quad (4.7)$$

Noting that $V(t) = \frac{1}{2} \sum_{i=2}^N \theta_{i-1} \|e_i(t)\|^2$ and employing the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we obtain

$$\sum_{i=2}^N \theta_{i-1} e_i(t)^T [f(t, x_i(t), \hat{x}_i(t)) - f(t, x_1(t), \hat{x}_1(t))] \leq \left(2\rho_0(t) + \sum_{r=1}^{\mathcal{R}} \rho_r(t) \right) V(t) + \sum_{r=1}^{\mathcal{R}} \rho_r(t) V(t - \tau_r). \quad (4.8)$$

Since $\Theta \widehat{L} + \widehat{L}^T \Theta$ is positive definite, we have

$$e(t)^T ((\Theta \widehat{L}) \otimes I_n) e(t) = \frac{1}{2} e(t)^T ((\Theta \widehat{L} + \widehat{L}^T \Theta) \otimes I_n) e(t) \geq \frac{\lambda_{\min}}{2} \|e(t)\|^2.$$

From the definition of $V(t)$, it holds that $\|e(t)\|^2 \geq \frac{2}{\theta_{\max}} V(t)$; therefore,

$$-c(t) e(t)^T ((\Theta \widehat{L}) \otimes I_n) e(t) \leq -\frac{\lambda_{\min}}{\theta_{\max}} c(t) V(t). \quad (4.9)$$

By Assumption 3, $q(e_i(t)) = (I_n + \Lambda_i(t))e_i(t)$ with $\|\Lambda_i(t)\| \leq \delta$, which implies $e_i(t)^T q(e_i(t)) \geq (1 - \delta) \|e_i(t)\|^2$. Consequently,

$$-\gamma \sum_{i=2}^N \theta_{i-1} e_i(t)^T q(e_i(t)) \leq -\gamma(1 - \delta) \sum_{i=2}^N \theta_{i-1} \|e_i(t)\|^2 = -2\gamma(1 - \delta) V(t). \quad (4.10)$$

Substituting Eqs (4.8)–(4.10) into Eq (4.7) yields

$$\dot{V}(t) \leq \left(2\rho_0(t) + \sum_{r=1}^{\mathcal{R}} \rho_r(t) - \frac{\lambda_{\min}}{\theta_{\max}} c(t) - 2\gamma(1 - \delta) \right) V(t) + \sum_{r=1}^{\mathcal{R}} \rho_r(t) V(t - \tau_r). \quad (4.11)$$

Define

$$m_1(t) := \frac{\lambda_{\min}}{\theta_{\max}} c(t) + 2\gamma(1 - \delta) - 2\rho_0(t) - \sum_{r=1}^{\mathcal{R}} \rho_r(t),$$

$$m_2(t) := \sum_{r=1}^{\mathcal{R}} \rho_r(t).$$

Then we have the differential inequality

$$\dot{V}(t) \leq -m_1(t) V(t) + m_2(t) \left(\sup_{s \in [-\tau, 0]} V(t + s) \right). \quad (4.12)$$

According to Eq (4.6), it follows that

$$\inf \text{avg}[\epsilon]_n \geq (e^{\tau\mu} - 1)\rho^\zeta,$$

that is,

$$\gamma > \frac{1}{2(1 - \delta)} \left[(e^{\tau\mu} - 1)\rho^\zeta - \inf_n \left(\frac{\lambda_{\min}}{\theta_{\max}} \text{avg}[c]_n - 2 \text{avg} \left[\sum_{r=0}^{\mathcal{R}} \rho_r \right]_n \right) \right].$$

By Lemma 4, when $\gamma > \gamma^*$ and $\tau < (1/\mu) \ln \left(1 + \frac{\inf \text{avg}[\epsilon]_n}{\rho^\zeta} \right)$, $V(t)$ decays exponentially; i.e., there exist constants $C, \alpha > 0$ such that

$$V(t) \leq C e^{-\alpha t} V(0).$$

Hence, the error $e(t)$ converges to zero exponentially. $\square$

Theorem 2. Consider the MAS (2.1) under Assumptions 1–3, with the adaptive controller $u_i(t) = -\gamma(t)q(e_i(t))$ and update law in Eq (4.4). If $\gamma^* > 0$ is the gain threshold defined in Theorem 1, then the following conclusions hold:

1. The adaptive gain $\gamma(t)$ is bounded, and $\lim_{t \rightarrow \infty} \gamma(t) = \gamma_\infty < \infty$.
2. Regardless of whether $\gamma(t)$ exceeds γ^* , $\lim_{t \rightarrow \infty} \|e(t)\| = 0$.
3. If there exists a time T such that $\gamma(T) > \gamma^*$, then for all $t \geq T$, the error $e(t)$ decays exponentially.

Proof. The proof is divided into five parts.

Part 1. Boundedness of $\gamma(t)$: From the update law in Eq (4.4), we have

$$\dot{\gamma}(t) = \eta q \left(\sum_{i=2}^N \|e_i(t)\|^2 \right) \geq \eta(1 - \delta) \sum_{i=2}^N \|e_i(t)\|^2 \geq 0,$$

hence $\gamma(t)$ is nondecreasing. We now show that $\gamma(t)$ is bounded. Suppose, to the contrary, that $\gamma(t)$ is unbounded. Then there exists a time T_0 such that $\gamma(T_0) > \gamma^*$, where γ^* is the gain threshold defined in Theorem 1. Since $\gamma(t)$ is nondecreasing, we have $\gamma(t) \geq \gamma(T_0) > \gamma^*$ for all $t \geq T_0$.

Consider the error system (2.4) with the time-varying gain $\gamma(t)$. Because $e_i(t)^T q(e_i(t)) \geq (1 - \delta)\|e_i(t)\|^2 \geq 0$ and $\gamma(t) \geq \gamma(T_0)$, it follows that for $t \geq T_0$,

$$-\gamma(t) \sum_{i=2}^N \theta_{i-1} e_i(t)^T q(e_i(t)) \leq -2\gamma(T_0)(1 - \delta)V(t). \quad (4.13)$$

Thus, replacing Eq (4.10) by Eq (4.13) and following the same analysis as in the proof of Theorem 1 (with constant gain $\gamma = \gamma(T_0) > \gamma^*$), we conclude that the error system (2.4) is exponentially stable. Consequently, there exist constants $C, \alpha > 0$ such that

$$V(t) \leq C e^{-\alpha(t-T_0)} V(T_0), \quad \forall t \geq T_0,$$

where C, α are defined in Theorem 1. Therefore,

$$\int_{T_0}^{\infty} \|e(t)\|^2 dt \leq \int_{T_0}^{\infty} \frac{2V(t)}{\theta_{\min}} dt \leq \frac{C\theta_{\max}\|e(T_0)\|^2}{\theta_{\min}} \int_{T_0}^{\infty} e^{-\alpha(t-T_0)} dt < \infty. \quad (4.14)$$

Now, from the update law in Eq (4.4) and the quantizer lower bound in Eq (4.5), we obtain

$$\dot{\gamma}(t) \leq \eta(1 + \delta) \sum_{i=2}^N \|e_i(t)\|^2 = \eta(1 + \delta)\|e(t)\|^2.$$

Integrating this inequality from T_0 to any $t > T_0$ yields

$$\gamma(t) = \gamma(T_0) + \int_{T_0}^t \dot{\gamma}(s) ds \leq \gamma(T_0) + \eta(1 + \delta) \int_{T_0}^{\infty} \|e(s)\|^2 ds < \infty,$$

which contradicts the assumption that $\gamma(t)$ is unbounded. Hence $\gamma(t)$ is bounded. Since $\gamma(t)$ is nondecreasing and bounded, $\lim_{t \rightarrow \infty} \gamma(t) = \gamma_\infty$ exists, and γ_∞ is finite. This establishes conclusion 1 of Theorem 2.

Part 2. Boundedness of $\|e(t)\|$: To obtain the boundedness of $\|e(t)\|$, we will first estimate $\dot{V}(t)$. From the error system (2.4) and $u_i(t) = -\gamma(t)q(e_i(t))$, we have

$$\dot{e}_i(t) = f(t, x_i(t), \hat{x}_i(t)) - f(t, x_1(t), \hat{x}_1(t)) - c(t) \sum_{j=2}^N \hat{l}_{ij} e_j(t) - \gamma(t) q(e_i(t)), \quad i = 2, \dots, N.$$

Using Assumption 2, conclusion 1 of Theorem 2 and $\|q(e_i)\| \leq (1 + \delta)\|e_i\|$ yields

$$\begin{aligned} \|\dot{e}_i(t)\| &\leq \rho_0(t)\|e_i(t)\| + \sum_{r=1}^{\mathcal{R}} \rho_r(t)\|e_i(t - \tau_r)\| + c(t)\|\widehat{L}\|\|e(t)\| + \gamma(t)(1 + \delta)\|e_i(t)\| \\ &\leq \bar{\rho}\|e_i(t)\| + \bar{\rho} \sum_{r=1}^{\mathcal{R}} \|e_i(t - \tau_r)\| + c_M\|\widehat{L}\|\|e(t)\| + \gamma_\infty(1 + \delta)\|e_i(t)\|. \end{aligned}$$

Let $F = \max\{\bar{\rho} + \gamma_\infty(1 + \delta) + c_M\|\widehat{L}\|, \bar{\rho}\}$ and $\bar{\rho} := \max_{\{r=0,1,\dots,\mathcal{R}\}} \sup_{t \geq 0} \rho_r(t)$. Using $\|e_i(t)\| \leq \|e(t)\|$ and $\|e_i(t - \tau_r)\| \leq \|e(t - \tau_r)\|$, one gets

$$\|\dot{e}_i(t)\| \leq F\left(\|e(t)\| + \sum_{r=1}^{\mathcal{R}} \|e(t - \tau_r)\|\right).$$

Consequently,

$$\|\dot{e}(t)\| \leq \sqrt{N-1} \max_{2 \leq i \leq N} \|\dot{e}_i(t)\| \leq \sqrt{N-1} F \left(\|e(t)\| + \sum_{r=1}^{\mathcal{R}} \|e(t - \tau_r)\| \right). \quad (4.15)$$

Then

$$\dot{V}(t) \leq \|\Theta \otimes I_n\| \|e(t)\| \|\dot{e}(t)\| \leq \theta_{\max} \|e(t)\| \|\dot{e}(t)\|.$$

Using $\|e(t)\| \leq \sqrt{2V(t)/\theta_{\min}}$ and the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we obtain

$$\dot{V}(t) \leq \widehat{F}\left(V(t) + \sum_{r=1}^{\mathcal{R}} V(t - \tau_r)\right), \quad (4.16)$$

where $\widehat{F} = (2 + \mathcal{R}) \sqrt{N-1} F \frac{\theta_{\max}}{\theta_{\min}}$.

Next we will show the boundedness of $\|e(t)\|$. Suppose that $\|e(t)\|$ is unbounded. Then $V(t) = \frac{1}{2} e^T(t)(\Theta \otimes I_n)e(t)$ is also unbounded because $V(t) \geq \frac{\theta_{\min}}{2} \|e(t)\|^2$. From Eq (4.14), we have

$$\int_{T_0}^{\infty} V(t) dt \leq \frac{\theta_{\max}}{2} \int_{T_0}^{\infty} \|e(t)\|^2 dt < \infty. \quad (4.17)$$

Hence there exist a constant $\varepsilon > 0$ and a strictly increasing sequence $\{a_m\}$ with $a_m \rightarrow \infty$ such that $V(a_m) \geq \varepsilon$. Let $M_m = \max_{t \in [a_m, a_m + 2\tau]} V(t) \geq \varepsilon$. Since $V(t)$ is unbounded,

$$\hat{a}_m = \inf\{t \geq a_m + 2\tau \mid V(t) \geq M_m\}$$

is well defined and $V(\hat{a}_m) = M_m$, $V(t) \leq M_m$ for all $t \in [a_m, \hat{a}_m]$.

For any $t \in [a_m + \tau, \hat{a}_m]$, we get from Eq (4.16) that

$$\dot{V}(t) \leq \widehat{F}(1 + \mathcal{R})M_m.$$

Hence, for $t \leq \hat{a}_m$, one gets

$$V(t) \geq V(\hat{a}_m) - \widehat{F}(1 + \mathcal{R})M_m(\hat{a}_m - t) \geq M_m[1 - \widehat{F}(1 + \mathcal{R})(\hat{a}_m - t)].$$

Case 1: $\tau \leq \frac{1}{2\widehat{F}(1+\mathcal{R})}$. Since $\widehat{F}(1 + \mathcal{R})\tau \leq \frac{1}{2}$, we get

$$V(t) \geq M_m(1 - \widehat{F}(1 + \mathcal{R})\tau) \geq \frac{M_m}{2}$$

for any $t \in [\hat{a}_m - \tau, \hat{a}_m]$. Hence

$$\int_{\hat{a}_m - \tau}^{\hat{a}_m} V(t) dt \geq \frac{M_m}{2}\tau \geq \frac{\varepsilon\tau}{2}.$$

Case 2: $\tau > \frac{1}{2\widehat{F}(1+\mathcal{R})}$. Let $\tau^\Delta = \frac{1}{2\widehat{F}(1+\mathcal{R})}$. We get

$$V(t) \geq M_m(1 - \widehat{F}(1 + \mathcal{R})\tau^\Delta) = \frac{M_m}{2}$$

for any $t \in [\hat{a}_m - \tau^\Delta, \hat{a}_m]$, so that

$$\int_{\hat{a}_m - \tau^\Delta}^{\hat{a}_m} V(t) dt \geq \frac{M_m}{2}\tau^\Delta \geq \frac{\varepsilon}{4\widehat{F}(1 + \mathcal{R})}.$$

In both cases, there exists a constant $\widetilde{\delta} > 0$ independent of m such that

$$\int_{\hat{a}_m - \tau}^{\hat{a}_m} V(t) dt \geq \widetilde{\delta},$$

which contradicts Eq (4.17). Therefore, $\|e(t)\|$ is bounded. Denote

$$M_e = \sup_{t \geq 0} \|e(t)\| < \infty.$$

Part 3. Boundedness of $\frac{d}{dt}\|e(t)\|^2$: From Eq (4.15), we obtain

$$\|\dot{e}(t)\| \leq \sqrt{N-1}F\left(M_e + \sum_{r=1}^{\mathcal{R}} M_e\right) = \sqrt{N-1}F(1 + \mathcal{R})M_e,$$

which leads to

$$\frac{d}{dt}\|e(t)\|^2 = 2e(t)^T \dot{e}(t) \leq 2\sqrt{N-1}F(1 + \mathcal{R})M_e^2,$$

i.e., $\frac{d}{dt}\|e(t)\|^2$ is bounded.

Part 4. Asymptotic consensus: From Part 1, $\gamma(t)$ is increasing and bounded, so $\int_0^\infty \dot{\gamma}(t)dt < \infty$. Using Eq (4.4) and $q(s) \geq (1 - \delta)s$, we have

$$\int_0^\infty \eta(1 - \delta) \sum_{i=2}^N \|e_i(t)\|^2 dt \leq \int_0^\infty \dot{\gamma}(t)dt < \infty,$$

hence $\int_0^\infty \|e(t)\|^2 dt < \infty$. Part 3 shows that $\frac{d}{dt}\|e(t)\|^2$ is bounded, which implies $\|e(t)\|^2$ is uniformly continuous. By Barbălat's lemma (Lemma 5), $\lim_{t \rightarrow \infty} \|e(t)\|^2 = 0$, i.e.,

$$\lim_{t \rightarrow \infty} \|e(t)\| = 0.$$

Therefore, regardless of whether $\gamma(t)$ exceeds γ^* , $\lim_{t \rightarrow \infty} \|e(t)\| = 0$. This proves conclusion 2 of Theorem 2.

Part 5. Exponential convergence when $\gamma(T) > \gamma^$:* If there exists a time T such that $\gamma(T) > \gamma^*$, then because $\gamma(t)$ is increasing, $\gamma(t) \geq \gamma(T) > \gamma^*$ for all $t \geq T$. For the error system (2.4) with time-varying gain $\gamma(t)$, by Part 1 and Theorem 1 (with γ replaced by $\gamma(T)$), we conclude that $\|e(t)\|$ decays exponentially for $t \geq T$. This proves conclusion 3 of Theorem 2.

Thus the proof of Theorem 2 is complete. $\square$

Remark 4. It is worth emphasising that the gain threshold γ^* defined in Theorem 1 is introduced solely as a theoretical sufficient condition for deriving exponential convergence via the generalized Halanay inequality. It is not a parameter that the agents need to know or pre-set in the control protocol. As shown in Theorem 2, asymptotic consensus is guaranteed regardless of whether $\gamma(t)$ ever exceeds γ^* ; if the gain does surpass the threshold at some time T , then exponential convergence is additionally obtained for all $t \geq T$. Thus, γ^* serves only as an analytical tool to characterize the convergence rate, not as a design parameter for the controller.

5. Numerical simulations

Consider a MAS consisting of one leader (agent 1) and seven followers (agents 2, ..., 8). Each agent is modeled as a three-dimensional Lorenz system with two self-delays, governed by

$$\dot{x}_i(t) = f(x_i(t), x_i(t - \tau_1), x_i(t - \tau_2)), \quad i = 1, \dots, 8, \quad (5.1)$$

where

$$f(v, v^{\tau_1}, v^{\tau_2}) = \begin{bmatrix} 10(v_2 - v_1) \\ 28v_1 - v_2 - v_1v_3 \\ v_1v_2 - \frac{8}{3}v_3 \end{bmatrix} + \kappa \sum_{r=1}^2 (v - v^{\tau_r}), \quad (5.2)$$

where $v = (v_1, v_2, v_3)^T$, $v^{\tau_r} = (v_1(t - \tau_r), v_2(t - \tau_r), v_3(t - \tau_r))^T$ ($r = 1, 2$), $\kappa = 0.2$, $\tau_1 = 0.2$, and $\tau_2 = 0.5$.

For the Lorenz-type node dynamics in Eq (5.2), the function f is not globally Lipschitz due to the quadratic terms. However, in the numerical simulations, all trajectories evolve within a bounded region. On such a bounded invariant set, Assumption 2 is satisfied with constant coefficients. Indeed, writing

$$f(v, v^{\tau_1}, v^{\tau_2}) = Av + g(v) + \kappa \sum_{r=1}^2 (v - v^{\tau_r}),$$

where

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, \quad g(v) = \begin{bmatrix} 0 \\ -v_1 v_3 \\ v_1 v_2 \end{bmatrix},$$

we have $\|A\| \leq 30$ (induced Euclidean norm). For the nonlinear term g , its Lipschitz constant on any ball of radius $M > 0$ (i.e., $\|v\| \leq M$) is bounded by $2M$. In the present simulations, taking a conservative uniform bound $M = 50$ for all state components yields $2M = 100$. Consequently, we may choose

$$\rho_0 = \|A\| + 2M \leq 130, \quad \rho_1 = \rho_2 = \kappa = 0.2,$$

which satisfy the Lipschitz condition in Eq (5.2) for all trajectories considered. The same boundedness argument applies when varying parameters such as η , δ , or $\tau_{\max}$, as the synchronization errors remain uniformly bounded.

The leader (agent 1) evolves freely without control. The followers are controlled by the adaptive law

$$u_i(t) = -\gamma(t) q(e_i(t)), \quad i = 2, \dots, 8, \quad (5.3)$$

where $e_i(t) = x_i(t) - x_1(t)$ is the synchronization error and $\gamma(t)$ is a shared adaptive gain updated by

$$\dot{\gamma}(t) = \eta q\left(\sum_{i=2}^8 \|e_i(t)\|^2\right), \quad \gamma(0) = 0.1, \quad (5.4)$$

with the initial setting $\eta = 0.01$. The function $q(\cdot)$ denotes a logarithmic quantizer with parameter $\delta = 0.2$, satisfying $(1 - \delta)|s| \leq |q(s)| \leq (1 + \delta)|s|$ for any $s \in \mathbb{R}$.

The interaction among agents is described by a directed graph whose Laplacian matrix is

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 4 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

Partitioning L as in Eq (2.3) yields a nonsingular M-matrix $\widehat{L} \in \mathbb{R}^{7 \times 7}$, which confirms Assumption 1. From the first column of L , agents 2, 5, 7, and 8 have direct access to the leader. Computing Θ according to Lemma 3 gives $\lambda_{\min} = 1.5577$, $\theta_{\max} = 3.5000$, and $\theta_{\min} = 0.3636$. The coupling strength is taken as $c(t) = 1 + 0.5 \sin t$ (average 1 over 2π), fulfilling the time-average condition in Theorem 1. The initial states are set to $x_1(0) = [0, -1, 2]^T$ and $x_i(0) = [i, -2i, -3i]^T$ for $i = 2, \dots, 8$. All simulations are performed using the MATLAB solver dde23 over the interval $[0, 10]$ s, unless stated otherwise.

5.1. Basic synchronization performance

We first demonstrate the effectiveness of the proposed adaptive quantized controller with $\delta = 0.2$ and $\eta = 0.01$. The global error norm is defined as $\|e(t)\| = (\sum_{i=2}^8 \|e_i(t)\|^2)^{1/2}$. It decays rapidly and becomes

very small after about 1.5 s, indicating successful synchronization. Figure 1 shows the individual error norms $\|e_i(t)\|$ for $i = 2, \dots, 8$; all followers quickly approach the leader, and the transient differences are clearly visible.

Figure 2 shows the adaptive gain $\gamma(t)$ on $[0, 2]$ s, with a zoomed view on $[0, 0.6]$ s in the inset. The gain increases monotonically and converges to $\gamma_\infty \approx 4.837$ (dashed line), confirming the boundedness property stated in Theorem 2. The state trajectories of all agents appear in Figure 3; the leader (black) and the followers (colored) synchronize rapidly.

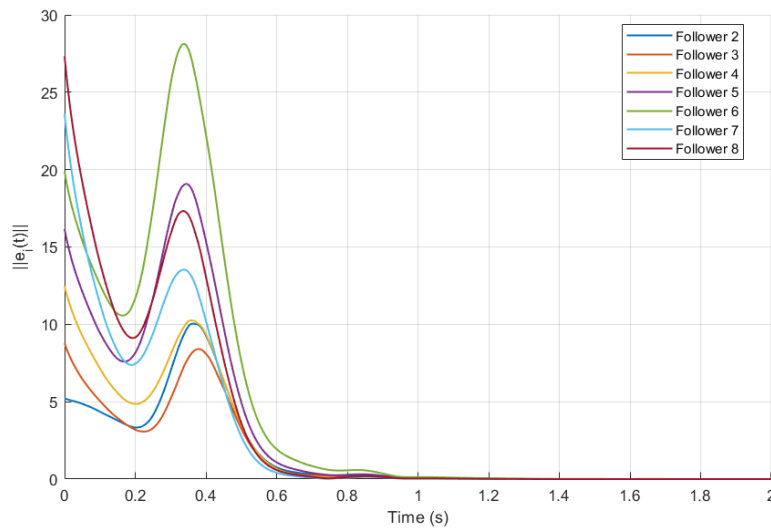


Figure 1. The individual error norms of the seven followers for $\eta = 0.01$ (first two seconds).

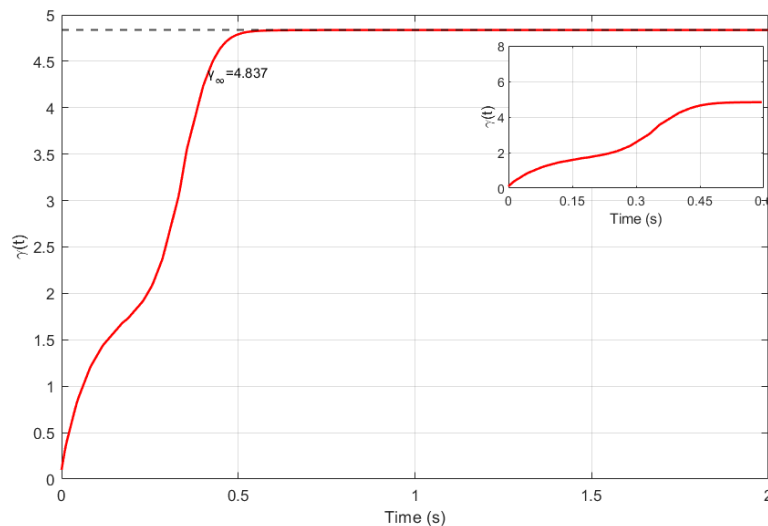


Figure 2. Adaptive gain $\gamma(t)$ for $\eta = 0.01$ (main plot on $[0, 2]$ s and a zoom on $[0, 0.6]$ s). The dashed line marks the final value γ_∞ .

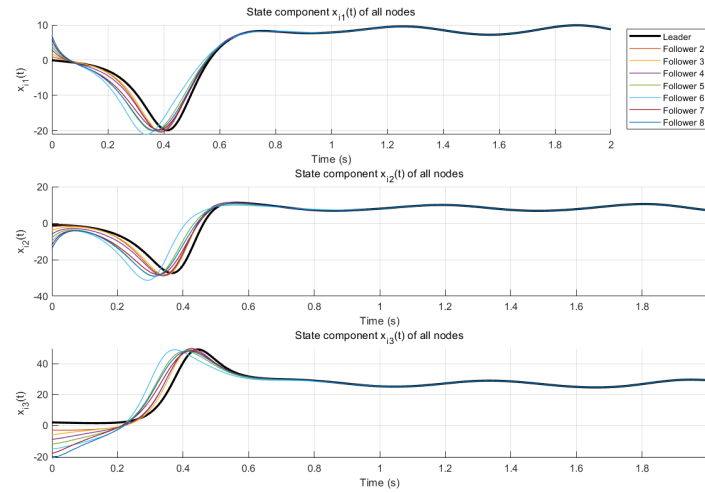


Figure 3. Time evolution of the three state components $x_{i1}(t)$, $x_{i2}(t)$ and $x_{i3}(t)$ of all eight nodes (leader in black, followers in color). The first two seconds are displayed.

5.2. Influence of the adaptation rate η

To investigate the effect of the adaptive rate, we repeat the simulation with $\eta = 0.5, 0.1, 0.02, 0.01$ while keeping other parameters unchanged. Figure 4 shows the gain trajectories for the four values of η on $[0, 2]$ s. A larger η leads to a faster increase of $\gamma(t)$, but all gains converge to bounded steady-state values. Table 1 lists the synchronization time T_{sync} (first instant with $\|e(t)\| < 10^{-4}$) and the final gain γ_{∞} (approximated by $\gamma(10)$). A smaller η yields a smaller final gain but a slightly longer sync time.

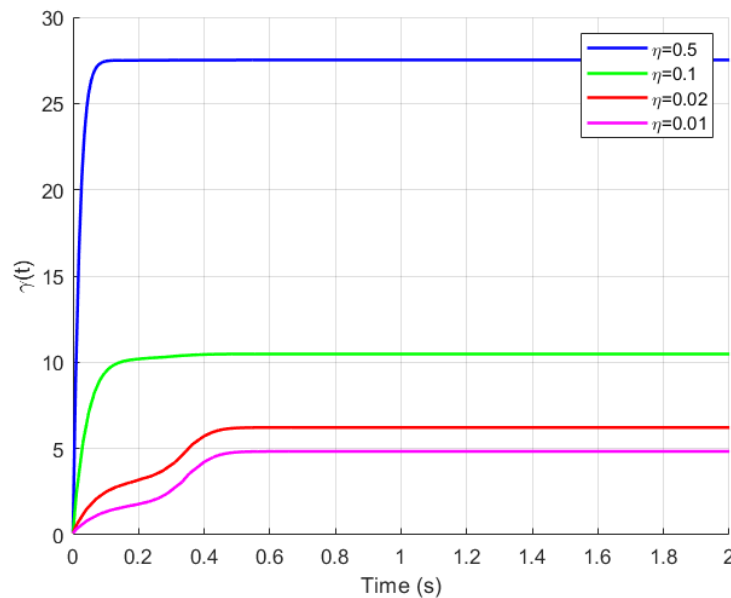


Figure 4. Adaptive gain $\gamma(t)$ for different adaptation rates η (first two seconds).

Table 1. Synchronization time T_{sync} and final adaptive gain γ_{∞} for different η .

η	T_{sync} (s)	γ_{∞}
0.50	1.214	27.518
0.10	1.694	10.480
0.02	2.094	6.222
0.01	2.629	4.837

5.3. Quantized vs. non-quantized control

To highlight the role of quantization, we compare the same system with and without the logarithmic quantizer ($\delta = 0.4$ vs. $\delta = 0$). Figure 5 shows the global error (left) and the adaptive gain (right). The quantized system exhibits a slightly slower convergence but still achieves asymptotic consensus, while the non-quantized system converges faster and with a lower final gain γ_{∞} (approx. 5.0688 vs. 4.8411). This trade-off confirms that quantization reduces communication burden at the cost of a higher control effort and a longer settling time, which is consistent with the theoretical analysis.

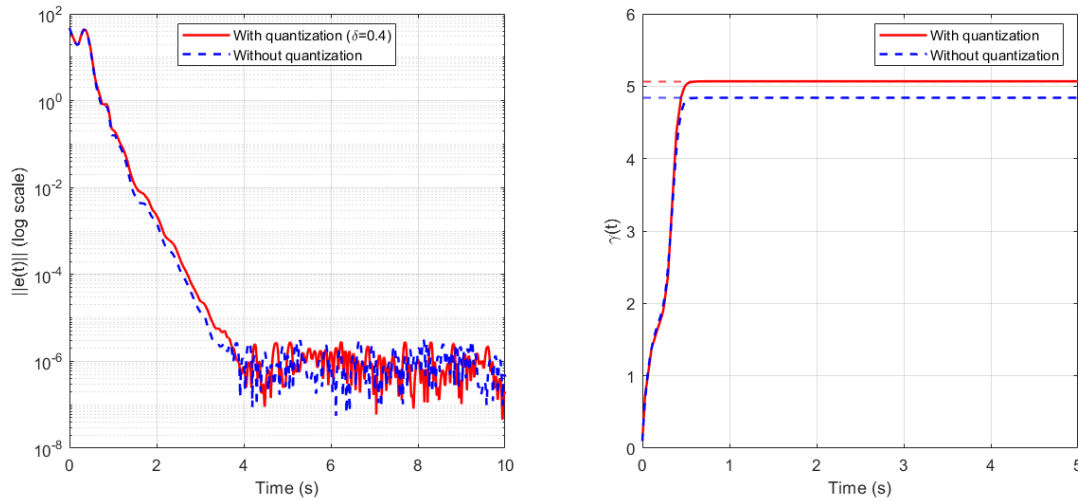


Figure 5. Comparison between quantized ($\delta = 0.4$) and non-quantized control: (left) the global error norm; (right) the adaptive gain. Dashed lines indicate the final gain values.

5.4. Existence of the gain threshold gain γ^*

Theorem 2 guarantees exponential consensus whenever the adaptive gain exceeds the gain threshold γ^* defined in Theorem 1. To verify this, we fix the controller gain to constant values and simulate the system with $\gamma = 23.76$ (below the estimated $\gamma^* \approx 25.76$) and $\gamma = 30.76$ (above γ^*). Figure 6 shows the resulting error evolution. With $\gamma = 30.76 > \gamma^*$, the error converges rapidly, in agreement with the exponential bound. For $\gamma = 23.76 < \gamma^*$, the error still goes to zero but at a slower rate, which corroborates the asymptotic consensus conclusion of Theorem 2.

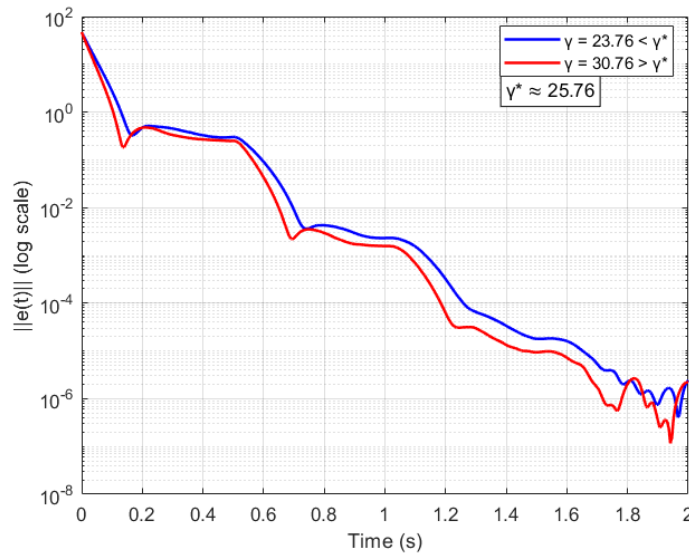


Figure 6. Fixed-gain quantized control: error norm for $\gamma = 23.76 < \gamma^*$ (blue) and $\gamma = 30.76 > \gamma^*$ (red). The estimated $\gamma^* \approx 25.76$ is marked.

5.5. Influence of quantization density δ

We further study the effect of the quantization parameter δ by choosing $\delta = 0.2, 0.4$, and 0.6 . Figure 7 shows that as δ increases, the convergence slows down ($T_{\text{sync}} = 2.628, 2.708, 3.411$ s), yet the final gain γ_∞ does not increase monotonically ($4.8370, 5.0688, 4.0836$). In fact, $\delta = 0.6$ yields the smallest γ_∞ because a very coarse quantizer can cause the adaptive law to underestimate the error and stop increasing the gain too early. Hence, coarser quantization does not necessarily demand a higher control effort.

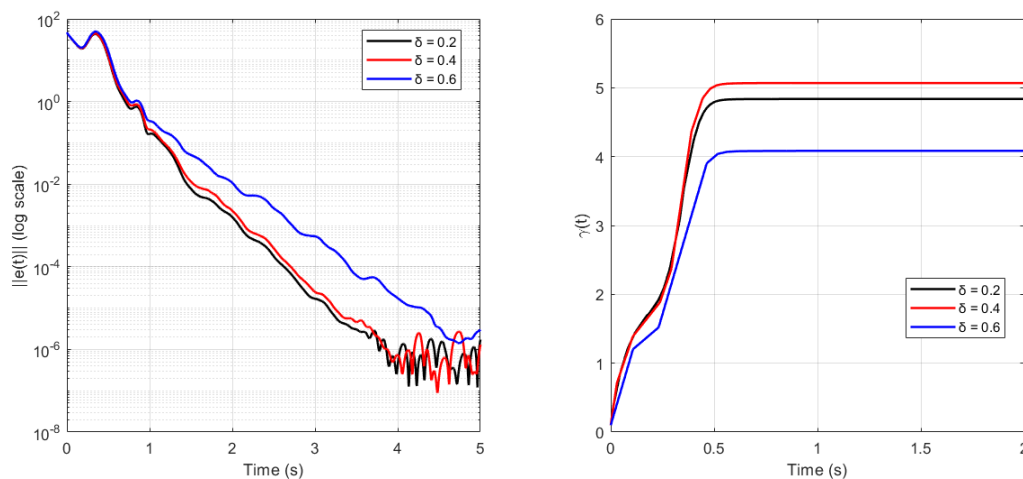


Figure 7. Effect of quantization density δ : (left) the global error; (right) the adaptive gain. Increasing δ slows down synchronization.

5.6. Robustness to multiple self-delays

Finally, we test the robustness of the scheme against different self-delay configurations. Three cases are considered: $\tau_{\max} = 0.2$ (delays $[0.1, 0.2]$), $\tau_{\max} = 0.5$ ($[0.2, 0.5]$), and $\tau_{\max} = 1.5$ ($[0.5, 1.5]$). Figure 8 shows the global error evolutions (main plot on $[0, 2]$ s and a zoom on $[1.0, 1.5]$ s). Larger delays cause slower convergence, but the system still achieves consensus in all cases. The corresponding adaptive gains are plotted in Figure 9 (with a zoom on $[0.5, 1.0]$ s). The final gains increase with the maximum delay, demonstrating that the adaptive mechanism compensates for the detrimental effect of longer delays. These simulations confirm the robustness of the proposed approach to multiple self-delays as claimed in the theoretical analysis.

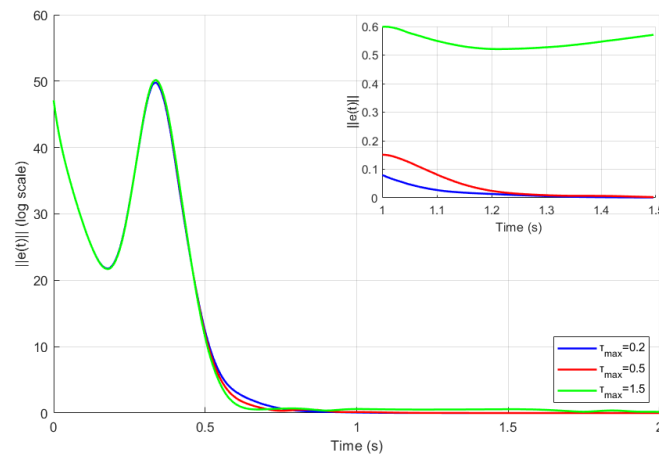


Figure 8. The global synchronization error for different maximum self-delays $\tau_{\max} = 0.2, 0.5, 1.5$.

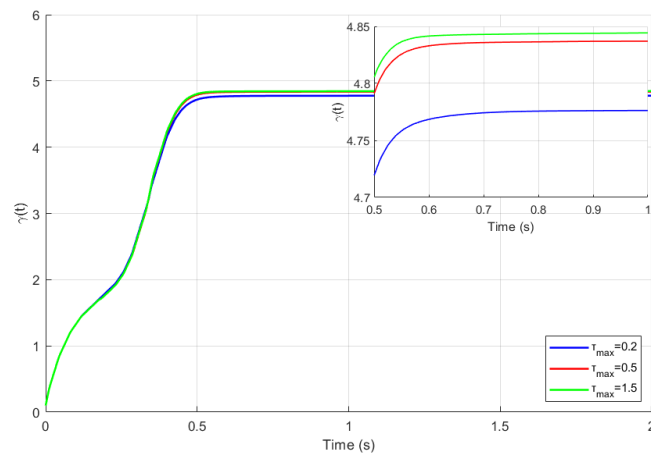


Figure 9. The adaptive gain evolution for the three delay configurations.

6. Conclusions

This paper establishes a novel average-valued condition for consensus of MASs via a quantized adaptive control protocol, considering multiple self-delays, time-varying coupling, and logarithmic quantizers. By constructing a suitable Lyapunov functional and applying a generalized Halanay inequality that exploits the time-average property of the coupling strength, sufficient conditions for exponential consensus are derived, and the boundedness of the adaptive gain is proved. It is shown that asymptotic consensus is always achieved, and exponential convergence occurs whenever the adaptive gain exceeds the gain threshold defined by the proposed average-valued condition. Numerical simulations validate the theoretical results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

The work presented here was performed in collaboration among both authors. Yufang Chen: Designed and analyzed the research, wrote the paper, and conducted the simulations. Zhenkun Huang: Conceived the research idea, and performed theoretical analysis and verification. All authors contributed to and approved the manuscript.

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Conflict of interest

The authors declare there is no conflict of interest.

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