



Research article

The soliton solution and periodic lump solution of the (3+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation with variable coefficients

Wenjing Yang¹, Kaili Liu¹, Bo Tang^{1,2,3,4}, Shijie Deng¹ and Fengyun Lv^{1,*}

¹ School of Mathematics and Statistics, Hubei University of Arts and Science, Xiangyang 441053, China

² Hubei Key Laboratory of Power System Design and Test for Electrical Vehicle, Hubei University of Arts and Science, Xiangyang 441053, China

³ School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China

⁴ School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

* **Correspondence:** Email: mathslv@163.com.

Abstract: This paper investigates the soliton solution and periodic lump solution of the $(3 + 1)$ -dimensional generalized Bogoyavlensky–Konopelchenko equation with variable coefficients. First, by introducing specific constraint conditions, a novel bilinear expression for the aforementioned equation is derived. Subsequently, with the aid of symbolic computation, the corresponding soliton and periodic lump solution are successfully constructed. Finally, three-dimensional plots, density plots, and x-axis projection diagrams are presented to analyze the interaction behaviors of the obtained solutions, which provides intuitive visual evidence for the physical characteristics of these solutions.

Keywords: the $(3+1)$ -dimensional generalized Bogoyavlensky-Konopelchenko equation with variable coefficient; periodic lump solution; soliton solution

1. Introduction

Partial differential equations (PDEs) are primarily used to characterize propagation and diffusion phenomena in continuous media, and are extensively applied in physics, engineering, finance, and other fields. Compared to constant-coefficient PDEs, variable coefficients partial differential equations (vcPDEs) can more accurately simulate physical processes in inhomogeneous media. In recent years, numerous scholars have conducted in-depth studies on vcPDEs [1–5]. Various methods have been successfully applied to solve variable coefficients problems, including the tanh-function

expansion method [6–8], the Bäcklund transformation technique [9–11], the Lie group symmetry analysis [12, 13], and so on.

In this paper, we consider the (3+1)-dimensional generalized Bogoyavlensky-Konopelchenko (BK) equation with variable coefficients [14]:

$$\begin{aligned} &\alpha_1(t)u_{xt} + \alpha_2(t)u_{yt} + \alpha_3(t)u_{zt} + \alpha_4(t)u_{xy} + \alpha_5(t)u_{yy} + \alpha_6(t)u_{yz} + \alpha_7(t)u_{xxx} + \alpha_8(t)u_{xxy} + \\ &\alpha_9(t)u_{xxz} + \alpha_{10}(t)u_x u_{xx} + \alpha_{11}(t)u_x u_{xy} + \alpha_{12}(t)u_{xx} u_y + \alpha_{13}(t)u_x u_{xz} + \alpha_{14}(t)u_{xx} u_z = 0, \end{aligned} \quad (1.1)$$

where the coefficients $\alpha_i(t)$ ($i = 1, 2, \dots, 14$) are functions of time t . The BK equation is mainly applied to simulate the intricate propagation and interaction of nonlinear waves (such as solitons and rogue waves) in three-dimensional space within physical systems including fluids, plasmas, and nonlinear optics. The introduction of variable coefficients enables the equation to account for spatial and temporal inhomogeneities of the physical medium, thus improving its capability to describe complex nonlinear wave propagation phenomena.

In recent years, numerous analytical studies have been carried out on the BK equation. In 2024, Karim K. Ahmed et al. obtained several analytical solutions for the (2+1)-dimensional generalized BK equation using an improved extended tanh-function method [15]. In 2025, Waseem Razzaq and Asim Zafar adopted the Hirota bilinear method to construct three-wave solutions, block-wave solutions, and rogue wave soliton solutions of the generalized BK equation [16]. In 2026, Ma et al. further investigated the dynamical structures of lump waves in a refined (2+1)-dimensional BK model [17]. Overall, most existing research focuses on constructing localized wave solutions for low-dimensional constant-coefficient BK equations, while analytical investigations into the (3+1)-dimensional variable-coefficient generalized BK equation remain scarce. Compared with constant-coefficient models, the space-time-dependent variable-coefficient form is capable of characterizing wave discrepancies induced by medium inhomogeneity and hence possesses superior theoretical adaptability. Accordingly, this paper employs the Hirota bilinear method to derive two types of analytical solutions, namely periodic lump solutions and soliton solutions of the target equation. The physical properties of these two kinds of solutions are comparatively analyzed, which enriches the system of exact solutions for the equation.

Some cases of Eq (1.1) are as follows.

(1) When $\alpha_1(t) = \alpha_9(t) = 1, \alpha_{13}(t) = 4, \alpha_{14}(t) = 2, \alpha_2(t) = \alpha_3(t) = \alpha_4(t) = \alpha_5(t) = \alpha_6(t) = \alpha_7(t) = \alpha_8(t) = \alpha_{10}(t) = \alpha_{11}(t) = \alpha_{12}(t) = 0$, Eq (1.1) can be simplified to a (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation [18]:

$$u_{xt} + u_{xxz} + 4u_x u_{xz} + 2u_{xx} u_z = 0.$$

The CBS equation is transformed into an ordinary differential equation using the symmetry method, and the exact solutions are subsequently obtained.

(2) When $\alpha_1(t) = \alpha_8(t) = 1, \alpha_{11}(t) = \alpha_{12}(t) = 3, \alpha_4(t) = \delta_1, \alpha_5(t) = \delta_2, \alpha_2(t) = \alpha_3(t) = \alpha_6(t) = \alpha_7(t) = \alpha_9(t) = \alpha_{10}(t) = \alpha_{13}(t) = \alpha_{14}(t) = 0$, Eq (1.1) can be degenerated to the generalized Calogero-Bogoyavlenskii-Schiff equation (gCBSE) [19]:

$$u_{xt} + u_{xxy} + 3u_x u_{xy} + 3u_{xx} u_y + \delta_1 u_{xy} + \delta_2 u_{yy} = 0,$$

where δ_i are constants ($i=1,2$). By employing a new extended direct algebraic method, Alwan et al. obtained analytical solitary wave solutions to the gCBSE.

(3) When $\alpha_2(t) = \alpha_3(t) = \alpha_8(t) = \alpha_9(t) = 1, \alpha_{11}(t) = \alpha_{12}(t) = \alpha_{13}(t) = \alpha_{14}(t) = -3, \alpha_1(t) = \alpha_4(t) = \alpha_5(t) = \alpha_6(t) = \alpha_7(t) = \alpha_{10}(t) = 0$, Eq (1.1) can be reduced to the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation [20]:

$$u_{yt} + u_{zt} + u_{xxxxy} + u_{xxxz} - 3u_x u_{xy} - 3u_{xx} u_y - 3u_x u_{xz} - 3u_{xx} u_z = 0.$$

Wazwaz conducted a Painlevé analysis of the BLMP equation and derived multiple complex soliton solutions.

In this paper, we consider the following integrability constraints:

$$\alpha_{10}(t) = 6\alpha_7(t), \alpha_{11}(t) = \alpha_{12}(t) = 3\alpha_8(t), \alpha_{13}(t) = \alpha_{14}(t) = 3\alpha_9(t).$$

Through the bilinear transformation

$$u(x, y, z, t) = 2(\ln f(x, y, z, t))_x, \quad (1.2)$$

we can derive the bilinear form of Eq (1.1):

$$\begin{aligned} & [(\alpha_1(t)D_x D_t + \alpha_2(t)D_y D_t + \alpha_3(t)D_z D_t + \alpha_4(t)D_x D_y + \alpha_5(t)D_y D_y \\ & + \alpha_6(t)D_y D_z + \alpha_7(t)D_x^4 + \alpha_8(t)D_x^3 D_y + \alpha_9(t)D_x^3 D_z)]f \cdot f \\ & = \alpha_1(t)(f f_{xt} - f_x f_t) + \alpha_2(t)(f f_{yt} - f_y f_t) + \alpha_3(t)(f f_{zt} - f_z f_t) + \alpha_4(t)(f f_{xy} - f_x f_y) + \\ & \quad \alpha_5(t)(f f_{yy} - f_y f_y) + \alpha_6(t)(f f_{yz} - f_y f_z) + \alpha_7(t)(3f_{xx} f_{xx} - 4f_x f_{xxx} + f f_{xxxx}) + \\ & \quad \alpha_8(t)(3f_{xy} f_{xx} - 3f_x f_{xxy} - f_y f_{xxx} + f f_{xxxy}) + \alpha_9(t)(3f_{xz} f_{xx} - 3f_x f_{xxz} - f_z f_{xxx} + f f_{xxxz}) \\ & = 0, \end{aligned} \quad (1.3)$$

where D denotes the bilinear differential operator, which is defined as follows [21]:

$$D_x^p D_y^q f \cdot g = (\partial_x - \partial_{x'})^p (\partial_y - \partial_{y'})^q f(x, y) g(x', y') \Big|_{x'=x, y'=y}.$$

The structure of this paper is organized as follows: In Section 2, we obtain the periodic lump solution of Eq (1.1); in Section 3, we obtain soliton solutions for Eq (1.1); and in Section 4, we conclude the paper with future research prospects.

2. Periodic lump solution

In this section, we obtain the periodic lump solution of Eq (1.1). To begin with, we assume the following:

$$\begin{aligned} f &= k_1 \cos(\zeta_{1i}) + k_2 \cosh(\zeta_2) + k_3 \cosh(\zeta_3) + k_4, \\ \zeta_i &= a_i x + b_i y + c_i z + d_i(t). \end{aligned} \quad (2.1)$$

Periodic lump solutions describe localized high-energy wave packets periodically emerging and drifting on a periodic background wave field, which physically correspond to intermittent extreme localized waves on periodic tidal internal waves in oceans, periodic clustered density perturbations in magnetized plasmas, and oscillating localized light spots in periodically modulated optical media. Such analytical solutions are mathematically self-consistent under ideal integrable conservative conditions

and can qualitatively interpret the periodic excitation of intermittent intense disturbances in nature. Nevertheless, they cannot directly reproduce quantitative wave propagation in practical environments due to the neglect of dissipation, random turbulence, finite boundary effects, and wave breaking; they only serve as a theoretical benchmark and require additional dissipative and environmental corrections to match real observational data.

Then, by substituting Eq (2.1) into Eq (1.1), after solving the system of equations, we can obtain the following:

$$\begin{aligned}
 a_1 &= \frac{a_2 c_1}{c_2}, \quad c_3 = \frac{a_3 c_2}{a_2}, \quad b_1 = \frac{-a_2 c_1 \alpha_7(t) - c_1 c_2 \alpha_9(t)}{c_2 \alpha_8(t)}, \\
 b_2 &= \frac{-a_2 \alpha_7(t) - c_2 \alpha_9(t)}{\alpha_8(t)}, \quad b_3 = \frac{-a_3 c_2 \alpha_9(t) - a_2 a_3 \alpha_7(t)}{a_2 \alpha_8(t)}, \\
 d_1(t) &= \int \frac{B_1 B_2 c_1}{B_3 c_2} dt, \quad d_2(t) = \int \frac{B_1 B_2}{B_3} dt, \quad d_3(t) = \int \frac{a_3 B_1 B_2}{a_2 B_3} dt \\
 B_1 &= a_2 \alpha_7(t) + c_2 \alpha_9(t), \\
 B_2 &= a_2 (\alpha_4(t) \alpha_8(t) - \alpha_5(t) \alpha_7(t)) + c_2 (\alpha_6(t) \alpha_8(t) - \alpha_5(t) \alpha_9(t)), \\
 B_3 &= \alpha_8(t) (a_2 (\alpha_1(t) \alpha_8(t) - \alpha_2(t) \alpha_7(t)) + c_2 (\alpha_3(t) \alpha_8(t) - \alpha_2(t) \alpha_9(t))).
 \end{aligned} \tag{2.2}$$

By substituting Eq (2.2) into Eqs (2.1) and (1.2), the solution of Eq (1.1) is reducible to the following:

$$u_1 = \frac{2(a_1 k_1 (-\sin(\zeta_1)) + a_2 k_2 \sinh(\zeta_2) + a_3 k_3 \sinh(\zeta_3))}{k_1 \cos(\zeta_1) + k_2 \cosh(\zeta_2) + k_3 \cosh(\zeta_3) + k_4}. \tag{2.3}$$

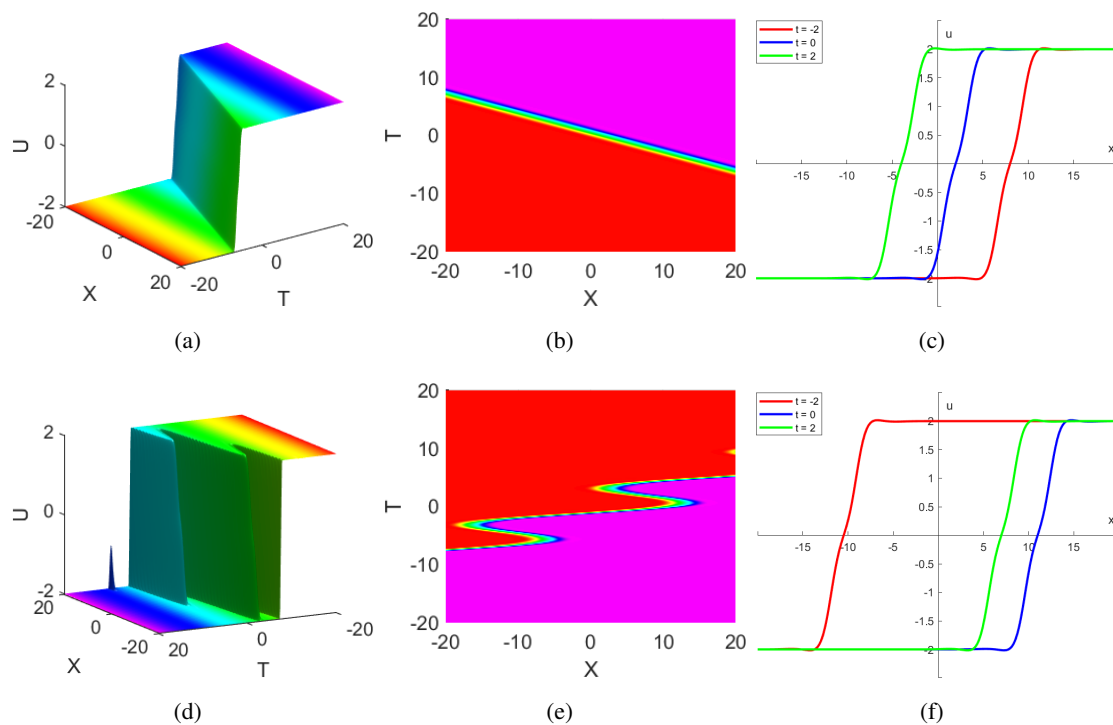


Figure 1. Solution u_1 for $k_2 = 1, k_3 = 1$. (a), (b), (c): $\alpha_4(t) = 1, \alpha_5(t) = 1$; (d), (e), (f): $\alpha_4(t) = \cos(t), \alpha_5(t) = \sin(t)$.

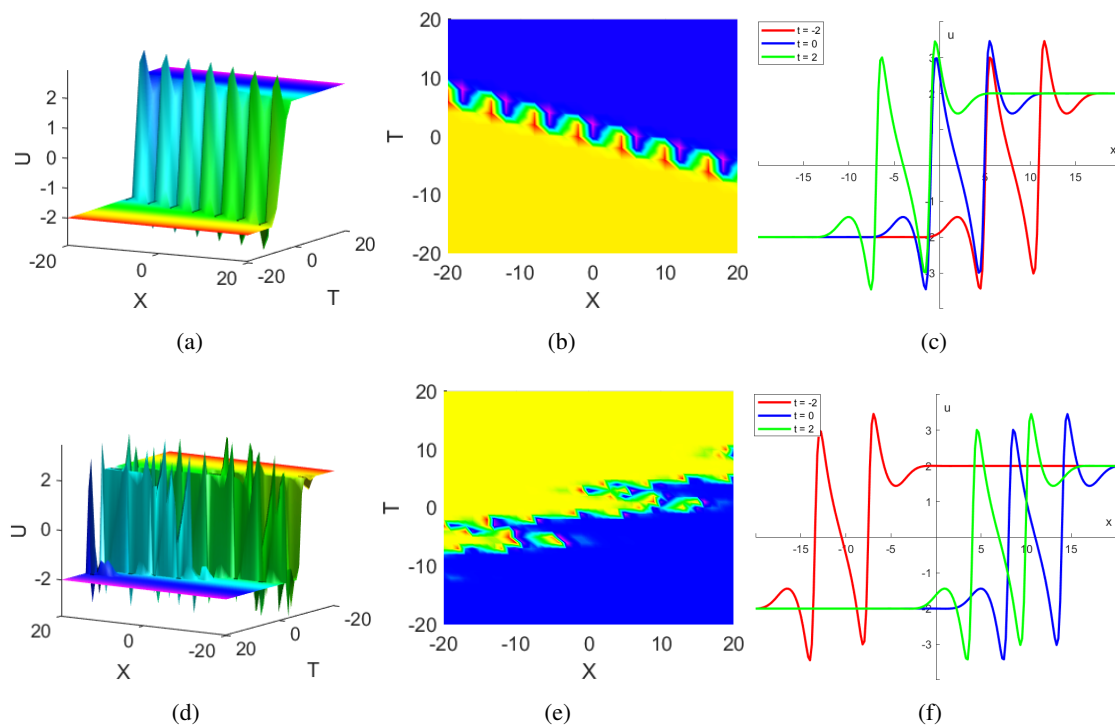


Figure 2. Solution u_1 for $k_2 = \frac{1}{100}, k_3 = \frac{1}{100}$. (a), (b), (c): $\alpha_4(t) = 1, \alpha_5(t) = 1$; (d), (e), (f): $\alpha_4(t) = \cos(t), \alpha_5(t) = \sin(t)$.

The profiles of solution u_1 are illustrated in Figures 1 and 2, where the adopted parameters are set as $a_2 = 1, a_3 = 1, c_1 = 1, c_2 = 1, k_1 = 1, k_2 = 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, \alpha_7(t) = 2, \alpha_8(t) = 1, \alpha_9(t) = 1, y = 1$, and $z = 1$. It can be observed from the figures that in the constant-coefficient case, the solitary wave propagates at a constant speed in a stable manner with a regular waveform profile and a fixed localized energy region. By contrast, after introducing variable coefficients, the propagation trajectory bends and deviates, the waveform undergoes compression and distortion, the original stable localized distribution is disrupted, the regular spatiotemporal periodic structure is destroyed, and the overall dynamic evolution becomes unsteady and non-uniform.

3. Soliton solution

Solitons are self-sustained localized waves stabilized by the balance between nonlinearity and dispersion, featuring invariant waveforms during propagation and elastic interactions upon collision. Physically, they correspond to experimentally verified classical structures, including ocean internal solitary waves, steady density perturbations in plasmas, and spatial optical solitons in optical waveguides. For practical applications, the variable coefficients of the equation are mapped to actual physical properties of seawater, plasmas, and optical media. Dissipation terms and boundary constraints are introduced to eliminate the ideal lossless hypothesis, and parameters are calibrated against field measurements. The modified soliton solutions can clarify the inherent evolution mechanism of nonlinear waves, and deliver qualitative and semi-quantitative references for coastal engineering protection, optical transmission design, and plasma confinement regulation.

3.1. Single soliton solution

In this section, we will solve Eq (1.1), based on the following assumptions:

$$\begin{aligned} f &= 1 + e^{\zeta_1}, \\ \zeta_i &= a_i x + b_i y + c_i z + d_i(t). \end{aligned} \quad (3.1)$$

By substituting Eq (3.4) into Eq (1.3), we collect the coefficients of each term and obtain a system of equations. By solving this system of equations, we can get the following:

$$d_1(t) = \int \frac{-a_1^3 b_1 \alpha_8(t) - a_1 b_1 \alpha_4(t) - a_1^3 c_1 \alpha_9(t) + a_1^4 (-\alpha_7(t)) - b_1 c_1 \alpha_6(t) - b_1^2 \alpha_5(t)}{a_1 \alpha_1(t) + b_1 \alpha_2(t) + c_1 \alpha_3(t)} dt. \quad (3.2)$$

Through the transformation of Eq (1.2), we can obtain the following single soliton solution:

$$u_2 = \frac{2a_1 e^{a_1 x + b_1 y + c_1 z + d_1(t)}}{e^{a_1 x + b_1 y + c_1 z + d_1(t)} + 1}. \quad (3.3)$$

3.2. Two soliton solution

To generate a two soliton solution, we can assume the following:

$$\begin{aligned} f &= 1 + e^{\zeta_1} + e^{\zeta_2} + k_1 \exp(\zeta_1 + \zeta_2), \\ \zeta_i &= a_i x + b_i y + c_i z + d_i(t). \end{aligned} \quad (3.4)$$

In the same way, we can obtain the following two soliton solution:

$$\begin{aligned} d_1(t) &= \int -A_1 dt, \quad d_2(t) = \int -A_2 dt \\ k_1 &= -\frac{G(a_1 - a_2, b_1 - b_2, c_1 - c_2, A_1 - A_2)}{G(a_1 + a_2, b_1 + b_2, c_1 + c_2, A_1 + A_2)}, \\ G(p, q, r, A) &= \alpha_7(t) p^4 + \alpha_8(t) q p^3 + \alpha_9(t) r p^3 + \alpha_4(t) q p - A \alpha_1(t) p - \\ &\quad A q \alpha_2(t) - A r \alpha_3(t) + q^2 \alpha_5(t) + q r \alpha_6(t), \\ A_1 &= \frac{\alpha_7(t) a_1^4 + (b_1 \alpha_8(t) + c_1 \alpha_9(t)) a_1^3 + b_1 \alpha_4(t) a_1 + b_1 (b_1 \alpha_5(t) + c_1 \alpha_6(t))}{a_1 \alpha_1(t) + b_1 \alpha_2(t) + c_1 \alpha_3(t)}, \\ A_2 &= \frac{\alpha_7(t) a_2^4 + (b_2 \alpha_8(t) + c_2 \alpha_9(t)) a_2^3 + b_2 \alpha_4(t) a_2 + b_2 (b_2 \alpha_5(t) + c_2 \alpha_6(t))}{a_2 \alpha_1(t) + b_2 \alpha_2(t) + c_2 \alpha_3(t)}. \end{aligned} \quad (3.5)$$

Correspondingly,

$$u_3 = \frac{2 \left((a_1 + a_2) k_1 \exp(\zeta_1 + \zeta_2) + a_1 e^{\zeta_1} + a_2 e^{\zeta_2} \right)}{k_1 \exp(\zeta_1 + \zeta_2) + e^{\zeta_1} + e^{\zeta_2} + 1}. \quad (3.6)$$

The profiles of solution u_2 are illustrated in Figures 3 and 4, where the adopted parameters are set as $a_1 = 1, b_1 = 1, c_1 = 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, \alpha_6(t) = 1, \alpha_7(t) = 1, \alpha_8(t) = 1, \alpha_9(t) = 1, y = 1$, and $z = 1$. Figures 5 and 6 depict solution u_3 with parameters $a_1 = 2, b_1 = 2, c_1 = 2, a_2 = 1, b_2 = 1, c_2 = 1, \alpha_1(t) = 1, \alpha_2(t) = 1, \alpha_3(t) = 1, \alpha_6(t) = 1, \alpha_7(t) = 1, \alpha_8(t) = 1, \alpha_9(t) = 1, y = 1$, and $z = 1$.

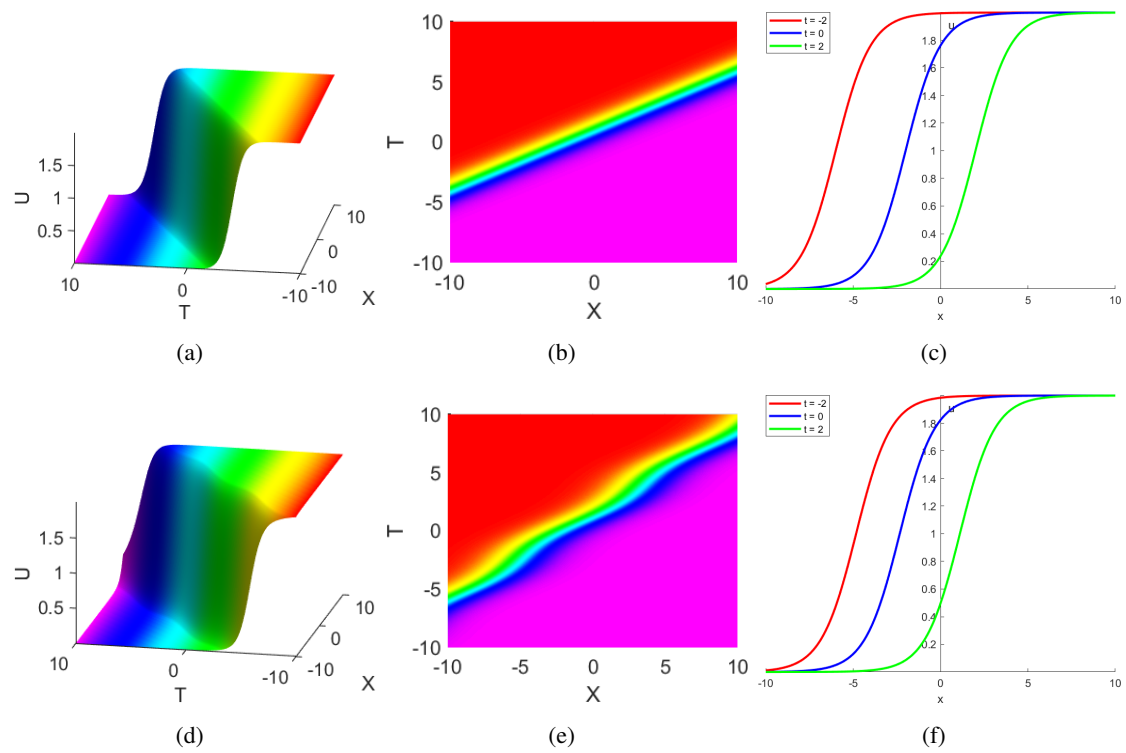


Figure 3. (a), (b), (c): $\alpha_4(t) = 1, \alpha_5(t) = 1$; (d), (e), (f): $\alpha_4(t) = \cos(t), \alpha_5(t) = \sin(t)$.

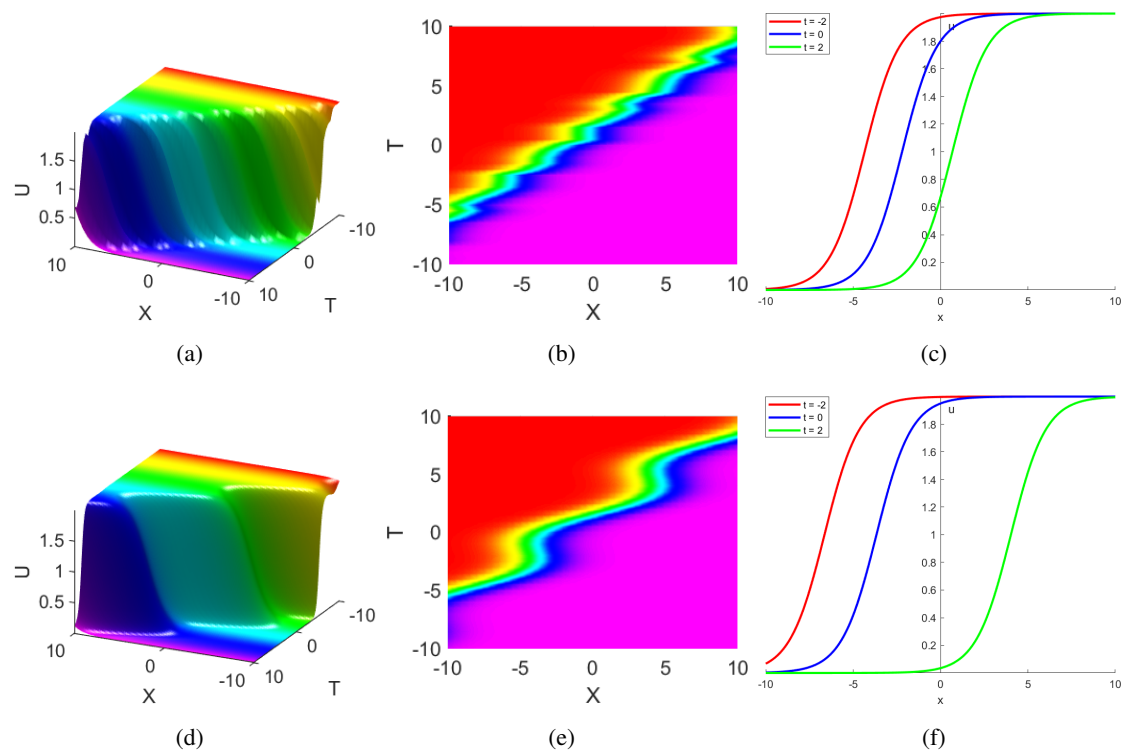


Figure 4. (a), (b), (c): $\alpha_4(t) = 5 \cos(3t), \alpha_5(t) = 3 \sin(5t)$; (d), (e), (f): $\alpha_4(t) = 3t^2, \alpha_5(t) = 5 \sin(t)$.

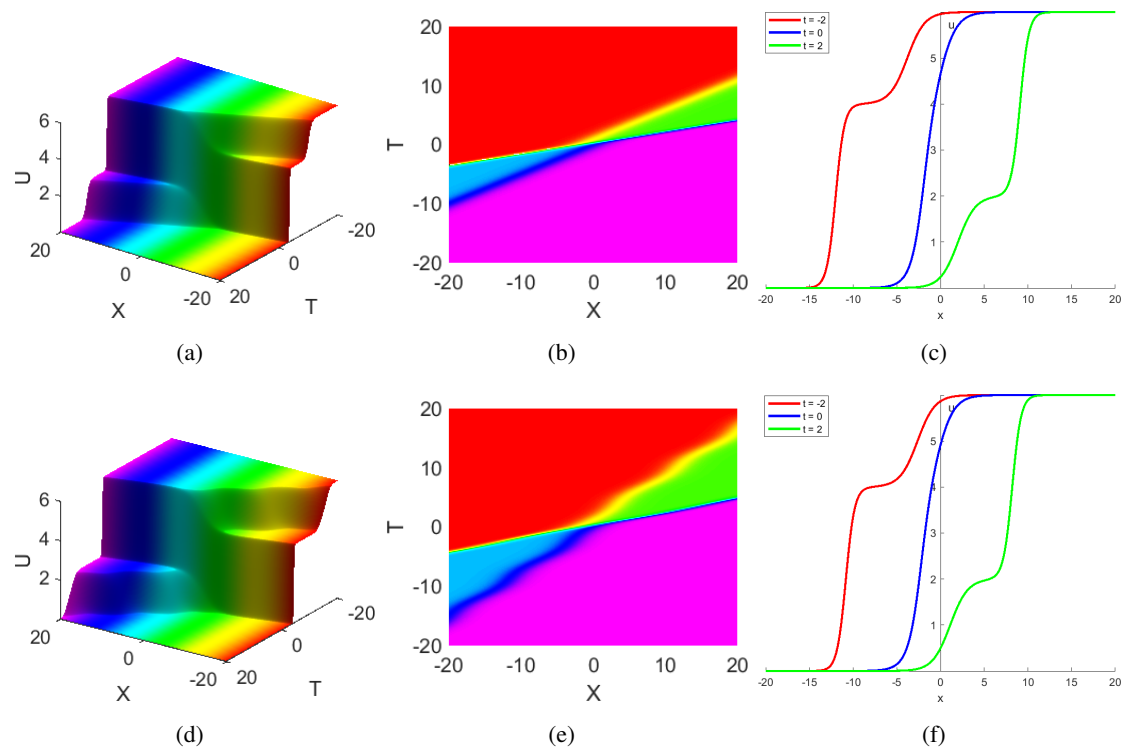


Figure 5. (a), (b), (c): $\alpha_4(t) = 1, \alpha_5(t) = 1$; (d), (e), (f): $\alpha_4(t) = \cos(t), \alpha_5(t) = \sin(t)$.

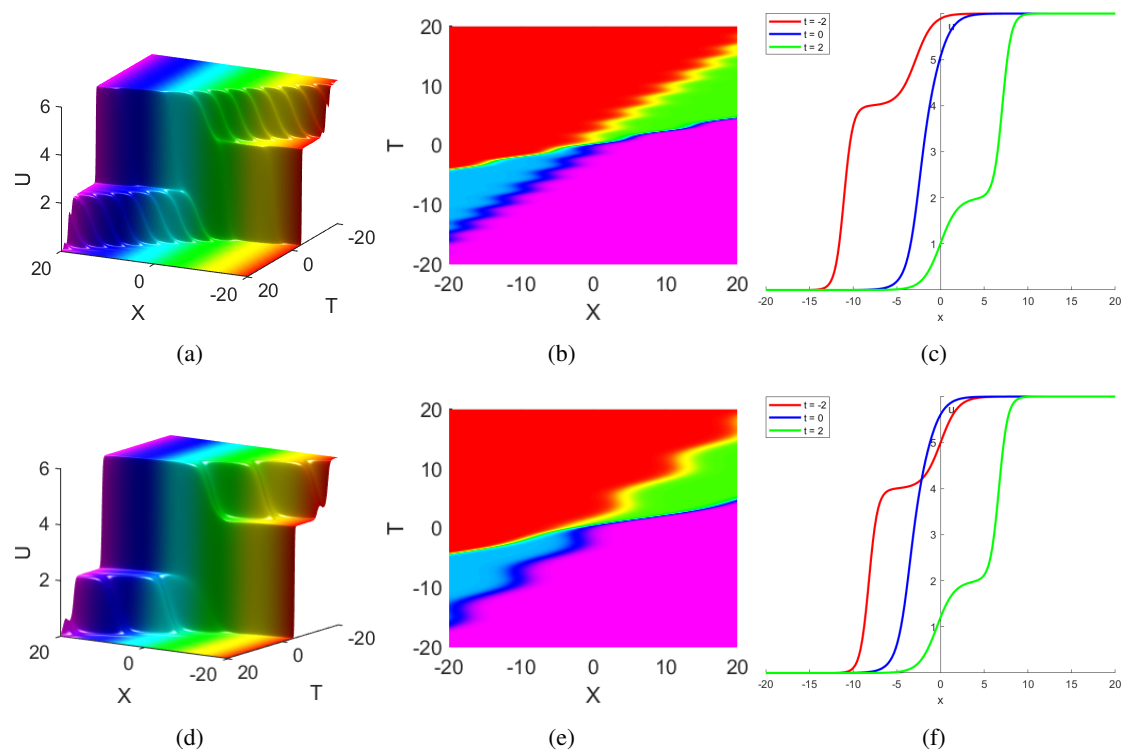


Figure 6. (a), (b), (c): $\alpha_4(t) = 5 \cos(3t), \alpha_5(t) = 5 \sin(3t)$; (d), (e), (f): $\alpha_4(t) = -2t^2, \alpha_5(t) = 5 \sin(t)$.

It can be observed from the figures that the dynamical behaviors of the solutions under four groups of coefficient configurations are compared as follows. In the constant-coefficient scenario, the solitary wave steadily propagates at a constant velocity with a highly stable dynamical evolution. When low-frequency and small-amplitude periodic variable coefficients are adopted, the solitary wave is subject to slight periodic disturbances. Its propagation trajectory undergoes small periodic oscillations and the waveform only presents a mild pulsation, while the overall propagation structure remains essentially unchanged. For periodic variable coefficients with a large amplitude and a high oscillation frequency, the propagation path exhibits obvious wavy bending, and the waveform suffers periodic compression and stretching deformation with reciprocating oscillatory disturbances. In the case of variable coefficients combining a quadratic increasing term and a periodic term, the trajectory of the solitary wave continuously and substantially drifts, and the waveform distortion gradually intensifies. In conclusion, the functional form, oscillation amplitude, and frequency of variable coefficients can effectively modulate the propagation velocity and evolutionary characteristics of solitary waves.

4. Conclusions

In this paper, soliton solutions and periodic lump solutions were derived for the (3+1)-dimensional generalized BK equation with variable coefficients. Most existing studies focused on the (2+1)-dimensional BK equation with constant coefficients, while systematic investigations into localized waves of this high-dimensional variable-coefficient model are quite limited, which constitutes the primary motivation of this work. Relying on the bilinear Eq (1.3), we successfully constructed novel analytical solutions to the original Eq (1.1), as expressed in Eqs (2.3), (3.3), and (3.6). To explore the physical characteristics of the acquired solutions, multiple sets of variable coefficients were selected, and comparative analyses were performed via three-dimensional diagrams and density contour plots (Figures 1–6). The Hirota bilinear method adopted in this work can readily convert the original nonlinear equation into a tractable bilinear form; explicit analytical solutions of localized waves can be directly obtained by introducing appropriate trial solutions. This study performed analytical investigations on the (3+1)-dimensional variable-coefficient generalized BK equation and enriched the system of exact solutions for this equation. Nowadays, the integration of artificial intelligence and mathematical analyses has attracted growing attention. In future research, the present model can be generalized to its fractional-order counterpart; intelligent optimization algorithms will be utilized to automatically optimize trial functions, and a hybrid solution framework integrating analytical derivation and machine learning will be established to further improve the solving system of nonlinear PDEs.

Author contributions

The first author was responsible for writing the original draft and revising the manuscript. The second author participated in reviewing and editing the writing. The third author was in charge of writing, reviewing and editing. The fourth author conducted the software implementation. The fifth author participated in reviewing and editing the writing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. X. Y. Gao, In plasma physics and fluid dynamics: Symbolic computation on a (2+1)-dimensional variable-coefficient Sawada-Kotera system, *Appl. Math. Lett.*, **159** (2025), 109262. <https://doi.org/10.1016/j.aml.2024.109262>
2. S. F. Tian, H. Q. Zhang, On the integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation, *J. Phys. A: Math. Theor.*, **45** (2012), 055203. <https://doi.org/10.1088/1751-8113/45/5/055203>
3. X. T. Gao, B. Tian, Similarity reductions on a (2+1)-dimensional variable-coefficient modified Kadomtsev-Petviashvili system describing certain electromagnetic waves in a thin film, *Int. J. Theor. Phys.*, **63** (2024), 99. <https://doi.org/10.1007/s10773-024-05629-4>
4. S. Kumar, M. Niwas, Analyzing multi-peak and lump solutions of the variable-coefficient Boiti–Leon–Manna–Pempinelli equation: A comparative study of the Lie classical method and unified method with applications, *Nonlinear Dyn.*, **111** (2023), 22457–22475. <https://doi.org/10.1007/s11071-023-09012-6>
5. Z. Z. Lan, Multi-soliton solutions for a (2+1)-dimensional variable-coefficient nonlinear Schrödinger equation, *Appl. Math. Lett.*, **86** (2018), 243–248. <https://doi.org/10.1016/j.aml.2018.05.014>
6. M. Shakeel, S. T. Mohyud-Din, Improved (G'/G)-expansion and extended tanh methods for (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff equation, *Alex. Eng. J.*, **54** (2015), 27–33. <https://doi.org/10.1016/j.aej.2014.11.003>
7. W. B. Rabie, H. M. Ahmed, A. Darwish, H. H. Hussein, Construction of new solitons and other wave solutions for a concatenation model using modified extended tanh-function method, *Alex. Eng. J.*, **74** (2023), 445–451. <https://doi.org/10.1016/j.aej.2023.05.046>

8. M. S. Ghayad, H. M. Ahmed, N. M. Badra, H. Rezazadeh, M. A. Hosseinzadeh, W. B. Rabie, New analytical wave structures for generalized B-type Kadomtsev–Petviashvili equation by improved modified extended tanh function method, *Phys. Scripta*, **99** (2024), 125224. <https://doi.org/10.1088/1402-4896/ad8aa4>
9. W. X. Ma, A. Abdeljabbar, A bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation, *Appl. Math. Lett.*, **25** (2012), 1500–1504. <https://doi.org/10.1016/j.aml.2012.01.003>
10. Y. H. Yin, X. Lü, W. X. Ma, Bäcklund transformation, exact solutions and diverse interaction phenomena to a (3+1)-dimensional nonlinear evolution equation, *Nonlinear Dyn.*, **108** (2022), 4181–4194. <https://doi.org/10.1007/s11071-021-06531-y>
11. L. N. Gao, Y. Y. Zi, Y. H. Yin, W. X. Ma, X. Lü, Bäcklund transformation, multiple wave solutions and lump solutions to a (3+1)-dimensional nonlinear evolution equation, *Nonlinear Dyn.*, **89** (2017), 2233–2240. <https://doi.org/10.1007/s11071-017-3581-3>
12. S. Sahoo, G. Garai, S. Saha Ray, Lie symmetry analysis for similarity reduction and exact solutions of modified KdV–Zakharov–Kuznetsov equation, *Nonlinear Dyn.*, **87** (2017), 1995–2000. <https://doi.org/10.1007/s11071-016-3169-3>
13. V. Jadaun, S. Kumar, Lie symmetry analysis and invariant solutions of (3+1)-dimensional Calogero–Bogoyavlenskii–Schiff equation, *Nonlinear Dyn.*, **93** (2018), 349–360. <https://doi.org/10.1007/s11071-018-4196-z>
14. P. F. Han, T. Bao, Novel hybrid-type solutions for the (3+1)-dimensional generalized Bogoyavlensky–Konopelchenko equation with time-dependent coefficients, *Nonlinear Dyn.*, **107** (2022), 1163–1177. <https://doi.org/10.1007/s11071-021-07019-5>
15. K. K. Ahmed, H. H. Hussein, H. M. Ahmed, W. B. Rabie, W. Alexan, Analysis of the dynamical behaviors for the generalized Bogoyavlensky–Konopelchenko equation and its analytical solutions occurring in mathematical physics, *Ain Shams Eng. J.*, **15** (2024), 103000. <https://doi.org/10.1016/j.asej.2024.103000>
16. W. Razzaq, A. Zafar, Bilinearization of generalized Bogoyavlensky–Konopelchenko equation with neural networks: Painlevé analysis, *Nonlinear Dyn.*, **113** (2025), 25083–25096. <https://doi.org/10.1007/s11071-025-11384-w>
17. W. X. Ma, Lump waves in a generalized Bogoyavlensky–Konopelchenko model with spatially balanced derivatives, *Math. Eng.*, **8** (2026), 140–149. <https://doi.org/10.3934/mine.2026005>
18. G. M. Moatimid, R. M. El-Shiekh, A. G. A. Al-Nowehy, Exact solutions for Calogero–Bogoyavlenskii–Schiff equation using symmetry method, *Appl. Math. Comput.*, **220** (2013), 455–462. <https://doi.org/10.1016/j.amc.2013.06.034>
19. B. Al Alwan, M. Abu Bakar, W. A. Faridi, A. C. Turcu, A. Akgül, M. Sallah, The propagating exact solitary waves formation of generalized Calogero–Bogoyavlenskii–Schiff equation with robust computational approaches, *Fractal. Fract.*, **7** (2023), 191. <https://doi.org/10.3390/fractalfract7020191>
20. A. M. Wazwaz, Painlevé analysis for new (3+1)-dimensional Boiti–Leon–Manna–Pempinelli equations with constant and time-dependent coefficients, *Int. J. Numer. Method. H.*, **30** (2020), 4259–4266. <https://doi.org/10.1108/HFF-10-2019-0760>

21. R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, 2004.
<https://doi.org/10.1017/CBO9780511543043>

Appendix

We verify the correctness of the derived solutions using the symbolic computation software Mathematica. The detailed verification procedure is as follows: The analytically derived solutions are directly substituted into the original BK equation for symbolic simplification; and the simplified expression is identically equal to zero, which confirms that these solutions strictly satisfy the original governing equation and verifies their accuracy. Taking the single-soliton solution as an example, the Mathematica verification code is presented below:

$$\begin{aligned}
 u &:= 2 \frac{\partial \log(f)}{\partial x}, \\
 d_1(t) &:= \int \frac{-a_1^3 b_1 \alpha_8(t) - a_1 b_1 \alpha_4(t) - a_1^3 c_1 \alpha_9(t) + a_1^4 (-\alpha_7(t)) - b_1 c_1 \alpha_6(t) - b_1^2 \alpha_5(t)}{a_1 \alpha_1(t) + b_1 \alpha_2(t) + c_1 \alpha_3(t)} dt, \\
 f &:= e^{a_1 x + b_1 y + c_1 z + d_1(t)} + 1. \\
 &\text{Simplify}[\alpha_7(t) \frac{\partial^4 u}{\partial x \partial x \partial x \partial x} + \alpha_1(t) \frac{\partial^2 u}{\partial x \partial t} + 6\alpha_7(t) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial x} + \alpha_8(t) \frac{\partial^4 u}{\partial x \partial x \partial x \partial y} + \alpha_4(t) \frac{\partial^2 u}{\partial x \partial y} + \\
 &\quad 3\alpha_8(t) \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial x} + 3\alpha_8(t) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} + \alpha_9(t) \frac{\partial^4 u}{\partial x \partial x \partial x \partial z} + 3\alpha_9(t) \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial x \partial x} + \\
 &\quad 3\alpha_9(t) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial z} + \alpha_2(t) \frac{\partial^2 u}{\partial y \partial t} + \alpha_5(t) \frac{\partial^2 u}{\partial y \partial y} + \alpha_6(t) \frac{\partial^2 u}{\partial y \partial z} + \alpha_3(t) \frac{\partial^2 u}{\partial z \partial t}].
 \end{aligned}$$



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