



Research article

Infinitely many solutions of biharmonic-Kirchhoff systems on locally finite graphs

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Abstract: In recent years, the study of partial differential equations (PDEs) and their systems in graph settings has emerged as a prominent research area, driven by their significant applications in image processing and data clustering. Motivated by these developments, we aim to extend certain existence results for biharmonic-Kirchhoff systems and biharmonic systems from the Euclidean setting (continuous models) to their counterparts in locally finite graphs (discrete models). Our primary focus is on establishing the existence of infinitely many solutions for a biharmonic-Kirchhoff system defined on a locally finite graph by employing the symmetric mountain pass theorem as our key analytical tool. Under the assumption that the nonlinear term is super-quartic, we prove that the system admits infinitely many solutions. Additionally, we present analogous results for the biharmonic system. A notable observation from our study is that the results obtained in the locally finite graph setting surpass those in the Euclidean setting. This improvement is fundamentally attributed to the more favorable embedding properties inherent to the locally finite graph framework.

Keywords: biharmonic-Kirchhoff system; symmetric mountain pass theorem; locally finite graph; infinitely many solutions; biharmonic system

1. Introduction and main results

In this paper, we investigate the existence of infinitely many solutions for the following biharmonic-Kirchhoff system on a locally finite graph $G = (V, E)$ (see below for the precise definition of the

graph setting):

$$\begin{cases} \Delta^2 u - (a_1 + b_1 \int_V |\nabla u|^2 d\mu) \Delta u + H_1(x)u = F_u(x, u, v), & x \in V, \\ \Delta^2 v - (a_2 + b_2 \int_V |\nabla v|^2 d\mu) \Delta v + H_2(x)v = F_v(x, u, v), & x \in V, \end{cases} \quad (1.1)$$

where V is the vertices set of G , E is the edges set of G , $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $a_i > 0$, $b_i \geq 0$, $H_i : V \rightarrow \mathbb{R}$, $i = 1, 2$, $F : V \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $F_u(x, u, v) = \frac{\partial F(x, u, v)}{\partial u}$, and $F_v(x, u, v) = \frac{\partial F(x, u, v)}{\partial v}$.

Next, we provide the precise explanations on the notations in system (1.1). We refer to [1, 2] for these notations. $xy \in E$ denotes the edge which links x to its neighbor y in V and denote $\omega_{xy} > 0$ by its weight. For any edge $xy \in E$, it is supposed that $\omega_{xy} = \omega_{yx}$. The graph $G = (V, E)$ is also supposed to be a connected graph which means that any two vertices x and y in V have to be linked by finite edges. Moreover, $G = (V, E)$ is a locally finite graph which means that the number of edges $xy \in E$ is finite for any given $x \in V$. $dist(x, y)$ denotes the distance of two vertices x, y , which means the minimal number of edges that link x with y . Let $C(V) = \{u | u : V \rightarrow \mathbb{R}\}$ and define the Laplacian operator $\Delta : C(V) \rightarrow C(V)$ by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)),$$

where $\mu : V \rightarrow \mathbb{R}^+$ is a finite measure with positive lower bound $\mu_{\min}$, that is, $\mu(x) \geq \mu_{\min} > 0$, and $y \sim x$ denotes the vertex y satisfying the edge $xy \in E$. Then,

$$\Delta^2 u(x) := \Delta(\Delta u(x)) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(\Delta u(y) - \Delta u(x)).$$

Define the gradient ∇u of u at $x \in V$ by a vector

$$\nabla u(x) := \left(\sqrt{\frac{w_{xy}}{2\mu(x)}}(u(y) - u(x)) \right)_{y \sim x, y \in V},$$

which is indexed by y . Let

$$\Gamma(u, v)(x) := \nabla u(x) \cdot \nabla v(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)).$$

Then

$$|\nabla u|(x) = \sqrt{\Gamma(u, u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}.$$

For any $u \in C(V)$, we denote

$$\int_V u(x) d\mu = \sum_{x \in V} u(x) \mu(x).$$

The biharmonic operator $\Delta^2 u$ of $u : V \rightarrow \mathbb{R}$ satisfies the following identity (see [2]):

$$\int_V (\Delta^2 u) \phi d\mu = \int_V \Delta u \Delta \phi d\mu, \quad \forall \phi \in C_c(V),$$

where $C_c(V) := \{u \in C(V) \mid \text{the set } \{x \in V : u(x) \neq 0\} \text{ is of finite cardinality}\}$.

In recent years, the analysis on locally finite graph is becoming a hot topic because of its own mathematical interest and its applications to image processing and data clustering (see [3–7]). Especially, the development of analysis on locally finite graphs has greatly promoted the investigation of nonlinear partial differential equations on graphs. In particular, the existence, multiplicity and convergence of solutions for p -Laplacian, quasilinear Laplacian and (p, q) -Laplacian equations and systems on locally finite or weighted graphs have been extensively studied by variational methods and critical point theory (see [8–13]). Beyond these Laplacian-type problems, increasing attention has also been paid to Kirchhoff-type equations and more general nonlinear systems on graphs. Various results on least-energy sign-changing solutions, parameter dependence and multiplicity have been obtained for Kirchhoff-type equations, generalized poly-Laplacian systems and (p, q) -Kirchhoff systems (see [14–17]). In the Euclidean setting, variational methods and critical point theory have also been widely applied to the existence and multiplicity of nontrivial solutions of differential equations (for example, see [18–23]). Of particular relevance to the present work are fourth-order and biharmonic problems. The existence and multiplicity of solutions for fourth-order elliptic equations, biharmonic equations and biharmonic Kirchhoff equations have been established by mountain pass methods, fountain-type theorems and related variational techniques (see [24–29]). Higher-order Kirchhoff-type problems with exponential critical growth have also been investigated in [30]. Naturally, it is of interest to ask whether these variational and critical point methods can be further developed for biharmonic-Kirchhoff systems on locally finite graphs. Indeed, some progress has already been made for nonlinear elliptic equations and systems in the graph setting, while the multiplicity theory for fourth-order Kirchhoff systems on locally finite graphs remains much less developed. For example, in [1], Grigor'yan et al. used the mountain pass theorem to prove that there is a positive solution to a Laplacian equation with a Dirichlet boundary. They also applied the method to p -Laplacian equations and *poly*-Laplace equations. In [8], Han and Shao used the mountain pass theorem and Nehari manifold, under suitable conditions, it is proved that when $\lambda > 1$, there is a positive solution to a p -Laplacian equation with a parameter λ . And they proved the convergence of solutions when $\lambda \rightarrow \infty$. In [9], Pang et al. studied the existence of nontrivial solutions of three quasilinear Laplacian systems with parameters on finite graphs or locally finite graphs. The existence of infinitely many solutions of systems were obtained by using a critical point theorem without compactness condition. In [12], Yang and Zhang generalized two embedding theorems and investigated the existence and multiplicity of nontrivial solutions for a (p, q) -Laplacian coupled system with perturbations and two parameters λ_1 and λ_2 on locally finite graph. By using mountain pass theorem and Ekeland's variational principle, they obtained that system has at least one nontrivial solution when the nonlinear term satisfies the sub- (p, q) conditions and obtain that system has at least one solution of positive energy and one solution of negative energy when the nonlinear term satisfies the super- (p, q) conditions which is weaker than the well known Ambrosetti-Rabinowitz condition. In [15], Pan and Ji used the constrained variational method to prove the existence of a least energy sign-changing solution u_b of a Kirchhoff equation if the nonlinear term f satisfies certain

assumptions, and to show the energy of u_b is strictly larger than twice that of the least energy solutions. In [17], by using the well-known mountain pass theorem and Ekeland's variational principle, Yu et al. proved that there exist at least two fully nontrivial solutions for a (p, q) -Kirchhoff elliptic system with convex-concave nonlinearity and perturbation terms on a locally finite graph. They also presented a necessary condition of the existence of semitrivial solutions for the system. In [14], Ou and Zhang used a direct non-Nehari manifold method to obtain the existence results of least energy sign-changing solutions and ground state solutions for a class of Kirchhoff-type equations with general power law, logarithmic nonlinearity and Dirichlet boundary value on a locally finite graph, and they obtained the sign-changing least energy is strictly larger than twice of the ground state energy. In [31], Ji and Rădulescu obtained the existence of ground state solutions for a biharmonic equation on lattice graph by variational methods when the reaction is periodic, and they also extended their results to quasi-transitive graphs. In [32], by variational methods, Ji and Miyagaki obtained the existence of ground states for an elliptic equation with zero mass on lattice graphs, an asymptotically periodic potential and vanishing potential at infinity. In [33], He and Ji developed the theory of Musielak-Orlicz spaces on locally finite graphs, some elementary properties of double phase operators on locally finite graphs and presented some existence results of solutions to double phase problems on lattice graphs. In [34], He and Ji obtained infinitely many solutions for the logarithmic Schrödinger equation on lattice graphs when the potential is coercive by the Fountain theorem, and they also obtained the existence of ground state solutions via the variational methods in the cases of periodic potential, asymptotically periodic potential and bounded potential. Moreover, they extended the results to the p -Laplacian equation with the logarithmic nonlinearity.

In this paper, our works are mainly inspired by a recent work due to Han et al. [2] where the following nonlinear biharmonic equation was investigated:

$$\Delta^2 u - \Delta u + (\lambda a + 1)u = |u|^{p-2}u,$$

on a locally graph $G = (V, E)$. Under some suitable assumptions of Sobolev spaces, they proved that the equation admits a ground state solution u_λ for any $\lambda > 1$ and $p > 2$. Moreover, they also demonstrated an asymptotic result of solutions $\{u_\lambda\}$ as $\lambda \rightarrow +\infty$. We are also inspired by the works due to Zhang et al. [29] and Yuan and Liu [27]. In [29], Zhang et al. studied the following fourth-order elliptic equation in the Euclidean setting:

$$\begin{cases} \Delta^2 u - \Delta u + H(x)u = f(x, u), & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (1.2)$$

where the potential $H \in C(\mathbb{R}^N, \mathbb{R})$ is allowed to be sign-changing. Under the weak superquadratic conditions, they established the existence of infinitely many solutions via variational method and symmetry mountain pass theorem for Eq (1.2). In [27], Yuan and Liu considered the following biharmonic-Kirchhoff equation in the Euclidean setting:

$$\begin{cases} \Delta^2 u - \left(a + b \int_{\mathbb{R}^5} |\nabla u|^2 dx\right) \Delta u + H(x)u = f(u), \\ u(x) = u(|x|) \in H^2(\mathbb{R}^5), \end{cases} \quad (1.3)$$

where $H \in C(\mathbb{R}^5, \mathbb{R})$ and $f \in C(\mathbb{R}, \mathbb{R})$. By using a perturbation approach and the symmetric mountain pass theorem, the existence and multiplicity of radially symmetric solutions for Eq (1.3) are obtained.

Based on the works in [2, 27, 29], it is a natural motivation to apply the variational method and symmetry mountain pass theorem to investigate the existence of infinitely many solutions for system (1.1) which can be seen as a discrete analogy on locally finite graph for Eqs (1.2) and (1.3). Compared to [27, 29], a notable observation from our study is that the results obtained in the locally finite graph setting surpass those in the Euclidean setting. This improvement is fundamentally attributed to the more favorable embedding properties inherent to the locally finite graph framework. Specifically, as established in Lemma 2.1, the embedding $E_i \hookrightarrow L^\infty(V)$ is continuous, and any function $b \in L^1(V)$ is automatically in $L^\infty(V)$. This stands in stark contrast to the Euclidean setting $\mathbb{R}^N$ ($N \geq 2$ is an interger), where such a uniform boundedness estimate is generally unavailable. This subtle but crucial difference allows us to impose weaker assumptions on the nonlinear term F . For instance, our growth condition (F_1) only requires F to be controlled by a function $a(|(s, t)|)b(x)$ with $b \in L^1(V)$. This framework readily accommodates nonlinearities with exponential growth, such as the example given in Remark 4.2, which is excluded in the Euclidean case where subcritical polynomial growth (e.g., condition (4.3)) is mandatory to ensure compactness. Moreover, compared to Eqs (1.2) and (1.3) which are the single equation case, system (1.1) possesses the difficulties caused by the coupling by u and v in nonlinear term so that we have to make more delicate analysis. Compared to [2], system (1.1) involves the nonlocal Kirchhoff terms $(\int_V |\nabla u|^2 d\mu)\Delta u$ and $(\int_V |\nabla v|^2 d\mu)\Delta v$, the existence of infinitely many solutions is investigated rather than the existence of ground state solutions, the symmetric mountain pass theorem is used rather than the Nehari manifold method. In Section 4, we also present some similar results for the biharmonic system ($b_i = 0, i = 1, 2$) on locally finite graph corresponding to system (1.1). We firstly make the following assumptions:

- (H) $H_i \in C(V)$, $i = 1, 2$, $\inf_V H_i(x) > 0$ for all $x \in V$, and there exists a vertex $x_0 \in V$ such that $H_i(x) \rightarrow +\infty$ as $d(x, x_0) \rightarrow +\infty$;
 (F_0) $F(x, 0, 0) \equiv 0$ for all $x \in V$, and $F(x, s, t)$ is continuously differentiable in $(s, t) \in \mathbb{R}^2$ for all $x \in V$;
 (F_1) there exists a function $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a function $b : V \rightarrow \mathbb{R}^+$ with $b \in L^1(V)$ such that

$$|F_s(x, s, t)|, |F_t(x, s, t)|, |F(x, s, t)| \leq a(|(s, t)|)b(x)$$

for all $x \in V$, $(s, t) \in \mathbb{R}^2$;

$$(C_1) \lim_{|(s, t)| \rightarrow 0} \frac{a(|(s, t)|)}{|(s, t)|^2} = h < +\infty;$$

$$(C_2) \text{ there exists a } \rho_* > 0 \text{ such that } \alpha_* := \frac{1}{4}\rho_*^2 - \max_{|s|+|t| \leq (\gamma_{\infty,1} + \gamma_{\infty,2})\rho_*} a(|(s, t)|) \int_V b(x) d\mu > 0, \text{ where } \gamma_{\infty,i} =$$

$$\left(\frac{1}{\mu_{\min} \inf_V H_i(x)} \right)^{\frac{1}{2}}, i = 1, 2;$$

$$(F_2) \lim_{|(s, t)| \rightarrow \infty} \frac{F(x, s, t)}{|(s, t)|^4} = +\infty \text{ for all } x \in V;$$

(F_3) there exists $r_0 > 0$ such that

$$F(x, s, t) \geq 0 \text{ for all } x \in V, (s, t) \in \mathbb{R}^2 \text{ with } |(s, t)| \geq r_0;$$

$$(F_4) \mathcal{F}(x, s, t) := \frac{1}{4} (F_s(x, s, t)s + F_t(x, s, t)t) - F(x, s, t) \geq 0, \text{ and there exist } c > 0 \text{ and } k > 1 \text{ such that}$$

$$|F(x, s, t)|^k \leq c|(s, t)|^{2k} \mathcal{F}(x, s, t) \text{ for all } x \in V, (s, t) \in \mathbb{R}^2 \text{ with } |(s, t)| \geq r_0;$$

$$(F_5) \text{ there exist } \mu > 4 \text{ and } \sigma \in \left(0, \frac{\mu-2}{2 \max\{\gamma_{2,1}^2, \gamma_{2,2}^2\}} \right) \text{ such that}$$

$$\mu F(x, s, t) \leq sF_s(x, s, t) + tF_t(x, s, t) + \sigma(s^2 + t^2) \text{ for all } x \in V, (s, t) \in \mathbb{R}^2,$$

where $\gamma_{2,i} = \mu_{\min}(\inf_V H_i(x))^{\frac{1}{2}}$;

(F_6) $F(x, s, t) = F(x, -s, -t)$ for all $x \in V$ and $(s, t) \in \mathbb{R}^2$.

We briefly comment on the roles of the above assumptions. Conditions (F_1) and (C_1) (or [C_2]) control the local growth of F and are used to verify the geometric condition in Lemma 3.7 (using [C_1]) or Lemma 3.8 (using [C_2]). The super-quartic growth condition (F_2), along with the technical Assumptions (F_3) and (F_4) (for Lemma 3.1) or (F_3)–(F_5) (for Lemma 3.3) is crucial for proving the boundedness of Cerami sequences. The evenness condition (F_6) is needed for the symmetric mountain pass theorem.

Next, we state our main results in this paper.

Theorem 1.1. *Suppose that $b_i > 0, i = 1, 2$, (H), (F_0), (F_1), (C_1)(or [C_2]), (F_2), (F_3), (F_4), and (F_6) are satisfied. Then, problem (1.1) has infinitely many solutions $\{(u_k, v_k)\}_{k=1}^\infty$ such that*

$$\begin{aligned} & \frac{1}{2} \int_V (|\Delta u_k|^2 + a_1 |\nabla u_k|^2 + H_1(x) u_k^2) d\mu + \frac{1}{2} \int_V (|\Delta v_k|^2 + a_2 |\nabla v_k|^2 + H_2(x) v_k^2) d\mu \\ & + \frac{b_1}{4} \left(\int_V |\nabla u_k|^2 d\mu \right)^2 + \frac{b_2}{4} \left(\int_V |\nabla v_k|^2 d\mu \right)^2 - \int_V F(x, u_k, v_k) d\mu \rightarrow +\infty, \text{ as } k \rightarrow +\infty. \end{aligned} \quad (1.4)$$

Theorem 1.2. *Suppose that $b_i > 0, i = 1, 2$, (H), (F_0), (F_1), (C_1)(or [C_2]), (F_2), (F_3), (F_5), and (F_6) are satisfied. Then, problem (1.1) has infinitely many solutions $\{(u_k, v_k)\}_{k=1}^\infty$ such that Eq (1.4) holds.*

Remark 1.1 The conditions (F_3) and (F_4) origin from [35, 36], respectively. They are weaker than the well known Ambrosetti-Rabinowitz condition:

(AR) *There exist $\mu > 4$ and $r_0 > 0$ such that*

$$\mu F(x, s, t) \leq s F_s(x, s, t) + t F_t(x, s, t), \text{ for all } x \in V, (s, t) \in \mathbb{R}^2 \text{ with } |(s, t)| \geq r_0.$$

There are examples satisfying our assumptions in Theorems 1.1 and 1.2. For example, let $G = (V, E)$ be a locally finite graph satisfying that there exist constants $C > 0$ and $\alpha > 0$ such that for some fixed $x_0 \in V$,

$$|S_n| := \#\{x \in V : \text{dist}(x, x_0) = n\} \leq C(1 + n)^\alpha, \text{ for all } n \in \mathbb{N} \cup \{0\},$$

where $\#\{\cdot\}$ denotes the number of elements of the set $\{\cdot\}$. Let $\mu(x) \equiv 1$, $H_i(x) = 1 + \text{dist}(x, x_0)$, $i = 1, 2$, for all $x \in V$ and

$$F(x, s, t) = b(x) \left(\frac{5}{6} s^6 + \frac{5}{6} t^6 - \frac{3}{4} s^4 - \frac{3}{4} t^4 \right), \text{ for all } x \in V,$$

where $b(x) = \frac{1}{[1 + \text{dist}(x, x_0)]^m}$ with $m > \alpha + 1$.

Remark 1.2. To the best of our knowledge, there are few papers to investigate the existence of infinitely many solutions for the partial differential equations on the locally finite graph except for [9] where the poly-Laplacian systems with parameters and Dirichlet boundary value as well as (p, q) -Laplacian systems were investigated. Compared with [9], Theorems 1.1 and 1.2 in the present paper contain two essential novelties. First, we consider a biharmonic-Kirchhoff system (1.1), which involves the nonlocal Kirchhoff terms $(\int_V |\nabla u|^2 d\mu) \Delta u$ and $(\int_V |\nabla v|^2 d\mu) \Delta v$. These terms introduce some additional difficulties in verifying the (C)-condition, since they give rise to quartic terms $(\int_V |\nabla u|^2 d\mu)^2$

and $(\int_V |\nabla v|^2 d\mu)^2$ in the energy functional. Second, the two works employ different critical point theorems: In [9], the Bonanno-Bisci theorem [37] is used, which does not require a compactness condition, while we apply the symmetric mountain pass theorem (Lemma 2.3) which requires the verification of the compactness (C)-condition so that our assumptions on the nonlinear term F are different from those in [9].

2. Variational setting

Define $L^r(V) = \{u : V \rightarrow \mathbb{R} \mid \int_V |u|^r d\mu < +\infty\}$ ($1 \leq r < +\infty$) with the norm defined by

$$\|u\|_r = \left(\int_V |u(x)|^r d\mu \right)^{\frac{1}{r}}.$$

Then $(L^r(V), \|\cdot\|_r)$ is a Banach space ($1 \leq r < +\infty$) (see [38]). Define $L^\infty(V) = \left\{ u : V \rightarrow \mathbb{R} \mid \sup_{x \in V} |u(x)| < +\infty \right\}$ with the norm defined by

$$\|u\|_\infty = \sup_{x \in V} |u(x)|.$$

Let $W^{2,2}(V)$ be the completion of $C_c(V)$ under the norm

$$\|u\|_{W^{2,2}(V)} = \left(\int_V (|\Delta u|^2 + |\nabla u|^2 + u^2) d\mu \right)^{\frac{1}{2}}.$$

$W^{2,2}(V)$ is a Hilbert space with the inner product

$$(u, v)_{W^{2,2}(V)} = \int_V (\Delta u \Delta v + \nabla u \nabla v + uv) d\mu, \quad u, v \in W^{2,2}(V)$$

(see [38]). We work in the following subspace of $W^{2,2}(V)$:

$$E_i := \{u \in W^{2,2}(V) : \int_V (|\Delta u|^2 + a_i |\nabla u|^2 + H_i(x) u^2) d\mu < +\infty\}, \quad i = 1, 2,$$

equipped with the inner product

$$(u, v)_{E_i} = \int_V (\Delta u \Delta v + a_i \nabla u \nabla v + H_i(x) uv) d\mu, \quad u, v \in E_i, \quad i = 1, 2$$

and the norm

$$\|u\|_{E_i} = \left(\int_V (|\Delta u|^2 + a_i |\nabla u|^2 + H_i(x) u^2) d\mu \right)^{\frac{1}{2}}, \quad u, v \in E_i, \quad i = 1, 2.$$

Lemma 2.1. *Assume that (H) holds. Then, $E_i, i = 1, 2$ are continuously embedded into $L^r(V)$ for $2 \leq r \leq +\infty$ and*

$$\|u\|_r \leq \gamma_{r,i} \|u\|_{E_i}, \quad \forall u \in E_i, \quad i = 1, 2, \quad 2 \leq r < +\infty, \quad (2.1)$$

$$\|u\|_\infty \leq \gamma_{\infty,i} \|u\|_{E_i}, \quad \forall u \in E_i, \quad i = 1, 2, \quad (2.2)$$

where $\gamma_{r,i} = \left(\mu_{\min} \inf_V H_i(x) \right)^{\frac{2-r}{2r}} (\inf_V H_i(x))^{-\frac{1}{r}}$ if $2 \leq r < +\infty$ and $\gamma_{\infty,i} = \left(\frac{1}{\mu_{\min} \inf_V H_i(x)} \right)^{\frac{1}{2}}$ if $r = \infty$, and the embedding $E_i \hookrightarrow L^s(V)$, $i = 1, 2$ are compact for any $2 \leq r \leq +\infty$, that is, for any bounded sequence $\{u_k\} \subset E_i$, there exists $u \in E_i$, $i = 1, 2$ such that, up to a subsequence,

$$\begin{cases} u_k \rightharpoonup u & \text{in } E_i, i = 1, 2; \\ u_k(x) \rightarrow u(x) & \forall x \in V; \\ u_k \rightarrow u & \text{in } L^r(V). \end{cases}$$

Proof. This proof is similar to Lemma 2.5 in [2] with some differences caused by $H_i(x)$, $i = 1, 2$. For any given $u \in E_i$, $i = 1, 2$ and any given vertex $x_0 \in V$, we have

$$\begin{aligned} \|u\|_{E_i}^2 &= \int_V (|\Delta u|^2 + a_i |\nabla u|^2 + H_i(x) u^2) d\mu \\ &\geq \int_V H_i(x) u^2 d\mu \\ &= \sum_{x \in V} \mu(x) H_i(x) u^2(x) \\ &\geq \mu_{\min} H_i(x_0) u^2(x_0) \\ &\geq \mu_{\min} \inf_V H_i(x) u^2(x_0), \end{aligned}$$

which implies

$$u(x_0) \leq \left(\frac{1}{\mu_{\min} \inf_V H_i(x)} \right)^{\frac{1}{2}} \|u\|_{E_i}.$$

It follows from the arbitrariness of $x_0 \in V$ that $E_i \hookrightarrow L^\infty(V)$ is continuous embedding and Eq (2.2) holds. Furthermore, we can get $E_i \hookrightarrow L^r(V)$ continuously for any $2 \leq r < \infty$. In fact, for any $u \in E_i$, by (V), it is easy to see that $u \in L^2(V)$. Then, for any $2 \leq s < \infty$,

$$\int_V |u|^r d\mu = \int_V |u|^2 |u|^{r-2} d\mu \leq (\mu_{\min} \inf_V H_i(x))^{\frac{2-r}{2}} \|u\|_{E_i}^{r-2} \int_V |u|^2 d\mu < +\infty,$$

which implies that $u \in L^r(V)$ for any $2 \leq r < \infty$ and

$$\|u\|_r = \left(\int_V |u|^r d\mu \right)^{\frac{1}{r}} \leq \left(\mu_{\min} \inf_V H_i(x) \right)^{\frac{2-r}{2r}} (\inf_V H_i(x))^{-\frac{1}{r}} \|u\|_{E_i} = \gamma_{r,i} \|u\|_{E_i}.$$

Since E_i ($i = 1, 2$) are Hilbert spaces, they are reflexive. Thus for any bounded sequence $\{u_k\}$ in E_i , there is, up to a subsequence, $u_k \rightharpoonup u_i$ in E_i , $i = 1, 2$. On the other hand, $\{u_k\} \subset E_i$ is also bounded in $L^2(V)$ and we also get $u_k \rightharpoonup u_i$ in $L^2(V)$, which tells us that, for any $\phi \in L^2(V)$,

$$\lim_{k \rightarrow \infty} \int_V (u_k - u_i) \phi d\mu = \lim_{k \rightarrow \infty} \sum_{x \in V} \mu(x) (u_k(x) - u_i(x)) \phi(x) = 0. \quad (2.3)$$

For any fixed $x_0 \in V$, let

$$\phi_0(x) = \begin{cases} 1 & x = x_0, \\ 0 & x \neq x_0. \end{cases}$$

Obviously, ϕ_0 belongs to $L^2(V)$. By substituting ϕ_0 into ϕ in Eq (2.3), we get

$$\lim_{k \rightarrow \infty} \mu(x_0)(u_k(x_0) - u(x_0)) = 0,$$

which implies that

$$\lim_{k \rightarrow \infty} u_k(x) = u(x) \text{ for any } x \in V. \quad (2.4)$$

Next we prove $u_k \rightarrow u$ in $L^s(V)$ for all $2 \leq s \leq \infty$. Since $\{u_k\}$ are bounded in E_i , $i = 1, 2$ and $u \in E_i$, $i = 1, 2$, there exist positive constants C_i , $i = 1, 2$ such that $\|u_k - u\|_{E_i}^2 \leq C_i$, $i = 1, 2$. Let $x_0 \in V$ be fixed. By (V), for any $\epsilon > 0$, there exists some $r_0 > 0$ such that $H_i(x) \geq \frac{C_i}{\epsilon}$ as $d(x, x_0) > r_0$. Then,

$$\begin{aligned} \int_{d(x, x_0) > r_0} |u_k - u|^2 d\mu &\leq \frac{\epsilon}{C_i} \int_{d(x, x_0) > r_0} H_i(x) |u_k - u|^2 d\mu \\ &\leq \frac{\epsilon}{C_i} \|u_k - u\|_{E_i}^2 \\ &\leq \epsilon. \end{aligned} \quad (2.5)$$

Moreover, by Eq (2.4), we have

$$\lim_{k \rightarrow \infty} \int_{d(x, x_0) \leq r_0} |u_k - u|^2 d\mu = 0. \quad (2.6)$$

Combining Eqs (2.5) and (2.6), we conclude that

$$\lim_{k \rightarrow \infty} \int_V |u_k - u|^2 d\mu = 0,$$

that is, $u_k \rightarrow u$ in $L^2(V)$. Since

$$\|u_k - u\|_\infty^2 \leq \frac{1}{\mu_{\min}} \int_V |u_k - u|^2 d\mu,$$

for any $2 < r < \infty$, we have

$$\int_V |u_k - u|^r d\mu \leq \|u_k - u\|_\infty^{r-2} \int_V |u_k - u|^2 d\mu \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Therefore, up to a subsequence, $u_k \rightarrow u$ in $L^s(V)$ for all $2 \leq r \leq \infty$. The proof is completed. $\square$

Let $E := E_1 \times E_2$. Then, E is a Hilbert space with the inner product $\langle (u, v), (\phi, \psi) \rangle = (u, \phi)_{E_1} + (v, \psi)_{E_2}$.

The corresponding norm is $\|(u, v)\|_E = (\|u\|_{E_1}^2 + \|v\|_{E_2}^2)^{\frac{1}{2}}$. The equivalent norm is $\|(u, v)\| = \|u\|_{E_1} + \|v\|_{E_2}$ which will be used in the sequel. We define the functional $\Phi : E \rightarrow \mathbb{R}$ by

$$\begin{aligned}\Phi(u, v) &= \frac{1}{2} \int_V (|\Delta u|^2 + a_1 |\nabla u|^2 + H_1(x) u^2) d\mu + \frac{1}{2} \int_V (|\Delta v|^2 + a_2 |\nabla v|^2 + H_2(x) v^2) d\mu \\ &\quad + \frac{b_1}{4} \left(\int_V |\nabla u|^2 d\mu \right)^2 + \frac{b_2}{4} \left(\int_V |\nabla v|^2 d\mu \right)^2 - \int_V F(x, u, v) d\mu \\ &= \frac{1}{2} \|u\|_{E_1}^2 + \frac{1}{2} \|v\|_{E_2}^2 + \frac{b_1}{4} \left(\int_V |\nabla u|^2 d\mu \right)^2 + \frac{b_2}{4} \left(\int_V |\nabla v|^2 d\mu \right)^2 - \int_V F(x, u, v) d\mu. \quad (2.7)\end{aligned}$$

Lemma 2.2. ([2, 12]) *If (H) and (F₁) hold, then $\Phi \in C^1(E, \mathbb{R})$ and*

$$\begin{aligned}\langle \Phi'(u, v), (\phi_1, \phi_2) \rangle &= (u, \phi_1) + (v, \phi_2) + b_1 \int_V |\nabla u|^2 d\mu \int_V \nabla u \nabla \phi_1 d\mu + b_2 \int_V |\nabla v|^2 d\mu \int_V \nabla v \nabla \phi_2 d\mu \\ &\quad - \int_V F_u(x, u, v) \phi_1 d\mu - \int_V F_v(x, u, v) \phi_2 d\mu, \quad (2.8)\end{aligned}$$

for all $(u, v), (\phi_1, \phi_2) \in E$. Moreover, $\Psi' : E \rightarrow E^*$ is compact, where $\Psi(u, v) = \int_V F(x, u, v) d\mu$.

By Eq (2.8), it is easy to obtain that

$$\begin{aligned}\langle \Phi'(u, v), (u, v) \rangle &= \|u\|_{E_1}^2 + \|v\|_{E_2}^2 + b_1 \left(\int_V |\nabla u|^2 d\mu \right)^2 + b_2 \left(\int_V |\nabla v|^2 d\mu \right)^2 \\ &\quad - \int_V F_u(x, u, v) u d\mu - \int_V F_v(x, u, v) v d\mu, \quad \forall (u, v) \in E. \quad (2.9)\end{aligned}$$

Let X be a Banach space. We say that $\Phi \in C^1(X, \mathbb{R})$ satisfies (C)-condition if any sequence $\{u_n\} \subset X$ such that

$$\Phi(u_n) \text{ is bounded, } \|\Phi'(u_n)\|_{X^*} (1 + \|u_n\|) \rightarrow 0 \quad (2.10)$$

has a convergent subsequence, where $\|\cdot\|_{X^*}$ is the norm in the dual space X^* (see [39]).

Lemma 2.3. ([40, 41]) *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finitely dimensional. If $\Phi \in C^1(X, \mathbb{R})$ satisfies (C)-condition, and*

(I₁) $\Phi(0) = 0$, $\Phi(-u) = \Phi(u)$ for all $u \in X$;

(I₂) there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;

(I₃) for any finite dimensional subspace $\tilde{X} \subset X$, there exists $R = R(\tilde{X}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{X} \setminus B_R$, then Φ possesses an unbounded sequence of critical values.

3. Proofs for the biharmonic-Kirchhoff case ($b_i > 0, i = 1, 2$)

In this section, we aim to investigate system (1.1) with ($b_i > 0, i = 1, 2$) and prove Theorems 1.1 and 1.2. It is easy to see that $(u, v) \in E$ is a critical point of Φ if and only if

$$(u, \phi_1) + b_1 \int_V |\nabla u|^2 d\mu \int_V \nabla u \nabla \phi_1 d\mu - \int_V F_u(x, u, v) \phi_1 d\mu = 0,$$

and

$$(v, \phi_2) + b_2 \int_V |\nabla v|^2 d\mu \int_V \nabla v \nabla \phi_2 d\mu - \int_V F_v(x, u, v) \phi_2 d\mu = 0,$$

for all $(\phi_1, \phi_2) \in E$.

In what follows, we will present some lemmas to prove our main theorems.

Lemma 3.1. *Under (H) , (F_1) , (F_2) , (F_3) , and (F_4) , any sequence $\{(u_n, v_n)\} \subset E$ satisfying*

$$\Phi(u_n, v_n) \text{ is bounded, } \|\Phi'(u_n, v_n)\|(1 + \|(u_n, v_n)\|) \rightarrow 0 \quad (3.1)$$

is bounded in E .

Proof. To prove the boundness of $\{(u_n, v_n)\}$, arguing by contradiction, assume that $\|(u_n, v_n)\| \rightarrow \infty$. Let $\phi_{1n} = \frac{u_n}{\|(u_n, v_n)\|}$ and $\phi_{2n} = \frac{v_n}{\|(u_n, v_n)\|}$. Then, $\|\phi_{1n}\|_{E_1} \leq 1$, $\|\phi_{2n}\|_{E_2} \leq 1$, $\|\phi_{1n}\|_r \leq \gamma_{r,1} \|\phi_{1n}\|_{E_1} \leq \gamma_{r,1}$ and $\|\phi_{2n}\|_r \leq \gamma_{r,2} \|\phi_{2n}\|_{E_2} \leq \gamma_{r,2}$ for all $r \geq 2$. Then, passing to a subsequence, we may assume that $\phi_{1n} \rightharpoonup \phi_1^*$ in E_1 , $\phi_{2n} \rightharpoonup \phi_2^*$ in E_2 . Furthermore, by Lemma 2.1, $\phi_{1n} \rightarrow \phi_1^*$, $\phi_{2n} \rightarrow \phi_2^*$ in $L^r(V)$ for all $r \geq 2$, and $\phi_{1n}(x) \rightarrow \phi_1^*(x)$, $\phi_{2n}(x) \rightarrow \phi_2^*(x)$ for all $x \in V$.

Firstly, we claim that $\phi_1^* \not\equiv 0$ or $\phi_2^* \not\equiv 0$. Indeed, if $\phi_1^* \equiv \phi_2^* \equiv 0$ for all $x \in V$, then $\phi_{1n}, \phi_{2n} \rightarrow 0$ in $L^r(V)$ for all $r \geq 2$, and $\phi_{1n}(x), \phi_{2n}(x) \rightarrow 0$ for all $x \in V$. Combining Eq (2.7) with Eq (3.1), we get

$$2 \limsup_{n \rightarrow \infty} \int_V \frac{|F(x, u_n, v_n)|}{\|(u_n, v_n)\|^2} d\mu \geq \limsup_{n \rightarrow \infty} \int_V \frac{|F(x, u_n, v_n)|}{\|u_n\|_{E_1}^2 + \|v_n\|_{E_2}^2} d\mu \geq \frac{1}{2}. \quad (3.2)$$

For the constants $0 \leq d < e$, we denote

$$\Omega_n[d, e] = \{x \in V : d \leq |(u_n(x), v_n(x))| < e\}.$$

Then, it follows from (F_1) that

$$\begin{aligned} & \int_{\Omega_n[0, r_0]} \frac{|F(x, u_n, v_n)|}{\|(u_n, v_n)\|^2} d\mu \\ & \leq \frac{1}{\|(u_n, v_n)\|^2} \int_{\Omega_n(0, r_0)} a(|(u_n, v_n)|) b(x) d\mu \\ & \leq \frac{1}{\|(u_n, v_n)\|^2} \max_{0 \leq |(s, t)| \leq r_0} a(|(s, t)|) \int_V b(x) d\mu \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.3)$$

It follows from Eqs (2.7), (2.8), and (3.1) that there exists a constant $c_1 > 0$ such that

$$|\Phi(u_n, v_n)| \leq c_1, \text{ for all } n \in \mathbb{N}, \quad (3.4)$$

and for all large n ,

$$c_1 + 1 \geq \Phi(u_n, v_n) - \frac{1}{4} \langle \Phi'(u_n, v_n), (u_n, v_n) \rangle = \int_V \mathcal{F}(x, u_n, v_n) d\mu + \frac{1}{4} \|u_n\|^2 + \frac{1}{4} \|v_n\|^2, \quad (3.5)$$

where $\mathcal{F}$ is defined in assumptions (F_3) . Set $k' = \frac{k}{k-1}$. Since $k > 1$, we get $2k' \in (2, \infty)$. Hence, from (F_3) and Eq (3.5), we have

$$\int_{\Omega_n[r_0, \infty)} \frac{|F(x, u_n, v_n)|}{\|(u_n, v_n)\|^2} d\mu$$

$$\begin{aligned}
&= \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^2} |(\phi_{1n}, \phi_{2n})|^2 d\mu \\
&\leq \left(\int_{\Omega_n[r_0, \infty)} \left(\frac{F(x, u_n, v_n)}{|(u_n, v_n)|^2} \right)^k d\mu \right)^{\frac{1}{k}} \left(\int_{\Omega_n[r_0, \infty)} |(\phi_{1n}, \phi_{2n})|^{2k'} d\mu \right)^{\frac{1}{k'}} \\
&\leq c^{\frac{1}{k}} \left(\int_{\Omega_n[r_0, \infty)} \mathcal{F}(x, u_n, v_n) d\mu \right)^{\frac{1}{k}} \left(\int_{\Omega_n[r_0, \infty)} |(\phi_{1n}, \phi_{2n})|^{2k'} d\mu \right)^{\frac{1}{k'}} \\
&\leq [c(c_1 + 1)]^{\frac{1}{k}} \left(\int_{\Omega_n[r_0, \infty)} |(\phi_{1n}, \phi_{2n})|^{2k'} d\mu \right)^{\frac{1}{k'}} \\
&\leq 2^{\frac{2k'-1}{k'}} [c(c_1 + 1)]^{\frac{1}{k}} \left(\int_V |\phi_{1n}|^{2k'} d\mu + \int_V |\phi_{2n}|^{2k'} d\mu \right)^{\frac{1}{k'}} \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.6}$$

Combining (F_0) , Eq (3.3) with Eq (3.6), we get

$$\int_V \frac{|F(x, u_n, v_n)|}{\|(u_n, v_n)\|^2} d\mu = \int_{\Omega_n[0, r_0)} \frac{|F(x, u_n, v_n)|}{\|(u_n, v_n)\|^2} d\mu + \int_{\Omega_n[r_0, \infty)} \frac{|F(x, u_n, v_n)|}{\|(u_n, v_n)\|^2} d\mu \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which contradicts with Eq (3.2). So $\phi_1^* \neq 0$ or $\phi_2^* \neq 0$.

Next, we prove $\{(u_n, v_n)\}$ is bounded in E . Without loss of generality, we assume that $\phi_1^* \neq 0$. Set

$$A := \{x \in V : \phi_1^*(x) \neq 0\}.$$

Then $A \neq \emptyset$. Thus, for all $x \in A$, we have

$$\lim_{n \rightarrow \infty} |u_n(x)| = +\infty.$$

Hence,

$$A \subset \Omega_n[r_0, \infty) \text{ for large } n \in \mathbb{N}.$$

It follows from Eqs (2.7) and (3.4), (F_2) , and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{n \rightarrow \infty} \frac{-c}{\|(u_n, v_n)\|^4} \\
&\leq \lim_{n \rightarrow \infty} \frac{\Phi(u_n, v_n)}{\|(u_n, v_n)\|^4} \\
&\leq \lim_{n \rightarrow \infty} \left[\max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \int_V \frac{F(x, u_n, v_n)}{\|(u_n, v_n)\|^4} d\mu \right] \\
&= \lim_{n \rightarrow \infty} \left[\max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \int_{\Omega_n[0, r_0)} \frac{F(x, u_n, v_n)}{\|(u_n, v_n)\|^4} d\mu - \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} |(\phi_{1n}, \phi_{2n})|^4 d\mu \right] \\
&\leq \limsup_{n \rightarrow \infty} \left[\max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} + \frac{1}{\|(u_n, v_n)\|^4} \max_{0 \leq |(s, t)| \leq r_0} a(|(s, t)|) \int_V b(x) d\mu \right. \\
&\quad \left. - \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} |(\phi_{1n}, \phi_{2n})|^4 d\mu \right] \\
&= \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \liminf_{n \rightarrow \infty} \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} |(\phi_{1n}, \phi_{2n})|^4 d\mu
\end{aligned}$$

$$\begin{aligned}
&\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \int_V \liminf_{n \rightarrow \infty} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \chi_{\Omega_n[r_0, \infty)} |(\phi_{1n}, \phi_{2n})|^4 d\mu \\
&\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \int_A \liminf_{n \rightarrow \infty} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \chi_{\Omega_n[r_0, \infty)} |(\phi_{1n}, \phi_{2n})|^4 d\mu \\
&= -\infty,
\end{aligned} \tag{3.7}$$

where $\chi_{\Omega_n[r_0, \infty)} = \begin{cases} 1, & x \in \Omega_n[r_0, \infty) \\ 0, & x \in V \setminus \Omega_n[r_0, \infty) \end{cases}$, which is a contradiction. Thus, $\{(u_n, v_n)\}$ is bounded in E .

Lemma 3.2. Under (H), (F_1) , (F_2) , (F_3) and (F_4) , any sequence $\{(u_n, v_n)\} \subset E$ satisfying Eq (3.1) has a convergent subsequence in E .

Proof. From Lemma 3.1, we know that $\{(u_n, v_n)\}$ is bounded in E , that is, there exists a positive constant M_1 such that $\|(u_n, v_n)\|_{E_1} \leq M_1$. Then, by Lemma 2.1, $\|u_n\|_\infty \leq M_1 \gamma_{\infty,1}$ and $\|v_n\|_\infty \leq M_1 \gamma_{\infty,2}$. Going if necessary to a subsequence, we can assume that $(u_n, v_n) \rightharpoonup (u, v)$ in E . By Lemma 2.1, $u_n \rightarrow u$ and $v_n \rightarrow v$ in $L^s(V)$ for all $2 \leq s \leq +\infty$. Thus,

$$\begin{aligned}
&\int_V |F_u(x, u_n, v_n) - F_u(x, u, v)| |u_n - u| d\mu \\
&\leq \int_V (|F_u(x, u_n, v_n)| + |F_u(x, u, v)|) |u_n - u| d\mu \\
&\leq \int_V 2a(|(u_n, v_n)|) b(x) |u_n - u| d\mu \\
&\leq 2 \max_{|(s,t)| \leq M_1 \gamma_{\infty,1} + M_1 \gamma_{\infty,2}} a(|(s, t)|) \int_V b(x) d\mu \|u_n - u\|_\infty \\
&\rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.8}$$

Similarly, we also obtain

$$\int_V |F_v(x, u_n, v_n) - F_v(x, u, v)| |v_n - v| d\mu \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.9}$$

Moreover, by Lemma 2.1, we have $u_n(x) \rightarrow u(x)$ and $v_n(x) \rightarrow v(x)$ for all $x \in V$. Then, by the definition of ∇u , it is easy to see that $\nabla u_n(x) \rightarrow \nabla u(x)$ for all $x \in V$, as $n \rightarrow \infty$. Then, by the boundness of $\{u_n\}$ in E , we have

$$\begin{aligned}
&b_1 \left(\int_V |\nabla u|^2 d\mu \int_V \nabla u \nabla(u_n - u) d\mu - \int_V |\nabla u_n|^2 d\mu \int_V \nabla u_n \nabla(u_n - u) d\mu \right) \\
&\leq b_1 \left(\int_V |\nabla u|^2 d\mu \int_V |\nabla u| |\nabla(u_n - u)| d\mu + \int_V |\nabla u_n|^2 d\mu \int_V |\nabla u_n| |\nabla(u_n - u)| d\mu \right) \\
&\leq b_1 \left[\left(\int_V |\nabla u|^2 d\mu \right)^{\frac{3}{2}} + \left(\int_V |\nabla u_n|^2 d\mu \right)^{\frac{3}{2}} \right] \left(\int_V |\nabla u_n - \nabla u|^2 d\mu \right)^{\frac{1}{2}} \\
&\rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.10}$$

Observe that

$$\|u_n - u\|_{E_1}^2 + \|v_n - v\|_{E_2}^2$$

$$\begin{aligned}
&= \langle \Phi'(u_n, v_n) - \Phi'(u, v), (u_n - u, v_n - v) \rangle \\
&\quad + \int_V [F_u(x, u_n, v_n) - F_u(x, u, v)](u_n - u) d\mu + \int_V [F_v(x, u_n, v_n) - F_v(x, u, v)](v_n - v) d\mu \\
&\quad + b_1 \left(\int_V |\nabla u|^2 d\mu \int_V \nabla u \nabla (u_n - u) d\mu - \int_V |\nabla u_n|^2 d\mu \int_V \nabla u_n \nabla (u_n - u) d\mu \right) \\
&\quad + b_2 \left(\int_V |\nabla v|^2 d\mu \int_V \nabla v \nabla (v_n - v) d\mu - \int_V |\nabla v_n|^2 d\mu \int_V \nabla v_n \nabla (v_n - v) d\mu \right). \tag{3.11}
\end{aligned}$$

By Eq (3.1), it is clear that $\langle \Phi'(u_n, v_n) - \Phi'(u, v), (u_n - u, v_n - v) \rangle \rightarrow 0$ as $n \rightarrow \infty$. From Eqs (3.8)–(3.11), we get

$$\|u_n - u\|_{E_1} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and

$$\|v_n - v\|_{E_2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The proof is completed. $\square$

Lemma 3.3. Under (H), (F₁), (F₂), (F₃), and (F₅), any sequence $\{(u_n, v_n)\} \subset E$ satisfying Eq (3.1) has a convergent subsequence in E .

Proof. Firstly, we prove that $\{(u_n, v_n)\}$ is bounded in E . To prove the boundedness of $\{(u_n, v_n)\}$, arguing by contradiction, suppose that $\|(u_n, v_n)\| \rightarrow \infty$. By Eqs (3.1), (2.9) and (F₄), for large n , there exists a constant $c > 0$ such that

$$\begin{aligned}
c + 1 &\geq \Phi(u_n, v_n) - \frac{1}{\mu} \langle \Phi'(u_n, v_n), (u_n, v_n) \rangle \\
&= \frac{\mu - 2}{2\mu} (\|u_n\|_{E_1}^2 + \|v_n\|_{E_2}^2) + \frac{(\mu - 4)b_1}{4\mu} \left(\int_V |\nabla u_n|^2 d\mu \right)^2 + \frac{(\mu - 4)b_2}{4\mu} \left(\int_V |\nabla v_n|^2 d\mu \right)^2 \\
&\quad + \int_V \left[\frac{1}{\mu} F_u(x, u_n, v_n) u_n + \frac{1}{\mu} F_v(x, u_n, v_n) v_n - F(x, u_n, v_n) \right] d\mu \\
&\geq \frac{\mu - 2}{2\mu} (\|u_n\|_{E_1}^2 + \|v_n\|_{E_2}^2) + \frac{(\mu - 4)b_1}{4\mu} \left(\int_V |\nabla u_n|^2 d\mu \right)^2 + \frac{(\mu - 4)b_2}{4\mu} \left(\int_V |\nabla v_n|^2 d\mu \right)^2 \\
&\quad - \frac{\sigma}{\mu} (\|u_n\|_2^2 + \|v_n\|_2^2) \\
&\geq \frac{\mu - 2}{2\mu} (\|u_n\|_{E_1}^2 + \|v_n\|_{E_2}^2) - \frac{\sigma}{\mu} (\gamma_{2,1}^2 \|u_n\|_{E_1}^2 + \gamma_{2,2}^2 \|v_n\|_{E_2}^2) \\
&\geq \left(\frac{\mu - 2}{2\mu} - \frac{\sigma}{\mu} \max\{\gamma_{2,1}^2, \gamma_{2,2}^2\} \right) (\|u_n\|_{E_1}^2 + \|v_n\|_{E_2}^2). \tag{3.12}
\end{aligned}$$

Note that $\sigma \in \left(0, \frac{\mu - 2}{2 \max\{\gamma_{2,1}^2, \gamma_{2,2}^2\}}\right)$. Thus, Eq (3.12) implies that $\{(u_n, v_n)\}$ is bounded in E . The rest proof is the same as that in Lemma 3.2. $\square$

Lemma 3.4. Under (H), (F₁), and (F₂), for any finite dimensional subspace $\widetilde{E} = \widetilde{E}_1 \times \widetilde{E}_2 \subset E$, where $\widetilde{E}_1 \subset E_1$ and $\widetilde{E}_2 \subset E_2$, there holds

$$\Phi(u, v) \rightarrow -\infty, \text{ as } \|(u, v)\| \rightarrow \infty, \forall (u, v) \in \widetilde{E}. \tag{3.13}$$

Proof. Arguing indirectly, assume that for some sequence $\{(u_n, v_n)\} \subset \widetilde{E}$ with $\|(u_n, v_n)\| \rightarrow \infty$, there exists $M_1 > 0$ such that $\Phi(u_n, v_n) \geq -M_1$ for all $n \in \mathbb{N}$. Next, we argue from three cases.

Case (I): assume that $\|u_n\|_{E_1} \rightarrow \infty$ and $\|v_n\|_{E_2} \rightarrow \infty$. Let $\varphi_{1n} = \frac{u_n}{\|u_n\|_{E_1}}$ and $\varphi_{2n} = \frac{v_n}{\|v_n\|_{E_2}}$. Then, $\|\varphi_{1n}\|_{E_1} = 1$ and $\|\varphi_{2n}\|_{E_2} = 1$. Then, we may assume that there exists a subsequence, still denoted by $\{\varphi_{1n}\}$ and $\{\varphi_{2n}\}$, such that $\varphi_{1n} \rightarrow \varphi_1^*$ and $\varphi_{2n} \rightarrow \varphi_2^*$ in E . Since $\widetilde{E}$ is finite dimensional, then $\varphi_{1n} \rightarrow \varphi_1^*$, $\varphi_{2n} \rightarrow \varphi_2^* \in \widetilde{E}$ in E , and $\varphi_{1n}(x) \rightarrow \varphi_1^*(x)$, $\varphi_{2n}(x) \rightarrow \varphi_2^*(x)$ for all $x \in V$, and so $\|\varphi_1^*\|_{E_1} = 1$ and $\|\varphi_2^*\|_{E_2} = 1$. Set

$$B := \{x \in V : \varphi_1^*(x) \neq 0\}.$$

We get $B \neq \emptyset$. For all $x \in B$, we have

$$\lim_{n \rightarrow \infty} |u_n(x)| = +\infty.$$

Hence,

$$B \subset \Omega_n[r_0, \infty) \text{ for large } n \in \mathbb{N}.$$

Similar to the argument of Eq (3.7), we have the following contradiction:

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{-M_1}{\|(u_n, v_n)\|^4} \\ &\leq \lim_{n \rightarrow \infty} \frac{\Phi(u_n, v_n)}{\|(u_n, v_n)\|^4} \\ &\leq \lim_{n \rightarrow \infty} \left[\max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \int_V \frac{F(x, u_n, v_n)}{\|(u_n, v_n)\|^4} d\mu \right] \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \liminf_{n \rightarrow \infty} \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \frac{|\varphi_{1n}\|_{E_1} \|\varphi_{2n}\|_{E_2}}{\|(u_n, v_n)\|^4} d\mu \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \frac{1}{8} \liminf_{n \rightarrow \infty} \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \min\{|\varphi_{1n}|^4, |\varphi_{2n}|^4\} d\mu \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \frac{1}{8} \int_V \liminf_{n \rightarrow \infty} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \chi_{\Omega_n[r_0, \infty)} \min\{|\varphi_{1n}|^4, |\varphi_{2n}|^4\} d\mu \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \frac{1}{8} \int_B \liminf_{n \rightarrow \infty} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \chi_{\Omega_n[r_0, \infty)} \min\{|\varphi_{1n}|^4, |\varphi_{2n}|^4\} d\mu \\ &= -\infty, \end{aligned} \tag{3.14}$$

where we have used the inequality that

$$|(\varphi_{1n}\|u_n\|_{E_1}, \varphi_{2n}\|v_n\|_{E_2})|^4 \geq |\varphi_{1n}|^4 \|u_n\|_{E_1}^4 + |\varphi_{2n}|^4 \|v_n\|_{E_2}^4 \geq \frac{1}{8} \min\{|\varphi_{1n}|^4, |\varphi_{2n}|^4\} \|(u_n, v_n)\|^4.$$

Case (II): assume that $\|u_n\|_{E_1} \rightarrow \infty$ and $\|v_n\|_{E_2}$ is bounded. Let $\hat{\varphi}_{1n} = \frac{u_n}{\|u_n\|_{E_1}}$. Then $\|\hat{\varphi}_{1n}\|_{E_1} = 1$ and there exists a constant $M_2 > 0$ such that $\|v_n\|_{E_2} \leq M_2$. Then, we may assume that there exists a subsequence, still denoted by $\{\hat{\varphi}_{1n}\}$, such that $\hat{\varphi}_{1n} \rightarrow \hat{\varphi}_1^*$ in E_1 . Since $\widetilde{E}_1$ is finite dimensional, then $\hat{\varphi}_{1n} \rightarrow \hat{\varphi}_1^* \in \widetilde{E}_1$ in E , and $\hat{\varphi}_{1n}(x) \rightarrow \hat{\varphi}_1^*(x)$ for all $x \in V$, and so $\|\hat{\varphi}_1^*\|_{E_1} = 1$. Set

$$D := \{x \in V : \hat{\varphi}_1^*(x) \neq 0\}.$$

We get $D \neq \emptyset$. For all $x \in D$, we have

$$\lim_{n \rightarrow \infty} |u_n(x)| = +\infty.$$

Hence,

$$D \subset \Omega_n[r_0, \infty) \text{ for large } n \in \mathbb{N}.$$

Similar to the argument of Eq (3.7), we have the following contradiction:

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{-M_1}{\|(u_n, v_n)\|^4} \\ &\leq \lim_{n \rightarrow \infty} \frac{\Phi(u_n, v_n)}{\|(u_n, v_n)\|^4} \\ &\leq \lim_{n \rightarrow \infty} \left[\max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \int_V \frac{F(x, u_n, v_n)}{\|(u_n, v_n)\|^4} d\mu \right] \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \liminf_{n \rightarrow \infty} \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \frac{|\hat{\varphi}_{1n}|^4 \|u_n\|_{E_1}^4 \|v_n\|_{E_2}^4}{\|(u_n, v_n)\|^4} d\mu \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \frac{1}{16} \liminf_{n \rightarrow \infty} \int_{\Omega_n[r_0, \infty)} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} |\hat{\varphi}_{1n}|^4 d\mu \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \frac{1}{16} \int_V \liminf_{n \rightarrow \infty} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \chi_{\Omega_n[r_0, \infty)} |\hat{\varphi}_{1n}|^4 d\mu \\ &\leq \max \left\{ \frac{b_1}{4a_1^2}, \frac{b_2}{4a_2^2} \right\} - \frac{1}{16} \int_D \liminf_{n \rightarrow \infty} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^4} \chi_{\Omega_n[r_0, \infty)} |\hat{\varphi}_{1n}|^4 d\mu \\ &= -\infty, \end{aligned} \tag{3.15}$$

where we have used the inequality that

$$\frac{|\hat{\varphi}_{1n}|^4 \|u_n\|_{E_1}^4 \|v_n\|_{E_2}^4}{(\|u_n\|_{E_1} + \|v_n\|_{E_2})^4} \geq \frac{\hat{\varphi}_{1n}^4 \|u_n\|_{E_1}^4}{(\|u_n\|_{E_1} + M_2)^4} \geq \frac{\hat{\varphi}_{1n}^4 \|u_n\|_{E_1}^4}{(\|u_n\|_{E_1} + \|u_n\|_{E_1})^4} = \frac{\hat{\varphi}_{1n}^4}{2^4} \text{ for all sufficiently large } n.$$

Case (III): assume that $\|u_n\|_{E_1}$ is bounded and $\|v_n\|_{E_2} \rightarrow \infty$. This case is similar to Case (II) by exchanging u_n with v_n . We omit the details. The proof is completed. $\square$

Corollary 3.5. Under (H), (F_1) , and (F_2) , for any finite dimensional subspace $\tilde{E} \subset E$, there exists $R = R(\tilde{E}) > 0$, such that $\Phi(u, v) \leq 0$, $\forall (u, v) \in \tilde{E}$, $\|(u, v)\| \geq R$.

Let $\{e_{ij}\}_{j=1}^\infty, i = 1, 2$ are two total orthonormal bases of $E_i, i = 1, 2$, respectively. For any given $k \in \mathbb{N}$, let

$$X_j = (\mathbb{R}e_{1j}, \mathbb{R}e_{2j}), Y_k = \bigoplus_{j=1}^k X_j = \left(\bigoplus_{j=1}^k \mathbb{R}e_{1j}, \bigoplus_{j=1}^k \mathbb{R}e_{2j} \right), Z_k = \bigoplus_{j=k+1}^\infty X_j = \left(\bigoplus_{j=k+1}^\infty \mathbb{R}e_{1j}, \bigoplus_{j=k+1}^\infty \mathbb{R}e_{2j} \right), k \in \mathbb{Z}.$$

Then, $E = Y_k \oplus Z_k$ and Y_k is of finite dimension.

Lemma 3.6. Under (H), for $r \geq 2$, it holds that

$$\beta_k := \sup_{(u,v) \in Z_k, \|(u,v)\|=1} (\|u\|_r + \|v\|_r) \rightarrow 0, \text{ as } k \rightarrow \infty. \tag{3.16}$$

Proof. It is easy to see that $0 < \beta_{k+1} \leq \beta_k$. Hence, $\beta_k \rightarrow \beta \geq 0$ ($k \rightarrow \infty$). By the definition of β_k , we obtain that for every $k \in \mathbb{N}$, there exists $(u_k, v_k) \in Z_k$ with $\|(u_k, v_k)\| = 1$ such that

$$\|u\|_r + \|v\|_r > \frac{\beta_k}{2}. \quad (3.17)$$

For any given $(\phi_1, \phi_2) \in E$, write $\phi_1 = \sum_{j=1}^{\infty} c_{1j} e_{1,j}$ and $\phi_2 = \sum_{j=1}^{\infty} c_{2j} e_{2,j}$, where $c_{1j} = (\phi_1, e_{1,j})_{E_1}$ and $c_{2j} = (\phi_2, e_{2,j})_{E_2}$. By Parseval equality, we have $\|\phi_1\|_{E_1}^2 = \sum_{i=1}^{\infty} c_{1i}^2$ and $\|\phi_2\|_{E_2}^2 = \sum_{i=1}^{\infty} c_{2i}^2$. Furthermore, by $\phi_1 \in E_1$ and $\phi_2 \in E_2$, we get

$$\left(\sum_{i=k+1}^{\infty} c_{1i}^2 \right)^{\frac{1}{2}} + \left(\sum_{j=k+1}^{\infty} c_{2j}^2 \right)^{\frac{1}{2}} \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (3.18)$$

Then, it follows from the Cauchy-Schwartz inequality and Eq (3.18) that

$$\begin{aligned} | \langle (u_k, v_k), (\phi_1, \phi_2) \rangle | &= \left| \left(u_k, \sum_{j=1}^{\infty} c_{1j} e_{1,j} \right) \right| + \left| \left(v_k, \sum_{j=1}^{\infty} c_{2j} e_{2,j} \right) \right| \\ &= \left| \left(u_k, \sum_{i=k+1}^{\infty} c_{1i} e_{1,i} \right) \right| + \left| \left(v_k, \sum_{j=k+1}^{\infty} c_{2j} e_{2,j} \right) \right| \\ &\leq \|u_k\|_{E_1} \cdot \left\| \sum_{i=k+1}^{\infty} c_{1i} e_{1,i} \right\|_{E_1} + \|v_k\|_{E_2} \cdot \left\| \sum_{j=k+1}^{\infty} c_{2j} e_{2,j} \right\|_{E_2} \\ &= \left(\sum_{i=k+1}^{\infty} c_{1i}^2 \right)^{\frac{1}{2}} + \left(\sum_{j=k+1}^{\infty} c_{2j}^2 \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

which, together with the Riesz lemma, implies that $(u_k, v_k) \rightarrow (0, 0)$ in E and then $u_k \rightarrow 0$ in E_1 and $v_k \rightarrow 0$ in E_2 . By Lemma 2.1, the compact embedding of $E_i \hookrightarrow L^r(V)$ ($r \geq 2, i = 1, 2$) implies that $u_k \rightarrow 0$ and $v_k \rightarrow 0$ in $L^r(V)$. Hence, letting $k \rightarrow \infty$, it follows from Eq (3.17) that $\beta = 0$. Thus, the proof is completed. $\square$

Lemma 3.7. Under (H), (F_1) , and (C_1) , for any given large k , there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z_k} \geq \alpha$.

Proof. It follows from (C_1) that for any given $\varepsilon > 0$, there exists a $0 < \delta < \frac{\max\{\gamma_{\infty,1}, \gamma_{\infty,2}\}}{2^{\frac{p+1}{p-2}}(h+\varepsilon)\|b\|_{\infty}^{\frac{1}{p-2}}}$ such that

$$a(|(s, t)|) \leq (h + \varepsilon)|(t, s)|^2 \leq 2(h + \varepsilon)(|t|^2 + |s|^2), \text{ for } \forall |(t, s)| \leq \delta. \quad (3.19)$$

For all $(u, v) \in Z_k$ with $\|(u, v)\| = \rho = \frac{\delta}{\max\{\gamma_{\infty,1}, \gamma_{\infty,2}\}}$, by Lemma 2.1, we have

$$|(u(x), v(x))| \leq \|u\|_{\infty} + \|v\|_{\infty} \leq \gamma_{\infty,1}\|u\|_{E_1} + \gamma_{\infty,2}\|v\|_{E_2} \leq \max\{\gamma_{\infty,1}, \gamma_{\infty,2}\}\|(u, v)\| \leq \delta. \quad (3.20)$$

Moreover, by Lemma 3.6, for any given large k , it holds that

$$(\|u\|_2 + \|v\|_2)^2 \leq \frac{1}{16(h + \varepsilon)\|b\|_{\infty}} \|(u, v)\|^2, \quad \forall (u, v) \in Z_k. \quad (3.21)$$

Note that the fact that $b \in L^1(V)$ implies that $b \in L^\infty(V)$. Then, for all $(u, v) \in Z_k$ with $\|(u, v)\| = \rho$, by (F_1) and Eqs (3.19)–(3.21), we obtain that

$$\begin{aligned}
 \Phi(u, v) &= \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 + \frac{b_1}{4} \left(\int_V |\nabla u|^2 d\mu \right)^2 + \frac{b_2}{4} \left(\int_V |\nabla v|^2 d\mu \right)^2 - \int_V F(x, u, v) d\mu \\
 &\geq \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 - \int_V a(|(u, v)|) b(x) d\mu \\
 &\geq \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 - 2(h + \varepsilon) \|b\|_\infty (\|u\|_2^2 + \|v\|_2^2) \\
 &\geq \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 - 2(h + \varepsilon) \|b\|_\infty (\|u\|_2 + \|v\|_2)^2 \\
 &\geq \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 - 2(h + \varepsilon) \|b\|_\infty \cdot \frac{1}{16(h + \varepsilon) \|b\|_\infty} \|(u, v)\|^2 \\
 &\geq \frac{1}{2} \cdot \frac{1}{2} \|(u, v)\|^2 - \frac{1}{8} \|(u, v)\|^2 \\
 &= \frac{1}{8} \rho^2 := \alpha > 0.
 \end{aligned}$$

Thus, the proof is complete. $\square$

Lemma 3.8. Under (H) , (F_1) and (C_2) , $\Phi|_{\partial B_{\rho_*} \cap Z_k} \geq \alpha_*$, where ρ_* and α_* are given in (C_2) .

Proof. By (F_1) and (C_2) , for $(u, v) \in Z_k$ with $\|(u, v)\| = \rho$, we have

$$\begin{aligned}
 \Phi(u, v) &= \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 + \frac{b_1}{4} \left(\int_V |\nabla u|^2 d\mu \right)^2 + \frac{b_2}{4} \left(\int_V |\nabla v|^2 d\mu \right)^2 - \int_V F(x, u, v) d\mu \\
 &\geq \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 - \int_V a(|(u, v)|) b(x) d\mu \\
 &\geq \frac{1}{2}\|u\|_{E_1}^2 + \frac{1}{2}\|v\|_{E_2}^2 - \max_{|s|+|t| \leq (\gamma_{\infty,1} + \gamma_{\infty,2})\rho} a(|(s, t)|) \int_V b(x) d\mu \\
 &\geq \frac{1}{2} \cdot \frac{1}{2} (\|u\|_{E_1} + \|v\|_{E_2})^2 - \max_{|s|+|t| \leq (\gamma_{\infty,1} + \gamma_{\infty,2})\rho} a(|(s, t)|) \int_V b(x) d\mu \\
 &= \alpha_* > 0.
 \end{aligned}$$

Thus, the proof is completed. $\square$

Proof of Theorem 1.1. Let $X = E$, $Y = Y_k$, $Z = Z_k$. By Lemmas 3.1, 3.2, 3.7 and Corollary 3.5, all conditions of Lemma 2.4 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial solutions $\{(u_k, v_k)\}$ such that Eq (1.4) holds. $\square$

Proof of Theorem 1.2. Let $X = E$, $Y = Y_k$, $Z = Z_k$. By Lemmas 3.1, 3.2, 3.8 and Corollary 3.5, all conditions of Lemma 2.4 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial solutions $\{(u_k, v_k)\}$ such that Eq (1.4) holds. $\square$

4. The results corresponding to the biharmonic case ($b_i = 0, i = 1, 2$)

In this section, corresponding to Theorems 1.1 and 1.2, we present some results for system (1.1) with $b_i = 0, i = 1, 2$, that is,

$$\begin{cases} \Delta^2 u - a_1 \Delta u + H_1(x)u = F_u(x, u, v), & x \in V, \\ \Delta^2 v - a_2 \Delta v + H_2(x)v = F_v(x, u, v), & x \in V, \end{cases} \quad (4.1)$$

and their proofs are almost same as Theorems 1.1 and 1.2 by simply changing 4 into 2 in Eqs (3.5), (3.7), (3.12), (3.14), (3.15) and so on. Hence, we omit the detail proofs. Next, we state the results.

Theorem 4.1. Suppose that (H) , (F_0) , (F_1) , (C_1) (or $[C_2]$), (F_3) , and (F_6) and the following conditions hold:

$$(F_2)' \lim_{|(s,t)| \rightarrow \infty} \frac{F(x,s,t)}{|(s,t)|^2} = +\infty \text{ for all } x \in V;$$

$$(F_4)' \mathcal{F}(x, s, t) = \frac{1}{2} (F_s(x, s, t)s + F_t(x, s, t)t) - F(x, s, t) \geq 0, \text{ and there exist } c_1 > 0 \text{ and } k > 1 \text{ such that}$$

$$|F(x, s, t)|^k \leq c_1 |(s, t)|^{2k} \mathcal{F}(x, s, t) \text{ for all } x \in V, (s, t) \in \mathbb{R}^2, |(s, t)| \geq r_0.$$

Then, problem (4.1) has infinitely many solutions $\{(u_k, v_k)\}_{k=1}^\infty$ such that

$$\begin{aligned} & \frac{1}{2} \int_V (|\Delta u_k|^2 + a_1 |\nabla u_k|^2 + H_1(x)u_k^2) d\mu + \frac{1}{2} \int_V (|\Delta v_k|^2 + a_2 |\nabla v_k|^2 + H_2(x)v_k^2) d\mu \\ & - \int_V F(x, u_k, v_k) d\mu \rightarrow +\infty, \text{ as } k \rightarrow +\infty. \end{aligned} \quad (4.2)$$

Theorem 4.2. Suppose that (H) , (F_0) , (F_1) , (C_1) (or $[C_2]$), $(F_2)'$, (F_3) , and (F_6) and the following condition hold:

$$(F_5)' \text{ there exist } \mu > 2 \text{ and } \sigma \in \left(0, \frac{\mu-2}{2 \max\{\gamma_{2,1}^2, \gamma_{2,2}^2\}}\right) \text{ such that}$$

$$\mu F(x, s, t) \leq sF_s(x, s, t) + tF_t(x, s, t) + \sigma(s^2 + t^2) \text{ for all } x \in V, (s, t) \in \mathbb{R}^2,$$

$$\text{where } \gamma_{2,i} = \mu_{\min}(\inf_V V_i(x))^{\frac{1}{2}};$$

Then, problem (4.1) has infinitely many solutions $\{(u_k, v_k)\}_{k=1}^\infty$ such that (4.2) holds.

Remark 4.1. From the proofs of Theorems 1.1 and 1.2, it is not difficult to see that Theorems 1.1 and 1.2 still hold even if one of $b_i, i = 1, 2$ is equal to zero, which implies that system (1.1) is a coupling system between biharmonic-Kirchhoff equation and biharmonic equation, and by comparing (F_2) , (F_4) , (F_5) with $(F_2)'$, $(F_4)'$, and $(F_5)'$, the biharmonic-Kirchhoff equation plays a leading role rather than the biharmonic equation.

Remark 4.2. One can easily establish the results corresponding to Theorems 4.1 and 4.2 for the following scalar biharmonic equation:

$$\Delta^2 u - a \Delta u + H(x)u = f(x, u), \quad x \in V,$$

by letting $u = v$, $a = a_1 = a_2$, $H = H_1 = H_2$, and $f(x, u) = F_u(x, u, v) = F_v(x, u, v)$ for all $x \in V$. At the moment, (F_1) and (C_1) reduce to the following conditions, respectively:

(f_1) There exists a function $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a function $b : V \rightarrow \mathbb{R}^+$ with $b \in L^1(V)$ such that

$$|F(x, s)|, |f(x, s)| \leq a(|s|)b(x)$$

for all $x \in V$ and $s \in \mathbb{R}$, where $F(x, s) = \int_0^s f(x, t)dt$;

$$(c_1) \lim_{|s| \rightarrow 0} \frac{a(|s|)}{|s|^2} = h < +\infty.$$

In [29], in the Euclidean setting, the nonlinear term f in system (1.2) is required to satisfy that there exist constants $c_1, c_2 > 0$ and $p \in (2, 2_*)$ such that

$$|f(x, s)| \leq c_1|s| + c_2|s|^{p-1}, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}, \quad (4.3)$$

where $2_* = +\infty$ if $N \leq 4$ and $2_* = \frac{2N}{N-4}$ if $N > 4$. It is easy to verify that condition (4.3) excludes those examples with exponential growth, such as

$$f(x, s) = \begin{cases} b(x)e^s(s^4 + 4s^3), & s \geq 0, \\ b(x)e^s(4s^3 - s^4), & s < 0, \end{cases} \quad (4.4)$$

where $b(x) \in L^1(V)$. However, our assumptions (f_1) and (c_1) allow f to be exponential growth and Eq (4.4) satisfies all our assumptions in Theorem 4.2. The basic reason caused such difference is that $L^1(V) \subset L^\infty(V)$ and Eq (2.2) holds in the locally finite graph setting, which in general does not hold in the Euclidean setting $\mathbb{R}^N (N \geq 2)$. This crucial distinction arises from the nature of the measure on a locally finite graph. Since $\int_V b d\mu = \sum_{x \in V} \mu(x)b(x)$ and $\mu_{\min} := \inf_{x \in V} \mu(x) > 0$, the summability of b immediately implies

$$|b(x)| \leq \frac{1}{\mu_{\min}} \sum_{y \in V} \mu(y)|b(y)| = \frac{\|b\|_1}{\mu_{\min}} < \infty, \quad \forall x \in V.$$

Hence, $L^1(V) \subset L^\infty(V)$ with $\|b\|_\infty \leq \mu_{\min}^{-1} \|b\|_1$. In the Euclidean setting $\mathbb{R}^N (N \geq 2)$, however, no such embedding exists: L^1 functions can have arbitrarily high spikes on sets of small Lebesgue measure. This explains why our framework accommodates nonlinearities with exponential growth, whereas the Euclidean counterpart must impose subcritical polynomial growth conditions to ensure compactness.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Authors contribution

The authors contribute to the main manuscript equally.

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Conflict of interest

The authors declare there is no conflict of interest.

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