



Research Article

Exponential stabilization of perturbed wave equations with localized internal damping: Theory and numerical validation

Kalifa Lassana Barry¹, Karima Laoubi¹, Seda İğret Araz² and Salah Boulaaras^{3,*}

¹ Department of Mathematics, M'Hamed Bougara University of Boumerdes, Boumerdes 35000, Algeria

² Department of Mathematics Education, Faculty of Education, Siirt University, Siirt 56100, Turkey

³ Department of Mathematics, College of Sciences, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: s.boulaaras@qu.edu.sa; Tel: +966559618327.

Abstract: This work investigates the exponential stability of a perturbed non-linear (or linear) wave equation in a sub-critical regime with respect to both the damping and perturbation terms. Using an energy-based analysis, we first establish the decay of the total energy over a finite time interval $T > 0$. When the damping region satisfies the geometric control condition (GCC), and the perturbations contribute to the dissipation mechanism, we prove that the energy of the system decays exponentially by combining the multiplier method with a unique continuation principle. In addition to the theoretical analysis, we present numerical simulations for one- and two-dimensional damped wave equations with localized internal damping. Finite difference schemes are employed to approximate the dynamics and quantify the dissipation of energy, and the results confirm the theoretical predictions: Introducing a localized damping region induces strong stabilization, and the total discrete energy decays exponentially over time.

Keywords: exponential stability; geometric control condition; multiplier method; nonlinear equations; subcritical regime

1. Introduction

The analysis of nonlinear wave equations involving interior damping, dynamic boundary conditions, and localized dissipation has attracted considerable attention. Foundational contributions by Lions [1, 2] established the framework for exact controllability and stabilization in distributed parameter systems. Furthermore, Adams [3] and Simon [4] developed core tools regarding Sobolev embeddings and compactness arguments, which remain essential for well-posedness theory.

Subsequent research has addressed nonlinear effects arising from damping and source terms. Georgiev and Todorova [5] proved existence results for wave equations with nonlinear damping and nonlinear sources, highlighting interactions between dissipation and forcing mechanisms. Vitillaro expanded the theory in several influential studies, treating global nonexistence in dissipative problems [6], global existence with nonlinear boundary damping and source terms [7], strong solutions under dynamic boundary conditions [8], and wave equations with interior and nonlinear boundary damping [9], including supercritical regimes [10]. Gazzola and Squassina [11] analyzed the competition between damping and source effects in semilinear wave models and identified conditions under which solutions exist globally or blow up in finite time. Related questions concerning global dynamics, energy behavior, and finite-time blow-up for nonlinear equations have also been investigated by Boulaaras [12, 13], and the nonexistence of global weak solutions for strongly damped p -Laplacian wave equations with logarithmic source terms was studied in [14]. Tools such as Carleman estimates and observability inequalities developed by Gal and Tebou [15] and unique continuation results by Ruiz [16] contributed to control and stabilization theory. Localized damping has also been the focus of many studies. Khemmoudj and Medjden [17] established exponential decay for semilinear Cauchy–Ventcel problems, and Addoun and Laoubi [18] investigated stabilization and regularity under localized internal damping combined with Ventcel boundary conditions. More recent developments involve nonlinear damping with variable-exponent structures. Al-Mahdi et al. studied asymptotic behavior under nonlinear boundary damping [19], thermoelastic Timoshenko systems [20], and viscoelastic wave models with distributed delays [21, 22]. Guo et al. [23] proved the existence of global attractors under Hardy potentials and nonlinear damping. Han et al. [24] obtained quantitative stability results for weak and strong damping, whereas Nicaise et al. [25] established indirect stabilization for weakly coupled hyperbolic systems. Cavalcanti et al. [26] analyzed stabilization under Zaremba boundary conditions using semiclassical defect measures. Analytical tools such as the Moser iteration method [27], together with Sobolev and trace embedding theorems, ensure that nonlinear perturbations and damping terms remain within a well-posed functional setting.

These contributions collectively demonstrate that localized interior or boundary damping plays a decisive role in achieving stabilization, energy decay, and global well-posedness for nonlinear wave equations, thus motivating the present work.

Let O_0 be a neighborhood of $0_{\mathbb{R}^N}$ with $N \geq 2$, and let $\Omega \subset \mathbb{R}^N$ be a connected, open, bounded domain with a C^2 boundary Γ . The boundary is decomposed into two disjoint nonempty parts, Γ_0 and Γ_1 :

$$\text{meas}(\Gamma_0), \text{meas}(\Gamma_1) \neq 0, \quad O_0 \cap \Omega = \emptyset, \quad \Gamma = \Gamma_0 \cup \Gamma_1, \quad \overline{O_0} \cap \overline{\Gamma_1} = \emptyset.$$

Let O be a neighborhood of Γ_1 , and define $\omega = O \cap \Omega$, representing the interior region where local damping is active. We study the wave system:

$$\begin{cases} \varphi_{tt} - \Delta \varphi + \alpha(x)\Theta(\varphi_t) = f(\varphi), & (0, \infty) \times \Omega, \\ \varphi = 0, & (0, \infty) \times \Gamma_0, \\ \varphi_{tt} + \partial_\nu \varphi - \Delta_T \varphi + \beta(x)\Upsilon(\varphi_t) = g(\varphi), & (0, \infty) \times \Gamma_1, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), & \overline{\Omega}, \end{cases} \quad (1.1)$$

where the non-negative functions α and β satisfy

$$\alpha(x) = \begin{cases} \alpha_0 > 0, & x \in \omega, \\ 0, & x \in \overline{\Omega} \setminus \omega, \end{cases} \quad \beta(x) \geq \beta_0 > 0, \quad x \in \Gamma_1. \quad (1.2)$$

Here, Δ denotes the Laplace operator in Ω , Δ_T the Laplace–Beltrami operator on Γ_1 , and ν the outward unit normal. The Laplace–Beltrami operator satisfies

$$\int_{\Gamma} u \Delta_T v \, d\Gamma = - \int_{\Gamma} \nabla_T u \nabla_T v \, d\Gamma, \quad \forall u, v \in H^1(\Gamma_1). \quad (1.3)$$

The damping and perturbation terms satisfy the following hypotheses.

(I) **Damping:** Θ and Υ are continuous, monotone increasing, and satisfy

$$\Theta(s)s \geq 0, \quad \Upsilon(s)s \geq 0, \quad \Theta(0) = \Upsilon(0) = 0.$$

(II) **Perturbations:** $f, g \in C_{loc}^{0,1}(\mathbb{R})$ satisfy

$$|f'(s)| = O(|s|^{p-2}), \quad |g'| = O(|s|^{q-2}), \quad |s| \rightarrow \infty, \quad (1.4)$$

for some $p, q \geq 2$.

The nonlinearities are defined by

$$\begin{cases} \Theta(s) = a|s|^{\tilde{m}-2}s + |s|^{m-2}s, & 1 < \tilde{m} \leq m, \, a \geq 0, \\ \Upsilon(s) = b|s|^{\tilde{\mu}-2}s + |s|^{\mu-2}s, & 1 < \tilde{\mu} \leq \mu, \, b \geq 0, \\ f(s) = \tilde{\gamma}|s|^{\tilde{p}-2}s + \gamma|s|^{p-2}s + c_1, & 2 \leq \tilde{p} \leq p, \\ g(s) = \tilde{\delta}|s|^{\tilde{q}-2}s + \delta|s|^{q-2}s + c_2, & 2 \leq \tilde{q} \leq q. \end{cases} \quad (1.5)$$

We also recall the critical Sobolev exponents:

$$r_{\Omega} = \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ \infty, & N = 2, \end{cases} \quad r_{\Gamma_1} = \begin{cases} \frac{2(N-1)}{N-3}, & N \geq 4, \\ \infty, & N = 2 \text{ or } 3. \end{cases} \quad (1.6)$$

The nonlinearities are subcritical when

$$\begin{cases} 2 \leq p < 1 + \frac{r_{\Omega}}{2}, \\ 2 \leq q < 1 + \frac{r_{\Gamma_1}}{2}, \end{cases} \quad \begin{cases} 1 \leq m < 1 + \frac{r_{\Omega}}{2}, \\ 1 \leq \mu < 1 + \frac{r_{\Gamma_1}}{2}. \end{cases} \quad (1.7)$$

In this work, we study the local-in-time exponential stability of system (1.1) in the subcritical regime. The domain $\Omega \subset \mathbb{R}^N$ is bounded and sufficiently smooth, and damping acts both inside Ω and on part of the boundary.

Our contributions are as follows:

1. Extend solutions globally through an energy-based argument and determine conditions under which blow-up may occur.

2. Prove exponential stabilization by constructing an appropriate energy approximation. Stability holds when (i) the geometric control condition (GCC) is satisfied on the damped region, and (ii) perturbations contribute to energy dissipation in addition of its superlinear behavior.

The case of fully nonlinear damping and perturbation functions with variable exponents has been investigated in [18–23]. When damping is linear, system (1.1) falls into the framework of [11]. Nonlinear damping for weak/strong dissipation was treated in [24]. Well-posedness in the Hadamard sense is discussed in [9, 10], including subcritical, critical, supercritical, and super-supercritical cases.

The main result of the present paper establishes the following:

1. An energy decay estimate for system (1.1), assuming well-defined initial energy;
2. For $1 < \mu$, $m \leq 2$, local or global exponential stability when f and g act as sinks, either for nonlinear perturbations when $N < 4$ or for linear perturbations ($p = q = 2$) when $N \geq 2$.

Exponential stability remains valid whenever the solution associated with the initial data is bounded in $L^\infty(\bar{\Omega})$.

We do not reprove general Hadamard well-posedness (already established in [10]); instead, we recall simplified versions of those results and concentrate on the characterization of energy dissipation leading to stability.

In Section 4, we prove exponential stability under the additional assumption that f and g are globally Lipschitz. The Appendix clarifies when this assumption holds depending on $p, q, \tilde{p}, \tilde{q}$, and N .

Numerical validation. A final section presents numerical simulations in one and two dimensions using finite differences. These results demonstrate that introducing a localized damping region leads to significant energy dissipation and confirms exponential decay of the total energy, in agreement with the theoretical predictions.

For reader information, as any numerical method designed to approximate the exact solution of a given problem, the finite difference method is inherently affected by discretization errors. The smaller these errors are, the closer the numerical approximation converges to the exact solution. In the present work, the standard second-order finite difference scheme was deliberately adopted because of its simplicity and robustness, as its primary purpose is to illustrate and validate the theoretical results established in this work rather than to develop a new numerical methodology.

Nevertheless, more sophisticated numerical approaches have been developed in recent years. In particular, higher-order and compact finite difference schemes, as well as meshless radial basis function finite difference (RBF–FD) methods, have demonstrated improved dispersion properties, leading to a more accurate representation of wave propagation, while also offering greater flexibility for nonuniform and irregular node distributions. Furthermore, in higher-dimensional settings (e.g., three-dimensional problems), implicit time discretizations or nonlinear damping models generally lead to large, sparse algebraic systems. When combined with highly refined computational meshes, these systems can still be solved efficiently using modern iterative solvers. For further information on these numerical techniques, the interested reader is referred to [28–31] and those cited therein.

2. Materials and methods

2.1. Our methodology

The main goals of this paper are as follows:

- (1) To demonstrate that the effect of nonlinear localized dampings $\alpha\Theta$ and $\beta\Upsilon$ is sufficient to exponentially stabilize the system (1.1) when f, g also contribute to dissipate the energy;
- (2) To show that this effect may still be efficient for a certain time when f, g lead the system to blow-up;
- (3) To conduct a numerical illustration of the exponential stability thus established.

As expected, such a system with nonlinear perturbation may experience blow-up; that is, the released energy which quantifies the wave's ability of motion reaches infinity for a finite time or becomes nonpositive. As a result, one needs to carefully analyze the system's well-posedness, including the existence and uniqueness of the system's solution and the positive boundedness of its energy.

For this aim, the paper uses the results of existence and uniqueness already stated by E. Vitillaro [9] and thus analyzes the sign of the energy released by the system (1.1) in order to state its stability. Next, the exponential stability is stated by the well-known method using the GCC. The technique is a combination of adequate weighted multipliers, leading to an observability result as in [17, 26, 32].

2.2. Statement of problem

In part analyzes the well-posedness of the system (1.1). Before this, here are some notations and key properties of f, g and Θ, Υ useful in this paper.

2.2.1. Preliminaries

- s' is the Hölder conjugate of s , given by

$$\frac{1}{s} + \frac{1}{s'} = 1.$$

- $\langle \cdot | \cdot \rangle$ denotes the inner product, sometimes written as $(\cdot)$.
- $C([0, T], X)$ with norm $\|f\|_{C([0, T]; X)} = \sup_{t \in [0, T]} \|f(t)\|_X$.
- $\|\cdot\|_p$ is the norm in $L^p(\Omega)$ for $p \geq 2$.
- For $A \subset \overline{\Omega}$, $\|\cdot\|_{p, A}$ is the norm in $L^p(A)$.
- The norm in $L^p(A \times (a, b))$ is

$$\|f\|_{p, (A \times (a, b))} = \left(\int_a^b \|f(s)\|_{p, A}^p ds \right)^{\frac{1}{p}}.$$

- $A \hookrightarrow B$ means $A \subset B$ with continuous embedding, and $A \hookrightarrow\hookrightarrow B$ means compact embedding.
- $\mathbf{I}_{n \times m}$ and $\mathbf{O}_{n \times m}$ denote the identity and zero matrices* of size $n \times m$, respectively.
- For $X, Y \in \mathbb{R}^n$, we define:

$$\begin{aligned} X > Y &\iff x_i > y_i, \quad \forall i = 1, \dots, n, \\ X < Y &\iff x_i < y_i, \quad \forall i = 1, \dots, n. \end{aligned}$$

Next, here are some key properties on f, g, Θ , and Υ , which have been implicitly used in [5, 9] for the existence and uniqueness result.

*See page 338 of [15].

Assumption 1. (Key properties of f, g) The perturbation functions f and g satisfy the following:

- (i) $f(0) = c_1$, $g(0) = c_2$. In particular, $f(0) = g(0) = 0$ if $c_1 = c_2 = 0$.
(ii) There exist constants $C_f, C_g > 0$ such that for all $s_1, s_2 \in \mathbb{R}$,

$$\begin{aligned} |f(s_1) - f(s_2)| &\leq C_f \left(|s_{1,2}|^{\bar{p}-2} + |s_{1,2}|^{p-2} \right) |s_1 - s_2|, \\ |g(s_1) - g(s_2)| &\leq C_g \left(|s_{1,2}|^{\bar{q}-2} + |s_{1,2}|^{q-2} \right) |s_1 - s_2|, \end{aligned}$$

where $|s_{1,2}|^r = |s_1|^r + |s_2|^r$, $r \geq 0$.

In particular, if $c_1 = c_2 = 0$,

$$|f(s_1)| \leq C_f \left(|s_1|^{\bar{p}-1} + |s_1|^{p-1} \right), \quad |g(s_1)| \leq C_g \left(|s_1|^{\bar{q}-1} + |s_1|^{q-1} \right).$$

- (iii) For all $i, j \in \mathbb{N}^*$, f and g satisfy

$$(f, g) \in \begin{cases} C^i(\mathbb{R}) \times C^j(\mathbb{R}), & \text{if } (i+1, j+1) < (\bar{p}, \bar{q}), \\ (C^\infty(\mathbb{R}))^2, & \text{if } (p, q) = (2, 2). \end{cases}$$

Moreover, there exist constants $C_{f,i}, C_{g,j} > 0$ such that

$$|f^{(i)}(s)| \leq C_{f,i} \left(|s|^{\bar{p}-(i+1)} + |s|^{p-(i+1)} \right), \quad |g^{(j)}(s)| \leq C_{g,j} \left(|s|^{\bar{q}-(j+1)} + |s|^{q-(j+1)} \right).$$

- (iv) For all $i, j \in \mathbb{N}^*$ satisfying (iii), we have $f^{(i)}(0) = g^{(j)}(0) = 0$.

Remark 2.1. (Key observations on Θ, Υ) If $2 \leq \bar{m}, \bar{\mu}$, then Θ and Υ satisfy the same properties as f, g with $c_1 = c_2 = 0$.

However, if $1 < \bar{m} \leq m < 2$ or $1 < \bar{\mu} \leq \mu < 2$, the properties hold only on $\mathbb{R} \setminus (-M, M)$ for some $M > 0$.

2.2.2. Well-posedness results

First, the space $L^2(\Gamma_1)$ will be considered in this paper as $\{z \in L^2(\Gamma) : z|_{\Gamma_0} = 0\}$. Now, consider the following spaces:

$$\begin{cases} \mathcal{V}_0 = \{(u, v) \in H^1(\Omega) \times (H^1(\Gamma) \cap L^2(\Gamma_1)) : v = u_{\Gamma_1}, u|_{\Gamma_0} = 0\}, \\ \mathcal{V}_1 = L^2(\Omega) \times L^2(\Gamma_1) \end{cases}$$

and

$$\begin{cases} L_\alpha^p(\Omega) = \{u \in L^2(\Omega) : \alpha^{\frac{1}{p}} u \in L^p(\Omega)\}, \\ L_\beta^q(\Gamma_1) = \{u \in L^2(\Gamma_1) : \beta^{\frac{1}{q}} u \in L^q(\Gamma_1)\}, \end{cases}$$

respectively endowed with the norms $\|\cdot\|_{\mathcal{V}_0}$, $\|\cdot\|_{\mathcal{V}_1}$, $\|\cdot\|_{p,\alpha}$, and $|\cdot|_{q,\beta}$ such that

$$\begin{cases} \|(u, v)\|_{\mathcal{V}_0}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_T v\|_{L^2(\Gamma_1)}^2, & \begin{cases} \|u\|_{p,\alpha} = \|u\|_2 + \|\alpha^{\frac{1}{p}} u\|_p, \\ \|u\|_{q,\beta} = \|u\|_{2,\Gamma_1} + \|\beta^{\frac{1}{q}} u\|_{q,\Gamma_1}. \end{cases} \\ \|(u, v)\|_{\mathcal{V}_1}^2 = \|u\|_2^2 + \|v\|_{2,\Gamma_1}^2, \end{cases}$$

Consider $\mathcal{H} = \mathcal{V}_0 \times \mathcal{V}_1$ equipped with the norm $\|\cdot\|_{\mathcal{H}}$ such that

$$\|(U, V)\|_{\mathcal{H}}^2 = \|U\|_{\mathcal{V}_0}^2 + \|V\|_{\mathcal{V}_1}^2$$

and

$$H_{\alpha, \beta}^{p, q} = \mathcal{V}_0 \cap \left(L_{\alpha}^p(\Omega) \times L_{\beta}^q(\Gamma_1) \right),$$

endowed with the norm

$$\|(u, v)\|_{\alpha, \beta}^{p, q} = \|(u, v)\|_{\mathcal{V}_0} + \|u\|_{p, \alpha} + \|v\|_{q, \beta}.$$

We also introduce, for all $p \in [1, \infty)$, the following Sobolev space when $\Omega, \partial\Gamma$ are of class C^2 :

$$W^{2, p} = \left(W^{2, p}(\Omega) \times W^{2, p}(\Gamma) \right) \cap \mathcal{V}_0, \quad W^{2, 2} = H^2(\Omega) \times (H^2(\Gamma) \cap H^1(\Gamma_1)).$$

In this paper, we will consider $w = (\varphi, \varphi_{\Gamma_1})^t \in \mathcal{V}_0$ such that $W = (w, w_t)^t \in \mathcal{H}$ satisfies the system.

Thus, by differentiating W with respect to the time variable t , the problem (1.1) is reformulated as follows:

$$W_t = AW - C(W) + S(W),$$

where

$$S(W) = \begin{pmatrix} O_{2 \times 1} \\ F(w) \end{pmatrix} \in \mathcal{H}, \quad C(W) = \begin{pmatrix} O_{2 \times 1} \\ G(w_t) \end{pmatrix} \in \mathcal{H}$$

and A is the linear operator defined on $D(A) = \{W = (w, w_t)^t \in \mathcal{H} : AW \in \mathcal{H}\}$ by:

$$A = \begin{pmatrix} O_{2 \times 2} & I_{2 \times 2} \\ A_1 & O_{2 \times 2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta_T - \partial_v \end{pmatrix} \quad (2.1)$$

and satisfies $\langle AZ, Z \rangle_{\mathcal{H}} = 0$ for all $W \in \mathcal{H}$.

The nonlinear fields are as follows:

$$F(w) = \begin{pmatrix} f(\varphi) \\ g(\varphi_{\Gamma_1}) \end{pmatrix}, \quad G(w_t) = \begin{pmatrix} \alpha \Theta(\varphi_t) \\ \beta \Upsilon(\varphi_{\Gamma_1}) \end{pmatrix}.$$

Thus, the system (1.1) is equivalent to the following first-order Cauchy problem:

$$\begin{cases} W_t = AW - C(W) + S(W), \\ W(0) = W_0 \in \mathcal{H}. \end{cases} \quad (2.2)$$

Theorem 2.2. (Local existence and uniqueness) *If the conditions (i) and (ii) in Assumption 1, and Eq (1.7) hold, then the system (2.2) admits a unique weak solution $W \in C([0, T]; \mathcal{H})$ such that*

$$w \in C([0, T]; \mathcal{V}_0), \quad w_t \in C([0, T]; \mathcal{V}_1) \cap \left(L_{\alpha}^m(\Omega) \times L_{\beta}^{\mu}(\Gamma_1) \right)$$

for some small $T > 0$.

Moreover, the energy associated to System (2.2) is given by:

$$E(t) = \frac{1}{2} \|W(t)\|_{\mathcal{H}}^2 - \int_{\Omega} \mathbb{F}(\varphi(x, t)) dx - \int_{\Gamma_1} \mathbb{G}(\varphi_{\Gamma_1}(x, t)) d\Gamma_1, \quad t \in (0, T), \quad (2.3)$$

where:

$$\mathbb{F}(r) = \int_0^r f(s) ds, \quad \mathbb{G}(r) = \int_0^r g(s) ds.$$

The following regularity theorem is a result obtained and studied in [10].

Theorem 2.3. (Regularity) Assume that the conditions (i) and (ii) in Assumption 1, and Eq (1.7) hold. If $W_0 \in W^{2,2} \times \mathcal{V}_0$, then the unique weak local solution $W = (w, w_t)^t$ to problem (2.2) satisfies the following regularity properties:

$$w \in C([0, T]; W^{2,2}) \cap C_w^1([0, T]; \mathcal{V}_0) \cap C_w^2([0, T]; \mathcal{V}_1),$$

where $C_w^i(I, X)$ is the space of functions weakly continuous and differentiable up to order $i \in \mathbb{N}$ from in time t over I into the Banach space X .

For simplicity, we just write here and below $W \in C([0, T]; \mathcal{H})$ as a solution of Eq (2.2), which is also true but not its complete regularity.

We now consider here and below $\mathcal{F}(w) = \begin{pmatrix} \mathbb{F}(\varphi) \\ \mathbb{G}(\varphi_{\Gamma_1}) \end{pmatrix}$, and the energy associated to the system (2.2) is rewritten as follows:

$$E(t) = \frac{1}{2} \|W(t)\|_{\mathcal{H}}^2 - \langle \mathcal{F}(w(t)), (1, 1)^T \rangle_{\mathcal{V}_1}. \quad (2.4)$$

Theorem 2.4. (Global existence and uniqueness) Assume that the condition (i) in Assumption 1 and Eq (1.7) are valid. Moreover, for $\bar{m} = \max\{2; m\}$ and $\bar{\mu} = \max\{2; \mu\}$, if one of the following conditions is satisfied:

- (i) $(\gamma, \delta) > (0, 0)$, $(p, q) \leq (\bar{m}, \bar{\mu})$ and $\text{ess inf}_{\Omega} \alpha > 0$ if $p > 2$,
- (ii) $(\gamma, \delta) \leq (0, 0)$, $(\tilde{p}, \tilde{q}) \leq (\bar{m}, \bar{\mu})$ and $\text{ess inf}_{\Omega} \alpha > 0$ if $p > 2$,
- (iii) $\gamma > 0$, $\delta \leq 0$, $(p, \tilde{q}) \leq (\bar{m}, \bar{\mu})$ and $\text{ess inf}_{\Omega} \alpha > 0$ if $p > 2$,
- (iv) $\gamma \leq 0$, $\delta > 0$, $(\tilde{p}, q) \leq (\bar{m}, \bar{\mu})$ and $\text{ess inf}_{\Omega} \alpha > 0$ if $p > 2$,
- (v) $\gamma, \delta, \tilde{\gamma}, \tilde{\delta} \leq 0$,

then for every $W_0 \in \mathcal{H}$, the system (1.1) admits a unique global solution in time; that is, the life span T_* of solution w is ∞ .

Remark 2.5. These results stem from an analysis of the energy dissipated by the system (2.2) as performed in [5]. In particular, the solution exists at any time where the energy is well-defined, that is, non-negative and finite (see Remark 2.7). In principle, when a global-in-time solution exists, no blow-up occurs; that is, the lifespan T_* of w is equal to $+\infty$. Then, the system (2.2) blows up if and only if all the conditions (i – v) are not satisfied.

We consider, for any $z \in C([0, T_*]; \mathcal{V}_0)$, the following functional:

$$I(z(t)) = \frac{1}{2} \|z(t)\|_{\mathcal{V}_0}^2 - \langle \mathcal{F}(z(t)), (1, 1)^T \rangle_{\mathcal{V}_1}, \quad t \geq 0.$$

We have the following stability result:

Theorem 2.6. (Energy decay)

Let $W \in C([0, T_*]; \mathcal{H})$ be the maximal solution[†] to the system (2.2). The energy of system (2.2) decreases over time for all $T \in (0, T_*)$ provided that

$$I(w_0) > 0.$$

[†]Here and throughout, a maximal solution refers to one that cannot be extended beyond its lifespan; that is, it is nonextendable in time. See [10] for details in our setting. We use the term “global-in-time solution” to mean that $T_* = +\infty$.

Proof. First, we observe that the energy functional associated with Eq (2.2) is given by

$$E(t) = I(w(t)) + \frac{1}{2} \|w_t(t)\|_{V_1}^2, \quad t \geq 0.$$

Hence, the energy decay is immediate whenever $I(w(t)) \geq 0$. Moreover, the continuity of $I(w(t))$ with respect to time and the assumption $I(w_0) = I(w(0)) > 0$ ensure the existence of some $T = T_{m,\mu} \in (0, T_*)$ such that $I(w(t)) \geq 0$ for all $t \in (0, T_{m,\mu})$.

Multiplying Eq (2.2)₁ by W in $\mathcal{H}$, we obtain, for all $t \in [0, T]$,

$$\frac{1}{2} \frac{d}{dt} \|W(t)\|_{\mathcal{H}}^2 = \langle AW(t), W(t) \rangle_{\mathcal{H}} - \langle C(W(t)), W(t) \rangle_{\mathcal{H}} + \langle S(W(t)), W(t) \rangle_{\mathcal{H}}.$$

Using Eq (2.1) along with the definition of $\mathcal{F}$, we deduce

$$\frac{d}{dt} \left[\frac{1}{2} \|W(t)\|_{\mathcal{H}}^2 - \langle \mathcal{F}(w(t)), (1, 1)^\top \rangle_{V_1} \right] = -\langle G(w_t(t)), w_t(t) \rangle_{V_1}.$$

Because G is monotone increasing, we conclude that $E'(t) \leq 0$ for all $t \in [0, T]$.

This completes the proof. $\square$

Remark 2.7. In the study of wave problems, a negative energy can result from various instability phenomena of the solution, such as the explosion of the solution, solution trapping, or the existence of strongly nonlinear steady states. Therefore, we must ensure that the energy remains positive, as not all of its terms are necessarily positive. A sufficient condition for the global positivity of the energy and its decay is

$$\langle \mathcal{F}(w), (1, 1)^\top \rangle_{V_1} \leq 0,$$

for instance, when

$$c_1 = c_2 = 0, \quad \widetilde{\gamma}, \widetilde{\delta}, \gamma, \delta \leq 0.$$

In this case, the stability of the energy (well-defined and decays) is global in time; that is, $T \rightarrow T_* = +\infty$.

In the case where $\langle \mathcal{F}(w), (1, 1)^\top \rangle_{V_1} > 0$, it becomes rather difficult to establish the long-term stability (i.e., energy decay along with its positivity as $T \rightarrow +\infty$) unless the damping exponents controlled those of the perturbations or the measure of the domain $\text{meas}(\Omega)$ is sufficiently small.

To keep things concise, we will not elaborate further on this point. However, any interested reader may verify this assertion by seeking a constant $l \in (0, 1/2)$ such that

$$\langle \mathcal{F}(w), (1, 1)^\top \rangle_{V_1} \leq l \|w\|_{V_0}^2,$$

through the use of the Sobolev embedding inequality and Poincaré's inequality, which both hold in this setting (see [19]).

3. Results

Consider that the perturbation maps f and g satisfy Eq (1.5) with $c_1 = c_2 = 0$. That is,

$$f(s) = \tilde{\gamma}|s|^{\tilde{p}-2}s + \gamma|s|^{p-2}s, \quad g(s) = \tilde{\delta}|s|^{\tilde{q}-2}s + \delta|s|^{q-2}s.$$

Consider also Θ and Υ , linear or sublinear; that is,

$$\Theta(s) = a|s|^{\tilde{m}-2}s + |s|^{m-2}s, \quad \Upsilon(s) = b|s|^{\tilde{\mu}-2}s + |s|^{\mu-2}s,$$

where $(1, 1) < (\tilde{m}, \tilde{\mu}) \leq (m, \mu) \leq (2, 2)$. What follows is therefore the main result of this paper:

Theorem 3.1 (Exponential stability). *Assume that the condition (1.7) holds for $m, \mu \leq 2$, and let $T > 0$ be a certain time of stability. Assume that $\tilde{\gamma}$, γ , $\tilde{\delta}$, and δ are such that*

$$f(s)s \leq \mathbb{F}(s) \leq 0, \quad g(s)s \leq \mathbb{G}(s) \leq 0, \quad s \in \mathbb{R}. \quad (3.1)$$

Then, the associated weak solution $W \in C([0, T]; \mathcal{H})$ to initial condition $W_0 \in \mathcal{H}$ is exponentially stable in $\Omega \times (0, T) \subset \mathbb{R}^N \times (0, T)$ for

- (i) $N \geq 2$ if f, g are linear ($\tilde{q} = \tilde{p} = q = p = 2$ or $f = g = 0$),
- (ii) $N = 1$ (see Section 5) if f, g are nonlinear ($p \geq \tilde{p} > 2$ or $q \geq \tilde{q} > 2$).

If in addition, $W_0 \in W^{2,2}$, then the associated strong solution W is exponentially stable for $N < 4$. The exponential stability is global-in-time if $T = +\infty$, and local-in-time otherwise.

Remark 3.2. *First, we add the fact that Theorem 3.1 still holds for some case where Eq (3.1) holds for some $S \subset \mathbb{R}$ such that $\text{meas}(S) \neq 0$.*

In order to prevent any confusion, the one-dimensional case has been preferably excluded in the problem statement because the use of the Laplace–Beltrami operator is not directly done as for the Laplace operator in spatial coordinates. For this, it is necessary to identify from a given spatial derivative, the normal derivative and the tangential derivative.

4. Proof of the main result

The purpose of this section is to study the exponential stability of system (2.2) under the conditions of Theorem 2.6. In view of Theorem 2.6, it is reasonable to investigate the exponential stability of system (2.2) on the extensible[‡] interval $[0, T]$, where $T = T_{m,\mu} < T_*$ is such that

$$I(w(t)) > 0 \quad \text{for all } t \in [0, T].$$

Further, there exist constants $C > 1$ and $\varsigma > 0$ such that

$$E(t) \leq Ce^{-\varsigma t} E(0), \quad t \in [0, T]. \quad (4.1)$$

[‡]We mean by “extensible” the fact that T could tend to $T_* = +\infty$ for a global solution.

To achieve this, we will establish an observability inequality and apply the unique continuation principle for the following system:

$$\begin{cases} \kappa_{tt} - \Delta \kappa + \alpha(x)\Theta(\kappa_t) = f(\kappa), & \text{in } (0, T) \times \Omega, \\ \kappa = 0, & \text{on } (0, T) \times \Gamma_0, \\ \kappa_{tt} + \partial_\nu \kappa - \Delta_T \kappa + \beta(x)\Upsilon(\kappa_t) = g(\kappa), & \text{on } (0, T) \times \Gamma_1, \\ \kappa(x, 0) = \kappa_0(x), \quad \kappa_t(x, 0) = \kappa_1(x), & \text{in } \overline{\Omega}, \end{cases} \quad (4.2)$$

which is equivalent to the first-order Cauchy problem defined as follows:

$$\begin{cases} K_t(t) = AK(t) - C(K(t)) + S(K(t)), & t \in (0, T), \\ K(0) = K_0 \in \mathcal{H}, \end{cases} \quad (4.3)$$

where the functions $\alpha, \beta \in L^\infty(\Omega) \times L^\infty(\Gamma_1)$ satisfy Eq (1.2) and

$$K = (k, k_t)^t, \quad k = (\kappa, \kappa_{\Gamma_1})^t.$$

Moreover, we add the following property:

$$\|K(t)\|_{C([0;T];\mathcal{H})} < R < +\infty \quad (4.4)$$

with $R > 0$.

The energy associated with system (4.2) at any instant $t \in (0, T)$ is given by

$$\begin{aligned} E_\kappa(t) = & \frac{1}{2} \left[\int_\Omega (|\kappa_t(t)|^2 + |\nabla \kappa(t)|^2) dx + \int_{\Gamma_1} (|\kappa_{\Gamma_1 t}(t)|^2 + |\nabla_T \kappa_{\Gamma_1}(t)|^2) d\Gamma_1 \right] \\ & - \int_\Omega \mathbb{F}(\kappa(t)) dx - \int_{\Gamma_1} \mathbb{G}(\kappa_{\Gamma_1}(t)) d\Gamma_1. \end{aligned} \quad (4.5)$$

Moreover, for all $t_1, t_2 \in (0, T)$ with $t_1 < t_2$, the following dissipation identity holds:

$$E_\kappa(t_2) - E_\kappa(t_1) = - \int_{t_1}^{t_2} \int_\Omega \alpha(x)\Theta(\kappa_t(t))\kappa_t(t) dx dt - \int_{t_1}^{t_2} \int_{\Gamma_1} \beta(x)\Upsilon(\kappa_{\Gamma_1 t}(t))\kappa_{\Gamma_1 t}(t) d\Gamma_1 dt. \quad (4.6)$$

For the moment, we assume the existence of $T(x_0) > 0$, which represents the time taken for each geodesic curve to enter the controlled region ω . The proof of its existence will be provided in the Appendix.

Theorem 4.1. (*Exponential decay*) *The system (4.2) is exponentially stable over the time interval $T > T(x_0)$ if and only if there exists a constant $C_R > 0$ such that the following observability inequality holds:*

$$E_\kappa(T) \leq C_R \left[\int_0^T \int_\Omega \alpha(x)\Theta(\kappa_t(t))\kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x)\Upsilon(\kappa_{\Gamma_1 t}(t))\kappa_{\Gamma_1 t}(t) d\Gamma_1 dt \right]. \quad (4.7)$$

Proof. By combining Eqs (4.6) and (4.7), we obtain the inequality

$$E_\kappa(T) \leq C_R(E_\kappa(0) - E_\kappa(T)),$$

which leads to

$$E_\kappa(T) \leq L_R E_\kappa(0), \quad (4.8)$$

where $L_R = C_R/(1 + C_R) < 1$.

Consider the sequence $\{a_n, n \in \mathbb{N}\}$ such that

$$a_n = \begin{cases} nT_0, & \text{if } n \in \mathbb{N}^*, \\ 0, & \text{if } n = 0 \end{cases}$$

with $0 < T_0 < T$.

Using the same argument as in the derivation of Eq (4.8) over the subintervals $[a_{n-1}, a_n] \subset [0, T]$ for $1 \leq n < N = \lfloor T/T_0 \rfloor$, we obtain

$$E_\kappa(a_n) \leq L_R E_\kappa(a_{n-1}) \leq \cdots \leq L_R^n E_\kappa(0).$$

Because $L_R^n = e^{n \ln(L_R)}$, it follows that

$$E_\kappa(a_n) \leq e^{-\sigma n T_0} E_\kappa(0),$$

where $\sigma = 1/T_0 \ln(1/L_R) > 0$.

Given that E_κ is a nonincreasing function over time, for all $t \in [a_{n-1}, a_n]$, we have

$$E_\kappa(t) \leq E_\kappa(a_{n-1}) \leq e^{-\sigma(n-1)T_0} E_\kappa(0) = e^{-\sigma n T_0} e^{\sigma T_0} E_\kappa(0).$$

Because $e^{\sigma T_0} = 1/L_R > 1$, and $e^{-\sigma n T_0} = e^{-\sigma a_n} \leq e^{-\sigma t}$, we deduce

$$E_\kappa(t) \leq C e^{-\sigma t} E_\kappa(0).$$

This establishes Eq (4.1) with $C = 1/L_R > 1$ and $\varsigma = \sigma > 0$, thus completing the proof. $\square$

We introduce the following notations:

$$\Omega_T = \Omega \times (0, T), \quad \Gamma_{0T} = \Gamma_0 \times (0, T), \quad \Gamma_{1T} = \Gamma_1 \times (0, T).$$

To establish the observability inequality (4.7), we require some additional results.

Assumption 2. (Geometric control condition) Let $x_0 \in \mathbb{R}^N$, with $N \geq 2$. We define the following vector:

$$m_0(x) = x - x_0, \quad x \in \mathbb{R}^N, \quad m_T^0(x) = m_0(x) - (m_0(x)|\nu(x)) \nu(x),$$

where m_T^0 is the tangential component of m_0 , and $\nu(x)$ denotes the outward unit normal vector to Ω along Γ_1 .

We also introduce the following notations:

- (i) $r_0 = \max_{x \in \bar{\Omega}} \left\{ \sqrt{\langle m_0(x), m_0(x) \rangle} \right\}$,
(ii) $\Gamma_1^* = \{x \in \Gamma_1 \mid (m_0(x)|\nu(x)) > 0\}$, $\Gamma_1 \setminus \Gamma_1^* = \{x \in \Gamma_1 \mid (m(x)|\nu(x)) \leq 0\}$,
(iii) $\Gamma_{1T}^* = \Gamma_1^* \times (0, T)$.

Proposition 4.2 (Θ and Υ). *Let $r > 1$ be such that the main map $s \mapsto h_r(s) = |s|^{r-2}s$ of Θ and Υ satisfy*

$$\begin{cases} c_0|s|^{r-1} \leq |h_r(s)| \leq c_1|s|, & \text{if } 0 < |s| \leq 1, \\ M_0|s|^2 \leq h_r(s)s \leq M_1|s|^r, & \text{otherwise,} \end{cases} \quad (4.9)$$

for some positive constants c_0, c_1, M_0, M_1 depending on r . Then, there exist two constants $C_0 = C_0(\tilde{m}, m)$ and $C_1 = C_1(\tilde{\mu}, \mu)$ such that Θ and Υ satisfy the following inequalities:

$$|s|^2 + |\Theta(s)|^2 \leq C_0\Theta(s)s, \quad |s|^2 + |\Upsilon(s)|^2 \leq C_1\Upsilon(s)s, \quad \forall s \in \mathbb{R}, (1, 1) < (\tilde{m}, \tilde{\mu}) \leq (m, \mu) \leq (2, 2). \quad (4.10)$$

Proof. First, one can easily check that the conditions in Eq (4.9) hold for any $r > 1$.

On one hand, using Eq (4.9)₁, it follows that

$$\frac{|s|^{r-1}}{|h_r(s)|} \leq \frac{1}{c_0}, \quad \frac{|h_r(s)|}{|s|} \leq c_1. \quad (4.11)$$

As a result,

$$c_1 + \frac{1}{c_0} \geq \frac{|s|^{r-1}}{|h_r(s)|} + \frac{|h_r(s)|}{|s|} = \frac{|s|^r + |h_r(s)|^2}{|h_r(s)s|}.$$

Since $h_r(s)s > 0$ for $s \neq 0$, then it follows that

$$|s|^r + |h_r(s)|^2 \leq \left(c_1 + \frac{1}{c_0}\right)h_r(s)s, \quad 0 < |s| \leq 1.$$

This is also true for $s = 0$.

On the other hand, using the same process on Eq (4.9)₂, we get

$$\frac{|s|^2}{h_r(s)s} \leq \frac{1}{M_0}, \quad \frac{h_r(s)s}{|s|^r} \leq M_1,$$

which implies that

$$|s|^{r+2} + |h_r(s)s|^2 \leq \left(M_1 + \frac{1}{M_0}\right)h_r(s)s|s|^r, \quad |s| > 1.$$

Therefore,

$$|s|^r + |h_r(s)|^2 \leq \left(M_1 + \frac{1}{M_0}\right)h_r(s)s|s|^{r-2}, \quad |s| > 1.$$

Note that $|s|^{r-2} \leq 1$ if $1 < r \leq 2$ and $|s| > 1$.

This finally concludes the following generalized inequality on h :

$$|s|^r + |h_r(s)|^2 \leq Ch_r(s)s, \quad s \in \mathbb{R}, 1 < r \leq 2,$$

where $C = \max\{c_1 + c_0^{-1}; M_1 + M_0^{-1}\}$.

From here, one can easily check this generalized inequality on Θ and Υ by simply observing that

$$\Theta = ah_{\tilde{m}} + h_m, \quad \Upsilon = bh_{\tilde{\mu}} + h_{\mu}, \quad a, b \geq 0,$$

and when $(1, 1) < (\tilde{\mu}, m) \leq (m, \mu) \leq (2, 2)$, the four maps $h_{\tilde{m}}, h_m, h_{\tilde{\mu}}$, and h_{μ} satisfies Eq (4.9) for some positive constants c_0, c_1, M_0 , and M_1 .

The proof is complete. $\square$

Proposition 4.3. Assume that $(1, 1) < (\tilde{m}, \tilde{\mu}) \leq (m, \mu) \leq (2, 2)$, and let $K = (k, k_t)^t \in C([0, T]; \mathcal{H})$ be a weak solution of system (4.3) satisfying the regularity Theorem 2.3. Then, there exists a constant $C_{m,\mu} > 0$ such that

$$\begin{aligned} & \left| \int_{\Omega_T} \alpha \Theta(\kappa_t) \left(m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right) d\Omega_T \right| + \left| \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \left(m_0^T \cdot \nabla_T \kappa_{\Gamma_1} + \frac{N-1}{2} \kappa_{\Gamma_1} \right) d\Gamma_{1T} \right| \\ & \leq C_{m,\mu,\tilde{m},\tilde{\mu}} \left[\int_{\Omega_T} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} + \int_0^T \|k(t)\|_{\mathcal{V}_0}^2 dt \right], \end{aligned} \quad (4.12)$$

where $d\Omega_T = dxdt$, and $d\Gamma_{1T} = d\Gamma_1 dt$.

Proof. We define for all (m, μ) satisfying $(1, 1) < (\tilde{m}, \tilde{\mu}) \leq (m, \mu) \leq (2, 2)$

$$I_m = \int_{\Omega_T} \alpha \Theta(\kappa_t) \left(m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right) d\Omega_T$$

and

$$J_{\mu} = \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \left(m_0^T \cdot \nabla_T \kappa_{\Gamma_1} + \frac{N-1}{2} \kappa_{\Gamma_1} \right) d\Gamma_{1T}.$$

We analyze each term separately.

• **Estimate of I_m :**

Applying the standard Hölder inequality, we obtain

$$\begin{aligned} |I_m| & \leq \|\alpha \Theta(\kappa_t)\|_{2,\Omega_T} \left\| m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right\|_{2,\Omega_T} \\ & \leq \|\alpha \Theta(\kappa_t)\|_{2,\Omega_T} \left(\|m_0 \cdot \nabla \kappa\|_{2,\Omega_T} + \frac{N-1}{2} \|\kappa\|_{2,\Omega_T} \right) \\ & \leq C_{r_0,N} \|\alpha \Theta(\kappa_t)\|_{2,\Omega_T} (\|\nabla \kappa\|_{2,\Omega_T} + \|\kappa\|_{2,\Omega_T}), \end{aligned}$$

where $C_{r_0,N} = \max\{r_0, (N-1)/2\}$.

From Poincaré's inequality, we have

$$\|\kappa\|_{2,\Omega} \leq \|\nabla \kappa\|_{2,\Omega}.$$

It then follows that

$$|I_m| \leq C \|\alpha \Theta(\kappa_t)\|_{2,\Omega_T} \|\nabla \kappa\|_{2,\Omega_T}$$

for some $C = C(r_0, N, c_p)$.

Now, applying the standard Young inequality, it follows that

$$|I_m| \leq C \left(\|\alpha\Theta(\kappa_t)\|_{2,\Omega_T}^2 + \|\nabla\kappa\|_{2,\Omega_T}^2 \right).$$

Because $\alpha \in L^\infty(\Omega)$ and Θ satisfy the inequality (4.10) of Proposition 4.2, then

$$\|\alpha\Theta(\kappa_t)\|_{2,\Omega_T}^2 \leq C_0 \int_{\Omega_T} \alpha\Theta(\kappa_t)\kappa_t d\Omega_T.$$

As a result,

$$|I_m| \leq C \left[\int_{\Omega_T} \alpha\Theta(\kappa_t)\kappa_t d\Omega_T + \int_0^T \|\kappa(t)\|_{V_0}^2 dt \right]$$

for some other $C_m = C(m, r_0, N, c_p, R) > 0$, and R satisfies Eq (4.4).

• **Estimate of J_μ :**

Using the same approach, we derive for some $C_\mu = C(\mu, r_0, N, R) > 0$ the following inequality:

$$|J_\mu| \leq C_\mu \left[\int_{\Gamma_{1T}} \beta\Upsilon(\kappa_{\Gamma_{1t}})\kappa_{\Gamma_{1t}} d\Gamma_{1T} + \int_0^T \|\kappa(t)\|_{V_0}^2 dt \right],$$

where R satisfies Eq (4.4).

Summing the estimates for I_m and J_μ , we obtain that for all $(1, 1) < (\tilde{m}, \tilde{\mu}) \leq (m, \mu) \leq (2, 2)$, we get the desired inequality (4.12).

The proof is thus complete. $\square$

In order to keep the article more concise, we will omit certain details, and we encourage readers to refer to the steps provided in [17].

Lemma 4.4. *Let $K \in C([0, T], \mathcal{H})$ be a weak solution of system (4.3). Then, for any $\psi \in (W^{1,\infty}(\overline{\Omega}))^N$, $N \geq 2$ that does not depend on the time variable t , the following identity holds:*

$$\begin{aligned} & 2 \left[\int_{\Omega} \kappa_t \psi \cdot \nabla \kappa dx + \int_{\Gamma_1} \kappa_{\Gamma_{1t}} \psi_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_1 \right]_0^T \\ & + \int_{\Omega_T} \operatorname{div}(\psi) \left[|\kappa_t|^2 - |\nabla \kappa|^2 \right] d\Omega_T \\ & + \int_{\Gamma_{1T}} \operatorname{div}_T(\psi_T) \left[|\kappa_{\Gamma_{1t}}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} \\ & - \int_{\Gamma_{1T}} (\psi \cdot \nu) \left[|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_\nu \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} \\ & + 2 \int_{\Omega_T} \nabla \kappa \cdot \nabla \psi \cdot \nabla \kappa d\Omega_T + 2 \int_{\Gamma_{1T}} \nabla_T \kappa_{\Gamma_1} \cdot \nabla_T \psi_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_{1T} \\ & + 2 \int_{\Omega_T} \alpha\Theta(\kappa_t) \psi \cdot \nabla \kappa d\Omega_T + 2 \int_{\Gamma_{1T}} \beta\Upsilon(\kappa_{\Gamma_{1t}}) \psi_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_{1T} \\ & + 2 \int_{\Omega_T} \operatorname{div}(\psi) \mathbb{F}(\kappa) d\Omega_T + 2 \int_{\Gamma_{1T}} \operatorname{div}_T(\psi_T) \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T} \\ & - 2 \int_{\Gamma_{1T}} (\psi \cdot \nu) \mathbb{F}(\kappa) d\Gamma_{1T} = 0, \end{aligned} \tag{4.13}$$

where ψ_T denotes the tangential component of ψ .

Proof. Here, use the following multiplier for the system (4.3):

$$L(K) = \begin{pmatrix} O_{2 \times 1} \\ l(k) \end{pmatrix}, \quad l(k) = \begin{pmatrix} 2\psi \cdot \nabla \kappa \\ 2\psi_T \cdot \nabla_T \kappa_{\Gamma_1} \end{pmatrix}, \quad \psi \in (W^{1,\infty}(\overline{\Omega}))^N, \quad N \geq 2,$$

where ψ_T denotes the tangential component of ψ .

The proof is achieved. $\square$

Lemma 4.5. *Let $K \in C([0, T]; \mathcal{H})$ be a weak solution of system (4.3). Then, for any $\psi \in (W^{1,\infty}(\overline{\Omega}))^N$, $N \geq 2$ independent of the time variable t , the following identity holds:*

$$\begin{aligned} & 2 \left[\int_{\Omega} \kappa_t m_0 \cdot \nabla \kappa \, dx + \int_{\Gamma_1} \kappa_{\Gamma_1 t} m_0^T \cdot \nabla_T \kappa_{\Gamma_1} \, d\Gamma_1 \right]_0^T + N \int_{\Omega_T} [|\kappa_t|^2 - |\nabla \kappa|^2] \, d\Omega_T \\ & + (N-1) \int_{\Gamma_{1T}} [|\kappa_{\Gamma_1 t}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2] \, d\Gamma_{1T} - \int_{\Gamma_{1T}} (m_0 \cdot \nu) \operatorname{tr}(\gamma_0) [|\kappa_{\Gamma_1 t}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2] \, d\Gamma_{1T} \\ & - \int_{\Gamma_{1T}} (m_0 \cdot \nu) [(|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_\nu \kappa_{\Gamma_1}|^2) + 2\gamma_0 \cdot \nabla_T \kappa_{\Gamma_1}] \, d\Gamma_{1T} \\ & + 2 \int_{\Omega_T} |\nabla \kappa|^2 \, d\Omega_T + 2 \int_{\Gamma_{1T}} |\nabla_T \kappa_{\Gamma_1}|^2 \, d\Gamma_{1T} + 2 \int_{\Omega_T} \alpha \Theta(\kappa_t) m_0 \cdot \nabla \kappa \, d\Omega_T \\ & + 2 \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_1 t}) m_0^T \cdot \nabla_T \kappa_{\Gamma_1} \, d\Gamma_{1T} + 2N \int_{\Omega_T} \mathbb{F}(\kappa) \, d\Omega_T + 2(N-1) \int_{\Gamma_{1T}} \mathbb{G}(\kappa_{\Gamma_1}) \, d\Gamma_{1T} \\ & - 2 \int_{\Gamma_{1T}} (m_0 \cdot \nu) \operatorname{tr}(\gamma_0) \mathbb{G}(\kappa_{\Gamma_1}) \, d\Gamma_{1T} - 2 \int_{\Gamma_{1T}} (m_0 \cdot \nu) \mathbb{F}(\kappa_{\Gamma_1}) \, d\Gamma_{1T} = 0. \end{aligned} \quad (4.14)$$

Proof. The result follows by choosing $\psi(x) = m_0(x)$ and $\psi_T(x) = m_0^T(x)$ because m_0 and m_0^T satisfy the following properties:

$$\begin{cases} \nabla m_0 = \underbrace{(1, \dots, 1)^t}_{N \text{ times}}, & \operatorname{div}(m_0) = N, \\ \nabla_T m_0^T = \underbrace{(1, \dots, 1)^t}_{N-1 \text{ times}} - (m_0 \cdot \nu) \gamma_0, & \operatorname{div}_T(m_0^T) = N-1 - (m_0 \cdot \nu) \operatorname{tr}(\gamma_0), \end{cases}$$

where γ_0 is the second fundamental form of Γ_1 (the curvature operator), and $\operatorname{tr}(\gamma_0)$ denotes its trace. $\square$

Lemma 4.6. *Let $K \in C([0, T]; \mathcal{H})$ be a weak solution of system (4.3). Then, for any $(\xi, \xi_{\Gamma_1}) \in C(\Omega) \times C(\Gamma_1)$ independent of the time variable t , the following identity holds:*

$$\begin{aligned} & \left[\int_{\Omega} \xi \kappa_t \, d\Omega + \int_{\Gamma_1} \xi_{\Gamma_1} \kappa_{\Gamma_1 t} \, d\Gamma_1 \right]_0^T - \int_{\Omega_T} \xi [|\kappa_t|^2 - |\nabla \kappa|^2] \, d\Omega_T - \int_{\Gamma_{1T}} \xi_{\Gamma_1} [|\kappa_{\Gamma_1 t}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2] \, d\Gamma_{1T} \\ & + \int_{\Omega_T} \kappa \nabla \kappa \cdot \nabla \xi \, d\Omega_T + \int_{\Gamma_{1T}} \kappa_{\Gamma_1} \nabla_T \kappa_{\Gamma_1} \cdot \nabla_T \xi_{\Gamma_1} \, d\Gamma_{1T} + \int_{\Omega_T} \alpha \xi \Theta(\kappa_t) \kappa \, d\Omega_T \\ & + \int_{\Gamma_{1T}} \beta \xi_{\Gamma_1} \Upsilon(\kappa_{\Gamma_1 t}) \kappa_{\Gamma_1} \, d\Gamma_{1T} - \int_{\Omega_T} \xi f(\kappa) \kappa \, d\Omega_T - \int_{\Gamma_{1T}} \xi_{\Gamma_1} g(\kappa_{\Gamma_1}) \kappa_{\Gamma_1} \, d\Gamma_{1T} = 0. \end{aligned} \quad (4.15)$$

Proof. The result follows using the same approach as in the proof of Lemma 4.4, applying the following multiplier:

$$L(K) = \begin{pmatrix} O_{2 \times 1} \\ l(k) \end{pmatrix}, \quad l(k) = \begin{pmatrix} \xi \kappa \\ \xi_{\Gamma_1} \kappa_{\Gamma_1} \end{pmatrix}.$$

□

Lemma 4.7. *Let $K \in C([0, T]; \mathcal{H})$ be a weak solution of system (4.3). Then, the following inequality holds:*

$$\begin{aligned} & 2 \left[\int_{\Omega} \kappa_t \left(m_0 \cdot \nabla \kappa + \frac{a_0}{2} \kappa \right) dx + \int_{\Gamma_1} \kappa_{\Gamma_1 t} \left(m_0^T \cdot \nabla_T \kappa_{\Gamma_1} + \frac{a_1}{2} \kappa_{\Gamma_1} \right) d\Gamma_1 \right]_0^T \\ & + (N - a_0) \int_{\Omega_T} |\kappa_t|^2 d\Omega_T + (N - 1 - a_1) \int_{\Gamma_{1T}} |\kappa_{\Gamma_1 t}|^2 d\Gamma_{1T} \\ & - (N - 2 - a_0) \int_{\Omega_T} |\nabla \kappa|^2 d\Omega_T - (N - 3 - a_1) \int_{\Gamma_{1T}} |\nabla_T \kappa_{\Gamma_1}|^2 d\Gamma_{1T} \\ & - \int_{\Gamma_{1T}} (m_0 \cdot \nu) \operatorname{tr}(\gamma_0) \left[|\kappa_{\Gamma_1 t}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} \\ & - \int_{\Gamma_{1T}} (m_0 \cdot \nu) \left[(|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_\nu \kappa_{\Gamma_1}|^2) + 2\gamma_0 \cdot |\nabla_T \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} \\ & + 2 \int_{\Omega_T} \alpha \Theta(\kappa_t) \left(m_0 \cdot \nabla \kappa + \frac{a_0}{2} \kappa \right) d\Omega_T + 2 \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_1 t}) \left(m_0^T \cdot \nabla_T \kappa_{\Gamma_1} + \frac{a_1}{2} \kappa_{\Gamma_1} \right) d\Gamma_{1T} \\ & + \int_{\Omega_T} [2N\mathbb{F}(\kappa) - a_0 f(\kappa)\kappa] d\Omega_T + \int_{\Gamma_{1T}} [2(N - 1)\mathbb{G}(\kappa_{\Gamma_1}) - a_1 g(\kappa_{\Gamma_1})\kappa_{\Gamma_1}] d\Gamma_{1T} \\ & - 2 \int_{\Gamma_{1T}} (m_0 \cdot \nu) \operatorname{tr}(\gamma_0) \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T} - 2 \int_{\Gamma_{1T}} (m_0 \cdot \nu) \mathbb{F}(\kappa_{\Gamma_1}) d\Gamma_{1T} = 0, \end{aligned} \tag{4.16}$$

where $a_0, a_1 > 0$ are constants.

Proof. It suffices to take $(\xi, \xi_{\Gamma_1}) = (a_0, a_1)$ in Eq (4.15), and then add both sides term by term with the identity (4.14). □

We are now in a position to establish the first crucial inequality to confirm the observability inequality in Eq (4.7).

Lemma 4.8. *Consider the elements defined in Assumption 2, and let $K \in C([0, T]; \mathcal{H})$ be a weak solution of system (4.3). Moreover, if $-f$ and $-g$ are superlinear, meaning that there exist constants $C_f, C_g \geq 2$ such that for all $s \in \mathbb{R}$,*

$$f(s)s \leq C_f \mathbb{F}(s) \leq 0, \quad g(s)s \leq C_g \mathbb{G}(s) \leq 0, \tag{4.17}$$

then the following inequality holds:

$$E_\kappa(T) \leq C_\kappa \left[\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t(t)) \kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}(t)) \kappa_{\Gamma_1 t}(t) d\Gamma_1 dt + \int_0^T \|k(t)\|_{\mathcal{V}_0}^2 dt \right], \tag{4.18}$$

where $C_\kappa > 0$.

Before proving Lemma 4.8, let us explore the different cases in which the pairs $(f, \mathbb{F})$ and $(g, \mathbb{G})$ satisfy the condition (4.17).

Remark 4.9. *The following remarks also hold for the pair $(g, \mathbb{G})$.*

From Eq (1.5), we can observe that for all $s \in \mathbb{R}$,

$$f(s)s = \tilde{\gamma}|s|^{\tilde{p}} + \gamma|s|^p, \quad \mathbb{F}(s) = \frac{\tilde{\gamma}}{p}|s|^{\tilde{p}} + \frac{\gamma}{p}|s|^p.$$

In this case, $f(s)s \leq C_f \mathbb{F}(s) \leq 0$ if and only if:

$$\tilde{\gamma} \left(1 - \frac{C_f}{\tilde{p}}\right) \leq 0, \quad \gamma \left(1 - \frac{C_f}{p}\right) \leq 0.$$

This aligns with at least the local stability of the system if and only if

- (i) $\tilde{\gamma}, \gamma \leq 0$, and $2 \leq C_f \leq \tilde{p}$,
- (ii) $\tilde{\gamma} \geq 0, \gamma \leq 0$, and $\tilde{p} \leq C_f \leq p$.

In particular, the case of linear perturbation ($p = 2$) is the only case where C_f should be 2 and $f(s)s = C_f \mathbb{F}(s)$.

The condition (i) is a choice where the exponential stability is global in time, whereas Condition (ii) could be nonglobal, especially when damping exponents m, μ do not control the perturbations' exponents $\tilde{p}, p$. This last scenario is the one called local-in-time exponential stability.

Proof. (Lemma 4.8) In order to have $E_k(T)$ as a lower bound in the inequality, it suffices that the sign of the coefficients of the terms in Eq (4.16), which appear in its expression, match the signs of those in $E_k(T)$. It is clear that for $(N-2, N-3) \leq (a_0, a_1) \leq (N, N-1)$, the coefficients $N - a_0$, $a_0 - (N-2)$, $N-1 - a_1$, $a_1 - (N-3)$ are positive. Therefore, we can choose in Eq (4.17):

$$(\tilde{p}, \tilde{q}) \geq (C_f, C_g) = \left(\frac{2N}{a_0}, \frac{2(N-1)}{a_1} \right) \geq (2, 2),$$

which leads to

$$2N\mathbb{F}(s) - a_0 f(s)s \geq 0, \quad 2(N-1)\mathbb{G}(s) - a_1 g(s)s \geq 0.$$

Consequently, there exist $c_f, c_g \geq 0$ such that

$$-c_f \mathbb{F}(s) \leq 2N\mathbb{F}(s) - a_0 f(s)s, \quad -c_g \mathbb{G}(s) \leq 2(N-1)\mathbb{G}(s) - a_1 g(s)s,$$

$c_f = c_g = 0$ only if $p = q = 2$.

Because $\langle m_0 | \nu \rangle \geq 0$ on Γ_{1T} , there exists $c > 0$ such that

$$0 \leq -2 \int_{\Gamma_{1T}} (m_0 | \nu) \text{tr}(\gamma_0) \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T} \leq -c \int_{\Gamma_{1T}} \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T}.$$

Consequently, linear or nonlinear perturbation, we can choose a_0 and a_1 such that

$$\begin{aligned} c_3 \int_0^T E_k(t) dt \leq & 2 \left| \int_{\Omega_T} \alpha \Theta(\kappa_t) \left(m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right) d\Omega_T \right| \\ & + 2 \left| \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \left(m_0^T \cdot \nabla_T \kappa_{\Gamma_1} + \frac{N-2}{2} \kappa_{\Gamma_1} \right) d\Gamma_{1T} \right| \\ & + P_1 + P_2 + P_3, \end{aligned} \quad (4.19)$$

where $c_3 > 0$, and P_1 , P_2 , and P_3 are defined as follows:

$$\begin{cases} P_1 = \left| 2 \left[\int_{\Omega} \kappa_t \left(m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right) dx + \int_{\Gamma_1} \kappa_{\Gamma_{1t}} \left(m_0^T \cdot \nabla_T \kappa_{\Gamma_1} + \frac{N-2}{2} \kappa_{\Gamma_1} \right) d\Gamma_1 \right]_0^T \right| \\ P_2 = \int_{\Gamma_{1T}} (m_0 |v|) (|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_v \kappa_{\Gamma_1}|^2) d\Gamma_{1T} + 2 \int_{\Gamma_{1T}} (m_0 |v|) \mathbb{F}(\kappa_{\Gamma_1}) d\Gamma_{1T} \\ P_3 = \left| \int_{\Gamma_{1T}} (m_0 |v|) \text{tr}(\gamma_0) [|\kappa_{\Gamma_{1t}}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 - 2\gamma_0 \cdot |\nabla_T \kappa_{\Gamma_1}|^2] d\Gamma_{1T} \right|. \end{cases}$$

Using Eq (4.12) on the two first terms of Eq (4.19), we obtain

$$\begin{aligned} c_3 \int_0^T E_k(t) dt \leq & C_{m,\mu,\tilde{m},\tilde{\mu}} \left[E_k(T) + \int_{\Omega_T} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} + \int_0^T \|k(t)\|_{\mathcal{V}_0}^2 dt \right] \\ & + P_1 + P_2 + P_3. \end{aligned} \quad (4.20)$$

For the remainder, the terms on the RHS of Eq (4.20) need to be estimable in terms of

$$\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t(t)) \kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_{1t}}(t)) \kappa_{\Gamma_{1t}}(t) d\Gamma_1 dt.$$

We proceed on a case-by-case basis:

- For P_1 :

By applying the Cauchy–Schwarz inequality followed by Young inequality, we obtain

$$\begin{aligned} 2 \left| \int_{\Omega} \kappa_t \left(m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right) dx \right| & \leq 2 \|\kappa_t\|_2 \left\| m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right\|_2 \\ & \leq C(\epsilon) \|\kappa_t\|_2^2 + \epsilon \left\| m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right\|_2^2 \\ & \leq C(\epsilon) \|\kappa_t\|_2^2 + \epsilon c \left(\|\nabla \kappa\|_2^2 + \|\kappa\|_2^2 \right), \end{aligned}$$

where $c = \max \{r_0, (N-1)/2\}$, and $\epsilon > 0$ is sufficiently small.

Using Poincaré's inequality, we obtain

$$2 \left| \int_{\Omega} \kappa_t \left(m_0 \cdot \nabla \kappa + \frac{N-1}{2} \kappa \right) dx \right| \leq c_{p_1} \left(\|\kappa_t\|_2^2 + \|\nabla \kappa\|_2^2 \right),$$

where $c_{P_1} > 0$.

From this, we deduce the existence of a constant $C_{P_1} > 0$ such that

$$|P_1| \leq C_{P_1}(E_k(0) + E_k(T)).$$

Using Eq (4.6) with $t_1 = 0$ and $t_2 = T$, we obtain

$$|P_1| \leq C_{P_1} \left[2E_k(T) + \int_{\Omega_T} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right]. \quad (4.21)$$

• For P_2 :

We adopt the same field constructions and cut-off functions as in [17], which are equally valid as multipliers for our system under homogeneous Dirichlet boundary conditions on Γ_0 . We begin by introducing ω_0 with a nonzero measure $\text{meas}(\omega_0) \neq 0$ as well as a neighborhood of $\overline{\Gamma_1^*}$ within ω , that is, $\overline{\omega_0} \cap \Omega \subset \omega$. We define the cut-off field[§] $\varrho \in (W^{1,\infty}(\overline{\Omega}))^N$, $N \geq 2$ such that

$$\varrho = \nu \quad \text{on } \Gamma_1^*, \quad \langle \varrho | \nu \rangle \leq 0 \quad \text{on } \Gamma_1 \setminus \Gamma_1^*,$$

and

$$\varrho = 0 \quad \text{in } \Omega \setminus \omega_0.$$

Applying identity (4.13) with $\psi = \varrho$, we immediately obtain the following identity:

$$\begin{aligned} P_2 = & 2 \left[\int_{\Omega} \kappa_t \varrho \cdot \nabla \kappa dx + \int_{\Gamma_1} \kappa_{\Gamma_{1t}} \varrho_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_1 \right]_0^T + \int_{\omega_{0T}} \text{div}(\varrho) [|\kappa_t|^2 - |\nabla \kappa|^2] d\Omega_T \\ & + \int_{\Gamma_{1T}} \text{div}_T(\varrho_T) [|\kappa_{\Gamma_{1t}}|^2 - |\nabla_T \kappa_{\Gamma_1}|^2] d\Gamma_{1T} + 2 \int_{\omega_{0T}} \nabla \kappa \cdot \nabla \varrho \cdot \nabla \kappa d\Omega_T \\ & + 2 \int_{\Gamma_{1T}} \nabla_T \kappa_{\Gamma_1} \cdot \nabla_T \varrho_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_{1T} + 2 \int_{\omega_{0T}} \alpha \Theta(\kappa_t) \varrho \cdot \nabla \kappa d\Omega_T \\ & + 2 \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \varrho_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_{1T} + 2 \int_{\omega_{0T}} \text{div}(\varrho) \mathbb{F}(\kappa) d\Omega_T \\ & + 2 \int_{\Gamma_{1T}} \text{div}_T(\varrho_T) \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T}, \end{aligned}$$

where $\omega_{0T} = \omega_0 \times (0, T)$.

Using the fact that $\varrho \in (W^{1,\infty}(\Omega))^N$, $\varrho_T \in (W^{1,\infty}(\Gamma))^N$ (tangential component of ϱ) and $\mathbb{F}(\kappa), \mathbb{G}(\kappa_{\Gamma_1}) \leq 0$, one can easily deduce for some $c_{\omega_0} > 0$ the following estimation:

$$\begin{aligned} P_2 \leq & P_\varrho + Q_\varrho \\ & + c_{\omega_0} \left(\|\kappa_t\|_{2,\omega_{0T}}^2 + \|\nabla \kappa\|_{2,\omega_{0T}}^2 + \|\kappa_{\Gamma_{1t}}\|_{2,\Gamma_{1T}}^2 + \|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2 - \int_{\omega_{0T}} \mathbb{F}(\kappa_{\Gamma_1}) d\Omega_T \right. \\ & \left. - \int_{\Gamma_{1T}} \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T} \right) \end{aligned}$$

[§]Here, the goal is to obtain an estimate involving $\partial_\nu \varphi_{\Gamma_1}$, as $\psi = \varrho = \nu$ on Γ_1^* , which amounts to analyzing the influence of Γ_1^* on the evolution of $k = (\kappa, \kappa_{\Gamma_1})^t$.

with

$$P_\varrho = 2 \int_{\omega_{0T}} \alpha \Theta(\kappa_t) \varrho \cdot \nabla \kappa d\Omega_T + 2 \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \varrho_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_{1T},$$

$$Q_\varrho = 2 \left[\int_{\Omega} \kappa_t \varrho \cdot \nabla \kappa dx + \int_{\Gamma_1} \kappa_{\Gamma_{1t}} \varrho_T \cdot \nabla_T \kappa_{\Gamma_1} d\Gamma_1 \right]_0^T.$$

The term P_ϱ satisfies on ω_{0T} the estimate (4.12) given in Proposition 4.3, omitting the terms $(N-1)\kappa/2$, $(N-2)\kappa_{\Gamma_1}/2$, that is

$$P_\varrho \leq C_{\tilde{m},m} \left[\int_{\omega_{0T}} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_0^T \|k(t)_{\omega_{0T}}\|_{\mathcal{V}_0}^2 dt \right].$$

Because $\varrho = \nu$ on Γ_{1T}^* , the contribution in energy of P_2 over $\Gamma_{1T}^* \subset \Gamma_{1T}$ is estimated as

$$\begin{aligned} & \int_{\Gamma_{1T}^*} (\varrho|\nu|) \left[|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_\nu \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} + 2 \int_{\Gamma_{1T}^*} \mathbb{F}(\kappa_{\Gamma_1}) d\Gamma_{1T} \\ & \leq Q_\varrho + C_{\omega_0} \left[\int_{\omega_{0T}} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right. \\ & \quad \left. + \int_0^T \|k(t)_{\omega_{0T}}\|_{\mathcal{V}_0}^2 dt - \int_{\omega_{0T}} \mathbb{F}(\kappa) d\Omega_T - \int_{\Gamma_{1T}} \mathbb{G}(\kappa_{\Gamma_1}) d\Gamma_{1T} \right], \end{aligned} \quad (4.22)$$

where $\|k_{\omega_{0T}}\|_{\mathcal{V}_0}^2 = \|\nabla \kappa\|_{2,\omega_{0T}}^2 + \|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2$.

From here and below, $\|k_D\|_{\mathcal{V}_0}^2 = \|\nabla \kappa\|_{2,D}^2 + \|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2$, $D \subset \Omega_T$.

Note that the integrals of Q_ϱ are over the whole domain Ω , whereas only a small enough neighborhood of Γ_{1T}^* is required. This is why, we need to introduce a cut-off function.

Because Ω is of class C^2 , we can construct a cut-off function (see Lemma 2.4, Chapter VII in [1]) $\varrho_0 \in W^{1,\infty}(\overline{\Omega})$ as follows:

$$\begin{cases} \varrho_0 \in [0, 1], & \text{almost everywhere in } \Omega, \\ \varrho_0 = 1, & \text{in } \omega_0, \\ \varrho_0 = 0, & \text{in } \Omega \setminus \omega, \end{cases} \quad \left(\frac{|\nabla \varrho_0|^2}{\varrho_0}, \frac{|\nabla_T \varrho_{0\Gamma_1}|^2}{\varrho_{0\Gamma_1}} \right) \in L^\infty(\omega) \times L^\infty(\Gamma_1).$$

By utilizing identity (4.15) with $(\xi, \xi_{\Gamma_1}) = (\varrho_0, \varrho_{0\Gamma_1})$, we obtain:

$$\begin{aligned} & \int_{\Omega_T} \varrho_0 \left[|\nabla \kappa|^2 - f(\kappa) \kappa \right] d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} \left[|\nabla_T \kappa_{\Gamma_1}|^2 - g(\kappa_{\Gamma_1}) \kappa_{\Gamma_1} \right] d\Gamma_{1T} \\ & \leq \int_{\Omega_T} \varrho_0 |\kappa_t|^2 d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} |\kappa_{\Gamma_{1t}}|^2 d\Gamma_{1T} + P_{\varrho_0} + Q_{\varrho_0} \\ & \quad + \left| \int_{\Omega_T} \alpha \varrho_0 \Theta(\kappa_t) \kappa d\Omega_T \right| + \left| \int_{\Gamma_{1T}} \beta \varrho_{0\Gamma_1} \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right|, \end{aligned}$$

where

$$P_{\varrho_0} = \left[\int_{\Omega} \varrho_0 \kappa_t \kappa d\Omega + \int_{\Gamma_1} \varrho_{0\Gamma_1} \kappa_{\Gamma_{1t}} \kappa_{\Gamma_1} d\Gamma_1 \right]_0^T$$

$$Q_{\varrho_0} = \int_{\Omega_T} \kappa \nabla \kappa \nabla \varrho_0 d\Omega_T + \int_{\Gamma_{1T}} \kappa_{\Gamma_1} \nabla_T \kappa_{\Gamma_1} \nabla_T \varrho_{0\Gamma_1} d\Gamma_{1T}.$$

The proof technique of Proposition 4.3 remains valid for the last two terms in the absence of terms $m_0 \cdot \nabla \kappa$ and $m_T^0 \cdot \nabla_T \kappa_{\Gamma_1}$. It then follows that

$$\begin{aligned} & \int_{\Omega_T} \varrho_0 \left[|\nabla \kappa|^2 - f(\kappa) \kappa \right] d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} \left[|\nabla_T \kappa_{\Gamma_1}|^2 - g(\kappa_{\Gamma_1}) \kappa_{\Gamma_1} \right] d\Gamma_{1T} \\ & \leq P_{\varrho_0} + Q_{\varrho_0} + C_{\varrho_0} \left[\int_{\Omega_T} \alpha \varrho_0 \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \varrho_{0\Gamma_1} \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right. \\ & \quad \left. + \int_0^T \|k(t)_{\omega_T}\|_{\mathcal{V}_0}^2 dt \right]. \end{aligned} \quad (4.23)$$

Similarly to the estimation of P_1 , P_{ϱ_0} satisfies

$$|P_{\varrho_0}| \leq C_{P_{\varrho_0}} \left[2E_\kappa(T) + \int_{\Omega_T} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right], \quad (4.24)$$

for some constant $C_{P_{\varrho_0}} > 0$.

Furthermore, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |Q_{\varrho_0}| &= \left| \int_{\omega_T} \left(\kappa \frac{\nabla \varrho_0}{\sqrt{\varrho_0}} \right) (\sqrt{\varrho_0} \nabla \kappa) d\Omega_T + \int_{\Gamma_{1T}} \left(\kappa_{\Gamma_1} \frac{\nabla_T \varrho_{0\Gamma_1}}{\sqrt{\varrho_{0\Gamma_1}}} \right) (\sqrt{\varrho_{0\Gamma_1}} \nabla_T \kappa_{\Gamma_1}) d\Gamma_{1T} \right| \\ &\leq \left(\int_{\omega_T} |\kappa|^2 \frac{|\nabla \varrho_0|^2}{\varrho_0} d\Omega_T \right)^{\frac{1}{2}} \left(\int_{\omega_T} \varrho_0 |\nabla \kappa|^2 d\Omega_T \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\Gamma_{1T}} |\kappa_{\Gamma_1}|^2 \frac{|\nabla_T \varrho_{0\Gamma_1}|^2}{\varrho_{0\Gamma_1}} d\Gamma_{1T} \right)^{\frac{1}{2}} \left(\int_{\Gamma_{1T}} \varrho_{0\Gamma_1} |\nabla_T \kappa_{\Gamma_1}|^2 d\Gamma_{1T} \right)^{\frac{1}{2}}. \end{aligned}$$

Using the standard Young inequality, the continuous embedding $\mathcal{V}_0 \hookrightarrow \mathcal{V}_1$, and the boundedness of $(|\nabla \varrho_0|^2 / \varrho_0, |\nabla_T \varrho_{0\Gamma_1}|^2 / \varrho_{0\Gamma_1})$ in $L^\infty(\omega) \times L^\infty(\Gamma_1)$, it then follows that:

$$\begin{aligned} |Q_{\varrho_0}| &\leq \frac{1}{2\epsilon} \left(\int_{\omega_T} \frac{|\nabla \varrho_0|^2}{\varrho_0} |\kappa|^2 d\Omega_T + \int_{\Gamma_{1T}} |\kappa_{\Gamma_1}|^2 \frac{|\nabla_T \varrho_{0\Gamma_1}|^2}{\varrho_{0\Gamma_1}} d\Gamma_{1T} \right) \\ &\quad + \epsilon \left(\int_{\omega_T} \varrho_0 |\nabla \kappa|^2 d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} |\nabla_T \kappa_{\Gamma_1}|^2 d\Gamma_{1T} \right) \\ &\leq C_{Q_{\varrho_0}} \left(\int_{\omega_T} \varrho_0 |\nabla \kappa|^2 d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} |\nabla_T \kappa_{\Gamma_1}|^2 d\Gamma_{1T} \right) \end{aligned} \quad (4.25)$$

for some constant $C_{Q_{\varrho_0}} > 0$.

By combining Eqs (4.24) and (4.25) into Eq (4.23), we deduce the validity of the following inequality:

$$\begin{aligned}
& \int_{\omega_T} \varrho_0 \left[|\nabla \kappa|^2 - f(\kappa)\kappa \right] d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} \left[|\nabla_T \kappa_{\Gamma_1}|^2 - g(\kappa_{\Gamma_1})\kappa_{\Gamma_1} \right] d\Gamma_{1T} \\
& \leq C(\omega_0) \left[2E_\kappa(T) + \int_{\Omega_T} \alpha \varrho_0 \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \varrho_{0\Gamma_1} \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right. \\
& \quad \left. + \|\varrho^{\frac{1}{2}} \nabla \kappa\|_{2,\Omega_T}^2 + \|\varrho_{\Gamma_1}^{\frac{1}{2}} \nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2 \right]
\end{aligned}$$

for some constant $C(\omega_0) > 0$.

Moreover, because $-f(\kappa)\kappa \geq -C_f \mathbb{F}(\kappa) \geq 0$, $-g(\kappa_{\Gamma_1})\kappa_{\Gamma_1} \geq -C_g \mathbb{G}(\kappa_{\Gamma_1}) \geq 0$, and $(C_f, C_g) > (1, 1)$, it then follows that

$$\begin{aligned}
& \int_{\omega_T} \varrho_0 \left[|\nabla \kappa|^2 - \mathbb{F}(\kappa) \right] d\Omega_T + \int_{\Gamma_{1T}} \varrho_{0\Gamma_1} \left[|\nabla_T \kappa_{\Gamma_1}|^2 - \mathbb{G}(\kappa_{\Gamma_1}) \right] d\Gamma_{1T} \\
& \leq C(\omega_0) \left[2E_\kappa(T) + \int_{\Omega_T} \alpha \varrho_0 \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \varrho_{0\Gamma_1} \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right. \\
& \quad \left. + \|\varrho^{\frac{1}{2}} \nabla \kappa\|_{2,\Omega_T}^2 + \|\varrho_{\Gamma_1}^{\frac{1}{2}} \nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2 \right]. \tag{4.26}
\end{aligned}$$

By combining Eqs (4.22) and (4.26) with the fact that the terms $-\int_{\omega_T} \mathbb{F}$, $-\int_{\Gamma_{1T}} \mathbb{G}$ are controlled by the energy $E_\kappa(T) + E_\kappa(0)$, we obtain

$$\begin{aligned}
& \int_{\Gamma_{1T}^*} (\varrho|\nu|) \left[|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_\nu \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} + 2 \int_{\Gamma_{1T}^*} \mathbb{F}(\kappa_{\Gamma_1}) d\Gamma_{1T} \\
& \leq \widetilde{C_{\omega_0}} \left[2E_\kappa(T) + \int_{\omega_{0T}} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T \right. \\
& \quad \left. + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} + \|\nabla \kappa\|_{2,\omega_T}^2 + \|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2 \right] + Q_\varrho
\end{aligned}$$

for some constant $\widetilde{C_{\omega_0}} > 0$.

Note that the term Q_ϱ satisfies the same estimate as P_1 in the absence of $(N-1)\kappa/2$ and $(N-2)\kappa_{\Gamma_1}/2$. Consequently, we obtain:

$$\begin{aligned}
& \int_{\Gamma_{1T}^*} (\varrho|\nu|) \left[|\kappa_t|^2 - |\nabla_T \kappa_{\Gamma_1}|^2 + |\partial_\nu \kappa_{\Gamma_1}|^2 \right] d\Gamma_{1T} + 2 \int_{\Gamma_{1T}^*} \mathbb{F}(\kappa_{\Gamma_1}) d\Gamma_{1T} \\
& \leq C_{\omega_0} \left[2E_\kappa(T) + \int_{\omega_{0T}} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right. \\
& \quad \left. + \|\nabla \kappa\|_{2,\omega_T}^2 + \|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2 \right]. \tag{4.27}
\end{aligned}$$

- For the term P_3 :

Because $|\langle m_0 | \nu \rangle| \leq \sqrt{|\langle m_0 | m_0 \rangle|} = r_0$, and $tr(\gamma_0) = tr(\nabla_T \nu) \in L^\infty(\Gamma_{1T})$, as Ω is of class C^2 , we obtain

$$\begin{aligned}
P_3 & \leq C \left(\|\kappa_{\Gamma_{1T}}\|_{2,\Gamma_{1T}}^2 + \|\nabla_T \kappa_{\Gamma_{1T}}\|_{2,\Gamma_{1T}}^2 \right) \\
& \leq C \left[2E_\kappa(T) + \int_{\Omega_T} \alpha \Theta(\kappa_t) \kappa_t d\Omega_T + \int_{\Gamma_{1T}} \beta \Upsilon(\kappa_{\Gamma_{1t}}) \kappa_{\Gamma_{1t}} d\Gamma_{1T} \right. \\
& \quad \left. + \|\nabla \kappa\|_{2,\Omega_T}^2 + \|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2 \right], \tag{4.28}
\end{aligned}$$

where $C > 0$.

By substituting P_1 of Eq (4.21), P_2 of Eq (4.22), and P_3 of Eq (4.28) into Eq (4.20), we obtain

$$E_k(T) \leq C_k \left[\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t(t)) \kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}(t)) \kappa_{\Gamma_1 t}(t) d\Gamma_1 dt + \int_0^T \|k(t)\|_{V_0}^2 dt \right],$$

where C_k depend on T and T large enough.

The proof is complete. $\square$

Note that in the whole process, one can preserve the zero-order terms $\|\kappa\|_{2,\omega_{0T}}^2$ and $\|\kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2$ in addition of the first-order terms $\|\nabla \kappa\|_{2,\omega_{0T}}^2$, $\|\nabla_T \kappa_{\Gamma_1}\|_{2,\Gamma_{1T}}^2$. The first-order terms are then absorbed by the left-hand side of the last combination between P_1 , P_2 , and P_3 in Eq (4.20). We therefore have

$$E_k(T) \leq C_k \left[\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t(t)) \kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}(t)) \kappa_{\Gamma_1 t}(t) d\Gamma_1 dt + \int_0^T \|k(t)\|_{V_1}^2 dt \right], \quad (4.29)$$

for $T > 0$ large enough.

Lemma 4.8 thus leads us to conclude that it is sufficient to establish a certain absorption of the term $\int_0^T \|k(t)\|_{V_1}^2 dt$ to confirm the observability inequality (4.7), and consequently, the exponential stability. To achieve this, we proceed similarly to theories in [17, 26], employing a contradiction argument based on the following unique continuation principle:

Lemma 4.10. *Let $h \in L^\infty(\Omega_T)$. The following system admits a unique solution $y \in C([0, T]; H^1(\Omega))$, which is trivial:*

$$\begin{cases} y_{tt} - \Delta y + h(x, t)y_t = 0, & \text{in } \Omega_T, \\ y = 0, & \text{in } \omega_T \cup (\Gamma_T \setminus \Gamma_{1T}^*), \\ \partial_\nu y = 0, & \text{on } \Gamma_{1T}^* \end{cases} \quad (4.30)$$

for any initial condition $y(0) = y_0 \in H^1(\Omega_T)$.

Proof. See A. Ruiz [16]. $\square$

Theorem 4.11. *Suppose that the conditions of Lemma 4.8 hold with $c_1 = c_2 = 0$ for perturbation maps f, g . Then, for some constant $C > 0$, the following inequality holds:*

$$\int_0^T \|k(t)\|_{V_1}^2 dt \leq C \left[\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t(t)) \kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}(t)) \kappa_{\Gamma_1 t}(t) d\Gamma_1 dt \right]. \quad (4.31)$$

Proof. First, it is evident that when the initial conditions of system (4.2) satisfy $K_0 = O_{4 \times 1}$, due to the positivity and dissipation of its energy over $(0, T)$, we have for all $t \in (0, T)$ that $E_k(t) = 0$. Consequently, $K \equiv 0$, and as a result, the inequality (4.31) holds.

Next, we consider the case where $0 < \|K_0\|_{\mathcal{H}} < R$ and a sequence of initial conditions $(K_0^{(n)})_{n \in \mathbb{N}^*}$ such that:

$$\|K_0^{(n)}\|_{\mathcal{H}} < R, \quad K_0^{(n)} \rightarrow K_0, \quad \text{strongly in } \mathcal{H}. \quad (4.32)$$

As previously mentioned, we proceed by contradiction, assuming that for any $C > 0$,

$$\int_0^T \|k(t)\|_{\mathcal{V}_1}^2 dt > C \left[\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t(t)) \kappa_t(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}(t)) \kappa_{\Gamma_1 t}(t) d\Gamma_1 dt \right].$$

This implies that we can extract a subsequence satisfying Eq (4.32), still denoted by $K_0^{(n)}$, whose associated solution $K^{(n)} = (k^{(n)}, \kappa_t^{(n)})^t \in C([0, T]; \mathcal{H})$, satisfies

$$\int_0^T \|k^{(n)}(t)\|_{\mathcal{V}_1}^2 dt > n \left[\int_0^T \int_{\Omega} \alpha(x) \Theta(\kappa_t^{(n)}(t)) \kappa_t^{(n)}(t) dx dt + \int_0^T \int_{\Gamma_1} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}^{(n)}(t)) \kappa_{\Gamma_1 t}^{(n)}(t) d\Gamma_1 dt \right].$$

Thus, we obtain

$$\lim_{n \rightarrow +\infty} \frac{\int_0^T \|k^{(n)}(t)\|_{\mathcal{V}_1}^2 dt}{I_n^K + J_n^K} = +\infty, \quad (4.33)$$

where

$$I_n^K = \int_{\Omega_T} \alpha(x) \Theta(\kappa_t^{(n)}(t)) \kappa_t^{(n)}(t) d\Omega_T, \quad J_n^K = \int_{\Gamma_{1T}} \beta(x) \Upsilon(\kappa_{\Gamma_1 t}^{(n)}(t)) \kappa_{\Gamma_1 t}^{(n)}(t) d\Gamma_{1T}.$$

Because $k^{(n)}$ is bounded in $L^\infty([0, T]; \mathcal{V}_0) \cap W^{1,\infty}([0, T]; \mathcal{V}_1)$, then we can extract a subsequence $(K^{(n_0)})_{n_0 \in \mathbb{N}^*}$ such that

$$\begin{aligned} k^{(n_0)} &\rightharpoonup k \text{ weak-}^* \text{ in } L^\infty([0, T]; \mathcal{V}_0) \quad \text{as } n_0 \rightarrow +\infty, \\ \kappa_t^{(n_0)} &\rightharpoonup \kappa_t \text{ weak-}^* \text{ in } L^\infty([0, T]; \mathcal{V}_1) \quad \text{as } n_0 \rightarrow +\infty. \end{aligned}$$

Moreover, because for all $(p_0, q_0) \in [2; r_\Omega) \times [2; r_{\Gamma_1})$, we have the following compact embeddings:

$$\mathcal{V}_0 \hookrightarrow L^{p_0}(\Omega) \times L^{q_0}(\Gamma_1) \hookrightarrow \mathcal{V}_1,$$

the compactness result from the Aubin–Lions–Simon theorem [4] then allows us to write

$$k^{(n_0)} \longrightarrow k \text{ strongly in } L^\infty([0, T]; L^p(\Omega) \times L^q(\Gamma_1)) \quad \text{as } n_0 \rightarrow +\infty, \quad (4.34)$$

where p, q satisfy Eq (1.7) such that

$$\begin{cases} \kappa_{tt}^{(n_0)} - \Delta \kappa^{(n_0)} + \alpha(x) \Theta(\kappa_t^{(n_0)}) = f(\kappa^{(n_0)}), & \text{in } \Omega_T, \\ \kappa^{(n_0)} = 0, & \text{on } \Gamma_{0T}, \\ \kappa_{tt}^{(n_0)} + \partial_\nu \kappa^{(n_0)} - \Delta_T \kappa^{(n_0)} + \beta(x) \Upsilon(\kappa_t^{(n_0)}) = g(\kappa^{(n_0)}), & \text{on } \Gamma_{1T}, \\ \kappa^{(n_0)}(x, 0) = \kappa_0^{(n_0)}(x), \quad \kappa_t^{(n_0)}(x, 0) = \kappa_1^{(n_0)}(x), & \text{in } \overline{\Omega}. \end{cases} \quad (4.35)$$

Now, we have two scenarios: $k \neq (0, 0)^\top$ or $k = (0, 0)^\top$. Let us explore each case.

(a) For $k \neq (0, 0)^\top$:

From Eq (4.33), it immediately yields that

$$\lim_{n \rightarrow +\infty} I_n^K + J_n^K = 0. \quad (4.36)$$

Thanks to Eq (1.2) and the monotonicity over $\mathbb{R}$ of the functions $s \mapsto \Theta(s)s$, $s \mapsto \Upsilon(s)s$, this implies that:

$$\lim_{n \rightarrow +\infty} k_t^{(n)} = k_t = (0, 0)^t, \quad \text{almost everywhere in } \omega_T \times \Gamma_{1T}. \quad (4.37)$$

Passing to the limit of Eq (4.35) and considering Eq (4.37) with the fact that $\alpha = 0$ over $\Omega \setminus \omega$, the following static system is obtained:

$$\begin{cases} \kappa_{tt} - \Delta \kappa = f(\kappa), & \text{in } \Omega_T, \\ \kappa = 0, & \text{on } \Gamma_{0T}, \\ \partial_\nu \kappa - \Delta_T \kappa = g(\kappa), & \text{on } \Gamma_{1T}, \\ \kappa = 0, & \text{on } \omega_{0T} \cup \Gamma_{1T}, \\ \kappa(x, 0) = \kappa_0(x), \quad \kappa_t(x, 0) = \kappa_1(x), & \text{in } \overline{\Omega}. \end{cases} \quad (4.38)$$

Differentiating in the distributional sense the system (4.38), it follows that

$$\begin{cases} y_{tt} - \Delta y - f'(\kappa)y_t = 0, & \text{in } \Omega_T, \\ \kappa = 0, & \text{on } \Gamma_{0T}, \\ \partial_\nu y - \Delta_T y - g'(\kappa)y_t = 0, & \text{on } \Gamma_{1T}, \\ y = 0, & \text{on } \omega_{0T} \cup \Gamma_{1T}, \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), & \text{in } \overline{\Omega}, \end{cases}$$

where $y = \kappa_t$ in $\overline{\Omega_T}$.

Note that $y = \kappa_t \equiv 0$ on Γ_{1T} , thus implying that $\Delta_T y = 0$ on Γ_{1T} .

Using the given unique continuation principle, it follows that $y = u_t = 0$ in $\overline{\Omega_T}$ when $(f'(\kappa), g'(\kappa_{\Gamma_1})) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_{1T})$ (see Appendix). Therefore, the system (4.38) becomes the following static system:

$$\begin{cases} -\Delta \kappa = f(\kappa), & \text{in } \Omega_T, \\ \kappa = 0, & \text{on } \Gamma_{0T}, \\ \partial_\nu \kappa - \Delta_T \kappa = g(\kappa), & \text{on } \Gamma_{1T}, \\ \kappa = 0, & \text{on } \omega_{0T} \cup \Gamma_{1T}, \\ \kappa(x, 0) = \kappa_0(x), \quad \kappa_t(x, 0) = \kappa_1(x), & \text{in } \overline{\Omega}. \end{cases} \quad (4.39)$$

Using the multiplier κ on the first equation of system (4.38), we get

$$\int_{\Omega_T} |\nabla \kappa|^2 dx + \int_{\Gamma_{1T}} |\nabla_T \kappa_{\Gamma_1}|^2 d\Gamma_{1T} - \int_{\Omega_T} f(\kappa) \kappa d\Omega_T - \int_{\Gamma_{1T}} g(\kappa_{\Gamma_1}) \kappa_{\Gamma_1} d\Gamma_{1T} = 0.$$

Because the maps $s \mapsto f(s)s$ and $s \mapsto g(s)s$ are nonpositive, it then follows that $k = (\kappa, \kappa_{\Gamma_1})^\top \equiv O_{2 \times 1}$, which contradicts the fact that $k \neq (0, 0)^\top$.

(b) For $k = (0, 0)^\top$:

We consider the sequence $(v^{(n_0)})_{n_0 \in \mathbb{N}^*}$ of normalized solutions in $L^2((0, T); \mathcal{V}_0)$ such that for all

$n_0 \in \mathbb{N}^*$,

$$v^{(n_0)} = (u^{(n_0)}, u_{\Gamma_1}^{(n_0)})^t = \left(\frac{\kappa^{(n_0)}}{\eta_{n_0}}, \frac{\kappa_{\Gamma_1}^{(n_0)}}{\eta_{n_0}} \right)^t,$$

where

$$\eta_{n_0} = \left(\int_0^T \|k^{(n_0)}(t)\|_{\mathcal{V}_1}^2 dt \right)^{\frac{1}{2}}, \quad \eta_{n_0} \rightarrow \eta = 0, \quad \text{as } n_0 \rightarrow +\infty.$$

Moreover, we have

$$\int_0^T \|v^{(n_0)}(t)\|_{\mathcal{V}_1}^2 dt = 1. \quad (4.40)$$

Dividing each equation of system (4.35) by η_{n_0} , we obtain that $V^{(n_0)} = (v^{(n_0)}, v_t^{(n_0)})^t$ is the solution of the following system:

$$\begin{cases} u_{tt}^{(n_0)} - \Delta u^{(n_0)} + \alpha(x) \frac{1}{\eta_{n_0}} \Theta(\eta_{n_0} u_t^{(n_0)}) = \frac{1}{\eta_{n_0}} f(\eta_{n_0} u^{(n_0)}), & \text{in } \Omega_T, \\ u^{(n_0)} = 0, & \text{on } \Gamma_{0T}, \\ u_{tt}^{(n_0)} + \partial_\nu u^{(n_0)} - \Delta_T u^{(n_0)} + \beta(x) \frac{1}{\eta_{n_0}} \Upsilon(\eta_{n_0} u_t^{(n_0)}) = \frac{1}{\eta_{n_0}} g(\eta_{n_0} u^{(n_0)}), & \text{on } \Gamma_{1T}, \\ u^{(n_0)}(x, 0) = u_0^{(n_0)}(x), \quad u_t^{(n_0)}(x, 0) = u_1^{(n_0)}(x), & \text{in } \overline{\Omega}, \end{cases} \quad (4.41)$$

and the corresponding energy is given by

$$E_v^{(n_0)}(t) = \frac{1}{2} \|V^{(n_0)}(t)\|_{\mathcal{H}}^2 - \frac{1}{\eta_{n_0}^2} \int_{\Omega} \mathbb{F}(\kappa^{(n_0)}(t)) dx - \frac{1}{\eta_{n_0}^2} \int_{\Gamma_1} \mathbb{G}(\kappa_{\Gamma_1}^{(n_0)}(t)) d\Gamma_1, \quad t \in (0, T).$$

Moreover, thanks to the linearity of the norm $\|\cdot\|_{\mathcal{H}}$, we can observe that for all $t \in (0, T)$ and $n_0 \in \mathbb{N}^*$

$$E_v^{(n_0)}(t) = \frac{1}{\eta_{n_0}^2} E_\kappa^{(n_0)}(t), \quad (4.42)$$

where $E_\kappa^{(n_0)}$ represents the energy of system (4.35).

In addition, dividing Eq (4.29) by $\eta_{n_0}^2$, one obtains

$$E_v^{(n_0)}(T) \leq C_\kappa \left[\frac{I_{n_0}^\kappa + J_{n_0}^\kappa}{\eta_{n_0}^2} + 1 \right] = C_\kappa \left[\frac{I_{n_0}^\kappa + J_{n_0}^\kappa}{\int_0^T \|k(t)\|_{\mathcal{V}_1}^2 dt} + 1 \right].$$

From Eq (4.33), it follows that $E_v^{(n_0)}$ is uniformly bounded in $L^\infty([0, T]; \mathcal{V}_0) \cap W^{1,\infty}([0, T]; \mathcal{V}_1)$. Then, there exists a subsequence of $(V^{(n_0)})_{n_0 \in \mathbb{N}^*}$, reindexed by n_0 such that:

$$\begin{aligned} v^{(n_0)} &\rightharpoonup v \text{ weak-* in } L^\infty([0, T]; \mathcal{V}_0), \quad \text{as } n_0 \rightarrow +\infty, \\ v_t^{(n_0)} &\rightharpoonup v_t \text{ weak-* in } L^\infty([0, T]; \mathcal{V}_1), \quad \text{as } n_0 \rightarrow +\infty, \\ v^{(n_0)} &\longrightarrow v \text{ strongly in } L^\infty([0, T]; L^p(\Omega) \times L^q(\Gamma_1)), \quad \text{as } n_0 \rightarrow +\infty, \end{aligned}$$

where p, q satisfy Eq (1.7), and the last strong convergence follows from the Aubin–Lions–Simon theorem. Moreover,

$$v^{(n_0)} \longrightarrow v \text{ strongly in } L^\infty([0, T]; H^s(\Omega) \times H^{s-\frac{1}{2}}(\Gamma_1)) \quad \text{as } n_0 \rightarrow +\infty$$

for any $s \in (1/2, 1)$. This implies that for $f(0) = c_1 = 0$ and $g(0) = c_2 = 0$,

$$\begin{aligned} f(u^{(n_0)}) &\rightarrow f(u) \text{ strongly in } L^2([0, T]; L^2(\Omega)), \\ g(u_{\Gamma_1}^{(n_0)}) &\rightarrow g(u_{\Gamma_1}) \text{ strongly in } L^2([0, T]; L^2(\Gamma_1)). \end{aligned}$$

In addition, one can easily deduce from Eq (4.33) that

$$u_t = 0 \text{ in } \Omega_T \cup \Gamma_{1T}. \quad (4.43)$$

In order to contradict Eq (4.40), we need the system (4.41) to converge to a system which satisfies the unique continuation's system in Lemma 4.10. For this aim, we need

$$\alpha \frac{1}{\eta_{n_0}} \Theta(\eta_{n_0} u_t^{(n_0)}) \rightarrow 0, \quad \beta \frac{1}{\eta_{n_0}} \Upsilon(\eta_{n_0} u_{\Gamma_1}^{(n_0)}) \rightarrow 0, \quad \text{when } n_0 \rightarrow +\infty.$$

As a result, we simply need

$$\alpha \frac{I_{n_0}^\kappa}{\eta_{n_0}} \rightarrow 0, \quad \beta \frac{J_{n_0}^\kappa}{\eta_{n_0}} \rightarrow 0, \quad \text{when } n_0 \rightarrow 0.$$

By a direct computation derived from the definition of the functions Θ and Υ given in Eq (1.5), we obtain, for (m, μ) satisfying Eq (1.7),

$$\begin{aligned} \min \{ \eta_{n_0}^{\bar{m}-1}; \eta_{n_0}^{m-1} \} I_{n_0}^\kappa &\leq \frac{1}{\eta_{n_0}} I_{n_0}^\kappa \leq \max \{ \eta_{n_0}^{\bar{m}-1}; \eta_{n_0}^{m-1} \} I_{n_0}^\kappa, \\ \min \{ \eta_{n_0}^{\bar{\mu}-1}; \eta_{n_0}^{\mu-1} \} J_{n_0}^\kappa &\leq \frac{1}{\eta_{n_0}} J_{n_0}^\kappa \leq \max \{ \eta_{n_0}^{\bar{\mu}-1}; \eta_{n_0}^{\mu-1} \} J_{n_0}^\kappa, \end{aligned}$$

where

$$I_{n_0}^\kappa = \int_{\Omega_T} \alpha(x) \Theta(\eta_{n_0} u_t^{(n_0)}) u_t^{(n_0)} d\Omega_T, \quad J_{n_0}^\kappa = \int_{\Gamma_{1T}} \beta(x) \Upsilon(\eta_{n_0} u_{\Gamma_1}^{(n_0)}) u_{\Gamma_1}^{(n_0)} d\Gamma_{1T}.$$

Because $E_{n_0}^v$ is uniformly bounded in n_0 and the local regularity theorem in Section 3, given in Theorem (2.2) to ensure that any local solution $W = (w, w_t)^\top \in C([0, T]; \mathcal{H})$ is bounded such that $w_t \in L_\alpha^m(\Omega) \times L_\beta^\mu(\Gamma_1)$, $I_{n_0}^\kappa$ and $J_{n_0}^\kappa$ are uniformly bounded in n_0 . Moreover, $(m, \mu) > (1, 1)$, and as a result, the zero-convergence of η_{n_0} when $n_0 \rightarrow +\infty$ implies that

$$\frac{1}{\eta_{n_0}} I_{n_0}^\kappa, \frac{1}{\eta_{n_0}} J_{n_0}^\kappa \rightarrow 0 \text{ when } n_0 \rightarrow +\infty.$$

Now, it suffices to show that if $(\bar{p}, \bar{q}) > (2, 2)$ or if $(p, q) = (2, 2)$, we have

$$\begin{aligned} \frac{1}{\eta_{n_0}} f(\eta_{n_0} u^{(n_0)}) &\rightarrow 0 \quad \text{in } L^2((0, T); L^2(\Omega)), \\ \frac{1}{\eta_{n_0}} g(\eta_{n_0} u_{\Gamma_1}^{(n_0)}) &\rightarrow 0 \quad \text{in } L^2((0, T); L^2(\Gamma_1)) \end{aligned} \quad (4.44)$$

as $n_0 \rightarrow +\infty$, ensuring that the terms $(1/\eta_{n_0})f(\eta_{n_0}u^{(n_0)})$ and $(1/\eta_{n_0})g(\eta_{n_0}u_{\Gamma_1}^{(n_0)})$ converge weakly to an element of $L^\infty(\Omega_T)$ and $L^\infty(\Gamma_{1T})$, respectively.

Using properties (i) and (ii) from Assumption 1, we can write for $c_1 = c_2 = 0$

$$|f(\eta_{n_0}u^{(n_0)})| \leq C_f \left(|\eta_{n_0}u^{(n_0)}|^{\bar{p}-1} + |\eta_{n_0}u^{(n_0)}|^{p-1} \right)$$

with $C_f > 0$.

This leads to

$$|f(\eta_{n_0}u^{(n_0)})|^2 \leq c_R \left(|\eta_{n_0}u^{(n_0)}|^{2(\bar{p}-1)} + |\eta_{n_0}u^{(n_0)}|^{2(p-1)} \right)$$

for some $c_R > 0$.

Thus

$$\begin{aligned} \frac{1}{\eta_{n_0}^2} \int_{\Omega_T} |f(\eta_{n_0}u^{(n_0)})|^2 d\Omega_T &\leq c_R \frac{1}{\eta_{n_0}^2} \left(\int_{\Omega_T} |\eta_{n_0}u^{(n_0)}|^{2(\bar{p}-1)} d\Omega_T + \int_{\Omega_T} |\eta_{n_0}u^{(n_0)}|^{2(p-1)} d\Omega_T \right) \\ &\leq c_R \left(\eta_{n_0}^{2(\bar{p}-2)} \int_{\Omega_T} |u^{(n_0)}|^{2(\bar{p}-1)} d\Omega_T + \eta_{n_0}^{2(p-2)} \int_{\Omega_T} |u^{(n_0)}|^{2(p-1)} d\Omega_T \right) \\ &\leq C_{n_0,R} \int_0^T \left(\|u^{(n_0)}(t)\|_{L^{2(\bar{p}-1)}(\Omega)}^{2(\bar{p}-1)} + \|u^{(n_0)}(t)\|_{L^{2(p-1)}(\Omega)}^{2(p-1)} \right) dt, \end{aligned}$$

where $C_{n_0,R} = c_R \max \left\{ \eta_{n_0}^{2(\bar{p}-2)}; \eta_{n_0}^{2(p-2)} \right\}$, which has the limit $\bar{C}_R \geq 0$ satisfying

$$\bar{C}_R = \begin{cases} 0, & \text{if } \bar{p} > 2, \\ c_R, & \text{if } p = 2 \end{cases} \quad (4.45)$$

as $n_0 \rightarrow +\infty$.

This implies that

$$\frac{1}{\eta_{n_0}^2} \int_{\Omega_T} |f(\eta_{n_0}u^{(n_0)})|^2 d\Omega_T \rightarrow \bar{C}_R \int_0^T \left(\|u(t)\|_{L^{2(\bar{p}-1)}(\Omega)}^{2(\bar{p}-1)} + \|u(t)\|_{L^{2(p-1)}(\Omega)}^{2(p-1)} \right) dt,$$

as $n_0 \rightarrow +\infty$.

Similarly, as $n_0 \rightarrow +\infty$

$$\frac{1}{\eta_{n_0}^2} \int_{\Omega_T} |g(\eta_{n_0}u^{(n_0)})|^2 d\Omega_T \leq \bar{C}_{g,R} \int_0^T \left(\|u^{(n_0)}(t)\|_{L^{2(\bar{q}-1)}(\Omega)}^{2(\bar{q}-1)} + \|u^{(n_0)}(t)\|_{L^{2(q-1)}(\Omega)}^{2(q-1)} \right) dt,$$

where:

$$\bar{C}_{g,R} \rightarrow \begin{cases} 0, & \text{if } \bar{q} > 2 \\ c_R, & \text{if } q = 2. \end{cases} \quad (4.46)$$

Consequently, Eq (4.44) holds in the case where $(p, q) \geq (\bar{p}, \bar{q}) > (2, 2)$.

Now, passing to the limit in system (4.41) with n_0 and differentiating each of its equations with respect to t (distributional sense), we obtain

$$\begin{cases} y_{tt} - \Delta y = 0, & \text{in } \Omega_T, \\ y_{tt} - \Delta_T y + \partial_\nu y = 0, & \text{on } \Gamma_{1T}, \\ y = 0, & \text{on } \omega_T \cup \Gamma_{0T}, \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), & \text{in } \overline{\Omega}, \end{cases} \quad (4.47)$$

where $y = u_t$.

Because u_t satisfies Eq (4.43), $y_{\Gamma_{1T}} = 0$.

In the case where $(p, q) = (2, 2)$, which also implies $(\bar{p}, \bar{q}) = (2, 2)$, it is evident that f and g are linear. Using their expressions given in Eq (1.5), we obtain

$$f(s) = \bar{c}_f s, \quad g(s) = \bar{c}_g s, \quad s \in \mathbb{R},$$

where $\bar{c}_f = \bar{\gamma} + \gamma$, and $\bar{c}_g = \bar{\delta} + \delta$.

Therefore,

$$\frac{1}{\eta_{n_0}} f(\eta_{n_0} u^{(n_0)}) = \bar{c}_f u^{(n_0)}, \quad \frac{1}{\eta_{n_0}} g(\eta_{n_0} u_{\Gamma_1}^{(n_0)}) = \bar{c}_g u_{\Gamma_1}^{(n_0)}.$$

As previously highlighted, by taking the limit of system (4.41) and then setting $y = u_t$, we obtain for $(p, q) = (2, 2)$

$$\begin{cases} y_{tt} - \Delta y - \bar{c}_f y_t = 0, & \text{in } \Omega_T, \\ y_{tt} - \Delta_T y + \partial_\nu y - \bar{c}_g y_t = 0, & \text{on } \Gamma_{1T}, \\ y = 0, & \text{on } \omega_T \cup \Gamma_{0T}, \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), & \text{in } \overline{\Omega}. \end{cases} \quad (4.48)$$

Applying Lemma 4.10 in both cases simultaneously, we immediately obtain that $y = u_t \equiv 0$ for $(2, 2) \leq (\bar{p}, \bar{q}) \leq (p, q)$ and $c_1 = c_2 = 0$. As in the previous case (a), this implies that $v = (u, u_{\Gamma_1})^t \equiv O_{2 \times 1}$ for p, q satisfying Eq (1.7), which also contradicts Eq (4.40) with the strong convergence result of $v^{(n_0)}$. The proof is complete. $\square$

5. Numerical simulation of damped wave equations with localized damping

This section presents a numerical study of the one- and two-dimensional damped wave equations with localized internal damping. Using finite difference schemes, we demonstrate how the introduction of an internal damping region induces strong stabilization of the wave energy. The results show that the total energy decays exponentially over time, confirming the theoretical predictions of uniform stabilization.

Wave equations with localized internal damping have received significant attention due to their rich mathematical structure and broad physical applications in mechanical vibration, acoustics, and control theory. These systems exhibit dissipative behavior in which the damping absorbs energy from oscillatory components, leading to stabilization. The numerical results presented in this section validate theoretical predictions concerning exponential decay of energy.

For the following, $B_s(a)$ denotes the ball of center $a \in \mathbb{R}^N$ and radius $s > 0$.

5.1. One-dimensional numerical analysis

To begin, note that the direct one-dimensional system of Eq (4.2) is not well-defined due to presence of the higher-order operator Δ_T over the zero-dimensional boundary Γ_1 . As a result, we primarily need to consider the two-dimensional system associated to Eq (1.1) and derive the one-dimensional model thanks to a partial Fourier series [33]. To do this, consider $\Omega = B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}$ (unit ring of half-internal radius hole) and the internal damping region $\omega = B_1(0) \setminus \overline{B_{\frac{3}{4}}(0)}$. We then reformulate the two-dimensional system by using the polar coordinates; that is, for all $X = (x, y) \in \Omega$,

$$\begin{cases} x = r \cos(\theta), \\ y = r \sin(\theta), \end{cases}$$

where $1/2 < r < 1$ represents the radius and $\theta \in (0, 2\pi)$, the angle between x -axis and $\vec{OX}$.

Note that $\Gamma_1 = \partial B_1(0)$, (external boundary) and $\Gamma_0 = \partial B_{\frac{1}{2}}(0)$ (internal boundary); in addition, $|\nu(X)| = |X| = 1$ for $X \in \Gamma_1$.

In polar coordinates, we have

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}, \quad \partial_{\nu/\Gamma_1} = \partial_r, \quad \Delta_{T/\Gamma_1} = \partial_{\theta\theta}.$$

Now, consider the following two-dimensional system associated with system (4.2):

$$\begin{cases} \kappa_{tt} - \kappa_{rr} - \frac{1}{r}\kappa_r + \frac{1}{r^2}\kappa_{\theta\theta} + \alpha(r, \theta)|\kappa_t|^{m-2}\kappa_t = \gamma|\kappa|^{p-2}\kappa, & \text{in } \Omega_T, \\ \kappa_{tt} + \kappa_r - \kappa_{\theta\theta} + \beta(\theta)|\kappa_t|^{\mu-2}\kappa_t = \delta|\kappa|^{q-2}\kappa, & \text{on } \Gamma_{1T}, \\ \kappa = 0, & \text{on } \Gamma_{0T}, \\ \kappa((r, \theta), 0) = \psi_0(r, \theta), \quad \kappa_t((r, \theta), 0) = \psi_1(r, \theta), & \text{in } \overline{\Omega_T}, \end{cases} \quad (5.1)$$

where $\kappa = \kappa((r, \theta), t)$, $(r, t) \times \in [1/2, 1] \times [0, T]$, $m, \mu \in]1, \infty[$ and $p, q \in]2, +\infty[$ (subcritical regime of $N = 2$).

Next, consider the following Fourier series:

$$\kappa((r, \theta), t) = \sum_{k \geq 0} \kappa^{(k)}(r, t) \cos(kl\theta), \quad l \in \mathbb{R}, \quad (5.2)$$

where $\kappa^{(k)} = \kappa^{(k)}(r, t)$ are the coefficients of the Fourier series.

Using the Eq (5.2) in system (5.1) and then choosing $\theta = 0$, one obtains the following one-dimensional system:

$$\begin{cases} \kappa_{tt}^{(k)} - \kappa_{rr}^{(k)} - \frac{1}{r}\kappa_r^{(k)} + \frac{(kl)^2}{r^2}\kappa^{(k)} + \alpha(r, 0)|\kappa_t^{(k)}|^{m-2}\kappa_t^{(k)} = \gamma|\kappa^{(k)}|^{p-2}\kappa^{(k)}, & \text{on } \Omega_r, t \in (0, T), \\ (\kappa_{tt}^{(k)} + \kappa_r^{(k)} + (kl)^2\kappa^{(k)} + \beta(0)|\kappa_t^{(k)}|^{\mu-2}\kappa_t^{(k)})(1, t) = \delta|\kappa^{(k)}(1, t)|^{q-2}\kappa^{(k)}(1, t), & \forall t \geq 0, \\ \kappa^{(k)}(\frac{1}{2}, t) = 0, & \forall t \geq 0, \\ (\kappa^{(k)}, \kappa_t^{(k)})(r, 0) = (\psi_0, \psi_1)(r, 0), & \forall r \in \overline{\Omega_r}, \end{cases} \quad (5.3)$$

where $\kappa^{(k)} = \kappa^{(k)}(r, t)$, $\Omega_r = [1/2, 1]$, $k \in \mathbb{N}^*$.

For more simplicity, we choose $\kappa = \kappa^{(k)}$, and l represents kl .

Now, let us use a numerical method commonly known as the finite difference method.

For this, we consider $n_r + 2$ (resp. $n_t + 2$) equidistant to $r_i \in \overline{\Omega_r}$ (resp. $t_j \in [0, T]$, $T > 0$) such that

$$\frac{1}{2} = r_0 < r_1 < \dots < r_{n_r+1} = 1, \quad 0 = t_0 < t_1 < \dots < t_{n_t+1} = T,$$

and for all $(i, j) \in \{0, \dots, n_r + 1\} \times \{0, \dots, n_t + 1\}$, we have

$$r_i = idr, \quad t_j = jdt,$$

where $dr = 1/(n_r + 1)$, and $dt = T/(n_t + 1)$.

Thus, a numerical approximation of system (5.3) is for all $j \in \{0, \dots, n_t + 1\}$:

$$\begin{cases} \frac{v_i^{j+1} - v_i^j}{dt} - \frac{w_{i+1}^j - w_i^j}{dr} - \frac{1}{r_i} w_i^j + \frac{l^2}{r_i^2} \kappa_i^j + \alpha_i |v_i^j|^{m-2} \left(\frac{v_i^{j+1} + v_i^j}{2} \right) = \gamma |\kappa_i^j|^{p-2} \kappa_i^j, & \forall i \in \{1, \dots, n_r\}, \\ \frac{v_{n_r+1}^{j+1} - v_{n_r+1}^j}{dt} + w_{n_r+1}^j + l^2 \kappa_{n_r+1}^j + \beta(0) |v_{n_r+1}^j|^{\mu-2} \left(\frac{v_{n_r+1}^{j+1} + v_{n_r+1}^j}{2} \right) = \delta |\kappa_{n_r+1}^j|^{q-2} \kappa_{n_r+1}^j, \\ \kappa_0^j = 0, \\ \kappa_i^0 = \psi_0(r_i), \quad \kappa_i^1 = \psi_i^0 + dt \psi_1(r_i), & \forall i \in \{0, \dots, n_r + 1\}, \end{cases} \quad (5.4)$$

where $\alpha_i = \alpha(r_i, 0)$, and

$$\kappa_i^j = \kappa(r_i, t_j), \quad v_i^j = \frac{\kappa_i^j - \kappa_i^{j-1}}{dt}, \quad w_i^j = \frac{\kappa_i^j - \kappa_{i-1}^j}{dr}.$$

Remark 5.1. First, note that v_i^j (resp. w_i^j) models a first time-derivative (resp. spatial-derivative) of κ . Moreover

$$\frac{v_i^{j+1} + v_i^j}{2} = \frac{\kappa_i^{j+1} - \kappa_i^{j-1}}{2dt}$$

is the central scheme for κ_t whereas

$$\frac{v_i^{j+1} - v_i^j}{dt}$$

models the second derivative in time of κ .

In order to have the energy expression associated to the numerical scheme system (5.4), one aims to state a similar identity as in Eq (4.6) of $E(t)$ in the theoretical section. That is, for two instants t_j and t_{j+1} ,

$$\frac{E(t_{j+1}) - E(t_j)}{dt} = -(\text{positive terms evolving with dampings terms } \alpha\Theta, \beta\Upsilon).$$

Because the damping terms express in $|v_i^j| \dots (v_i^{j+1} + v_i^j)$, the term $(v_i^{j+1} + v_i^j)$ is a perfect choice of multiplier, which aligns with the dissipative property of E . However, this multiplier gives as

perturbation energy $|\kappa_i^j| \cdots \kappa_i^j(v_i^{j+1} + v_i^j)$, which cannot be automatically expressed as a consecutive difference in j . Therefore, given that the perturbations f, g are local Lipschitz maps satisfying Assumption 1, an adequate way to solve this issue is to study a modified energy which approximates the energy of perturbations f, g .

In the spirit of Remark 5.1, the modified energy ξ^j associated with system (5.4) is

$$\begin{aligned} \xi^j = & \frac{r_{n_r+1} - dr}{2} \left[|v_{n_r+1}^{j+1}|^2 + l^2 \kappa_{n_y+1}^{j+1} \kappa_{n_y+1}^j - \frac{\delta}{q} (|\kappa_{n_r+1}^{j+1}|^q + |\kappa_{n_r+1}^j|^q) \right] \\ & + \frac{dr}{2} \left[l^2 \sum_{i=0}^{n_r} \frac{1}{r_i} \kappa_i^{j+1} \kappa_i^j + \sum_{i=0}^{n_r} r_i w_{i+1}^{j+1} w_{i+1}^j + \sum_{i=0}^{n_y} r_i |v_i^{j+1}|^2 - \frac{\gamma}{p} \sum_{i=0}^{n_r} r_i (|\kappa_i^{j+1}|^p + |\kappa_i^j|^p) \right]. \end{aligned}$$

Now, let us ensure the dissipative property of ξ^j .

Proposition 5.2. Assume that $dr, dt \in (0, 1)$ such that $0 < dt \leq dr < r_{n_r+1}$. Then, the modified energy ξ^j associated with system (5.4) is stable in term of energy dissipation; that is,

$$\xi^j \leq \xi^{j-1}, \quad \forall j \in (1, n_t + 1).$$

Moreover,

$$\frac{\xi^j - \xi^{j-1}}{dt} = -(r_{n_r+1} - dr)\beta(0) |v_{n_r+1}^j|^{\mu-2} \left| \frac{v_{n_r+1}^{j+1} + v_{n_r+1}^j}{2} \right|^2 - dr \sum_{i=1}^{n_r} r_i \alpha_i |v_i^j|^{m-2} \left| \frac{v_i^{j+1} + v_i^j}{2} \right|^2. \quad (5.5)$$

Proof. Multiplying Eq (5.4)₁ by $(dr/2)r_i(v_i^{j+1} + v_i^j)$ and then summing from $i = 1$ up to $i = n_r$, we obtain

$$\sum_{k=1}^5 a_k S_k = -dr \sum_{i=1}^{n_r} r_i \alpha_i |v_i^j|^{m-2} \left| \frac{v_i^{j+1} + v_i^j}{2} \right|^2, \quad (5.6)$$

where

$$\begin{aligned} S_1 &= \sum_{i=1}^{n_r} r_i (v_i^{j+1} - v_i^j)(v_i^{j+1} + v_i^j), \quad S_2 = - \sum_{i=1}^{n_r} r_i (w_{i+1}^j - w_i^j)(v_i^{j+1} + v_i^j), \\ S_3 &= - \sum_{i=1}^{n_r} w_i^j (v_i^{j+1} + v_i^j), \quad S_4 = \sum_{i=1}^{n_r} \frac{1}{r_i} \kappa_i^j (v_i^{j+1} + v_i^j), \quad S_5 = - \sum_{i=1}^{n_r} r_i |\kappa_i^j|^{p-2} \kappa_i^j (v_i^{j+1} + v_i^j) \end{aligned}$$

and

$$a_1 = \frac{dr}{2dt}, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{dr}{2}, \quad a_4 = \frac{l^2 dr}{2}, \quad a_5 = \frac{\gamma dr}{2}.$$

Here, our goal is to estimate each S_k in consecutive difference in j .

1. For S_1 :

Using the difference of squares identity and the fact that $v_0^j = 0$, it follows that

$$S_1 = \sum_{i=0}^{n_r} r_i |v_i^{j+1}|^2 - \sum_{i=0}^{n_r} r_i |v_i^j|^2. \quad (5.7)$$

2. For S_2 :

Extending the first factor of S_2 and considering the fact that $v_0^j = 0$, we obtain

$$\begin{aligned} S_2 &= - \sum_{i=1}^{n_r} r_i (w_{i+1}^j - w_i^j) (v_i^{j+1} + v_i^j) \\ &= \sum_{i=1}^{n_r} r_i w_i^j (v_i^{j+1} + v_i^j) - \sum_{i=0}^{n_r} r_i w_{i+1}^j (v_i^{j+1} + v_i^j). \end{aligned}$$

Taking i as $i + 1$ in the first term, it follows that

$$S_2 = \sum_{i=0}^{n_r-1} r_{i+1} w_{i+1}^j (v_{i+1}^{j+1} + v_{i+1}^j) - \sum_{i=0}^{n_r} r_i w_{i+1}^j (v_i^{j+1} + v_i^j).$$

Then, adding and subtracting the amount $L = r_{n+1} w_{n+1}^j (v_{n+1}^{j+1} + v_{n+1}^j)$ to the first term of S_2 , it follows that

$$\begin{aligned} S_2 &= -L + \sum_{i=0}^{n_r} r_{i+1} w_{i+1}^j (v_{i+1}^{j+1} + v_{i+1}^j) - \sum_{i=0}^{n_r} r_i w_{i+1}^j (v_i^{j+1} + v_i^j) \\ &= -L + \sum_{i=0}^{n_r} w_{i+1}^j [r_{i+1} (v_{i+1}^{j+1} + v_{i+1}^j) - r_i (v_i^{j+1} + v_i^j)]. \end{aligned}$$

Because $r_{i+1} = dr + r_i$, it follows that

$$S_2 = -L + \sum_{i=0}^{n_r} r_i w_{i+1}^j [(v_{i+1}^{j+1} - v_i^{j+1}) + (v_{i+1}^j - v_i^j)] + dr S_0,$$

where $S_0 = \sum_{i=0}^{n_r} w_{i+1}^j (v_{i+1}^{j+1} + v_{i+1}^j)$.

Using the expression of v_i^j , we have

$$v_{i+1}^{j+1} - v_i^{j+1} = \frac{dr}{dt} (w_{i+1}^{j+1} - w_{i+1}^j), \quad v_{i+1}^j - v_i^j = \frac{dr}{dt} (w_{i+1}^j - w_{i+1}^{j-1}),$$

which implies that

$$\begin{aligned} S_2 &= -L + dr S_0 + \frac{dr}{dt} \sum_{i=0}^{n_r} r_i w_{i+1}^j (w_{i+1}^{j+1} - w_{i+1}^{j-1}) \\ &= -L + dr S_0 + \frac{dr}{dt} \left(\sum_{i=0}^{n_r} r_i w_{i+1}^{j+1} w_{i+1}^j - \sum_{i=0}^{n_r} r_i w_{i+1}^j w_{i+1}^{j-1} \right). \end{aligned} \tag{5.8}$$

Let us keep this for the moment as S_2 .

3. For S_3 :

Taking i as $i + 1$, it follows that

$$\begin{aligned} S_3 &= - \sum_{i=0}^{n_r-1} w_{i+1}^j (v_{i+1}^{j+1} + v_{i+1}^j) \\ &= \frac{L}{r_{n_r+1}} - \sum_{i=0}^{n_r} w_{i+1}^j (v_{i+1}^{j+1} + v_{i+1}^j) \\ &= \frac{L}{r_{n_r+1}} - S_0 \end{aligned}$$

for $r_{n_r+1} \neq 0$.

Consequently,

$$a_2 S_2 + a_3 S_3 = -\frac{r_{n_r+1} - dr}{2r_{n_r+1}} L + \frac{dr}{2dt} \left(\sum_{i=0}^{n_r} r_i w_{i+1}^{j+1} w_{i+1}^j - \sum_{i=0}^{n_r} r_i w_{i+1}^j w_{i+1}^{j-1} \right), \quad (5.9)$$

for $dr < r_{n+1}$, which is true, anyway.

4. For S_4 :

Using the fact that $r_0 \neq 0$, $v_0^j = 0$, and

$$v_i^{j+1} + v_i^j = \frac{\kappa_i^{j+1} - \kappa_i^j}{dt}$$

it follows that

$$S_4 = \frac{1}{dt} \left[\sum_{i=0}^{n_r} \frac{1}{r_i} \kappa_i^{j+1} \kappa_i^j - \sum_{i=0}^{n_r} \frac{1}{r_i} \kappa_i^j \kappa_i^{j-1} \right]. \quad (5.10)$$

5. For S_5 :

As previously highlighted in Remark 5.1, we use the Lipschitz property of the map $s \mapsto \mathbb{F}_0(s) = (1/p)|s|^p$. That is,

$$|\mathbb{F}_0(s_0) - \mathbb{F}_0(s_1)| \leq |\mathbb{F}'_0(s)| |s_0 - s_1|, \quad s_0 < s < s_1 \in \mathbb{R}.$$

Assuming that the equality holds for $\kappa_i^{j-1} < \kappa_i^j < \kappa_i^{j+1}$ or conversely (may still be invalid for some value of κ_i^j), we have

$$\mathbb{F}_0(\kappa_i^{j+1}) - \mathbb{F}_0(\kappa_i^{j-1}) = \mathbb{F}'_0(\kappa_i^j) (\kappa_i^{j+1} - \kappa_i^{j-1}) = dt \mathbb{F}'_0(\kappa_i^j) (v_i^{j+1} + v_i^j).$$

Because $\kappa_0^j = 0$ and $s \mapsto \mathbb{F}'_0(s) = |s|^{p-2}s$, we have

$$\begin{aligned} S_5 &= - \sum_{i=0}^{n_r} r_i \mathbb{F}'_0(\kappa_i^j) (v_i^{j+1} + v_i^j) \\ &= -\frac{1}{dt} \left[\sum_{i=0}^{n_r} r_i \mathbb{F}_0(\kappa_i^{j+1}) - \sum_{i=0}^{n_r} r_i \mathbb{F}_0(\kappa_i^{j-1}) \right] \\ &= -\frac{1}{dt} \left[\sum_{i=0}^{n_r} r_i \mathbb{F}_1(\kappa_i^{j+1}) - \sum_{i=0}^{n_r} r_i \mathbb{F}_1(\kappa_i^j) \right], \end{aligned} \quad (5.11)$$

where

$$\mathbb{F}_1(\kappa_i^{j+1}) = \mathbb{F}_0(\kappa_i^{j+1}) + \mathbb{F}_0(\kappa_i^j).$$

Now, substituting S_k of Eqs (1.1) and (5.9)–(5.11) in Eq (5.6), it follows that

$$\begin{aligned} & \frac{1}{dt} \left[\frac{dr}{2} \sum_{i=0}^{n_r} r_i |v_i^{j+1}|^2 + \frac{dr}{2} \sum_{i=0}^{n_r} r_i w_{i+1}^{j+1} w_{i+1}^j + \frac{l^2 dr}{2} \sum_{i=0}^{n_r} \frac{1}{r_i} \kappa_i^{j+1} \kappa_i^j - \frac{\gamma dr}{2} \sum_{i=0}^{n_r} r_i \mathbb{F}_1(\kappa_i^{j+1}) \right] \\ & - \frac{1}{dt} \left[\frac{dr}{2} \sum_{i=0}^{n_r} r_i |v_i^j|^2 + \frac{dr}{2} \sum_{i=0}^{n_r} r_i w_{i+1}^j w_{i+1}^{j-1} + \frac{l^2 dr}{2} \sum_{i=0}^{n_r} \frac{1}{r_i} \kappa_i^j \kappa_i^{j-1} - \frac{\gamma dr}{2} \sum_{i=0}^{n_r} r_i \mathbb{F}_1(\kappa_i^j) \right] \\ & - \frac{r_{n_r+1} - dr}{2r_{n_r+1}} L = -dr \sum_{i=1}^{n_r} r_i \alpha_i |v_i^j|^{m-2} \left| \frac{v_i^{j+1} + v_i^j}{2} \right|^2, \end{aligned} \quad (5.12)$$

where $0 < dr < r_{n_r+1}$.

Now, let us write L as a consecutive difference in j .

From the second equation of system (5.4), we have at the boundary $i = n_r + 1$

$$w_{n_r+1}^j = -\frac{1}{dt} (v_{n_r+1}^{j+1} - v_{n_r+1}^j) - l^2 \kappa_{n_r+1}^j - \beta(0) |v_{n_r+1}^j|^{\mu-2} \left(\frac{v_{n_r+1}^{j+1} + v_{n_r+1}^j}{2} \right) + \delta |\kappa_{n_r+1}^j|^{q-2} \kappa_{n_r+1}^j.$$

Then, one can easily check that L expressed in the following consecutive difference in j :

$$\begin{aligned} L &= -L^{j+1} + L^j \\ & - \frac{\beta(0)r_{n_r+1}}{2} |v_{n_r+1}^j|^{\mu-2} \left| v_{n_r+1}^{j+1} + v_{n_r+1}^j \right|^2 + \delta r_{n_r+1} |\kappa_{n_r+1}^j|^{q-2} \kappa_{n_r+1}^j (v_{n_r+1}^{j+1} + v_{n_r+1}^j), \end{aligned}$$

where

$$L^j = \frac{r_{n_r+1}}{dt} \left(|v_{n_r+1}^j|^2 + l^2 \kappa_{n_r+1}^j \kappa_{n_r+1}^{j-1} \right).$$

Using the same process as for S_5 with local Lipschitz map $s \mapsto \mathbb{G}_0(s) = (1/q)|s|^q$, we get

$$\begin{aligned} L &= -L^{j+1} + L^j + \frac{\delta}{dt} (\mathbb{G}_1(\kappa_{n_r+1}^{j+1}) - \mathbb{G}_1(\kappa_{n_r+1}^j)) \\ & - \frac{\beta(0)r_{n_r+1}}{2} |v_{n_r+1}^j|^{\mu-2} \left| v_{n_r+1}^{j+1} + v_{n_r+1}^j \right|^2, \end{aligned}$$

where

$$\mathbb{G}_1(\kappa_i^j) = \mathbb{G}_0(\kappa_{n_r+1}^{j+1}) + \mathbb{G}_0(\kappa_{n_r+1}^j).$$

Substituting L in Eq (5.12), it follows that

$$\frac{\xi^j - \xi^{j-1}}{dt} = -(r_{n_r+1} - dr) \beta(0) |v_{n_r+1}^j|^{\mu-2} \left| \frac{v_{n_r+1}^{j+1} + v_{n_r+1}^j}{2} \right|^2 - dr \sum_{i=1}^{n_r} r_i \alpha_i |v_i^j|^{m-2} \left| \frac{v_i^{j+1} + v_i^j}{2} \right|^2 \leq 0, \quad 0 < dr < r_{n_r+1}.$$

Therefore, the dissipative equation (5.5) is satisfied. Moreover, we have $\xi^j - \xi^{j-1} \leq 0$, which gives $\xi^j \leq \xi^{j-1}$.

The proof is complete. $\square$

5.2. One-dimensional numerical example

In this part, we first illustrate the phenomenon thus studied in this work. Second, we illustrate how the disturbance of provoked by such a wave stabilizes over time (exponential stability) under previous the condition of Section 4. For this aim, consider $l = \pi$, and fix the mesh parameters as follows:

- number of subintervals on y-axis $n_r = 59$,
- number of subintervals over duration $T = 10$: $n_t = 9999$.

The subintervals' amplitude then follows:

$$dr = \frac{1}{120}, \quad dt = 10^{-3}.$$

The damping coefficients are assumed to be

$$\alpha(r) = \begin{cases} 3 + r, & \text{if } \frac{3}{4} < r < 1, \\ 0, & \text{otherwise.} \end{cases}, \quad \beta(0) = 1.$$

We consider the following classical cases:

1. Case where the wave's disturbance starts as a cosine-curve and with an initial speed linearly, but without any initial speed:

$$\psi_0(r) = \begin{cases} \cos(\pi r), & \text{if } \frac{1}{2} + dr < r \leq 1 \\ 0, & \text{if } r = \frac{1}{2} \end{cases}, \quad \psi_1(r) = 0. \quad (5.13)$$

2. Case where the wave's disturbance starts as an exponential curve and with an exponential speed:

$$\psi_0(r) = \begin{cases} e^r, & \text{if } \frac{1}{2} + dr < r \leq 1 \\ 0, & \text{if } r = \frac{1}{2} \end{cases}, \quad \psi_1(r) = \begin{cases} e^{1+r}, & \text{if } \frac{1}{2} < r \leq 1, \\ 0, & \text{if } r = \frac{1}{2}. \end{cases} \quad (5.14)$$

Remark 5.3. Note that the choice of $\psi_0(1/2) = 0$ aims to remain consistent with the boundary condition $\kappa_0^j = 0$. Therefore, there is no motion as this boundary (Γ_0).

Here are the different choices for $\gamma, \delta, m, \mu, p$, and q :

Table 1. Model parameters.

Coefs.	Global solution	Local solution	Maximal solution for $l = 0$
(γ, δ)	(-1, -2)	(1, 2)	(0.15, 0.15)
(m, μ)	(1.5, 1.5)	(1.5, 1.5)	(1.5, 1.5)
(p, q)	(2, 2)	(3, 3)	(3, 3)

The wave equation is discretized using central finite differences in space and time. Once the wave reaches the damped subregion, the amplitude decreases rapidly, and the energy plot confirms exponential decay.

Results and discussion (1D case)

The numerical solutions exhibit a clear attenuation of the wave amplitude once the propagating front enters the damped subregion. This behavior is illustrated in left side of Figures 1 and 2, where the solutions (global or local) show a rapid decrease in amplitude. As a result, the disturbed medium returns to rapidly into the rest state.

Moreover, their respective energy plot of the total discrete energy ξ^j , displayed in the respective right side of Figures 1 and 2, confirms that the energy decays exponentially over time, in full agreement with theoretical stabilization results.

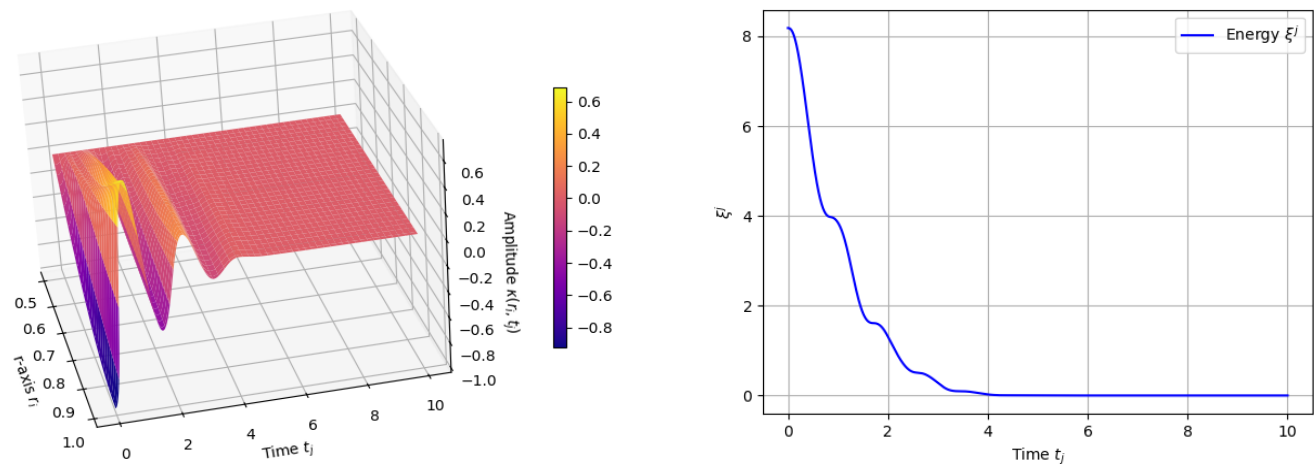


Figure 1. Global solution of systems (5.4)–(5.13) and its energy ξ^j .

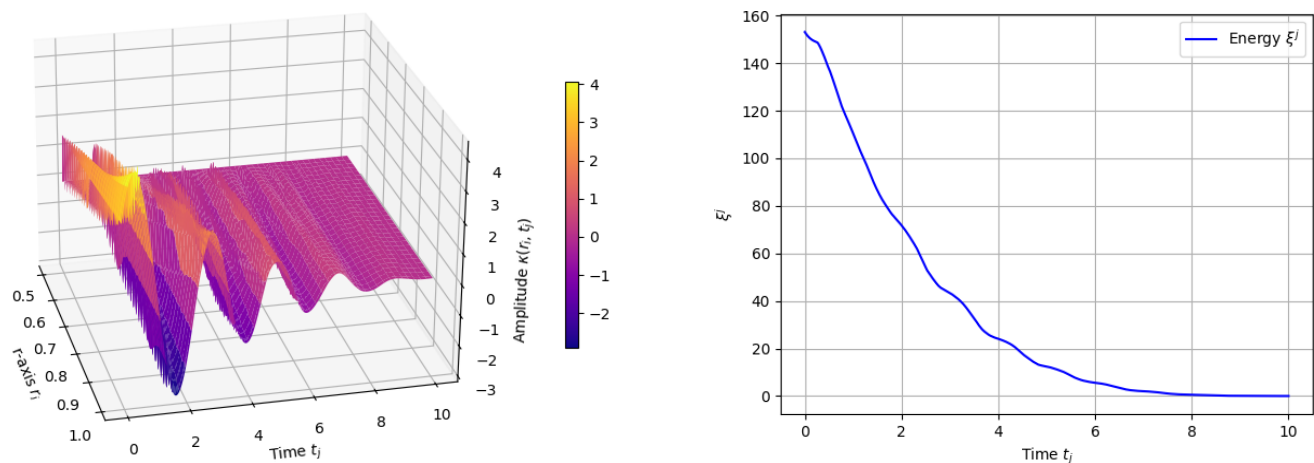


Figure 2. Local solution for systems (5.4)–(5.14) and its energy ξ^j .

Discussion

The numerical experiments in both one- and two-dimensional settings clearly demonstrate the stabilizing effect of localized internal damping. In the one-dimensional case, Figures 1 and 2 show that the wave amplitude and total energy decay rapidly once the solution enters the damped region.

Here are data which confirm these assertions:

Table 2. Energies data.

Figure	Initial energy	Time T_j where $\xi^j \leq 10^{-5}\xi^0$	Final energy
Figure 1	8.18	6.76 over 10	2.12×10^{-5}
Figure 2	153.08	none	3.84×10^{-3}

These data show that the energy in each case reach at least 0.004 only after 10 units of time. Using the percent formula of loss energy ξ_{loss} during the time interval $(0, t_j)$,

$$\xi_{loss} = 100 \times \frac{\xi^0 - \xi^j}{\xi^0},$$

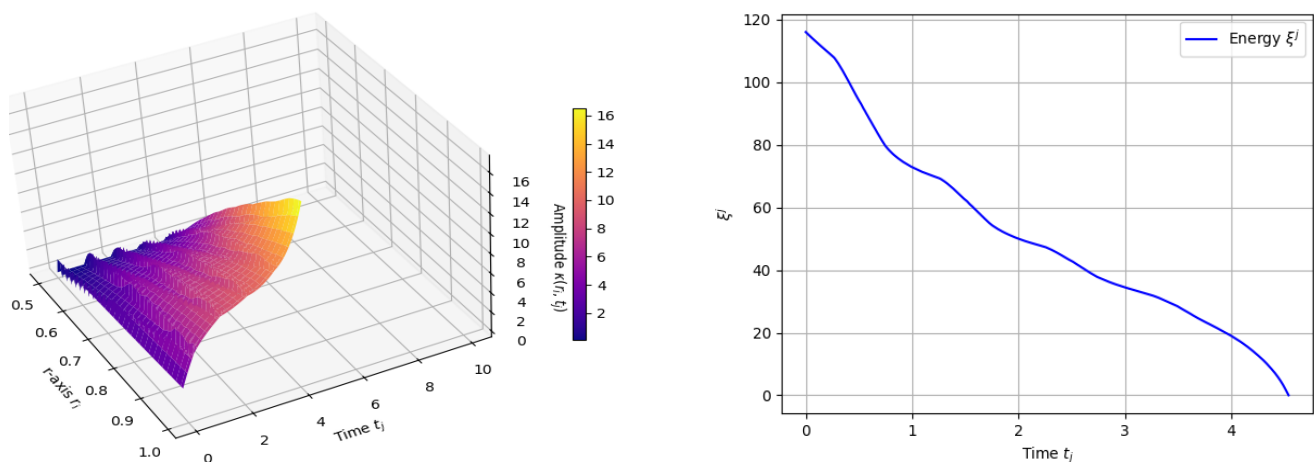
the data of Table 2 give that at least 99.95% of the full energy dissipates in both cases in 10 units of time.

An example of a maximal solution which not global in time is illustrated by Figure 3. Section 3 of the theoretical analysis states that an exponential stability result may still be held for $(1, 1) < (m, \mu) \leq (2, 2)$ with the superlinearity of $-f, -g$ even if it is local. That is,

$$f(s)s \leq C_f \mathbb{F}(s) \leq 0, \quad g(s)s \leq C_g \mathbb{G}(s) \leq 0.$$

The perturbation maps $s \mapsto f(s) = \gamma|s|^{p-2}s$ and $s \mapsto g(s) = \delta|s|^{q-2}s$ with such a choice of “maximal-solution” parameters from Table 1 are then locally superlinear only at $s = 0$, as $f(s)s \geq 0$, and $g(s)s \geq 0$. This implies that the only values κ for which the exponential stability holds is when $\kappa = 0$. This is perfectly illustrated in Figure 3, where the energy blows up. For further understanding of this superlinearity’s impact on the exponential stability, one can repeat the same process of numerical analysis and consider both γ and $\tilde{\gamma}$ such that $\gamma\tilde{\gamma} < 0$.

In two dimensions, it suffices to expand system (5.4) along with θ . A similar result with the one-dimensional case thus can emerge.

**Figure 3.** Maximal solution for systems (5.4)–(5.14) and its energy with $l = 0$.

6. Conclusions

In this work, we investigated the exponential stabilization of a perturbed wave equation with localized internal damping supported on a subregion $\omega \subset \Omega$ satisfying the geometric control condition (GCC). The well-posedness of the problem was established by combining classical results from the literature with an energy framework adapted to subcritical perturbations. Assuming that the nonlinear terms are sink maps, namely $f(s)s \leq 0$ and $g(s)s \leq 0$, we proved that the total energy decays over a finite time interval $T = T_{m,\mu}$ depending on the subcritical exponents $1 < m \leq 2$ and $1 < \mu \leq 2$, provided that the initial energy is positive and finite. By employing the multiplier method together with a unique continuation principle [16], we derived an observability inequality that guarantees exponential decay of the energy. As a consequence, the exponential stability of System (1.1) is global in time when $N < 4$ for strong solutions or $N = 2$ (weak solution) or when $p = q = 2$, and local in time otherwise. Moreover, the analysis suggests that exponential stabilization persists whenever the corresponding solution remains bounded in the $L^\infty(\bar{\Omega})$ norm. To complement the theoretical findings, numerical simulations were carried out for one- and two-dimensional damped wave equations with localized damping. Using finite difference discretizations, the simulations demonstrate that the presence of an internal damping region induces strong stabilization, as the total discrete energy decreases monotonically and exhibits exponential decay over time. These numerical results are in full agreement with the theoretical predictions and further illustrate the effectiveness of localized damping mechanisms for wave-energy dissipation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Kalifa Lassana Barry: Conceptualization, Investigation, Methodology, Software, Visualization, Writing - review and editing. **Karima Laoubi:** Conceptualization, Investigation, Methodology, Software, Visualization, Writing - review and editing. **Seda İğret Araz:** Conceptualization, Investigation, Methodology, Software, Visualization, Writing - review and editing. **Salah Boulaaras:** Conceptualization, Investigation, Methodology, Software, Visualization, Writing - review and editing. All authors discussed the results and contributed to the final manuscript.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (<https://www.qu.edu.sa>) for financial support (QU-APC-2026).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. J. L. Lions, *Exact Controllability, Stabilization and Perturbations for Distributed Systems*, Masson, Paris, **1** (1988).
2. J. L. Lions, *Some Methods of Solving Non-Linear Boundary Value Problems*, Dunod/Gauthier-Villars, Paris, 1969.
3. R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Academic Press, New York, **65** (1975).
4. J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96. <https://doi.org/10.1007/BF01762360>
5. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differ. Equ.*, **109** (1994), 295–308. <https://doi.org/10.1006/jdeq.1994.1051>
6. E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation and application, *Arch. Ration. Mech. Anal.*, **149** (1999), 155–182. <https://doi.org/10.1007/s002050050171>
7. E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, *J. Differ. Equ.*, **186** (2002), 259–298. [https://doi.org/10.1016/S0022-0396\(02\)00023-2](https://doi.org/10.1016/S0022-0396(02)00023-2)
8. E. Vitillaro, Strong solutions for the wave equation with a kinetic boundary condition, in *Recent Trends in Nonlinear Partial Differential Equations I. Evolution Problems*, *Contemp. Math.*, **594** (2013), 295–307.
9. E. Vitillaro, On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources, *J. Differ. Equ.*, **265** (2018), 4873–4941. <https://doi.org/10.1016/j.jde.2018.06.022>
10. E. Vitillaro, On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and source, *Arch. Ration. Mech. Anal.*, **223** (2017), 1183–1237. <https://doi.org/10.1007/s00205-016-1055-2>
11. F. Gazzola, M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **23** (2006), 185–207. <https://doi.org/10.1016/j.anihpc.2005.02.007>
12. S. Boulaaras, Global dynamics of the parabolic Choquard equation with asymptotically linear nonlinearity, *J. Inequal. Appl.*, **2026** (2026), 42. <https://doi.org/10.1186/s13660-026-03470-3>
13. S. Boulaaras, Existence, energy analysis, and finite-time blow-up for degenerate parabolic equations with mixed logarithmic sources, *J. Pseudo-Differ. Oper. Appl.*, **17** (2026), 19. <https://doi.org/10.1007/s11868-026-00772-4>
14. S. Boulaaras, Nonexistence of global weak solutions for strongly damped p -Laplacian wave equations with logarithmic source terms, *Appl. Anal.*, (2026), 1–37. <https://doi.org/10.1080/00036811.2026.2669577>
15. C. G. Gal, L. Tebou, Carleman inequalities for wave equations with oscillatory boundary conditions and application, *SIAM J. Control Optim.*, **55** (2017), 324–364. <https://doi.org/10.1137/15M1032211>

16. A. Ruiz, Unique continuation for weak solutions of the wave equation plus a potential, *J. Math. Pures Appl.*, **71** (1992), 455–467.
17. A. Khemmoudj, M. Medjden, Exponential decay for the semilinear Cauchy–Ventcel problem with localized damping, *Bol. Soc. Parana. Mat.*, **22** (2004), 97–116. <https://doi.org/10.5269/bspm.v22i2.7486>
18. R. I. Addoun, K. Laoubi, Stability and regularity analysis of nonlinear wave equations with localised internal and Ventcell boundary conditions, *Int. J. Control*, **98** (2025), 824–833. <https://doi.org/10.1080/00207179.2024.2368052>
19. A. M. Al-Mahdi, M. M. Al-Gharabli, M. Kafini, Asymptotic behavior of the wave equation solution with nonlinear boundary damping and source term of variable exponent-type, *AIMS Math.*, **9** (2024), 30638–30654. <https://doi.org/10.3934/math.20241479>
20. A. M. Al-Mahdi, M. Aouadi, M. M. Al-Gharabli, Stability of thermoelastic extensible Timoshenko system with nonlinear frictional damping of variable exponent-type, *J. Therm. Stresses*, **49** (2026), 149–166. <https://doi.org/10.1080/01495739.2025.2519939>
21. A. Choucha, S. Boulaaras, R. Jan, A. N. Tahat, Results of the asymptotic behavior for a viscoelastic wave equation with nonlinear distributed delay and acoustic conditions in boundary feedback, *J. Inequal. Appl.*, **2024** (2024), 156. <https://doi.org/10.1186/s13660-024-03241-y>
22. S. Boulaaras, A. Choucha, On a Hartree-type nonlinearity wave equation with distributed delay combined with a fractional condition, *Networks Heterogen. Media*, **21** (2026), 243–260. <https://doi.org/10.3934/nhm.2026012>
23. M. Guo, H. Dai, B. You, T. Caraballo, Global attractor of a wave equation with Hardy potentials and nonlinear damping, *J. Math. Phys.*, **66** (2025), 082701. <https://doi.org/10.1063/5.0263656>
24. J. Han, K. Wang, R. Xu, C. Yang, Global quantitative stability of wave equations with strong and weak dampings, *J. Differ. Equ.*, **390** (2024), 228–344. <https://doi.org/10.1016/j.jde.2024.01.033>
25. S. Nicaise, F. Trad, A. Wehbe, Indirect stabilization of locally weakly coupled second-order evolution systems of hyperbolic type, *Math. Methods Appl. Sci.*, **47** (2024), 14505–14562. <https://doi.org/10.1002/mma.10287>
26. M. M. Cavalcanti, P. Cornilleau, V. N. D. Cavalcanti, L. Robbiano, Semi-classical defect measure and internal stabilization for the semilinear wave equation subject to Zaremba boundary conditions, preprint, arXiv:2110.12689. <https://doi.org/10.48550/arXiv.2110.12689>
27. K. Sutrave, Parabolic moser iteration, Lecture notes.
28. T. Liu, B. Ding, A radial basis function neural network approach for solving a diffusion partial differential equation efficiently, *Appl. Math. Comput.*, **509** (2026), 129651. <https://doi.org/10.1016/j.amc.2025.129651>
29. T. Liu, R. Xue, A convergent multi-step efficient iteration method to solve nonlinear equation systems, *J. Appl. Math. Comput.*, **71** (2025), 2571–2588. <https://doi.org/10.1007/s12190-024-02324-9>
30. Y. Liu, Y. Li, T. Liu, An RBF–FD method for pricing under the Bates model: Handling stochastic volatility and jump processes, *Eng. Anal. Boundary. Elem.*, **183** (2026), 106622. <https://doi.org/10.1016/j.enganabound.2025.106622>

31. T. Liu, R. Xue, M. Barfeie, A unified framework for high-order compact finite differences using infinitely- and piecewise-smooth RBFs with polynomials, *Calcolo*, **63** (2026), 18. <https://doi.org/10.1007/s10092-026-00684-1>
32. C. Li, J. Liang, T. J. Xiao, Dynamical behaviors of solutions to nonlinear wave equations with vanishing local damping and Wentzell boundary conditions, *Z. Angew. Math. Phys.*, **69** (2018), 102. <https://doi.org/10.1007/s00033-018-0996-8>
33. K. Laoubi, D. Seba, Analysis of energy dissipation in hyperbolic problems influenced by internal and boundary control mechanisms, *Int. J. Control*, **98** (2025), 1263–1271. <https://doi.org/10.1080/00207179.2024.2390885>

Appendix

Proof. (Theorem 4.1: Existence of the time $T(x_0) > 0$) Considering the homogeneous system associated to system (4.2) with its initial condition $K_0 \in \mathcal{H}$, we obtain

$$\begin{cases} V_t = AV, \\ V(0) = K_0 \in \mathcal{H}, \end{cases} \quad (\text{A.1})$$

where the linear operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is defined in Eq (2.1).

Such a system is conservative, as $\langle AV, V \rangle_{\mathcal{H}} = 0$. Then,

$$E_v(t) = E_v(0) = \frac{1}{2} \|K_0\|_{\mathcal{H}}^2, \quad t \in (0, T).$$

Using the same GCC assumption stated in Assumption 2 and applying inequality (4.19), which holds for system (A.1), we obtain

$$c_3 T E_v(0) = c_3 \int_0^T E_v(t) dt \leq P_1 + P_2 + P_3,$$

where

$$\begin{cases} P_1 = \left| 2 \left[\int_{\Omega} v_t \left(m_0 \cdot \nabla v + \frac{N-1}{2} v \right) dx + \int_{\Gamma_1} v_{\Gamma_1 t} \left(m_0^T \cdot \nabla_T v_{\Gamma_1} + \frac{N-2}{2} v_{\Gamma_1} \right) d\Gamma_1 \right] \right|_0^T, \\ P_2 = \int_{\Gamma_{1T}} (m_0 |v|) (|v_t|^2 - |\nabla_T v_{\Gamma_1}|^2 + |\partial_\nu v_{\Gamma_1}|^2) d\Gamma_{1T}, \\ P_3 = \left| \int_{\Gamma_{1T}} (m_0 |v|) \text{tr}(\gamma_0) [|v_{\Gamma_1 t}|^2 - |\nabla_T v_{\Gamma_1}|^2 - 2\gamma_0 \cdot |\nabla_T v_{\Gamma_1}|^2] d\Gamma_{1T} \right|. \end{cases}$$

Using the estimate on P_1 from Eq (4.21) and the terms in P_3 while considering the conservation of energy E_v , we obtain the existence of constants $C_{r_0}, C_{1,3} > 0$ such that

$$|P_1| \leq C_{r_0} E_v(0), \quad |P_3| \leq C_{1,3} |P_2|.$$

Consequently, there exists $\tilde{C}_{1,3} > 0$ such that

$$(T - C_{r_0}) E_v(0) \leq \tilde{C}_{1,3} \int_{\Gamma_{1T}} (m_0 |v|) (|v_t|^2 - |\nabla_T v_{\Gamma_1}|^2 + |\partial_\nu v_{\Gamma_1}|^2) d\Gamma_{1T},$$

where C_{r_0} is proportionally dependent on r_0 .

Because our system (1.1) differs from the one studied in [17] only in its nonhomogeneous nature, we can proceed with the proof in the same manner as in [17]. Ultimately, it follows that inequality (4.18) in Lemma 4.8 holds if $T > C_{r_0} > 0$. Thus, it suffices to choose $T(x_0) = C_{r_0}^0$, where $C_{r_0}^0 > 0$ is the minimum value for which Eq (4.18) is satisfied with $T > T(x_0)$.

This concludes the proof. $\square$

Proof. (**Theorem 4.11:** Boundedness of $(f'(\kappa), g'(\kappa_{\Gamma_1}))$ in $L^\infty(\Omega_T) \times L^\infty(\Gamma_{1T})$)

The contradictions result in the first case (a) where $k \neq (0, 0)^\top$ in the proof of Theorem 4.11 is not valid if $f'(\kappa) \notin L^\infty(\Omega_T)$ or $g'(\kappa_{\Gamma_1}) \notin L^\infty(\Gamma_{1T})$. Therefore, it is necessary to examine the different cases where this is possible.

As we can see, $(f'(\kappa), g'(\kappa_{\Gamma_1})) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_{1T})$ if $p = q = 2$ (f, g are linear) or if $(\kappa, \kappa_{\Gamma_1}) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_{1T})$.

In the second case, we invoke the generalized Morrey's Theorem (see page 97 of [3]), which allows us to assert that for $N < 4$

$$W^{2,2} \hookrightarrow C^{0,2-\frac{N}{2}}(\overline{\Omega}),$$

where $C^{0,d}(\overline{\Omega})$, $d > 0$ is the space of d -Hölder continuous functions. This means that there exists a constant $C_d > 0$ such that

$$|h(x) - h(y)| \leq c|x - y|^d, \quad \forall x, y \in \mathbb{R}.$$

Thanks to the Weierstrass theorem, which guarantees that any continuous function on a compact set is bounded and attains its bounds, we can conclude that

$$C^{0,2-\frac{N}{2}}(\overline{\Omega}) \hookrightarrow L^\infty(\overline{\Omega}).$$

This allows us to conclude, by using the regularity Theorem 2.3, that the boundedness of $(f'(\kappa), g'(\kappa_{\Gamma_1}))$ in the L^∞ norm holds if $(m, \mu) \leq (2, 2)$ and $N < 4$.

As a reminder, the regularity $W^{2,2}$ is ensured by the strong solutions[¶] of system (1.1), not necessarily for weak solutions^{||} which evolve in $\mathcal{V}_0 \subset H^1(\Omega) \times H^1(\Gamma)$. In this case, the generalized Morrey's theorem (see page 97 of [3]) only gives for $N < 2$

$$\mathcal{V}_0 \hookrightarrow C^{0,1-\frac{N}{2}}(\overline{\Omega}) \hookrightarrow L^\infty(\overline{\Omega}).$$

Then, only the one-dimensional case holds for all weak solutions W of system (1.1).

The case where $N \geq 4$ is an open problem. The concern is about the possibility to have $(\kappa, \kappa_{\Gamma_1}) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_{1T})$. $\square$



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)

[¶]Solutions which satisfy the system (1.1) in classical sense; that is, it satisfies the equations of Eq (1.1) at any point of Ω_T .

^{||}A solution which has a well-defined energy in the integral sense.