



Research article

Symbolic trial-function method inspired by neural network architectures

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Abstract: This paper proposes a symbolic computational method using neural networks to solve Boussinesq equations. By combining activation functions such as sine, hyperbolic tangent, hyperbolic secant, and exponential, we constructed a single-hidden-layer model [3-3-1] and a double-hidden-layer model [3-3-2-1] to solve the (2+1)-dimensional and (1+1)-dimensional Boussinesq equations, respectively. This yielded multiple sets of novel, exact analytical solutions, which were visualized using 3D plots, contour plots, heatmaps, and evolution plots. To address the issues of insufficient direct neural network feature representation and poor generalization ability, this paper introduces a feature-enhanced neural network method. Model [(2-5)-3-1] was constructed for the (1+1)-dimensional equation, expanding the number of input neurons from 2 to 5; Model [(3-9)-3-1] was constructed for the (2+1)-dimensional equation, expanding the number of input neurons from 3 to 9. Both models retain 3 hidden-layer neurons and optimize the network structure based on the equation's dimension. By reconstructing the physical, spatiotemporal, and nonlinear characteristics of the equations and coupling them with inherent constraints, this method offers the advantages of richer solution results, higher accuracy, and significantly improved model generalization. It does not require bilinear transformations, features a simple structure, and has low computational cost, providing a new, viable solution for solving strongly nonlinear water wave equations.

Keywords: Boussinesq equation; nonlinear partial differential equations; neural network techniques; feature enhancement neural network; symbolic computation

1. Introduction

Nonlinear partial differential equations serve as the fundamental mathematical language for describing complex dynamical processes in fields such as fluid mechanics, electromagnetic

propagation, and financial modeling. The difficulty of finding their analytical solutions increases as the subject matter becomes more complex; consequently, numerical and symbolic computation methods occupy a central position in modern applied mathematics. Theoretical investigations into the asymptotic behavior of integrable systems, soliton decomposition, and stability also reveal the deep structure of nonlinear waves from another perspective.

The Boussinesq equation (Eq (1.1)) possesses significant physical relevance due to its description of long waves in shallow water [1]. Koç et al. constructed kink-type, singular-type, periodic-type, and periodic-lump-type composite subsolutions for Boussinesq-like equations using the $(m+1/G')$ -expansion method [2]. Regarding this family of equations, researchers have obtained lump solutions and rogue wave solutions based on the Hirota bilinear method [3], and Wazwaz used the bilinear method to reveal the solitary wave structures of three variants [4]. The Bäcklund transformation was used to construct conservation laws and periodic wave solutions for generalized equations [5], and the relationship between the elastic convergence coupling mechanism and new solutions to the Kadomtsev–Petviashvili (KP) equation was elucidated [6]; Lie symmetry analysis and fractional-order modeling expanded solution approaches from the perspectives of symmetry and memory effects, respectively [7–10]. Subsequently, expansion techniques such as the Sardar subequation method and the Kudryashov method [11–15], along with the theory of global existence and stability of solutions [16–18], collectively constitute the methodological lineage of this research direction.

$$u_{tt} - u_{xx} - u_{yy} - u_{xxx} - (u^2)_{xx} = 0. \quad (1.1)$$

The Wazwaz–Kaur-type Boussinesq equations have recently sparked a series of studies: Shakeel et al. were the first to introduce bifurcation theory and chaos analysis into their solution framework, obtaining periodic, bright–dark, and W-type soliton solutions and characterizing the bifurcation behavior [19]. This method was generalized to the $(2+1)$ -dimensional fractional-order case, yielding a richer family of traveling-wave solutions [20] and then were combined with Lyapunov exponents to characterize their chaotic properties and stability boundaries [21], demonstrating that the fractional-order framework can more accurately capture memory effects and is complementary to integer-order methods [22, 23]. Regarding asymptotic behavior, Yang et al. proved the soliton decomposition conjecture for the modified Camassa–Holm equation [24], and Zhou et al. characterized the long-time asymptotic structure of solutions to the nonlocal Lakshmanan–Porsezian–Daniel equation [25]. In terms of numerical methods, the finite volume/finite element method [26], the Galerkin finite element method [27], spectral methods [28], the multisymplectic method [29], the hybrid finite–element method [30], the reproducing kernel method [31], and meshless techniques [32] have all been applied to the classical Boussinesq model; Anbiya investigated the numerical stability of the “bad” Boussinesq equation by combining the Runge–Kutta method with the Strang splitting technique [33], and Chu et al. applied the variational iterative transformation method to fractional-order Navier–Stokes equations, verifying its reliability for solving problems involving memory-effect fluids [34].

Deep neural networks have demonstrated outstanding performance in computer vision tasks thanks to their powerful feature learning capabilities [35–37], and their application to solving partial differential equations has also been gaining momentum: The physics-informed neural network (PINN) proposed by Raissi et al. pioneered a new paradigm [38], and subsequent research has

expanded in the directions of adaptive mechanisms [39], conservation constraints [40], and the solution of higher-order soliton equations [41], with successful application to the Boussinesq equation itself [42,43]. In parallel, the bilinear neural network method (BNNM) proposed by Zhang and Bilge combines neural networks with bilinear transformations to solve the p-gBKP equation [44], and it was later generalized to various types of equations, including fractal solitons, periodic waves [45], lump solutions, and interference wave solutions [46–51]. Shen and Wu applied the BNNM to the Boiti–Leon–Manna–Pempinelli equation to obtain periodic-soliton solutions [52], and Shen et al. proposed the neural network symbolic computation approach (NNSCA) to solve the Yu–Toda–Sasa–Fukuyama equation [53], subsequently combining it with the symbolic approximation method (SANNA) to obtain a new family of solutions for the Harris–Benedict (HB) equation [54]. Isah et al. also investigated the influence of network structural parameters on the results of the BNNM solution to the Boussinesq equation [55].

To address the shortcomings of existing methods in terms of the balance between accuracy and efficiency, generalization, and the discovery of new solutions, this paper proposes a more direct and efficient neural network-symbolic method. Two types of architectures—Model [3-3-1] and Model [3-3-2-1]—are designed and integrated with multiple activation functions while also introducing two feature enhancement strategies to strengthen the models' ability to capture the physical properties and nonlinear structure of the equations. The (1+1)-dimensional and (2+1)-dimensional Boussinesq equations were re-solved, yielding multiple sets of novel, exact analytical solutions and 3D visualizations. The method eliminates the need for bilinear transformations, features a simple structure, and has low computational cost.

2. Neural network-based symbolic computation model

As illustrated in Figure 1, the neural network-driven symbolic computation workflow developed in this study comprises five essential steps, forming a complete architecture.

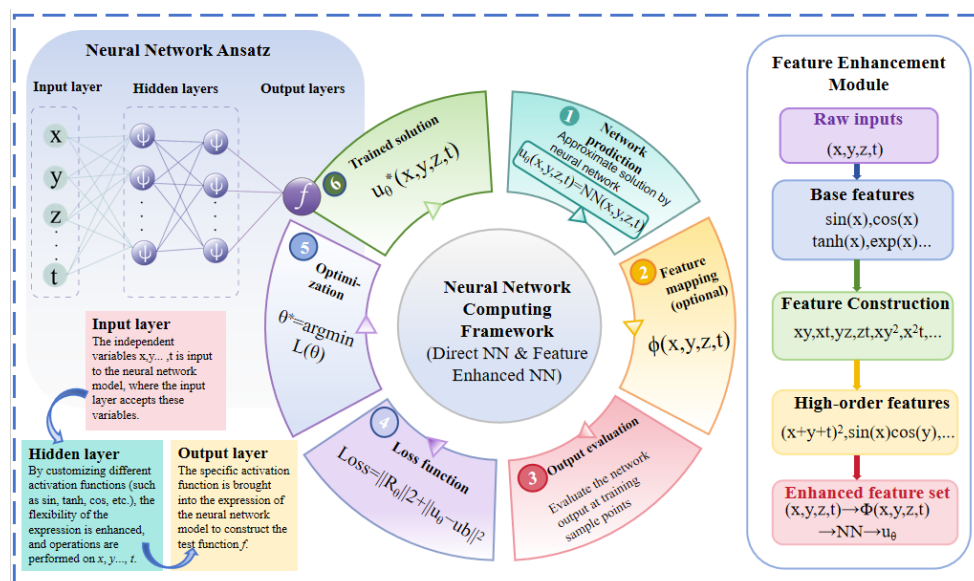


Figure 1. Neural network-driven symbolic computation workflow.

Step 1: Construct a feedforward neural network model, and use its explicit output expression to characterize the implicit solution function of the nonlinear partial differential equation under study.

Step 2: Substituting this expression into the control equation yields a complex algebraic expression describing the residuals of the equation.

Step 3: Extract the coefficients corresponding to the independent variable and the terms representing its partial derivatives of various orders from this algebraic expression, and formulate a system of nonlinear algebraic equations.

Step 4: Solve this system of equations to determine the constraints that the network parameters (weights W and bias b) must satisfy.

Step 5: Substitute the resulting constraints back into the initial network expression to obtain the exact analytical solution to this nonlinear partial differential equation.

The explicit mathematical expression of a single-layer neural network (Eq (2.1)) is as follows:

$$\begin{aligned} f &= w_{1,f}\psi_1(\zeta_1) + w_{2,f}\psi_2(\zeta_2) + w_{3,f}\psi_3(\zeta_3) + b_4, \\ \zeta_i &= w_{x,i}x + w_{y,i}y + w_{t,i}t + b_i, (i = 1, 2, 3). \end{aligned} \quad (2.1)$$

The explicit mathematical expression of a two-layer neural network (Eq (2.2)) is as follows:

$$\begin{aligned} f &= \sum_{n=1}^5 w_{n,f}\psi_n(\zeta_n) + b_6, \\ \zeta_i &= w_{1,i}\psi_1(\zeta_1) + w_{2,i}\psi_2(\zeta_2) + w_{3,i}\psi_3(\zeta_3), (i = 4, 5), \\ \zeta_1 &= w_{x,1}x + w_{t,1}t + b_5, \\ \zeta_2 &= w_{x,2}x + w_{t,2}t + b_4, \\ \zeta_3 &= w_{x,3}x + w_{t,3}t + b_3. \end{aligned} \quad (2.2)$$

The explicit mathematical representation of the feature-enhanced neural network of configuration [(2-5)-3-1] (Eq (2.3)) is as follows:

$$\begin{aligned} f &= w_{1,f}\psi_1(\zeta_1) + w_{2,f}\psi_2(\zeta_2) + w_{3,f}\psi_3(\zeta_3) + b_4, \\ \zeta_i &= xw_{x,i} + tw_{t,i} + xt w_{xt,i} + x^2 w_{x^2,i} + t^2 w_{t^2,i} + b_i, (i = 1, 2, 3). \end{aligned} \quad (2.3)$$

The explicit mathematical representation of the feature-enhanced neural network of configuration [(3-9)-3-1] (Eq (2.4)) is as follows:

$$\begin{aligned} f &= w_{1,f}\psi_1(\zeta_1) + w_{2,f}\psi_2(\zeta_2) + w_{3,f}\psi_3(\zeta_3) + b_4, \\ \zeta_i &= xw_{x,i} + yw_{y,i} + tw_{t,i} + x^2 w_{x^2,i} + y^2 w_{y^2,i} + t^2 w_{t^2,i} + xyw_{xy,i} + xt w_{xt,i} + yt w_{yt,i} + b_i, (i = 1, 2, 3). \end{aligned} \quad (2.4)$$

Equation (2.1) comes from the single-hidden-layer neural network model, while as Eq (2.2) follows from the double-hidden-layer counterpart. Equation (2.3) is constructed using the feature-enhanced neural network for the (1+1)-dimensional Boussinesq equation, and Eq (2.4) is built with the feature-enhanced neural network to solve the (2+1)-dimensional Boussinesq equation. This paper takes four core neural network architectures as shown in Figure 2: Model [3-3-1] (Figure 2(a)), Model [3-3-2-1] (Figure 2(b)), Model [(2-3)-3-1] (Figure 2(c)), and Model [(3-9)-3-1] (Figure 2(d)). This paper constructs seven different neural network configurations by adopting various combinations of activation functions, and seven distinct trial functions are derived correspondingly.

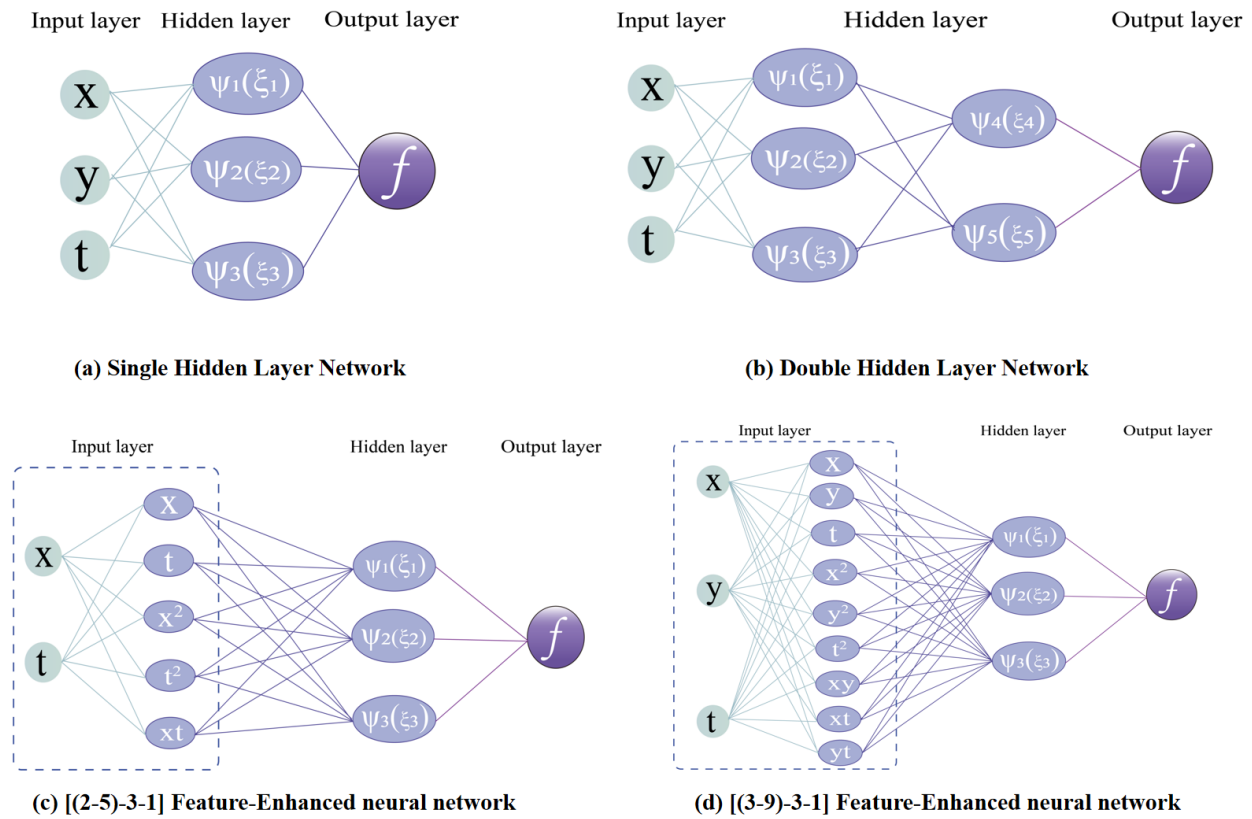


Figure 2. The neural network model.

3. Direct neural network

3.1. Single-hidden-layer model

3.1.1. Model [3-3-1]-1

To handle Eq (1.1), the (2+1)-dimensional Boussinesq equation, we employ a single-hidden-layer neural network (Figure 2(a)) with a 3-neuron input layer and a 3-neuron hidden layer.

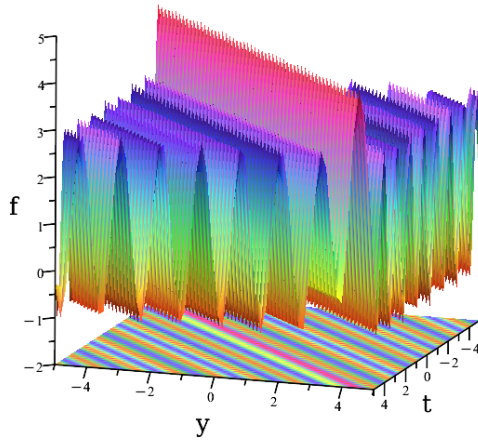
$$\psi_1(\zeta_1), \psi_2(\zeta_2), \psi_3(\zeta_3).$$

We order

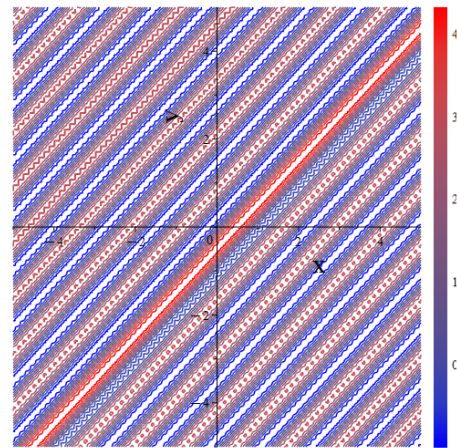
$$\psi_1(\zeta_1) = \text{sech}(\zeta_1)^2, \psi_2(\zeta_2) = \sin(\zeta_2), \psi_3(\zeta_3) = \cos(\zeta_3).$$

Utilizing the analytical expression derived from the neural network model Model [3-3-1]-1, the probe function can be expressed as follows:

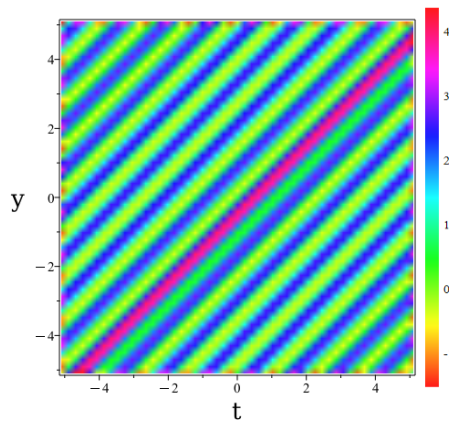
$$\begin{aligned} f = & b_4 + w_{1,u} \text{sech}(tw_{t,1} + xw_{x,1} + yw_{y,1} + b_1)^2 \\ & + w_{2,u} \sin(tw_{t,2} + xw_{x,2} + yw_{y,2} + b_2) \\ & + w_{3,u} \cos(tw_{t,3} + xw_{x,3} + yw_{y,3} + b_3). \end{aligned} \quad (3.1)$$



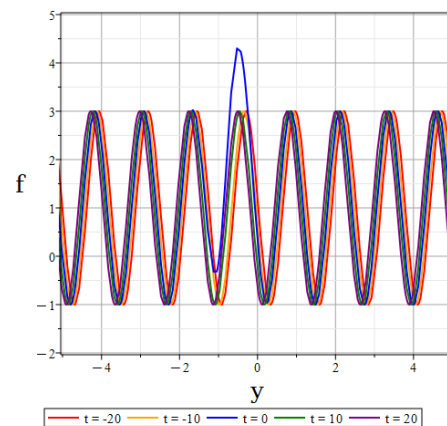
(a) Three-dimensional surface plots



(b) Contour diagrams



(c) Thermal maps



(d) The temporal evolution profile

Figure 3. Three-dimensional surface plots, contour diagrams, thermal maps, and the temporal evolution profile of Eq (3.3).

Plugging Eq (3.1) into the NLPDE, Eq (1.1), gives a solution:

$$\left\{ \begin{array}{l} b_2 = b_2, \quad b_3 = b_3, \quad t = t, \quad x = x, \quad y = y, \quad b_4 = b_4, \quad w_{1,u} = w_{1,u}, \quad w_{2,u} = 0, \\ w_{3,u} = w_{3,u}, \quad w_{t,1} = -w_{y,1}, \quad w_{t,2} = w_{t,2}, \quad w_{t,3} = -w_{t,3}, \quad w_{x,1} = 0, \\ w_{x,2} = w_{x,2}, \quad w_{x,3} = 0, \quad w_{y,1} = w_{y,1}, \quad w_{y,2} = w_{y,2}, \quad w_{y,3} = w_{y,3}. \end{array} \right\} \quad (3.2)$$

The solution to the original equation is obtained by substituting Eq (3.2) into Eq (3.1), and reads:

$$f = b_4 + w_{1,u} \operatorname{sech}((-t + y)w_{y,1} + b_1)^2 + w_{3,u} \cos((-t + y)w_{x,3} + b_3). \quad (3.3)$$

To illustrate the dynamic behavior of the solution, parameter values were assigned, and visualizations were generated using Maple. Figure 3 displays the three-dimensional surface, contour diagram, heat map, and temporal evolution of the solution. These plots reveal key properties and variation patterns of the solution to the NLPDE.

3.1.2. Model [3-3-1]-2

The activation functions in the hidden layer were changed to produce multiple analytical solutions using a single-hidden-layer neural network. Taking Model [3-3-1]-1 as a reference, the functions were chosen as

$$\psi_1(\zeta_1) = \tanh(\zeta_1)^2, \psi_2(\zeta_2) = (\zeta_2)^2, \psi_3(\zeta_3) = (\zeta_3)^2,$$

which defines Model [3-3-1]-2. Under this setting, the probe function is obtained as:

$$\begin{aligned} f = & b_4 + w_{1,u} \tanh(tw_{t,1} + xw_{x,1} + yw_{y,1} + b_2)^2 \\ & + w_{2,u} (tw_{t,2} + xw_{x,2} + yw_{y,2} + b_3)^2 \\ & + w_{3,u} (tw_{t,3} + xw_{x,3} + yw_{y,3} + b_5)^2. \end{aligned} \quad (3.4)$$

Substituting Eq (3.4) into the NLPDE (1.1) yields the following solution:

$$\left\{ \begin{array}{l} b_3 = \frac{b_5 w_{x,2}}{w_{x,3}}, b_5 = b_5, b_4 = b_4, w_{1,u} = w_{1,u}, w_{2,u} = -\frac{w_{3,u} w_{x,3}^2}{w_{x,2}^2}, \\ w_{3,u} = w_{3,u}, w_{t,1} = w_{y,1}, w_{t,2} = \frac{w_{t,3} w_{x,2}}{w_{x,3}}, w_{t,3} = w_{t,3}, w_{x,1} = 0, \\ w_{x,2} = w_{x,2}, w_{x,3} = w_{x,3}, w_{y,1} = w_{y,1}, w_{y,2} = \frac{w_{x,2} w_{y,3}}{w_{x,3}}, w_{y,3} = w_{y,3}. \end{array} \right\} \quad (3.5)$$

Plugging Eq (3.5) into Eq (3.4) leads to the solution of the original equation:

$$\begin{aligned} f = & b_4 + w_{1,u} \tanh(tw_{y,1} + yw_{y,1} + b_2)^2 - \frac{w_{3,u} w_{x,3}^2 \left(\frac{tw_{t,3} w_{x,2}}{w_{x,3}} + xw_{x,2} + \frac{yw_{x,2} w_{y,3}}{w_{x,3}} + \frac{b_5 w_{x,2}}{w_{x,3}} \right)^2}{w_{x,2}^2} \\ & + w_{3,u} (tw_{t,3} + xw_{x,3} + yw_{y,3} + b_5)^2. \end{aligned} \quad (3.6)$$

Simplifying equation Eq (3.6), we can obtain:

$$f := \frac{8w_{x,1}^4 + w_{t,1}^2 - w_{x,1}^2 - w_{y,1}^2}{2w_{x,1}^2} - 6w_{x,1}^2 \tanh(tw_{t,1} + xw_{x,1} + yw_{y,1} + b_2)^2. \quad (3.7)$$

Based on Eq (3.7), under given parameter conditions, we used Maple software to plot the three-dimensional surface, contour plots, heat maps, and evolution plots of this solution (Figure 4) to visually illustrate the structural characteristics and dynamic behavior of the solution.

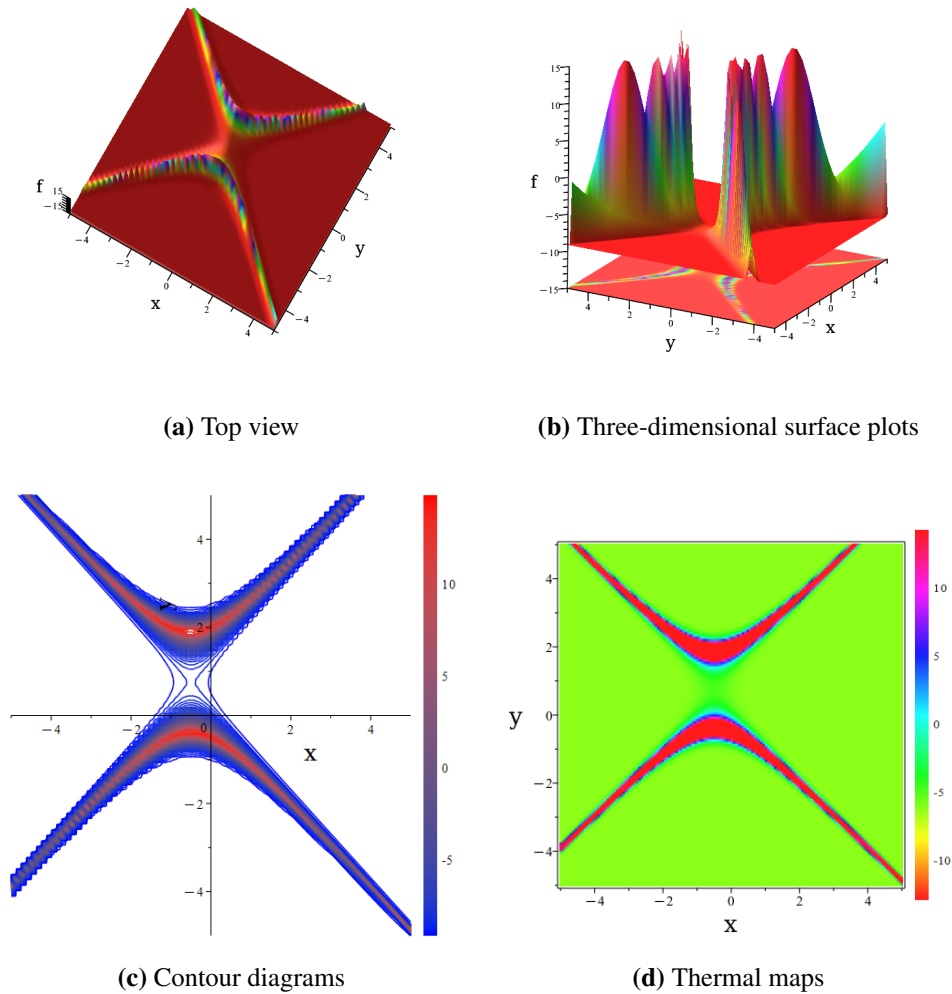


Figure 4. Top view, three-dimensional surface plots, contour diagrams, thermal maps of Eq (3.7).

3.1.3. Model [3-3-1]-3

In this subsection, we solve the (1+1)-dimensional Boussinesq equation.

$$u_{tt} - u_{xx} - u_{xxx} - (u^2)_{xx} = 0. \quad (3.8)$$

To achieve more accurate analytical solutions with a single-hidden-layer network, the activation functions in the hidden layer were partially modified. Keeping $\psi_2(\zeta_2) = \zeta_2^2$ and $\psi_3(\zeta_3) = \zeta_3^2$ unchanged from Model [3-3-1]-2, $\psi_1(\zeta_1)$ was replaced with an exponential function:

$$\psi_1(\zeta_1) = e^{\zeta_1}, \quad \psi_2(\zeta_2) = \zeta_2^2, \quad \psi_3(\zeta_3) = \zeta_3^2.$$

Under this configuration, designated as Model [3-3-1]-3, the trial function takes the form:

$$f = b_4 + w_{1,u} e^{(tw_{t,1} + xw_{x,1} + b_2)} + w_{2,u} (tw_{t,2} + xw_{x,2} + b_3)^2 + w_{3,u} (tw_{t,3} + xw_{x,3} + b_5)^2. \quad (3.9)$$

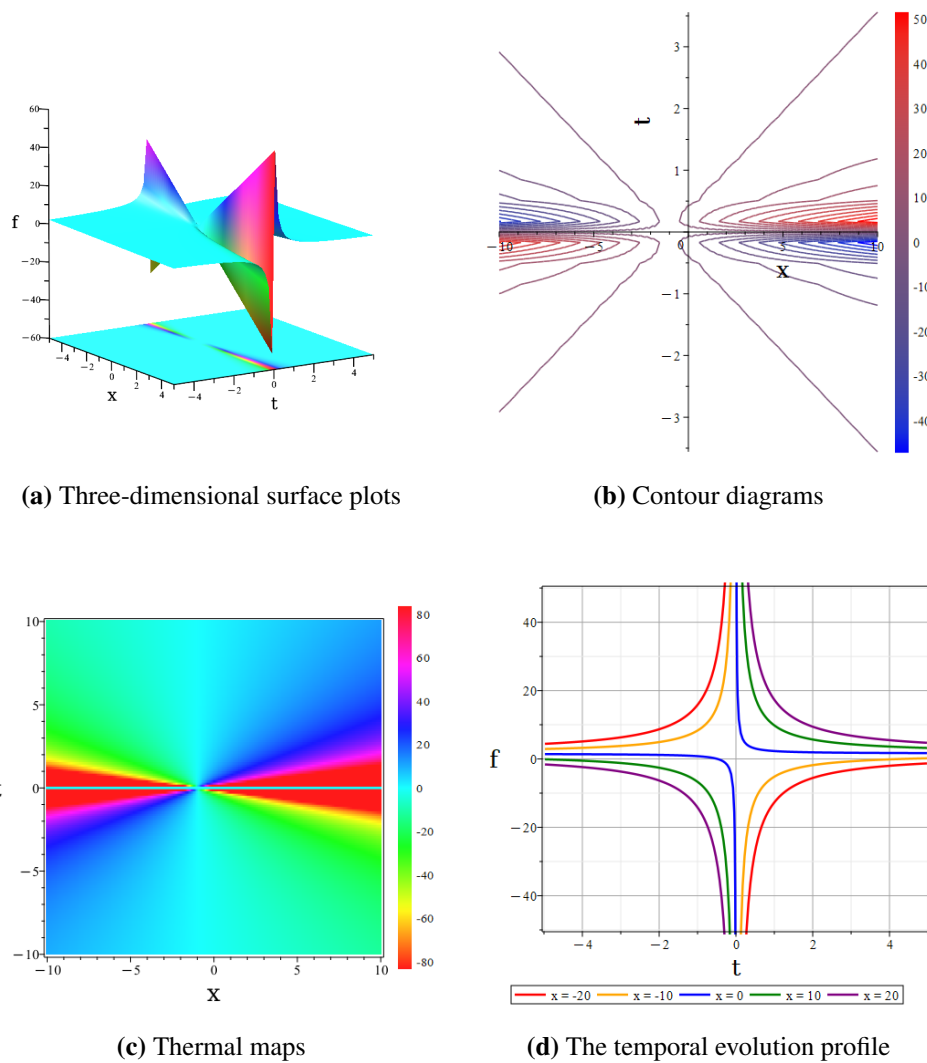


Figure 5. Three-dimensional surface plots, contour diagrams, thermal maps, and the temporal evolution profile of Eq (3.12).

Substituting Eq (3.9) into Eq (3.8) yields the following solution:

$$\left\{ \begin{array}{l} b_3 = b_3, \quad b_5 = b_5, \quad b_4 = b_4, \quad t = t, \quad x = x, \quad w_{1,u} = 0, \\ w_{2,u} = \frac{w_{t,2}^2 w_{x,3}^2 - w_{t,3}^2 w_{x,2}^2}{4(tw_{t,2}w_{x,3} - tw_{t,3}w_{x,2} + b_3w_{x,3} - b_5w_{x,2})^2 w_{x,2}^2}, \\ w_{3,u} = -\frac{w_{t,2}^2 w_{x,3}^2 - w_{t,3}^2 w_{x,2}^2}{4w_{x,3}^2 (tw_{t,2}w_{x,3} - tw_{t,3}w_{x,2} + b_3w_{x,3} - b_5w_{x,2})^2}, \\ w_{t,1} = w_{t,1}, \quad w_{t,2} = w_{t,2}, \quad w_{t,3} = w_{t,3}, \quad w_{x,1} = w_{x,1}, \quad w_{x,2} = w_{x,2}, \quad w_{x,3} = w_{x,3}. \end{array} \right. \quad (3.10)$$

The solution to the original equation is obtained by substituting Eq (3.10) into Eq (3.9), and reads:

$$f = b_4 + \frac{(w_{t,2}^2 w_{x,3}^2 - w_{t,3}^2 w_{x,2}^2)(tw_{t,2} + xw_{x,2} + b_3)^2}{4(tw_{t,2}w_{x,3} - tw_{t,3}w_{x,2} + b_3w_{x,3} - b_5w_{x,2})^2 w_{x,2}^2} - \frac{(w_{t,2}^2 w_{x,3}^2 - w_{t,3}^2 w_{x,2}^2)(tw_{t,3} + xw_{x,3} + b_5)^2}{4w_{x,3}^2 (tw_{t,2}w_{x,3} - tw_{t,3}w_{x,2} + b_3w_{x,3} - b_5w_{x,2})^2}. \quad (3.11)$$

Simplifying equation Eq (3.11), we can obtain

$$f = b_4 + \frac{(w_{t,2}^2 w_{x,3}^2 - w_{t,3}^2 w_{x,2}^2)(tw_{t,2} + xw_{x,2} + b_3)^2}{4((-tw_{t,3} - b_5)w_{x,2} + w_{x,3}(tw_{t,2} + b_3))^2 w_{x,2}^2} - \frac{(w_{t,2}^2 w_{x,3}^2 - w_{t,3}^2 w_{x,2}^2)(tw_{t,3} + xw_{x,3} + b_5)^2}{4w_{x,3}^2 (w_{x,3}(tw_{t,2} + b_3) - w_{x,2}(tw_{t,3} + b_5))^2}. \quad (3.12)$$

Once Eq (3.12) is obtained, parameter values are assigned and the solution is visualized using Maple. Figure 5 presents the three-dimensional surface, contour lines, heat map, and evolution graph of the solution.

3.2. Double-hidden-layer model

3.2.1. Model [3-3-2-1]-1

Unlike the single-hidden-layer model [3-3-1], the network architecture is extended by adding a second hidden layer. The input layer contains three neurons, the first hidden layer three neurons, the second hidden layer two neurons, and the output layer one neuron (Figure 2(b)).

This results in five neurons in total across the two hidden layers:

$$\psi_1(\zeta_1), \psi_2(\zeta_2), \psi_3(\zeta_3), \psi_4(\zeta_4), \psi_5(\zeta_5).$$

The first layer consists of three neurons, which are respectively

$$\psi_1(\zeta_1), \psi_2(\zeta_2), \psi_3(\zeta_3).$$

The two neurons in the second layer are

$$\psi_4(\zeta_4), \psi_5(\zeta_5).$$

To solve Eq (1.1), we use a double-hidden-layer neural network framework (Figure 2(b)), and we set the activation function in the hidden layer to

$$\psi_1(\zeta_1) = \zeta_1, \psi_2(\zeta_2) = \zeta_2, \psi_3(\zeta_3) = \zeta_3, \psi_4(\zeta_4) = \tanh(\zeta_4), \psi_5(\zeta_5) = \operatorname{sech}(\zeta_5).$$

Based on Model [3-3-2-1]-1, we can obtain the following probe function:

$$f = b_6 + w_{4,u} \tanh(w_{1,4}(tw_{t,1} + xw_{x,1} + yw_{y,1}) + w_{2,4}(tw_{t,2} + xw_{x,2} + yw_{y,2}) + w_{3,4}(tw_{t,3} + xw_{x,3} + yw_{y,3})) + w_{5,u} \operatorname{sech}(w_{1,5}(tw_{t,1} + xw_{x,1} + yw_{y,1}) + w_{2,5}(tw_{t,2} + xw_{x,2} + yw_{y,2}) + w_{3,5}(tw_{t,3} + xw_{x,3} + yw_{y,3})). \quad (3.13)$$

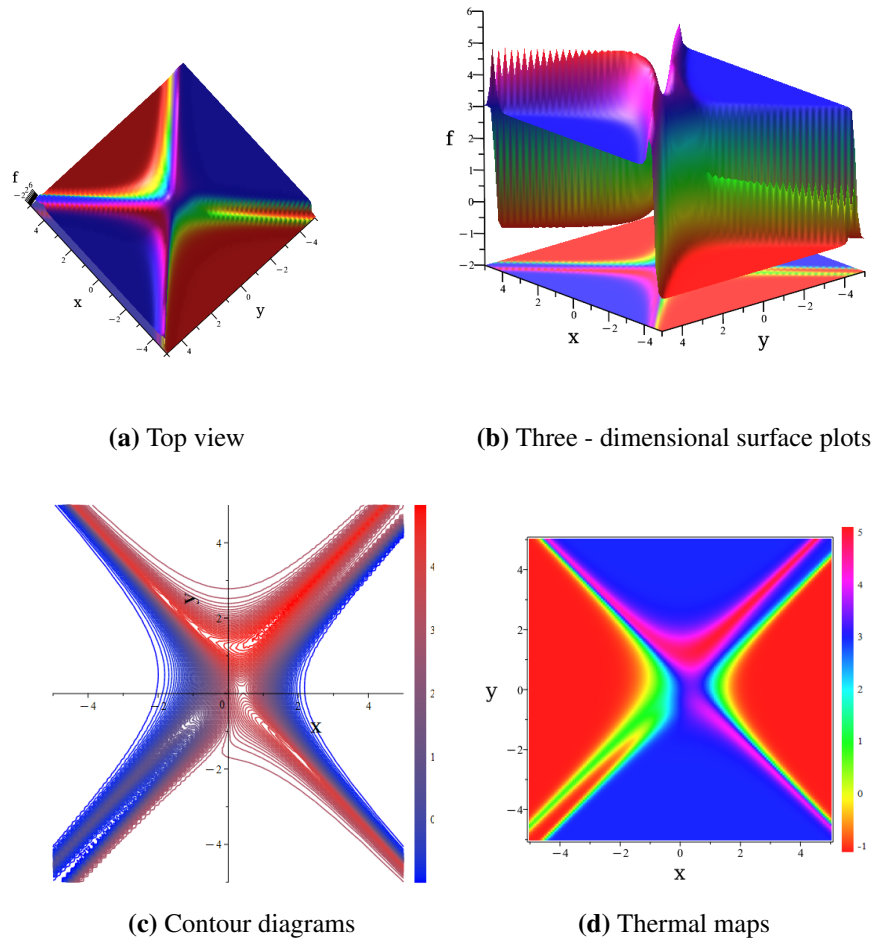


Figure 6. Top view, three-dimensional surface plots, contour diagrams, thermal maps of Eq (3.16).

Substituting Eq (3.13) into Eq (1.1), we obtain the following solution:

$$\left\{ \begin{array}{l} w_{1,4} = w_{1,4}, w_{1,5} = 0, w_{2,4} = w_{2,4}, w_{2,5} = w_{2,5}, \\ b_6 = b_6, w_{3,4} = w_{3,4}, w_{3,5} = w_{3,5}, w_{4,u} = 0, w_{5,u} = w_{5,u}, \\ w_{t,1} = w_{t,1}, w_{t,2} = -\frac{w_{2,5}w_{y,2} + w_{3,5}w_{t,3} + w_{3,5}w_{y,3}}{w_{2,5}}, \\ w_{t,3} = w_{t,3}, w_{x,1} = w_{x,1}, w_{x,2} = -\frac{w_{3,5}w_{x,3}}{w_{2,5}}, w_{x,3} = w_{x,3}, \\ w_{y,1} = w_{y,1}, w_{y,2} = w_{y,2}, w_{y,3} = w_{y,3}. \end{array} \right\} \quad (3.14)$$

Substituting Eq (3.15) into Eq (3.14), we can obtain the solution to the initial equation, which is expressed as follows:

$$f = b_6 + w_{4,u} \cdot \tanh\left(w_{1,4} \cdot (-t \cdot w_{y,1} + x \cdot w_{y,1}) + w_{3,4} \cdot (-t \cdot w_{y,3} + x \cdot w_{y,3})\right) + w_{5,u} \cdot \operatorname{sech}\left(w_{1,5} \cdot (-t \cdot w_{y,1} + y \cdot w_{y,1}) + w_{3,5} \cdot (-t \cdot w_{y,3} + y \cdot w_{y,3})\right). \quad (3.15)$$

After simplifying equation Eq (3.15), we can obtain

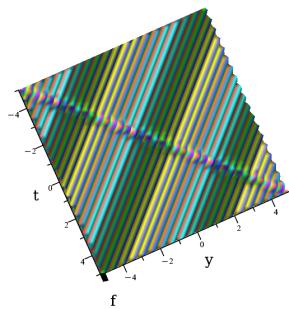
$$f = b_6 - w_{4,u} \cdot \tanh(w_{3,4} \cdot w_{y,3} \cdot (t - x)) + w_{5,u} \cdot \operatorname{sech}\left((t + y) \cdot \frac{(-w_{2,5} \cdot w_{t,2} + w_{3,5} \cdot w_{y,3}) \cdot w_{x,1} + w_{2,5} \cdot w_{t,1} \cdot w_{x,2}}{w_{x,1}}\right). \quad (3.16)$$

The dynamic characteristics of solution equation (3.16) are shown in Figure 6.

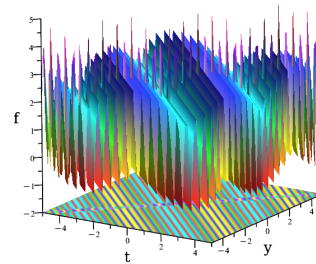
3.2.2. Model [3-3-2-1]-2

By increasing the number of hidden layers, the depth of the neural network is extended, thereby enhancing its feature representation capability and enabling the model to learn more complex nonlinear mappings. We set the activation function in the hidden layer to:

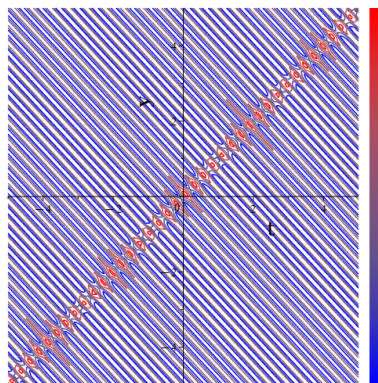
$$\psi_1(\zeta_1) = \zeta_1, \psi_2(\zeta_2) = \zeta_2, \psi_3(\zeta_3) = \zeta_3, \psi_4(\zeta_4) = \sin(\zeta_4), \psi_5(\zeta_5) = \operatorname{sech}(\zeta_5).$$



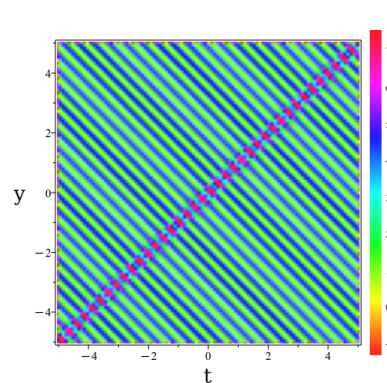
(a) Top view



(b) Three - dimensional surface plots



(c) Contour diagrams



(d) Thermal maps

Figure 7. Top view, three-dimensional surface plots, contour diagrams, and thermal maps of Eq (3.19).

The following trial function is obtained:

$$\begin{aligned}
 f = & b_6 + w_{4,u} \sin(w_{1,4}(tw_{t,1} + xw_{x,1} + yw_{y,1}) \\
 & + w_{2,4}(tw_{t,2} + xw_{x,2} + yw_{y,2}) + w_{3,4}(tw_{t,3} + xw_{x,3} + yw_{y,3})) \\
 & + w_{5,u} \operatorname{sech}(w_{1,5}(tw_{t,1} + xw_{x,1} + yw_{y,1}) \\
 & + w_{2,5}(tw_{t,2} + xw_{x,2} + yw_{y,2}) + w_{3,5}(tw_{t,3} + xw_{x,3} + yw_{y,3})).
 \end{aligned} \quad (3.17)$$

Substituting Eq (3.17) into Eq (1.1), we obtain the following solution:

$$\left\{ \begin{array}{l} t = t, w_{1,4} = w_{1,4}, w_{1,5} = w_{1,5}, w_{2,4} = w_{2,4}, w_{2,5} = 0, x = x, y = y, \\ b_6 = b_6, w_{3,4} = w_{3,4}, w_{3,5} = w_{3,5}, w_{4,u} = w_{4,u}, w_{5,u} = w_{5,u}, w_{t,1} = -\frac{w_{1,5}w_{y,1} + w_{3,5}w_{t,3} + w_{3,5}w_{y,3}}{w_{1,5}}, \\ w_{t,2} = \frac{2w_{1,4}w_{1,5}w_{y,1} + w_{3,5}w_{t,3}w_{1,4} + w_{1,4}w_{3,5}w_{y,3} + w_{1,5}w_{2,4}w_{y,2} - w_{1,5}w_{3,4}w_{t,3} + w_{1,5}w_{3,4}w_{y,3}}{w_{1,5}w_{2,4}}, \\ w_{t,3} = w_{t,3}, w_{x,1} = 0, w_{x,2} = 0, w_{x,3} = 0, w_{y,1} = w_{y,1}, w_{y,2} = w_{y,2}, w_{y,3} = w_{y,3}. \end{array} \right\} \quad (3.18)$$

Substituting Eq (3.18) into Eq (3.17), we can obtain the solution to the initial equation, which is expressed as follows:

$$f = b_6 + w_{4,u} \sin((w_{1,4}w_{y,1} + w_{2,4}w_{y,2} + w_{3,4}w_{y,3})(t + y)) + w_{5,u} \operatorname{sech}((w_{1,5}w_{y,1} + w_{3,5}w_{y,3})(t - y)). \quad (3.19)$$

To further observe the characteristics and variation patterns of the solution, we assign values to the parameters in the same way and used Maple to create three-dimensional plots, contour plots, heat maps, and evolution plots Figure 7 of the solution, thereby visualizing the solution.

4. Feature enhancement neural network

Although direct neural network methods can efficiently approximate solutions to the Boussinesq equations, they rely solely on modeling raw spatiotemporal features and do not fully incorporate the physical prior information of the equations. This limits their ability to characterize features and their generalization performance when addressing the evolution of water waves with strong nonlinearity and large gradients. To address this, this paper introduces a feature-enhanced neural network method. By reconstructing physical, spatiotemporal, and nonlinear features and coupling them with inherent physical constraints, this method improves the model's solution accuracy and generalization capability. The modeling approach and network architecture are detailed below.

4.1. Model [(2-5)-3-1]

To solve the (1+1)-dimensional Boussinesq equation (Eq (3.1)), this paper employs a feature-enhanced neural network approach: The input layer is initially set with two primitive neurons, and after free combination and selection via the feature-enhancement mechanism, the five neurons with the optimal representational capacity are retained to form the enhanced input layer. The output is then fed into a hidden layer containing three neurons for nonlinear mapping, in order to fully extract the deep dynamical features of the equation. Based on the above network architecture, the following test function is constructed:

$$\psi_1(\zeta_1) = \sin(\zeta_1), \psi_2(\zeta_2) = \cos(\zeta_2), \psi_3(\zeta_3) = e^{\zeta_3}.$$

Based on Model [(2-5)-3-1], we can obtain the following probe function:

$$\begin{aligned}
 f = & b_6 + w_{4,u} \sin \left(w_{1,4} (tw_{t,1} + xw_{x,1} + yw_{y,1}) \right. \\
 & + w_{2,4} (tw_{t,2} + xw_{x,2} + yw_{y,2}) + w_{3,4} (tw_{t,3} + xw_{x,3} + yw_{y,3}) \Big) \\
 & + w_{5,u} \operatorname{sech} \left(w_{1,5} (tw_{t,1} + xw_{x,1} + yw_{y,1}) \right. \\
 & + w_{2,5} (tw_{t,2} + xw_{x,2} + yw_{y,2}) + w_{3,5} (tw_{t,3} + xw_{x,3} + yw_{y,3}) \Big).
 \end{aligned} \quad (4.1)$$

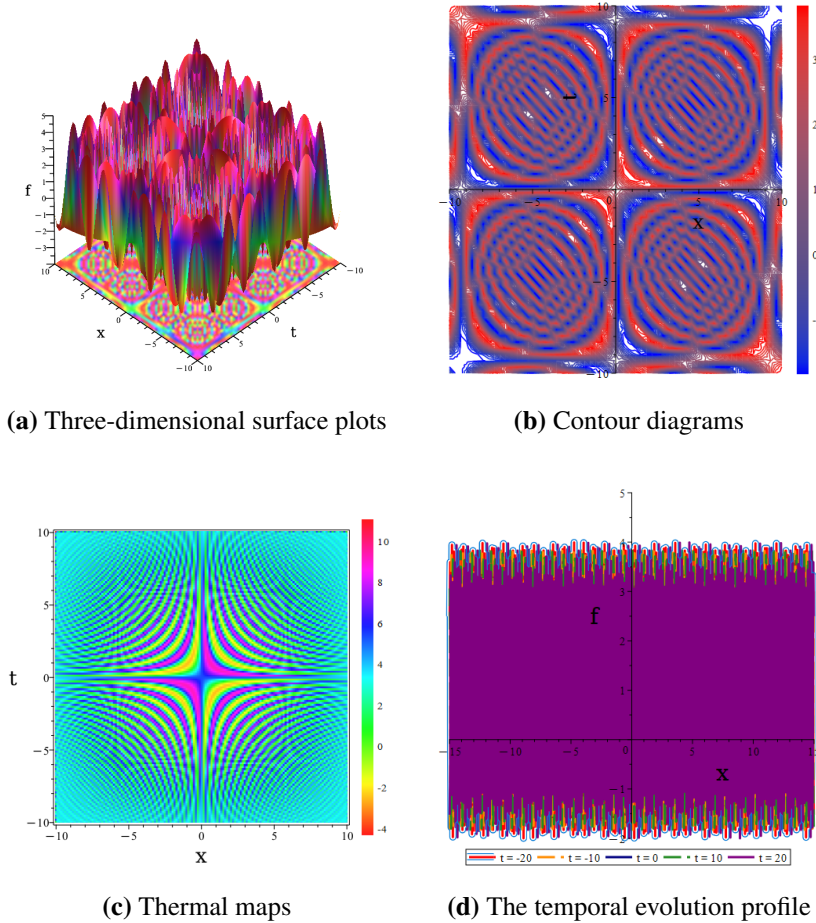


Figure 8. Three-dimensional surface plots, contour diagrams, thermal maps, and the temporal evolution profile of Eq (4.3).

Substituting Eq (4.1) into Eq (3.8), we obtain the following solution:

$$\left\{ \begin{array}{l} t = t, \ x = x, \ b_4 = b_4, \ w_{1,4} = w_{1,4}, \ w_{2,4} = w_{2,4}, \ w_{3,4} = 0, \ w_{t,1} = -x w_{xt,1}, \ w_{t,2} = -x w_{xt,2}, \\ w_{t,3} = w_{t,3}, \ w_{x,1} = -t w_{xt,1}, \ w_{x,2} = -t w_{xt,2}, \ w_{x,3} = w_{x,3}, \ w_{xt,1} = w_{xt,1}, \ w_{xt,2} = w_{xt,2}, \\ w_{xt,3} = w_{xt,3}, \ w_{t^2,1} = 0, \ w_{t^2,2} = 0, \ w_{t^2,3} = w_{t^2,3}, \ w_{x^2,1} = 0, \ w_{x^2,2} = 0, \ w_{x^2,3} = w_{x^2,3}. \end{array} \right\} \quad (4.2)$$

Substituting Eq (4.2) into Eq (4.1), we can obtain the solution to the initial equation, which is expressed as follows:

$$f := b_4 + w_{1,4} \sin(-xt w_{xt,1} + b_1) + w_{2,4} \cos(-xt w_{xt,2} + b_2). \quad (4.3)$$

The characteristics and variation patterns of the solution are further examined by assigning parameter values and generating plots in Maple. Figure 8 provides the three-dimensional surface plots, contour diagram, thermal map, and the temporal evolution profile of the solution.

4.2. Model [(3-9)-3-1]

Building on the solution of the (1+1)-dimensional equations, this paper generalizes the method to the (2+1)-dimensional Boussinesq equations: The three neurons in the input layer are expanded to nine through feature enhancement, resulting in the following trial function:

$$\psi_1(\zeta_1) = \sin(\zeta_1), \psi_2(\zeta_2) = \tanh(\zeta_2), \psi_3(\zeta_3) = e^{\zeta_3}.$$

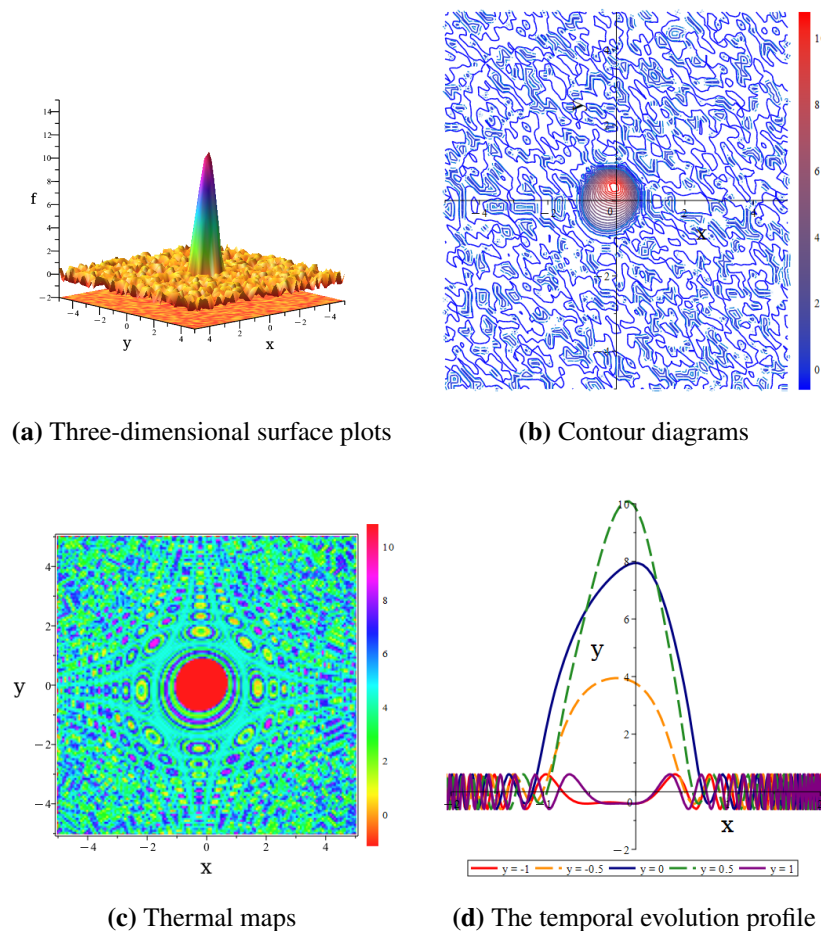


Figure 9. Three-dimensional surface plots, contour diagrams, thermal maps, and temporal evolution profile of Eq (4.6).

Based on Model [(3-9)-3-1], we can obtain the following probe function

$$f = b_4 + w_{1,4} \sin(xw_{x,1} + yw_{y,1} + tw_{t,1} + x^2w_{x^2,1} + y^2w_{y^2,1} + t^2w_{t^2,1} + xyw_{xy,1} + xtw_{xt,1} + ytw_{yt,1} + b_1) + w_{2,4} \tanh(xw_{x,1} + yw_{y,1} + tw_{t,1} + x^2w_{x^2,1} + y^2w_{y^2,1} + t^2w_{t^2,1} + xyw_{xy,1} + xtw_{xt,1} + ytw_{yt,1} + b_2) + w_{3,4} e^{(xw_{x,1} + yw_{y,1} + tw_{t,1} + x^2w_{x^2,1} + y^2w_{y^2,1} + t^2w_{t^2,1} + xyw_{xy,1} + xtw_{xt,1} + ytw_{yt,1} + b_3)}. \quad (4.4)$$

Substituting Eq (4.4) into Eq (1.1), we obtain the following solution:

$$\left\{ \begin{array}{l} t = t, \quad x = x, \quad y = y, \quad b_4 = b_4, \quad w_{1,4} = w_{1,4}, \quad w_{2,4} = w_{2,4}, \quad w_{3,4} = w_{3,4}, \quad w_{t,1} = -t, \\ w_{yt,1} - 2t w_{t^2,1} - x w_{xt,1} - x w_{xy,1} - y w_{yt,1} - 2y w_{y^2,1}, \quad w_{xt,1} = -t w_{xt,1} - y w_{xy,1}, \\ w_{xy,1} = w_{xy,1}, \quad w_{y,1} = w_{y,1}, \quad w_{yt,1} = w_{yt,1}, \quad w_{t^2,1} = w_{t^2,1}, \quad w_{y^2,1} = 0, \quad w_{x^2,1} = 0. \end{array} \right\} \quad (4.5)$$

Substituting Eq (4.5) into Eq (4.4), we can obtain the solution to the initial equation, which is expressed as follows:

$$f = b_4 + w_{1,4} \sin((-w_{yt,1} - w_{y^2,1})t^2 + (-2yw_{y^2,1} - w_{y,1} + (-w_{xt,1} - w_{xy,1})x)t + y^2w_{y^2,1} + yw_{y,1} + b_1) + w_{2,4} \tanh((-w_{yt,1} - w_{y^2,1})t^2 + (-2yw_{y^2,1} - w_{y,1} + (-w_{xt,1} - w_{xy,1})x)t + y^2w_{y^2,1} + yw_{y,1} + b_2) + w_{3,4} e^{((-w_{yt,1} - w_{y^2,1})t^2 + (-2yw_{y^2,1} - w_{y,1} + (-w_{xt,1} - w_{xy,1})x)t + y^2w_{y^2,1} + yw_{y,1} + b_3)}. \quad (4.6)$$

With parameter values assigned analogously, the solution is plotted in Maple. Figure 9 displays its 3D surface, contours, heat map, and evolution, illustrating the solution's features and trends.

5. Conclusions

This paper combines two methods—direct neural networks and feature-enhanced neural networks—to establish a novel symbolic computation framework for solving the (1+1)-dimensional and (2+1)-dimensional Boussinesq equations. In the study of the direct neural network method, we constructed two network models: a single-hidden-layer model [3-3-1] and a double-hidden-layer model [3-3-2-1]. By utilizing various activation functions—including sine, hyperbolic tangent, hyperbolic secant, and exponential—we obtained multiple sets of novel, exact analytical solutions, encompassing various solution types such as solitary waves, periodic waves, and rational function solutions. At the same time, this paper systematically verified the dynamical evolution characteristics of the solutions through 3D plots, contour plots, heatmaps, and evolution plots, confirming the outstanding advantages of this neural network model: A simple structure and high computational efficiency.

To address the shortcomings of direct neural network methods—namely, insufficient feature representation capability and weak generalization—this paper further employed feature-enhanced neural network methods to conduct optimization studies. For the (1+1)-dimensional Boussinesq equations, we constructed Model [(2-5)-3-1], expanding the number of input layer neurons from two to five; for the (2+1)-dimensional equation, we developed Model [(3-9)-3-1], expanding the number of input layer neurons from three to nine. Both dimensional models retained three hidden layer neurons and optimized the network structure based on the dimensional characteristics of the

equations. By reconstructing the physical, spatiotemporal, and nonlinear characteristics of the equations and solving them in conjunction with their inherent constraints, this method yields a larger number of analytical solutions with more complex structures. The results demonstrate that the feature enhancement mechanism effectively improves the solution accuracy and generalization performance of neural network models.

Overall, the method proposed in this paper does not require bilinear transformations, resulting in lower computational costs and greater adaptability. It provides a new research approach for the exact solution of strongly nonlinear water wave equations and can be extended in future research to the solution of fractional-order equations and various high-dimensional NLPDEs.

Author contributions

Conceptualization, J. S., Y. G., H. X. and K. D.; methodology, J. S.; validation, J. S. and Y. G.; writing—original draft preparation, J. L., J. S., H. X. and K. D.; writing—review and editing, J. S., H. X. and K. D.; supervision, J. S. and Y. G.; project administration, J. S., Y. G.; funding acquisition, J. S. and Y. G.

All authors have read and agreed to the published version of the manuscript.

Data availability statement

No data is associated in the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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