



Research article

Hopf bifurcation analysis of a delayed rumor propagation model in microblog networks

Wanmei Hou¹ and Xiangqing Zhao^{2,*}

¹ School of Marxism, Suqian University, Suqian 223800, China

² School of Mathematics and Physics, Suqian University, Suqian 223800, China

* **Correspondence:** Email: zho-xiangqing@163.com; Tel: +8618550722720.

Abstract: The rapid proliferation of online rumors undermines social stability, demanding dynamic analyses that transcend traditional steady-state assumptions. This study proposes a delayed Unknowns Neutrals Forwarders Refuters (UNFR) model that introduces discrete time delays into $R \rightarrow U$, $N \rightarrow R$, and $N \rightarrow F$ transitions to account for memory decay and decision-making hesitation. Employing characteristic equation analysis and Hopf bifurcation theory, alongside center manifold and normal form techniques, we examine the system's stability dynamics. We demonstrate that equilibria remain locally asymptotically stable below a critical delay; however, exceeding this threshold induces a Hopf bifurcation, giving rise to sustained periodic oscillations. Numerical simulations corroborate this transition from stability to periodicity, offering a theoretical basis for understanding the cyclical resurgence of rumors and informing effective, time-sensitive intervention strategies.

Keywords: online rumors; UNFR model; time delay; Hopf bifurcation; cyclical resurgence

1. Introduction

The rapid development of Online Social Networks (OSNs), such as WeChat and Twitter, has fundamentally accelerated information sharing but simultaneously provided a fertile ground for the spread of online rumors [1]. Mainstream platforms exhibit distinct propagation characteristics: For instance, WeChat's semi-closed social circles often amplify trust in misinformation within acquaintance networks, whereas Weibo's open, algorithm-driven nature can rapidly propel rumors to national prominence. Characterized by high velocity and broad coverage, these rumors can trigger social panic, damage reputations, and threaten public safety and national stability [2, 3]. Consequently, understanding the dynamic mechanisms of rumor spreading and formulating effective control strategies have become critical challenges in network science, sociology, and computer science.

To theoretically decode these mechanisms, researchers have increasingly drawn upon infectious

disease dynamics to construct mathematical models. While classical frameworks—such as the Daley–Kendall (DK) [4], Maki–Thompson (MK) [5], and various Susceptible–Infected (SI), Susceptible–Infected–Recovered (SIR), and Susceptible–Exposed–Infected–Recovered (SEIR) models [6, 7]—laid the foundation for rumor modeling, they often oversimplify user groups into broad categories, failing to capture the complex psychological states and behavioral decisions inherent in real networks. To achieve finer-grained modeling, recent studies have incorporated mechanisms like hesitation or stifling [8, 9] and developed advanced compartmental models (e.g., Susceptible–Exposed–Infected–Hesitant–Recovered (SEIHR) [10], Susceptible–Aware–Infected–Recovered (SAIR) [11]) to capture diverse transmission characteristics [12, 13]. A significant advancement in this direction is the UNFR model, which subdivides the population into Unknowns (U), Neutrals (N), Forwarders (F), and Refuters (R). By accounting for multiple transition paths, the UNFR framework provides a more realistic approximation of users’ psychological evolution—from initial unawareness and skepticism to active forwarding or refuting—than previous simpler models.

However, despite this structural sophistication, existing compartmental models largely focus on steady-state analysis and numerical simulations, typically assuming that all state transitions occur instantaneously. In reality, cognitive processes such as memory decay and information processing inherently consume time [14–16]. Within the UNFR framework, for instance, the reversion of Forwarders (F) to the Unknown state (U) due to lost interest, the decision-making process where Neutrals (N) evaluate rumors to become Forwarders (F) or Refuters (R), and the memory decay of Refuters (R) all involve significant temporal delays. These cognitive delays can be modeled as discrete time delays, which act as significant sources of system instability. When a delay crosses a critical threshold, the system may lose stability and undergo a Hopf bifurcation [17], transitioning from a stable equilibrium to sustained periodic oscillations [17–19]. From a macroscopic perspective, these mathematical oscillations correspond to the periodic resurgence of rumor popularity, explaining why rumors often “go viral” again even after initial debunking. Nevertheless, research on Hopf bifurcation remains scarce in models with complex user classifications like the UNFR framework. Furthermore, it remains unclear how this time-delay-induced instability manifests under different rumor prevalence scenarios—specifically, when the basic reproduction number $R_0 < 1$ versus $R_0 > 1$.

To bridge these gaps, this paper constructs a delayed UNFR model by introducing discrete time delays to comprehensively capture cognitive and memory decay mechanisms across different transition pathways. By combining analytical methods with symbolic computation to handle complex transcendental equations, we conduct a comprehensive Hopf bifurcation analysis. The core contributions of this study are as follows:

- **Existence of Hopf bifurcation thresholds:** We rigorously prove that time delay acts as a crucial switch affecting system stability. The system remains locally asymptotically stable when the delay is below a critical threshold but loses stability and undergoes a Hopf bifurcation exactly when the delay crosses that threshold.
- **Properties of bifurcating periodic solutions:** Beyond identifying the existence of the bifurcation, we apply the center manifold theorem and normal form theory to investigate its direction and stability. We demonstrate that the delayed UNFR model undergoes a supercritical Hopf bifurcation, yielding asymptotically stable periodic solutions with an increasing period, thus providing a rigorous theoretical explanation for sustained periodic rumor outbreaks.

- **Numerical validation:** We validate the theoretical predictions via MATLAB simulations, intuitively demonstrating the system's dynamic transition from stability to periodic oscillation as the delay crosses the bifurcation point.

The remainder of this paper is organized as follows: Section 2 formulates the delayed UNFR model. Section 3 analyzes the existence and local stability of the equilibria, detailing the Hopf bifurcation conditions for both rumor-free and rumor-endemic scenarios. Section 4 determines the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions using center manifold and normal form theories. Section 5 provides numerical examples to validate both bifurcation scenarios. Section 6 investigates the sensitivity of the basic reproduction number to system parameters to inform control strategies. Finally, Section 7 discusses the results and provides conclusions along with future research directions.

2. Mathematical model

Building on the UNFR framework proposed by Wang et al. [9], the total population is divided into four compartments: Unknowns (U), Neutrals (N), Forwarders (F), and Refuters (R). The model construction is based on the following assumptions regarding individual state transitions:

- A1: Upon initial exposure to a rumor,** an Unknown individual (U) undergoes a cognitive evaluation process leading to three possible states: becoming a Neutral (N) with probability α_2 , a Forwarder (F) with probability α_1 , or a Refuter (R) with probability α_3 .
- A2: Upon further exposure,** a Neutral individual (N) may make a definitive decision. They can be convinced and become a Forwarder (F) with probability β_1 , or become critical and transition to a Refuter (R) with probability β_2 .
- A3: Forwarders (F)** are subject to two competing mechanisms. Upon receiving debunking information, they transition to the Refuter (R) state with probability γ_1 . Concurrently, considering the timeliness of information memory, a Forwarder (F) may gradually lose interest and revert to the Unknown state (U) with probability γ_2 .
- A4: Over time,** members of the Refuter (R) group forget the specific details of the rumor or their refutation, thereby reverting to the Unknown state (U) with probability δ .

Based on Assumptions **A1–A4**, the original instantaneous UNFR model is derived as follows:

$$\begin{cases} \frac{du(t)}{dt} = -(\alpha_1 + \alpha_2 + \alpha_3)u(t)f(t) + \gamma_2f(t) + \delta r(t), \\ \frac{dn(t)}{dt} = \alpha_2u(t)f(t) - \beta_1n(t) - \beta_2n(t), \\ \frac{df(t)}{dt} = \alpha_1u(t)f(t) + \beta_1n(t) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3u(t)f(t) + \beta_2n(t) + \gamma_1f(t)r(t) - \delta r(t). \end{cases} \quad (2.1)$$

Although the aforementioned assumptions provide a foundational framework, the formulation of the model in Eq (2.1) presupposes that all state transitions occur instantaneously. To construct a more

realistic model that accounts for the time required for information processing and memory decay, we introduce the following assumption regarding time delays:

A5: Real-world rumor propagation involves inherent time delays due to cognitive processing and memory decay. Specifically, the reversion of Refuters to the Unknown state ($R \rightarrow U$) involves a memory decay delay τ_1 ; the transition of Neutrals to Refuters ($N \rightarrow R$) involves a decision-making delay τ_2 ; and the transition of Neutrals to Forwarders ($N \rightarrow F$) involves a decision-making delay τ_3 .

By incorporating these time-delay terms into the system in Eq (2.1) in accordance with Assumption **A5**, we derive the following delayed UNFR model:

$$\begin{cases} \frac{du(t)}{dt} = -(\alpha_1 + \alpha_2 + \alpha_3)u(t)f(t) + \gamma_2f(t) + \delta r(t - \tau_1), \\ \frac{dn(t)}{dt} = \alpha_2u(t)f(t) - \beta_1n(t - \tau_3) - \beta_2n(t - \tau_2), \\ \frac{df(t)}{dt} = \alpha_1u(t)f(t) + \beta_1n(t - \tau_3) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3u(t)f(t) + \beta_2n(t - \tau_2) + \gamma_1f(t)r(t) - \delta r(t - \tau_1). \end{cases} \quad (2.2)$$

The detailed descriptions of the parameters and time delays used in the model are provided in Table 1.

Table 1. Parameter notations and sociological meanings.

Symbol	Sociological Meaning
α_1	Probability of an unaware individual becoming a Forwarder upon exposure
α_2	Probability of an unaware individual becoming a Neutral
α_3	Probability of an unaware individual becoming a Refuter
β_1	Probability of a Neutral individual becoming a Forwarder
β_2	Probability of a Neutral individual becoming a Refuter
γ_1	Probability of a Forwarder becoming a Refuter upon seeing debunking info
γ_2	Rate at which Forwarders lose interest and revert to Unaware
δ	Rate at which Refuters forget details and revert to Unaware
τ_1	Time delay for the memory decay process ($R \rightarrow U$)
τ_2	Time delay for the decision-making process ($N \rightarrow R$)
τ_3	Time delay for the decision-making process ($N \rightarrow F$)

Before delving into the dynamics of the delayed model, it is insightful to first analyze the fundamental properties of the original non-delayed UNFR system in Eq (2.1). It is evident that there exists a rumor-free equilibrium $E_0 = (1, 0, 0, 0)$. By employing the Next-Generation Matrix method, the basic reproduction number is calculated as [9]:

$$R_0 = \frac{\alpha_2\beta_1 + \alpha_1(\beta_1 + \beta_2)}{\gamma_2(\beta_1 + \beta_2)}. \quad (2.3)$$

The threshold R_0 determines the fate of the rumor: If $R_0 < 1$, the rumor will eventually die out and the system will stabilize at the rumor-free equilibrium E_0 ; if $R_0 > 1$, the rumor will persist in the population. The subsequent sections investigate how the introduction of multiple time delays alters these steady states and potentially triggers bifurcations.

3. Analysis of Hopf bifurcation

The basic reproduction number R_0 determines the local stability of the rumor-free equilibrium E_0 and serves as the basis for parameter selection in subsequent numerical simulations. Beyond this steady-state analysis, our primary focus lies in the dynamic instability induced by time delays. As the delay varies, the roots of the system's transcendental characteristic equation may cross the imaginary axis, thereby inducing a Hopf bifurcation. The theoretical criterion for analyzing this phenomenon is provided by the following proposition:

Proposition 3.1 (See [20]). *Consider the system $\dot{x} = f(x, \theta)$, where $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}$. If the system satisfies:*

1. *For any θ such that $|\theta| < \theta_0$, $f(0, \theta) = 0$;*
2. *In the neighborhood of the point $(x, \theta) = (0, 0)$, the function $f(x, \theta)$ is analytic with respect to x and θ ;*
3. *The Jacobian matrix $A(\theta) = D_x f(0, \theta)$ has a pair of complex conjugate eigenvalues $\lambda(\theta)$ and $\bar{\lambda}(\theta)$ such that $\lambda(\theta) = \nu(\theta) + i\omega(\theta)$, where $\omega(0) > 0$, $\nu(0) = 0$, and the transversality condition $\nu'(0) \neq 0$ holds;*
4. *The remaining $n - 2$ eigenvalues of $A(0)$ all have strictly negative real parts;*

then the system undergoes a Hopf bifurcation at the equilibrium as θ crosses the critical value.

To investigate the Hopf bifurcation, we first determine the rumor-endemic equilibrium $E^* = (u^*, n^*, f^*, r^*)$.

By applying the conservation law $u(t) = 1 - n(t) - f(t) - r(t)$, the delayed the system in Eq (2.1) can be reduced to a three-dimensional system:

$$\begin{cases} \frac{dn(t)}{dt} = \alpha_2(1 - n(t) - f(t) - r(t))f(t) - (\beta_1 + \beta_2)n(t), \\ \frac{df(t)}{dt} = \alpha_1(1 - n(t) - f(t) - r(t))f(t) + \beta_1 n(t) - \gamma_2 f(t) - \gamma_1 f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3(1 - n(t) - f(t) - r(t))f(t) + \beta_2 n(t) + \gamma_1 f(t)r(t) - \delta r(t). \end{cases} \quad (3.1)$$

When $R_0 > 1$, the model in Eq (2.1) admits a unique rumor-endemic equilibrium $E^* = (u^*, n^*, f^*, r^*)$, where

$$n^* = \frac{\gamma_2 f^* + \delta r^*}{k},$$

$$f^* = \frac{g \delta r^*}{\gamma_1 r^* + \gamma_2(1 - g)},$$

$$r^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

$$u^* = 1 - n^* - f^* - r^*,$$

with the intermediate parameters A, B, C, k, e, g , and h defined as:

$$k = \frac{(\beta_1 + \beta_2)(\alpha_1 + \alpha_2 + \alpha_3)}{\alpha_2},$$

$$e = \frac{\alpha_1(\beta_1 + \beta_2) + \alpha_2\beta_1}{\alpha_2},$$

$$g = \frac{e}{k},$$

$$h = \frac{\beta_1 + \beta_2}{e},$$

$$A = \left[\left(\frac{\delta}{k} + 1 \right) \alpha_2 + h\gamma_1 \right] \gamma_1,$$

$$B = \alpha_2 \left(\gamma_2 + \frac{\delta\gamma_2}{k} + g\delta - g\gamma_2 - \gamma_1 \right) + h\gamma_1\gamma_2(2 - g),$$

$$C = (g - 1)(\alpha_2 - h\gamma_2)\gamma_2.$$

3.1. Hopf bifurcation associated with R-U delay

We first consider the effect of the time delay τ_1 in the R-U (memory decay delay) transition while keeping other delays zero ($\tau_2 = \tau_3 = 0$). The system equations are given by:

$$\begin{cases} \frac{du(t)}{dt} = -(\alpha_1 + \alpha_2 + \alpha_3)u(t)f(t) + \gamma_2f(t) + \delta r(t - \tau_1), \\ \frac{dn(t)}{dt} = \alpha_2u(t)f(t) - (\beta_1 + \beta_2)n(t), \\ \frac{df(t)}{dt} = \alpha_1u(t)f(t) + \beta_1n(t) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3u(t)f(t) + \beta_2n(t) + \gamma_1f(t)r(t) - \delta r(t - \tau_1). \end{cases} \quad (3.2)$$

Applying the conservation law $u(t) = 1 - n(t) - f(t) - r(t)$ to eliminate $u(t)$, the system reduces to:

$$\begin{cases} \frac{dn(t)}{dt} = \alpha_2 [1 - n(t) - f(t) - r(t)] f(t) - (\beta_1 + \beta_2)n(t), \\ \frac{df(t)}{dt} = \alpha_1 [1 - n(t) - f(t) - r(t)] f(t) + \beta_1n(t) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3 [1 - n(t) - f(t) - r(t)] f(t) + \beta_2n(t) + \gamma_1f(t)r(t) - \delta r(t - \tau_1). \end{cases} \quad (3.3)$$

To analyze the local stability of the endemic equilibrium $E^*(n^*, f^*, r^*)$, we linearize the system in Eq (3.3) around E^* . The Jacobian matrix consists of two parts: the instantaneous part A and the delayed part B . The entries a_{ij} of the instantaneous matrix A are computed by evaluating the partial derivatives

of the non-delayed terms at E^* . Specifically:

$$\begin{aligned} a_{11} &= \frac{\partial \dot{n}}{\partial n} \Big|_{E^*} = -\beta_1 + \beta_2 - \alpha_2 f^*, & a_{12} &= \frac{\partial \dot{n}}{\partial f} \Big|_{E^*} = \alpha_2 u^* - \alpha_2 f^*, & a_{13} &= \frac{\partial \dot{n}}{\partial r} \Big|_{E^*} = -\alpha_2 f^*, \\ a_{21} &= \frac{\partial \dot{f}}{\partial n} \Big|_{E^*} = \beta_1 - \alpha_1 f^*, & a_{22} &= \frac{\partial \dot{f}}{\partial f} \Big|_{E^*} = \alpha_1 u^* - \alpha_1 f^* - \gamma_2 - \gamma_1 r^*, & a_{23} &= \frac{\partial \dot{f}}{\partial r} \Big|_{E^*} = -(\alpha_1 + \gamma_1) f^*, \\ a_{31} &= \frac{\partial \dot{r}}{\partial n} \Big|_{E^*} = \beta_2 - \alpha_3 f^*, & a_{32} &= \frac{\partial \dot{r}}{\partial f} \Big|_{E^*} = \alpha_3 u^* - \alpha_3 f^* + \gamma_1 r^*, & a_{33} &= \frac{\partial \dot{r}}{\partial r} \Big|_{E^*} = (\gamma_1 - \alpha_3) f^*. \end{aligned}$$

Since the delay τ_1 appears in the third equation, the term $-\delta r(t - \tau_1)$ contributes $-\delta e^{-\lambda \tau_1}$ to the (3, 3) entry of the characteristic matrix. Thus, the linearized system matrix at E^* is:

$$J(E^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} - \delta e^{-\lambda \tau_1} \end{pmatrix}. \quad (3.4)$$

The characteristic equation is obtained from $\det(J(E^*) - \lambda I) = 0$:

$$\lambda^3 + P_3 \lambda^2 + P_2 \lambda + P_1 + e^{-\lambda \tau_1} (Q_3 \lambda^2 + Q_2 \lambda + Q_1) = 0, \quad (3.5)$$

where the coefficients are derived from the determinant expansion:

$$\begin{aligned} P_3 &= -(a_{11} + a_{22} + a_{33}), \\ P_2 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ P_1 &= -(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21}), \\ Q_3 &= -\delta, \\ Q_2 &= \delta(a_{11} + a_{22}), \\ Q_1 &= -\delta(a_{11}a_{22} - a_{12}a_{21}). \end{aligned}$$

Case 1: When $\tau_1 = 0$, the characteristic equation (3.5) reduces to $\lambda^3 + (P_3 + Q_3)\lambda^2 + (P_2 + Q_2)\lambda + (P_1 + Q_1) = 0$. By the Routh–Hurwitz criterion, E^* is locally asymptotically stable if:

$$P_3 + Q_3 > 0, \quad P_1 + Q_1 > 0, \quad (P_3 + Q_3)(P_2 + Q_2) > P_1 + Q_1. \quad (3.6)$$

Theorem 3.1. *If the conditions in Eq (3.6) hold, the rumor-endemic equilibrium E^* of the system with τ_1 is locally asymptotically stable when $\tau_1 = 0$.*

Case 2: When $\tau_1 > 0$, let $\lambda = i\omega$ ($\omega > 0$). Substituting $\lambda = i\omega$ into Eq (3.5) and separating the real and imaginary parts yields:

$$\begin{cases} (P_1 - P_3\omega^2) \cos(\tau_1\omega) - (P_2\omega - \omega^3) \sin(\tau_1\omega) = -(Q_1 - Q_3\omega^2), \\ (P_2\omega - \omega^3) \cos(\tau_1\omega) + (P_1 - P_3\omega^2) \sin(\tau_1\omega) = -Q_2\omega. \end{cases}$$

Squaring and adding these two equations to eliminate the trigonometric functions gives the equation for ω :

$$\omega^6 + C_2\omega^4 + C_1\omega^2 + C_0 = 0,$$

where $C_2 = P_3^2 - 2P_2 - Q_3^2$, $C_1 = P_2^2 - 2P_1P_3 + 2Q_1Q_3 - Q_2^2$, and $C_0 = P_1^2 - Q_1^2$. Assuming $C_0 = P_1^2 - Q_1^2 < 0$, there exists a unique positive root ω_0^* . The critical delay is then:

$$\tau_{10} = \frac{1}{\omega_0} \arccos \left(\frac{-(P_1 - P_3\omega_0^2)(Q_1 - Q_3\omega_0^2) - (P_2\omega_0 - \omega_0^3)Q_2\omega_0}{(P_1 - P_3\omega_0^2)^2 + (P_2\omega_0 - \omega_0^3)^2} \right).$$

To verify the transversality condition, we differentiate the characteristic equation (3.5) implicitly with respect to τ_1 :

$$\left[\frac{d\lambda}{d\tau_1} \right]^{-1} = \frac{3\lambda^2 + 2P_3\lambda + P_2}{\lambda e^{-\lambda\tau_1}(Q_3\lambda^2 + Q_2\lambda + Q_1)} + \frac{2Q_3\lambda + Q_2}{\lambda(Q_3\lambda^2 + Q_2\lambda + Q_1)} - \frac{\tau_1}{\lambda}.$$

Evaluating the real part at $\lambda = i\omega_0$ and $\tau_1 = \tau_{10}$, if $\operatorname{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_{10}}^{-1} \neq 0$ (or equivalently $\operatorname{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_{10}} \neq 0$), the transversality condition holds. By Proposition 3.1, we have the following theorem:

Theorem 3.2. Assume that $R_0 > 1$. If $C_0 < 0$ and the transversality condition $\operatorname{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_{10}}^{-1} \neq 0$ holds, then the following statements are true for the system in Eq (3.3):

- The rumor-endemic equilibrium E^* is locally asymptotically stable for $\tau_1 \in [0, \tau_{10})$;
- A Hopf bifurcation occurs at E^* when $\tau_1 = \tau_{10}$;
- The rumor-endemic equilibrium E^* is unstable for $\tau_1 > \tau_{10}$.

3.2. Hopf bifurcation associated with N-R delay

Next, we analyze the impact of the N-R delay τ_2 ($\tau_1 = 0$). The system is given by:

$$\begin{cases} \frac{du(t)}{dt} = -(\alpha_1 + \alpha_2 + \alpha_3)u(t)f(t) + \gamma_2f(t) + \delta r(t), \\ \frac{dn(t)}{dt} = \alpha_2u(t)f(t) - \beta_1n(t) - \beta_2n(t - \tau_2), \\ \frac{df(t)}{dt} = \alpha_1u(t)f(t) + \beta_1n(t) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3u(t)f(t) + \beta_2n(t - \tau_2) + \gamma_1f(t)r(t) - \delta r(t). \end{cases} \quad (3.7)$$

Applying the same conservation law, the system reduces to:

$$\begin{cases} \frac{dn(t)}{dt} = \alpha_2 [1 - n(t) - f(t) - r(t)] f(t) - \beta_1n(t) - \beta_2n(t - \tau_2), \\ \frac{df(t)}{dt} = \alpha_1 [1 - n(t) - f(t) - r(t)] f(t) + \beta_1n(t) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3 [1 - n(t) - f(t) - r(t)] f(t) + \beta_2n(t - \tau_2) + \gamma_1f(t)r(t) - \delta r(t). \end{cases} \quad (3.8)$$

*Let $f(x) = x^3 + C_2x^2 + C_1x + C_0$. Since $f(0) = C_0 < 0$ and $\lim_{x \rightarrow +\infty} f(x) = +\infty$, there exists a positive number x_0 such that $f(x_0) = 0$.

To analyze the local stability of E^* , we linearize the system in Eq (3.8) around E^* . The entries a_{ij} of the instantaneous matrix A are computed by evaluating the partial derivatives of the non-delayed terms at E^* . Note that the delayed term $\beta_2 n(t - \tau_2)$ does not contribute to the instantaneous coefficients a_{ij} :

$$\begin{aligned} a_{11} &= -\beta_1 - \alpha_2 f^*, & a_{12} &= \alpha_2 u^* - \alpha_2 f^*, & a_{13} &= -\alpha_2 f^*, \\ a_{21} &= \beta_1 - \alpha_1 f^*, & a_{22} &= \alpha_1 u^* - \alpha_1 f^* - \gamma_2 - \gamma_1 r^*, & a_{23} &= -(\alpha_1 + \gamma_1) f^*, \\ a_{31} &= -\alpha_3 f^*, & a_{32} &= \alpha_3 u^* - \alpha_3 f^* + \gamma_1 r^*, & a_{33} &= -(\alpha_3 - \gamma_1) f^* - \delta. \end{aligned}$$

Since the delay τ_2 appears in the first and third equations, the term $-\beta_2 n(t - \tau_2)$ contributes $-\beta_2 e^{-\lambda \tau_2}$ to the (1, 1) entry, and the term $\beta_2 n(t - \tau_2)$ contributes $\beta_2 e^{-\lambda \tau_2}$ to the (3, 1) entry of the characteristic matrix. Thus, the linearized system matrix at E^* is:

$$J(E^*) = \begin{pmatrix} a_{11} - \beta_2 e^{-\lambda \tau_2} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + \beta_2 e^{-\lambda \tau_2} & a_{32} & a_{33} \end{pmatrix}.$$

The characteristic equation is obtained from $\det(J(E^*) - \lambda I) = 0$:

$$\lambda^3 + P_3 \lambda^2 + P_2 \lambda + P_1 + e^{-\lambda \tau_2} (Q_3 \lambda^2 + Q_2 \lambda + Q_1) = 0, \quad (3.9)$$

where the coefficients are derived from the determinant expansion:

$$\begin{aligned} P_3 &= -(a_{11} + a_{22} + a_{33}), \\ P_2 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ P_1 &= -(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21}), \\ Q_3 &= -\beta_2, \\ Q_2 &= \beta_2(a_{22} + a_{33} + a_{13}), \\ Q_1 &= -\beta_2 [a_{22}(a_{33} + a_{13}) - a_{23}(a_{12} + a_{32})]. \end{aligned}$$

Case 1: When $\tau_2 = 0$, the characteristic equation (3.9) reduces to $\lambda^3 + (P_3 + Q_3)\lambda^2 + (P_2 + Q_2)\lambda + (P_1 + Q_1) = 0$. By the Routh–Hurwitz criterion, E^* is locally asymptotically stable if:

$$P_3 + Q_3 > 0, \quad P_1 + Q_1 > 0, \quad (P_3 + Q_3)(P_2 + Q_2) > P_1 + Q_1. \quad (3.10)$$

Theorem 3.3. *If the conditions in Eq (3.10) hold, the rumor-endemic equilibrium E^* of the system with τ_2 is locally asymptotically stable when $\tau_2 = 0$.*

Case 2: When $\tau_2 > 0$, let $\lambda = i\omega$ ($\omega > 0$). Substituting $\lambda = i\omega$ into Eq (3.9) and separating the real and imaginary parts yields:

$$\begin{cases} (P_1 - P_3\omega^2) \cos(\tau_2\omega) - (P_2\omega - \omega^3) \sin(\tau_2\omega) = -(Q_1 - Q_3\omega^2), \\ (P_2\omega - \omega^3) \cos(\tau_2\omega) + (P_1 - P_3\omega^2) \sin(\tau_2\omega) = -Q_2\omega. \end{cases}$$

Squaring and adding these two equations to eliminate the trigonometric functions gives the equation for ω :

$$\omega^6 + C_2\omega^4 + C_1\omega^2 + C_0 = 0,$$

where $C_2 = P_3^2 - 2P_2 - Q_3^2$, $C_1 = P_2^2 - 2P_1P_3 + 2Q_1Q_3 - Q_2^2$, and $C_0 = P_1^2 - Q_1^2$. Assuming $C_0 = P_1^2 - Q_1^2 < 0$, there exists a unique positive root ω_0 . The critical delay is then:

$$\tau_{20} = \frac{1}{\omega_0} \arccos \left(\frac{-(P_1 - P_3\omega_0^2)(Q_1 - Q_3\omega_0^2) - (P_2\omega_0 - \omega_0^3)Q_2\omega_0}{(P_1 - P_3\omega_0^2)^2 + (P_2\omega_0 - \omega_0^3)^2} \right).$$

To verify the transversality condition, we differentiate the characteristic equation (3.9) implicitly with respect to τ_2 :

$$\left[\frac{d\lambda}{d\tau_2} \right]^{-1} = \frac{3\lambda^2 + 2P_3\lambda + P_2}{\lambda e^{-\lambda\tau_2}(Q_3\lambda^2 + Q_2\lambda + Q_1)} + \frac{2Q_3\lambda + Q_2}{\lambda(Q_3\lambda^2 + Q_2\lambda + Q_1)} - \frac{\tau_2}{\lambda}.$$

Evaluating the real part at $\lambda = i\omega_0$ and $\tau_2 = \tau_{20}$, if $\operatorname{Re} \left[\frac{d\lambda}{d\tau_2} \right]_{\tau_{20}}^{-1} \neq 0$ (or equivalently $\operatorname{Re} \left[\frac{d\lambda}{d\tau_2} \right]_{\tau_{20}} \neq 0$), the transversality condition holds. By Proposition 3.1, we have the following theorem:

Theorem 3.4. Assume that $R_0 > 1$. If $C_0 < 0$ and the transversality condition $\operatorname{Re} \left[\frac{d\lambda}{d\tau_2} \right]_{\tau_{20}}^{-1} \neq 0$ holds, then the following statements are true for the system in Eq (3.8):

- The rumor-endemic equilibrium E^* is locally asymptotically stable for $\tau_2 \in [0, \tau_{20})$;
- A Hopf bifurcation occurs at E^* when $\tau_2 = \tau_{20}$;
- The rumor-endemic equilibrium E^* is unstable for $\tau_2 > \tau_{20}$.

3.3. Hopf bifurcation associated with N-F delay

Finally, we examine the impact of the N-F delay τ_3 ($\tau_1 = \tau_2 = 0$). The system is given by:

$$\begin{cases} \frac{du(t)}{dt} = -(\alpha_1 + \alpha_2 + \alpha_3)u(t)f(t) + \gamma_2f(t) + \delta r(t), \\ \frac{dn(t)}{dt} = \alpha_2u(t)f(t) - \beta_1n(t - \tau_3) - \beta_2n(t), \\ \frac{df(t)}{dt} = \alpha_1u(t)f(t) + \beta_1n(t - \tau_3) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3u(t)f(t) + \beta_2n(t) + \gamma_1f(t)r(t) - \delta r(t). \end{cases} \quad (3.11)$$

Applying the same conservation law, the system reduces to:

$$\begin{cases} \frac{dn(t)}{dt} = \alpha_2 [1 - n(t) - f(t) - r(t)] f(t) - \beta_1n(t - \tau_3) - \beta_2n(t), \\ \frac{df(t)}{dt} = \alpha_1 [1 - n(t) - f(t) - r(t)] f(t) + \beta_1n(t - \tau_3) - \gamma_2f(t) - \gamma_1f(t)r(t), \\ \frac{dr(t)}{dt} = \alpha_3 [1 - n(t) - f(t) - r(t)] f(t) + \beta_2n(t) + \gamma_1f(t)r(t) - \delta r(t). \end{cases} \quad (3.12)$$

Similarly, the entries a_{ij} of the instantaneous matrix A are obtained by differentiating the non-delayed terms at E^* . The delayed term $\beta_1n(t - \tau_3)$ affects the delayed matrix entries (1, 1) and (2, 1),

but not the instantaneous coefficient a_{21} :

$$\begin{aligned} a_{11} &= -\beta_2 - \alpha_2 f^*, & a_{12} &= \alpha_2 u^* - \alpha_2 f^*, & a_{13} &= -\alpha_2 f^*, \\ a_{21} &= -\alpha_1 f^*, & a_{22} &= \alpha_1 u^* - \alpha_1 f^* - \gamma_2 - \gamma_1 r^*, & a_{23} &= -(\alpha_1 + \gamma_1) f^*, \\ a_{31} &= \beta_2 - \alpha_3 f^*, & a_{32} &= \alpha_3 u^* - \alpha_3 f^* + \gamma_1 r^*, & a_{33} &= -(\alpha_3 - \gamma_1) f^* - \delta. \end{aligned}$$

The term $-\beta_1 n(t - \tau_3)$ contributes $-\beta_1 e^{-\lambda \tau_3}$ to the (1, 1) entry, and $\beta_1 n(t - \tau_3)$ contributes $\beta_1 e^{-\lambda \tau_3}$ to the (2, 1) entry. Thus, the linearized system matrix at E^* is:

$$J(E^*) = \begin{pmatrix} a_{11} - \beta_1 e^{-\lambda \tau_3} & a_{12} & a_{13} \\ a_{21} + \beta_1 e^{-\lambda \tau_3} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The characteristic equation is obtained from $\det(J(E^*) - \lambda I) = 0$:

$$\lambda^3 + P_3 \lambda^2 + P_2 \lambda + P_1 + e^{-\lambda \tau_3} (Q_3 \lambda^2 + Q_2 \lambda + Q_1) = 0, \quad (3.13)$$

where the coefficients are derived from the determinant expansion:

$$\begin{aligned} P_3 &= -(a_{11} + a_{22} + a_{33}), \\ P_2 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ P_1 &= -(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21}), \\ Q_3 &= -\beta_1, \\ Q_2 &= \beta_1(a_{22} + a_{12} + a_{33}), \\ Q_1 &= -\beta_1 [(a_{22} + a_{12})a_{33} - (a_{13} + a_{23})a_{32}]. \end{aligned}$$

Case 1: When $\tau_3 = 0$, the characteristic equation (3.13) reduces to $\lambda^3 + (P_3 + Q_3)\lambda^2 + (P_2 + Q_2)\lambda + (P_1 + Q_1) = 0$. By the Routh–Hurwitz criterion, E^* is locally asymptotically stable if:

$$P_3 + Q_3 > 0, \quad P_1 + Q_1 > 0, \quad (P_3 + Q_3)(P_2 + Q_2) > P_1 + Q_1. \quad (3.14)$$

Theorem 3.5. *If the conditions in Eq (3.14) hold, the rumor-endemic equilibrium E^* of the system with τ_3 is locally asymptotically stable when $\tau_3 = 0$.*

Case 2: When $\tau_3 > 0$, let $\lambda = i\omega$ ($\omega > 0$). Substituting $\lambda = i\omega$ into Eq (3.13) and separating the real and imaginary parts yields:

$$\begin{cases} (P_1 - P_3\omega^2) \cos(\tau_3\omega) - (P_2\omega - \omega^3) \sin(\tau_3\omega) = -(Q_1 - Q_3\omega^2), \\ (P_2\omega - \omega^3) \cos(\tau_3\omega) + (P_1 - P_3\omega^2) \sin(\tau_3\omega) = -Q_2\omega. \end{cases}$$

Squaring and adding these two equations to eliminate the trigonometric functions gives the equation for ω :

$$\omega^6 + C_2\omega^4 + C_1\omega^2 + C_0 = 0,$$

where $C_2 = P_3^2 - 2P_2 - Q_3^2$, $C_1 = P_2^2 - 2P_1P_3 + 2Q_1Q_3 - Q_2^2$, and $C_0 = P_1^2 - Q_1^2$. Assuming $C_0 = P_1^2 - Q_1^2 < 0$, there exists a unique positive root ω_0 . The critical delay is then:

$$\tau_{30} = \frac{1}{\omega_0} \arccos \left(\frac{-(P_1 - P_3\omega_0^2)(Q_1 - Q_3\omega_0^2) - (P_2\omega_0 - \omega_0^3)Q_2\omega_0}{(P_1 - P_3\omega_0^2)^2 + (P_2\omega_0 - \omega_0^3)^2} \right).$$

To verify the transversality condition, we differentiate the characteristic equation (3.13) implicitly with respect to τ_3 :

$$\left[\frac{d\lambda}{d\tau_3} \right]^{-1} = \frac{3\lambda^2 + 2P_3\lambda + P_2}{\lambda e^{-\lambda\tau_3}(Q_3\lambda^2 + Q_2\lambda + Q_1)} + \frac{2Q_3\lambda + Q_2}{\lambda(Q_3\lambda^2 + Q_2\lambda + Q_1)} - \frac{\tau_3}{\lambda}.$$

Evaluating the real part at $\lambda = i\omega_0$ and $\tau_3 = \tau_{30}$, if $\operatorname{Re} \left[\frac{d\lambda}{d\tau_3} \right]_{\tau_{30}}^{-1} \neq 0$ (or equivalently $\operatorname{Re} \left[\frac{d\lambda}{d\tau_3} \right]_{\tau_{30}} \neq 0$), the transversality condition holds. By Proposition 3.1, we have the following theorem:

Theorem 3.6. Assume that $R_0 > 1$. If $C_0 < 0$ and the transversality condition $\operatorname{Re} \left[\frac{d\lambda}{d\tau_3} \right]_{\tau_{30}}^{-1} \neq 0$ holds, then the following statements are true for the system in Eq (3.12):

- The rumor-endemic equilibrium E^* is locally asymptotically stable for $\tau_3 \in [0, \tau_{30})$;
- A Hopf bifurcation occurs at E^* when $\tau_3 = \tau_{30}$;
- The rumor-endemic equilibrium E^* is unstable for $\tau_3 > \tau_{30}$.

4. Numerical examples

To validate the theoretical results derived in the previous section, we conduct numerical simulations using the MATLAB solver `dde23`. We select specific parameter sets for each delay scenario (τ_1, τ_2, τ_3) satisfying the conditions $R_0 > 1$ and $C_0 < 0$. For each case, we demonstrate the dynamic transition of the system as the delay varies across the critical threshold, confirming the stability, bifurcation, and instability predicted by the theorems.

4.1. Simulation for system with R-U delay

In this section, we utilize the MATLAB solver `dde23` to conduct numerical simulations, verifying the existence of the Hopf bifurcation induced by the R-U delay τ_1 . Substituting the selected parameters $\alpha_1 = 0.4, \alpha_2 = 0.2, \alpha_3 = 0.4, \beta_1 = 0.15, \beta_2 = 0.2, \gamma_1 = 0.03, \gamma_2 = 0.4$, and $\delta = 0.6$ into the model, the delayed system is given by:

$$\begin{cases} \frac{du(t)}{dt} = -1.0u(t)f(t) + 0.4f(t) + 0.6r(t - \tau_1), \\ \frac{dn(t)}{dt} = 0.2u(t)f(t) - 0.35n(t), \\ \frac{df(t)}{dt} = 0.4u(t)f(t) + 0.15n(t) - 0.4f(t) - 0.03f(t)r(t), \\ \frac{dr(t)}{dt} = 0.4u(t)f(t) + 0.2n(t) + 0.03f(t)r(t) - 0.6r(t - \tau_1). \end{cases} \quad (4.1)$$

Using these parameters, the basic reproduction number is calculated as $R_0 \approx 1.2143 > 1$. Numerically solving the equilibrium equations yields the rumor-endemic equilibrium:

$$E^* \approx (0.8270, 0.0374, 0.0792, 0.0564).$$

Based on the stability analysis, we compute the characteristic equation coefficients at E^* . At $\tau_1 = 0$, the Routh–Hurwitz conditions yield $Y_3 \approx 1.0977 > 0$, $Y_1 \approx 0.0179 > 0$, and $Y_3 Y_2 - Y_1 \approx 0.3416 > 0$. The corresponding eigenvalues are approximately -0.6062 , -0.4213 , and -0.0702 , all of which have negative real parts. This indicates that the system is locally asymptotically stable in the absence of the R-U delay.

Furthermore, we verify the conditions for the Hopf bifurcation. The coefficient C_0 is calculated as:

$$C_0 = P_1^2 - Q_1^2 \approx -1.05 \times 10^{-4} < 0.$$

Solving the polynomial equation for ω gives the positive root $\omega_0 \approx 0.6199$. The corresponding critical delay is calculated as $\tau_{10} \approx 2.62$. Moreover, the transversality condition is evaluated as $\text{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_{10}}^{-1} \approx 2.896 \neq 0$, which implies that $\text{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_{10}} > 0$. This positive transversality value indicates that the real parts of the eigenvalues increase as τ_1 crosses the critical threshold. Consequently, the system undergoes a Hopf bifurcation at $\tau_1 = \tau_{10}$, losing stability and giving rise to periodic solutions for $\tau_1 > 2.62$.

Physically, the critical delay τ_{10} represents the maximum allowable time for the memory decay of Refuters (R) that maintains system stability. If the process of forgetting rumor details takes longer than this threshold, the delayed feedback disrupts the system's balance. This lag prevents the population from settling into a steady state, causing the rumor prevalence to oscillate indefinitely rather than stabilizing at a constant level.

To observe the system dynamics, we perform simulations with the constant history functions $u(t) = 0.7, n(t) = 0.1, f(t) = 0.1, r(t) = 0.1$ for $t \in [-\tau_1, 0]$.

Case 1: $\tau_1 = 2.46 < \tau_{10}$ (Stability). The system trajectories converge to the equilibrium E^* . As shown in Figure 1(a), the time series indicate that the system eventually stabilizes at an endemic level. The phase portrait in Figure 1(b) depicts trajectories spiraling towards the equilibrium point, confirming that E^* is locally asymptotically stable.

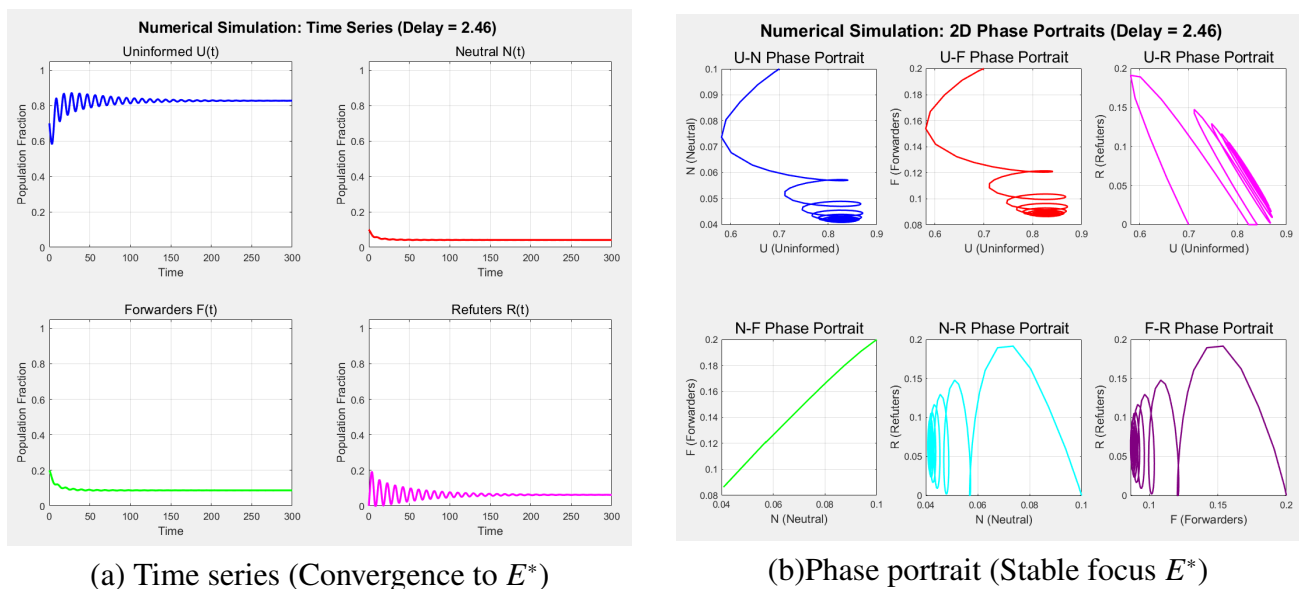


Figure 1. Simulation results when $\tau_1 = 2.46 < \tau_{10}$. The system converges to the endemic equilibrium E^* .

Case 2: $\tau_1 = 2.62 = \tau_{10}$ (Hopf bifurcation). As the delay reaches the critical threshold, the characteristic equation has a pair of purely imaginary eigenvalues, indicating the onset of a Hopf bifurcation. As shown in Figure 2(a),(b), the system exhibits the critical transition behavior where the stability of the equilibrium changes.

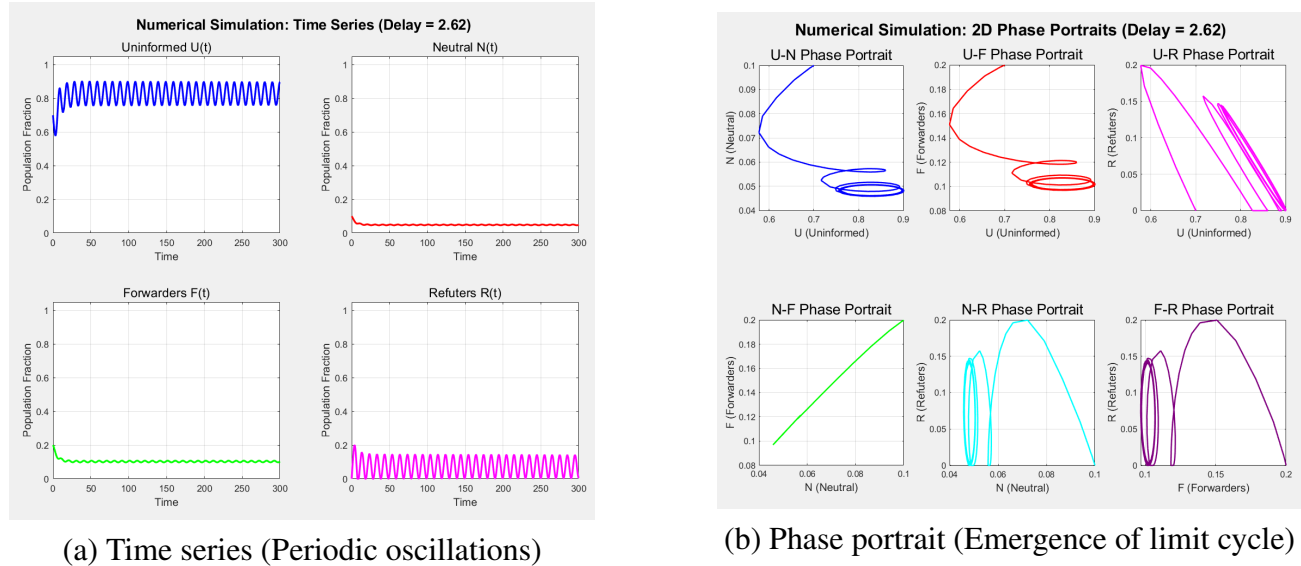


Figure 2. Simulation results when $\tau_1 = 2.62 = \tau_{10}$. A Hopf bifurcation occurs.

Case 3: $\tau_1 = 2.72 > \tau_{10}$ (Instability). When the delay exceeds the critical value, the equilibrium E^* loses stability and becomes an unstable focus. The system trajectories diverge from E^* and converge to a stable limit cycle. As shown in Figure 3(a),(b), sustained periodic oscillations emerge, confirming the instability of the endemic equilibrium.

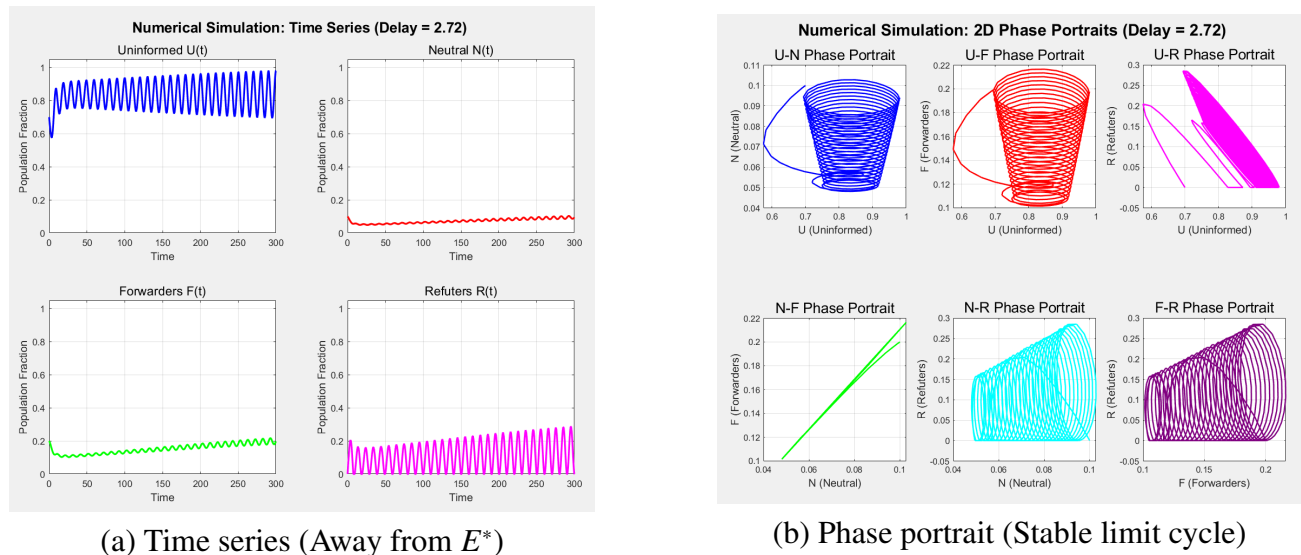


Figure 3. Simulation results when $\tau_1 = 2.72 > \tau_{10}$. The equilibrium E^* is unstable, and a stable limit cycle emerges.

These numerical results confirm the theoretical analysis, demonstrating that the R-U delay τ_1 can destabilize the rumor-endemic equilibrium when it exceeds a critical value, leading to persistent periodic rumor spreading.

4.2. Simulation for system with N-R delay

In this section, we utilize the MATLAB solver dde23 to conduct numerical simulations, verifying the existence of the Hopf bifurcation induced by the N-R delay τ_2 . Substituting the selected parameters $\alpha_1 = 0.3$, $\alpha_2 = 0.2$, $\alpha_3 = 0.5$, $\beta_1 = 0.1$, $\beta_2 = 0.15$, $\gamma_1 = 0.05$, $\gamma_2 = 0.15$, and $\delta = 0.1$ into the model, the delayed system is given by:

$$\begin{cases} \frac{du(t)}{dt} = -1.0u(t)f(t) + 0.15f(t) + 0.1r(t), \\ \frac{dn(t)}{dt} = 0.2u(t)f(t) - 0.1n(t) - 0.15n(t - \tau_2), \\ \frac{df(t)}{dt} = 0.3u(t)f(t) + 0.1n(t) - 0.15f(t) - 0.05f(t)r(t), \\ \frac{dr(t)}{dt} = 0.5u(t)f(t) + 0.15n(t - \tau_2) + 0.05f(t)r(t) - 0.1r(t). \end{cases} \quad (4.2)$$

Using these parameters, the basic reproduction number is calculated as $R_0 \approx 2.5333 > 1$. Numerically solving the equilibrium equations yields the rumor-endemic equilibrium:

$$E^* \approx (0.4448, 0.0459, 0.1290, 0.3803).$$

Based on the stability analysis, we compute the characteristic equation coefficients at E^* . At $\tau_2 = 0$, the Routh–Hurwitz conditions yield $Y_3 \approx 0.5081 > 0$, $Y_1 \approx 0.0055 > 0$, and $Y_3Y_2 - Y_1 \approx 0.0362 > 0$. The corresponding eigenvalues are approximately -0.2910 , -0.1086 , and -0.1086 , all of which have negative real parts. This indicates that the system is locally asymptotically stable in the absence of the N-R delay.

Furthermore, we verify the conditions for the Hopf bifurcation. The coefficient C_0 is calculated as:

$$C_0 = P_1^2 - Q_1^2 \approx -1.0 \times 10^{-5} < 0.$$

Solving the polynomial equation for ω gives the positive root $\omega_0 \approx 0.1223$. The corresponding critical delay is calculated as $\tau_{20} \approx 16.96$. Moreover, the transversality condition is evaluated as $\text{Re} \left[\frac{d\lambda}{d\tau_2} \right]_{\tau_{20}}^{-1} \approx 98.81 \neq 0$, which implies that $\text{Re} \left[\frac{d\lambda}{d\tau_2} \right]_{\tau_{20}} > 0$. This positive transversality value indicates that the real parts of the eigenvalues increase as τ_2 crosses the critical threshold. Consequently, the system undergoes a Hopf bifurcation at $\tau_2 = \tau_{20}$, losing stability and giving rise to periodic solutions for $\tau_2 > 16.96$.

Physically, the critical delay τ_{20} represents a threshold for the decision-making time of Neutrals (N) to become Refuters (R). When this cognitive process is delayed beyond this limit, the effect of refutation lags behind the propagation. This insufficient timely feedback causes the system to lose its steady state, resulting in periodic fluctuations in rumor activity rather than a constant endemic level.

To observe the system dynamics, we perform simulations with the constant history functions $u(t) = 0.7$, $n(t) = 0.1$, $f(t) = 0.1$, $r(t) = 0.1$ for $t \in [-\tau_2, 0]$.

Case 1: $\tau_2 = 16.0 < \tau_{20}$ (Stability). The system trajectories converge to the equilibrium E^* . As shown in Figure 4(a), the time series indicate that the system eventually stabilizes at an endemic level. The phase portrait in Figure 4(b) depicts trajectories spiraling towards the equilibrium point, confirming that E^* is locally asymptotically stable.

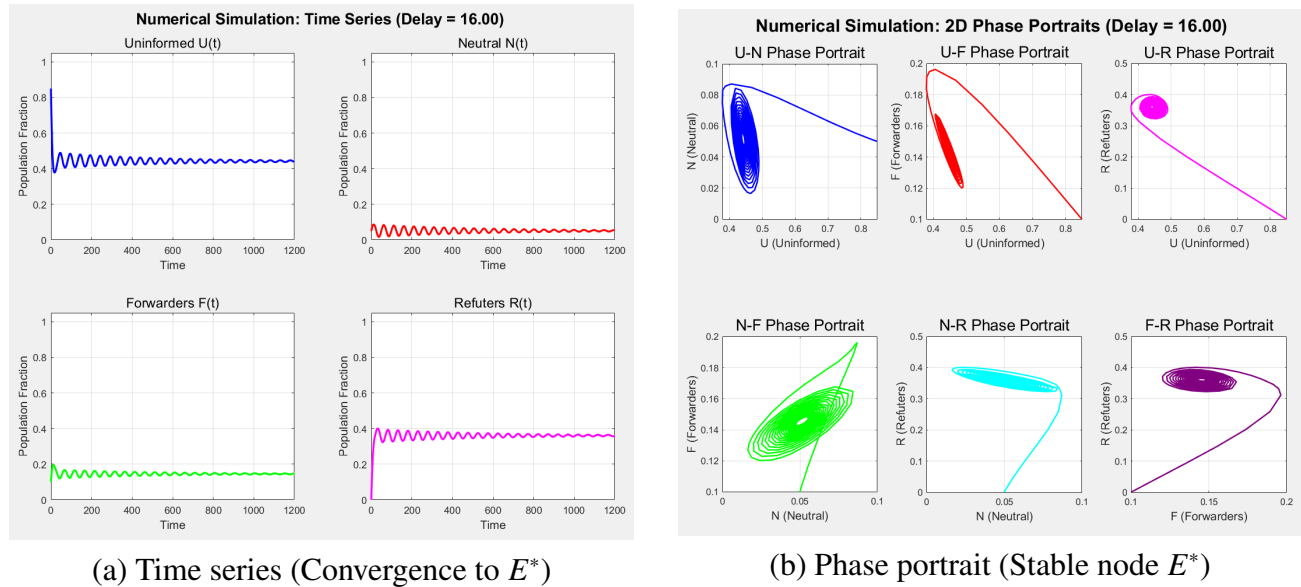


Figure 4. Simulation results when $\tau_2 = 16.0 < \tau_{20}$. The system converges to the endemic equilibrium E^* .

Case 2: $\tau_2 = 16.96 = \tau_{20}$ (Hopf bifurcation). As the delay reaches the critical threshold, the system undergoes a Hopf bifurcation. As shown in Figure 5(a),(b), the time series and phase portrait depict the critical state where the equilibrium changes stability.

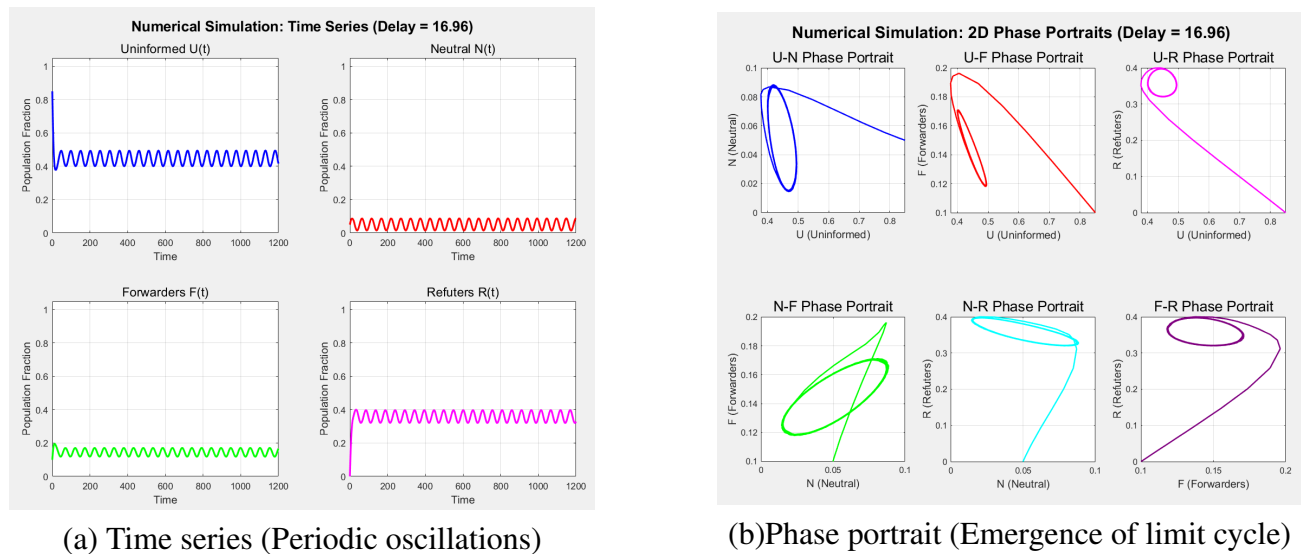


Figure 5. Simulation results when $\tau_2 = 16.96 = \tau_{20}$. A Hopf bifurcation occurs.

Case 3: $\tau_2 = 18.90 > \tau_{20}$ (**Instability**). When the delay exceeds the critical value, the equilibrium E^* becomes unstable. The system trajectories diverge from the unstable equilibrium and settle into a stable limit cycle. As shown in Figure 6(a),(b), sustained periodic oscillations are observed.

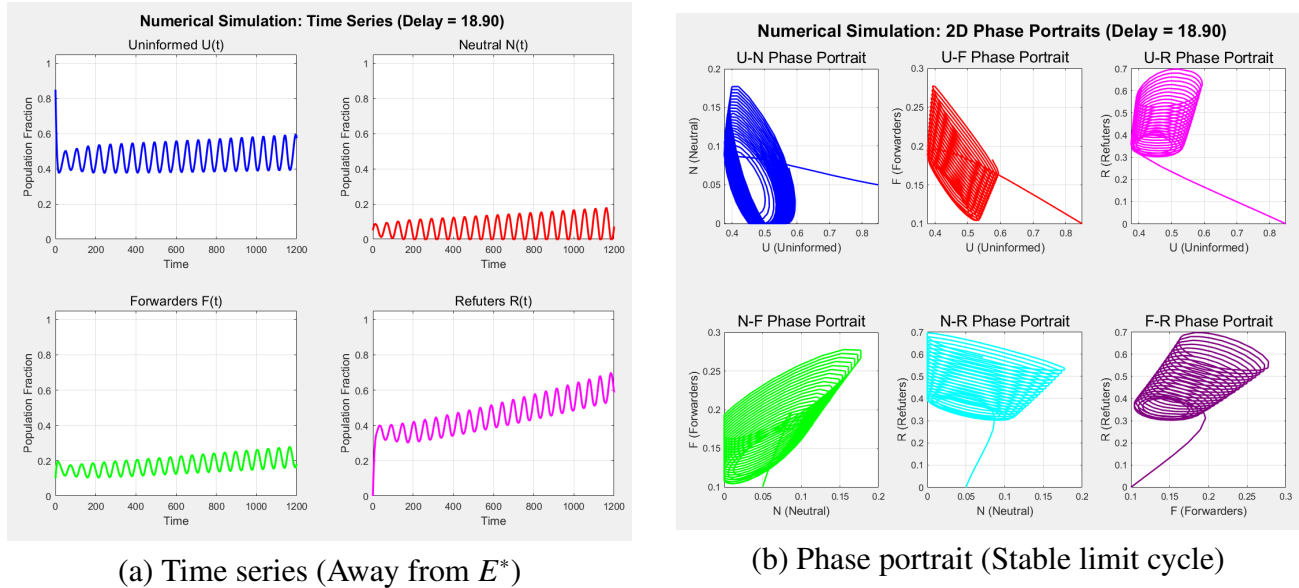


Figure 6. Simulation results when $\tau_2 = 18.90 > \tau_{20}$. The equilibrium E^* is unstable, and a stable limit cycle emerges.

These numerical results confirm the theoretical analysis, demonstrating that the N-R delay τ_2 can destabilize the rumor-endemic equilibrium when it exceeds a critical value, leading to persistent periodic rumor spreading.

4.3. Simulation for system with N-F delay

In this section, we utilize the MATLAB solver dde23 to conduct numerical simulations, verifying the existence of the Hopf bifurcation induced by the N-F delay τ_3 . Substituting the selected parameters $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.4$, $\beta_1 = 0.25$, $\beta_2 = 0.2$, $\gamma_1 = 0.05$, $\gamma_2 = 0.2$, and $\delta = 0.1$ into the model, the delayed system is given by:

$$\begin{cases} \frac{du(t)}{dt} = -1.0u(t)f(t) + 0.2f(t) + 0.1r(t), \\ \frac{dn(t)}{dt} = 0.3u(t)f(t) - 0.25n(t - \tau_3) - 0.2n(t), \\ \frac{df(t)}{dt} = 0.3u(t)f(t) + 0.25n(t - \tau_3) - 0.2f(t) - 0.05f(t)r(t), \\ \frac{dr(t)}{dt} = 0.4u(t)f(t) + 0.2n(t) + 0.05f(t)r(t) - 0.1r(t). \end{cases} \quad (4.3)$$

Using these parameters, the basic reproduction number is calculated as $R_0 \approx 2.3333 > 1$. Numerically solving the equilibrium equations yields the rumor-endemic equilibrium:

$$E^* \approx (0.4669, 0.0417, 0.1340, 0.3575).$$

Based on the stability analysis, we compute the characteristic equation coefficients at E^* . At $\tau_3 = 0$, the Routh–Hurwitz conditions yield $Y_3 \approx 0.6051 > 0$, $Y_1 \approx 0.0077 > 0$, and $Y_3 Y_2 - Y_1 \approx 0.0754 > 0$. The corresponding eigenvalues are approximately -0.2620 , -0.2620 , and -0.0811 , all of which have negative real parts. This indicates that the system is locally asymptotically stable in the absence of the N-F delay.

Furthermore, we verify the conditions for the Hopf bifurcation. The coefficient C_0 is calculated as:

$$C_0 = P_1^2 - Q_1^2 \approx -2.5 \times 10^{-5} < 0.$$

Solving the polynomial equation for ω gives the positive root $\omega_0 \approx 0.2920$. The corresponding critical delay is calculated as $\tau_{30} \approx 12.28$. Moreover, the transversality condition is evaluated as $\text{Re} \left[\frac{d\lambda}{d\tau_3} \right]_{\tau_{30}}^{-1} \approx 12.28 \neq 0$, which implies that $\text{Re} \left[\frac{d\lambda}{d\tau_3} \right]_{\tau_{30}} > 0$. This positive transversality value indicates that the real parts of the eigenvalues increase as τ_3 crosses the critical threshold. Consequently, the system undergoes a Hopf bifurcation at $\tau_3 = \tau_{30}$, losing stability and giving rise to periodic solutions for $\tau_3 > 12.28$.

Physically, the critical delay τ_{30} represents a threshold for the decision-making time of Neutrals (N) to become Forwarders (F). If the hesitation or evaluation time before forwarding is too long, it introduces a significant lag in the information transmission chain. This delay prevents the system from reaching equilibrium, causing the rumor spreading intensity to oscillate continuously over time.

To observe the system dynamics, we perform simulations with the constant history functions $u(t) = 0.7, n(t) = 0.1, f(t) = 0.1, r(t) = 0.1$ for $t \in [-\tau_3, 0]$.

Case 1: $\tau_3 = 11.24 < \tau_{30}$ (**Stability**). The system trajectories converge to the equilibrium E^* . As shown in Figure 7(a), the time series indicate that the system eventually stabilizes at an endemic level. The phase portrait in Figure 7(b) depicts trajectories spiraling towards the equilibrium point, confirming that E^* is locally asymptotically stable.

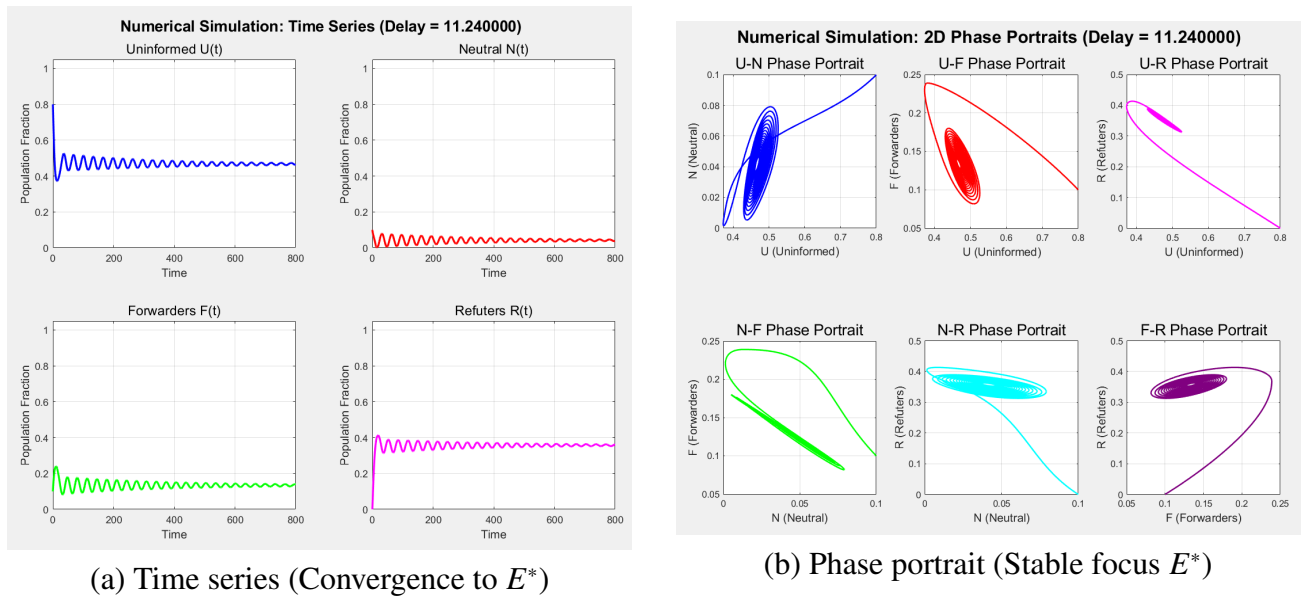


Figure 7. Simulation results when $\tau_3 = 11.24 < \tau_{30}$. The system converges to the endemic equilibrium E^* .

Case 2: $\tau_3 = 12.28 = \tau_{30}$ (**Hopf bifurcation**). As the delay reaches the critical threshold, the system undergoes a Hopf bifurcation. As shown in Figure 8(a),(b), the time series and phase portrait depict the critical transition where the stability changes.

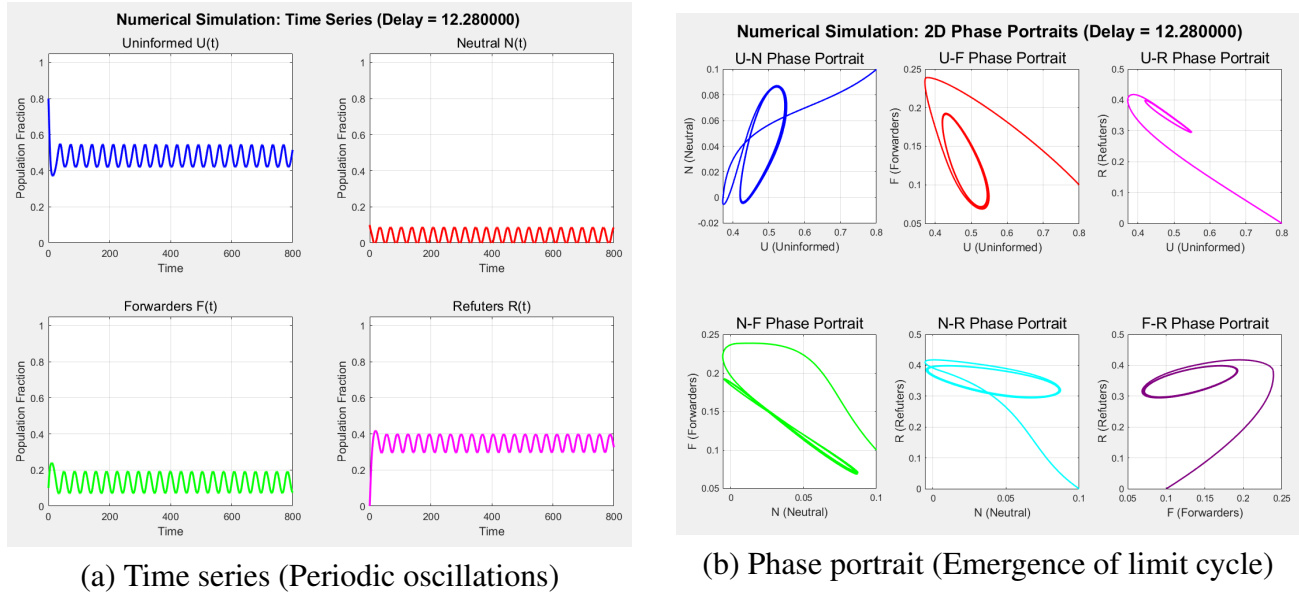


Figure 8. Simulation results when $\tau_3 = 12.28 = \tau_{30}$. A Hopf bifurcation occurs.

Case 3: $\tau_3 = 13.03 > \tau_{30}$ (**Instability**). When the delay exceeds the critical value, the equilibrium E^* loses stability and becomes an unstable focus. The system trajectories diverge from E^* and settle into a stable limit cycle. As shown in Figure 9(a),(b), sustained periodic oscillations are observed, confirming the instability of the endemic equilibrium.

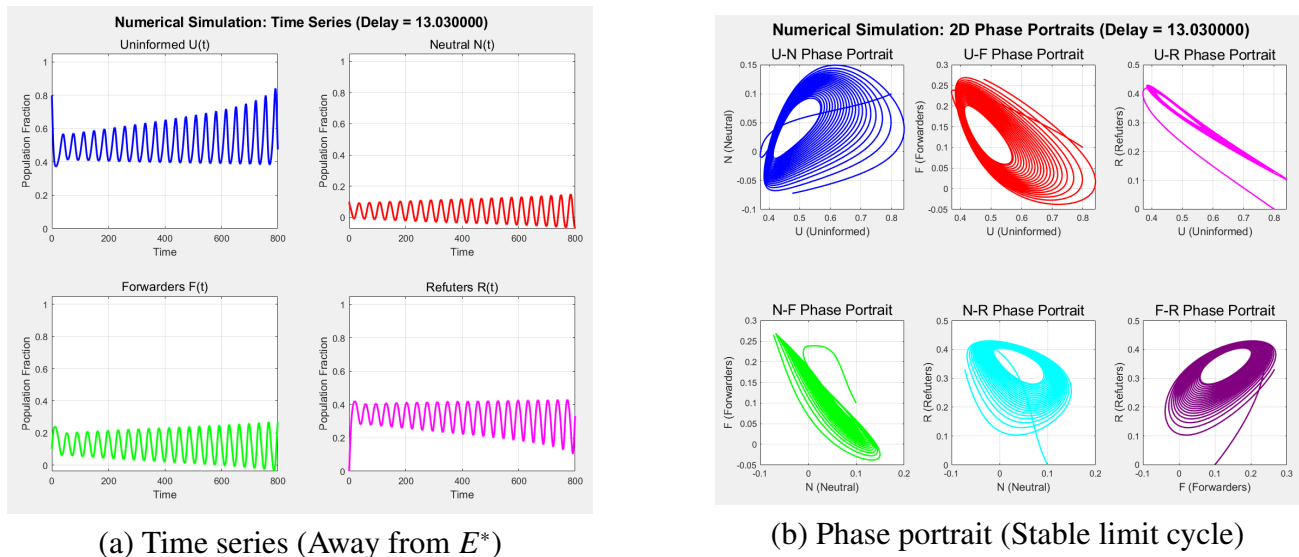


Figure 9. Simulation results when $\tau_3 = 13.03 > \tau_{30}$. The equilibrium E^* is unstable, and a stable limit cycle emerges.

These numerical results confirm the theoretical analysis, demonstrating that the N-F delay τ_3 can destabilize the rumor-endemic equilibrium when it exceeds a critical value, leading to persistent periodic rumor spreading.

Remark 4.1. *Note that there is a slight discrepancy between the theoretically calculated critical time delay and the actual value observed in numerical simulations. For instance, regarding the R-U delay, the theoretical value is $\tau_{10} = 2.450854$, whereas the actual value is 2.62. Despite this deviation, the theoretical result provides sufficient guidance for selecting the critical delay in simulations, thereby significantly saving computational time.*

5. Hopf bifurcation direction and stability of periodic solutions

5.1. Analysis of Hopf bifurcation properties

Without loss of generality, we analyze the Hopf bifurcation properties for the case in Eq (3.2) involving the R-U delay τ_1 . The procedures for the N-R delay τ_2 and N-F delay τ_3 are analogous and are omitted for brevity.

To explicitly determine the properties of the Hopf bifurcation, we investigate the direction of the bifurcation and the stability of the bifurcating periodic solutions. To this end, we employ the center manifold theorem and normal form theory.

Let $\tau = \tau_0 + \varpi$, where $\varpi \in \mathbb{R}$ represents a small perturbation parameter. For notational simplicity, we denote the critical values ω_0 and τ_0 as ω^* and τ^* , respectively, in the subsequent calculations. Substituting $u(t) = 1 - n(t) - f(t) - r(t)$ into the system eliminates $u(t)$, yielding a three-dimensional system in Eq (3.3) for (n, f, r) .

By defining the state variables $\chi_1(t) = n(t)$, $\chi_2(t) = f(t)$, $\chi_3(t) = r(t)$ and rescaling the time $t \rightarrow t/\tau$, the model in Eq (3.2) can be reformulated as a functional differential equation in the phase space $C = C([-1, 0], \mathbb{R}^3)$:

$$\dot{\chi}(t) = L_{\varpi}(\chi_t) + F(\varpi, \chi_t),$$

where $\dot{\chi}(t) = (\chi_1, \chi_2, \chi_3)^T$. The linear operator $L_{\varpi} : C \rightarrow \mathbb{R}^3$ and the nonlinear operator $F : \mathbb{R} \times C \rightarrow \mathbb{R}^3$ are defined as:

$$\begin{aligned} L_{\varpi}\varphi &= (\tau_0 + \varpi)(P'\varphi(0) + Q'\varphi(-1)), \\ F(\varpi, \varphi) &= (\tau_0 + \varpi)(F_1, F_2, F_3)^T. \end{aligned}$$

Here, Q' represents the delay matrix, and P' is the Jacobian matrix of the system evaluated at the endemic equilibrium $E^*(n^*, f^*, r^*)$, with $u^* = 1 - n^* - f^* - r^*$ denoting the equilibrium value of the ignorant individuals. The coefficient matrices are given by:

$$P' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad Q' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta \end{pmatrix}.$$

The nonlinear terms F_k correspond to the quadratic interactions of the system variables. Based on the reduced system dynamics where $\chi_1 = n$, $\chi_2 = f$, and $\chi_3 = r$, these terms are explicitly

formulated as:

$$\begin{aligned} F_1 &= h_{11}\varphi_1(0)\varphi_2(0) + h_{12}\varphi_2^2(0) + h_{13}\varphi_2(0)\varphi_3(0), \\ F_2 &= h_{21}\varphi_1(0)\varphi_2(0) + h_{22}\varphi_2^2(0) + h_{23}\varphi_2(0)\varphi_3(0), \\ F_3 &= h_{31}\varphi_1(0)\varphi_2(0) + h_{32}\varphi_2^2(0) + h_{33}\varphi_2(0)\varphi_3(0), \end{aligned}$$

where the coefficients h_{ij} are derived from the parameters of the original model to ensure consistency with the state definitions:

$$\begin{aligned} h_{11} &= -\alpha_2, & h_{12} &= -\alpha_2, & h_{13} &= -\alpha_2, \\ h_{21} &= -\alpha_1, & h_{22} &= -\alpha_1, & h_{23} &= -(\alpha_1 + \gamma_1), \\ h_{31} &= -\alpha_3, & h_{32} &= -\alpha_3, & h_{33} &= -(\alpha_3 - \gamma_1). \end{aligned}$$

By the Riesz representation theorem, there exists a matrix-valued function $\eta(\vartheta, \varpi)$ of bounded variation for $\vartheta \in [-1, 0]$ such that

$$L_{\varpi}\varphi = \int_{-1}^0 d\eta(\vartheta, \varpi)\varphi(\vartheta).$$

For computational convenience, we select

$$\eta(\vartheta, \varpi) = (\tau_0 + \varpi)(P'\delta(\vartheta) + Q'\delta(\vartheta + 1)),$$

where $\delta(\vartheta)$ denotes the Dirac delta function. Consequently, for $\phi \in C([-1, 0], \mathbb{R}^3)$, the extended linear operator and the nonlinear operator can be formulated as:

$$A(\varpi)\phi = \begin{cases} \frac{d\phi(\vartheta)}{d\vartheta}, & -1 \leq \vartheta < 0, \\ \int_{-1}^0 d\eta(\vartheta, \varpi)\phi(\vartheta), & \vartheta = 0, \end{cases} \quad R(\varpi)\phi = \begin{cases} 0, & -1 \leq \vartheta < 0, \\ F(\varpi, \phi), & \vartheta = 0. \end{cases}$$

Thus, the model in Eq (3.2) is transformed into the abstract differential equation:

$$\dot{\chi}(t) = A(\varpi)\chi_t + R(\varpi)\chi_t.$$

Considering the adjoint space $\psi \in C^1([0, 1], (\mathbb{R}^3)^*)$, the adjoint operator A^* is defined by:

$$A^*(\psi) = \begin{cases} -\frac{d\psi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 d\eta^T(\zeta, 0)\psi(-\zeta) d\zeta, & s = 0. \end{cases}$$

The associated bilinear inner product for $A(0)$ and A^* is given by:

$$\langle \psi(s), \varphi(\vartheta) \rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^0 \int_0^{\vartheta} \bar{\psi}(\zeta - \vartheta)\varphi(\zeta) d\zeta d\eta(\vartheta), \quad (5.1)$$

where $\eta(\vartheta) = \eta(\vartheta, 0)$.

Let $u(\vartheta) = (1, u_2, u_3)^T e^{i\omega^* \tau^* \vartheta}$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega^* \tau^*$, and $u^*(s) = \bar{D}(1, u_2^*, u_3^*)^T e^{i\omega^* \tau^* s}$ be the eigenvector of $A^*(0)$ corresponding to $-i\omega^* \tau^*$. Substituting these into the respective eigenvalue problems yields:

$$P' \begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix} + e^{-i\omega^* \tau^*} Q' \begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix} = i\omega^* \begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix}, \quad P'^T \begin{pmatrix} 1 \\ u_2^* \\ u_3^* \end{pmatrix} + e^{i\omega^* \tau^*} Q'^T \begin{pmatrix} 1 \\ u_2^* \\ u_3^* \end{pmatrix} = -i\omega^* \begin{pmatrix} 1 \\ u_2^* \\ u_3^* \end{pmatrix}.$$

Solving the above linear systems algebraically, we obtain the eigenvector components:

$$\begin{aligned} u_2 &= \frac{i\omega^* - a_{11} - a_{13}u_3}{a_{12}}, & u_3 &= \frac{a_{31} + a_{32}u_2}{i\omega^* - a_{33} + \delta e^{-i\omega^*\tau^*}}, \\ u_2^* &= \frac{-i\omega^* - a_{11} - a_{31}u_3^*}{a_{21}}, & u_3^* &= \frac{-a_{13} - a_{23}u_2^*}{i\omega^* + a_{33} - \delta e^{i\omega^*\tau^*}}. \end{aligned}$$

The normalization constant $\bar{D}$ is chosen to satisfy $\langle u^*, u \rangle = 1$. Applying the inner product definition in Eq (5.1), we compute:

$$\bar{D} = \left[1 + u_2 \bar{u}_2^* + u_3 \bar{u}_3^* - \tau^* \delta e^{-i\tau^*\omega^*} \bar{u}_3^* u_3 \right]^{-1}.$$

Following the standard procedures outlined in [19], the critical coefficients for the normal form expansion are calculated as:

$$\begin{aligned} g_{20} &= 2\tau^* \bar{D} \sum_{m=1}^3 \bar{u}_m^* \left[h_{m1} u_2 + h_{m2} u_2^2 + h_{m3} u_2 u_3 \right], \\ g_{11} &= \tau^* \bar{D} \sum_{m=1}^3 \bar{u}_m^* \left[h_{m1} (u_2 + \bar{u}_2) + 2h_{m2} |u_2|^2 + h_{m3} (u_2 \bar{u}_3 + \bar{u}_2 u_3) \right], \\ g_{02} &= 2\tau^* \bar{D} \sum_{m=1}^3 \bar{u}_m^* \left[h_{m1} \bar{u}_2 + h_{m2} \bar{u}_2^2 + h_{m3} \bar{u}_2 u_3 \right], \\ g_{21} &= 2\tau^* \bar{D} \sum_{m=1}^3 \bar{u}_m^* \left[h_{m1} \left(2W_{11}^{(1)}(0) u_2 + W_{20}^{(1)}(0) \bar{u}_2 \right) \right. \\ &\quad \left. + h_{m2} \left(2W_{11}^{(2)}(0) u_2^2 + W_{20}^{(2)}(0) \bar{u}_2^2 \right) + h_{m3} \left(2W_{11}^{(3)}(0) u_2 u_3 + W_{20}^{(3)}(0) \bar{u}_2 u_3 \right) \right], \end{aligned}$$

where the functions $W_{20}(\vartheta)$ and $W_{11}(\vartheta)$ are expressed as:

$$\begin{aligned} W_{20}(\vartheta) &= \frac{ig_{20}u(0)}{\tau^*\omega^*} e^{i\tau^*\omega^*\vartheta} + \frac{i\bar{g}_{02}\bar{u}(0)}{3\tau^*\omega^*} e^{-i\tau^*\omega^*\vartheta} + P_1 e^{2i\tau^*\omega^*\vartheta}, \\ W_{11}(\vartheta) &= -\frac{ig_{11}u(0)}{\tau^*\omega^*} e^{i\tau^*\omega^*\vartheta} + \frac{i\bar{g}_{11}\bar{u}(0)}{\tau^*\omega^*} e^{-i\tau^*\omega^*\vartheta} + P_2. \end{aligned}$$

The vectors P_1 and P_2 are determined by solving the following linear systems:

$$\begin{aligned} P_1 &= \begin{bmatrix} 2i\omega^* - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 2i\omega^* - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 2i\omega^* - a_{33} + \delta e^{-2i\tau^*\omega^*} \end{bmatrix}^{-1} \begin{bmatrix} h_{11}u_2 + h_{12}u_2^2 + h_{13}u_2u_3 \\ h_{21}u_2 + h_{22}u_2^2 + h_{23}u_2u_3 \\ h_{31}u_2 + h_{32}u_2^2 + h_{33}u_2u_3 \end{bmatrix}, \\ P_2 &= -\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} - \delta \end{bmatrix}^{-1} \begin{bmatrix} 2h_{11} + 4h_{12}|u_2|^2 + 2h_{13}(u_2\bar{u}_3 + \bar{u}_2u_3) \\ 2h_{21} + 4h_{22}|u_2|^2 + 2h_{23}(u_2\bar{u}_3 + \bar{u}_2u_3) \\ 2h_{31} + 4h_{32}|u_2|^2 + 2h_{33}(u_2\bar{u}_3 + \bar{u}_2u_3) \end{bmatrix}. \end{aligned}$$

Finally, the crucial quantities determining the bifurcation properties are computed as follows:

$$C_1(0) = \frac{i}{2\tau^*\omega^*} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau^*)\}}, \quad b_2 = 2\operatorname{Re}\{C_1(0)\}, \quad T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau^*)\}}{\tau^*\omega^*}.$$

Here, μ_2 dictates the direction of the Hopf bifurcation, b_2 determines the stability of the bifurcating periodic solutions, and T_2 indicates the period shift of these solutions.

Theorem 5.1. *For the model in Eq (3.2), the following properties hold:*

- When $\mu_2 > 0$, the system undergoes a supercritical Hopf bifurcation;
- When $b_2 < 0$, the bifurcated periodic solution is asymptotically stable;
- When $T_2 > 0$, the period of the bifurcated periodic solution increases.

5.2. Numerical validation and consistency with simulation results

We select the parameters as follows:

$$\alpha_1 = 0.400, \alpha_2 = 0.200, \alpha_3 = 0.400, \beta_1 = 0.150, \beta_2 = 0.200, \gamma_1 = 0.030, \gamma_2 = 0.40, \delta = 0.60.$$

With these parameters, we compute the coefficients determining the properties of the Hopf bifurcation using the normal form theory derived earlier:

$$\mu_2 \approx 0.612588, \quad b_2 \approx -0.423094, \quad T_2 \approx 0.756860.$$

According to Theorem 5.1, since $\mu_2 > 0$, the Hopf bifurcation at $\tau = \tau_0$ is supercritical. Additionally, $b_2 < 0$ indicates that the bifurcating periodic solutions are asymptotically stable, and $T_2 > 0$ implies that the period of these solutions increases as the delay τ increases past τ_0 . These theoretical predictions are fully consistent with the numerical simulation results presented in Section 3.1.

Specifically, the supercritical nature of the Hopf bifurcation ($\mu_2 > 0$) explains why stable limit cycles emerge when the delay exceeds the critical threshold, as observed in Figure 3(a),(b) where the system transitions from stable equilibrium to sustained periodic oscillations. The negative value of b_2 confirms the asymptotic stability of these periodic solutions, which is evident from the convergence of system trajectories to the limit cycle in the phase portraits. Furthermore, the positive T_2 value is consistent with the increasing oscillation period observed in the time series as the delay increases beyond τ_0 .

This theoretical analysis provides a rigorous mathematical foundation for the numerical observations, establishing a clear connection between the bifurcation properties and the dynamic behavior of the rumor propagation system. The consistency between theoretical predictions and simulation results validates the accuracy of our model and the effectiveness of the Hopf bifurcation analysis in characterizing the system's transition from stability to periodic behavior.

6. Sensitivity of basic reproduce number to system parameters

To formulate more scientific and effective rumor intervention strategies, it is essential to understand how the basic reproduction number R_0 responds to variations in system parameters. We compute the

normalized forward sensitivity indices of R_0 with respect to five key parameters: α_1 , α_2 , β_1 , β_2 , and γ_2 . The sensitivity index, defined as $\Upsilon_p = \frac{\partial R_0}{\partial p} \frac{p}{R_0}$, measures the relative change in R_0 for a given relative change in the parameter p .

Recall the basic reproduction number derived in Section 2:

$$R_0 = \frac{\alpha_2 \beta_1 + \alpha_1 (\beta_1 + \beta_2)}{\gamma_2 (\beta_1 + \beta_2)}.$$

The calculations yield the following sensitivity indices:

$$\begin{aligned}\Upsilon_{\alpha_1} &= \frac{\partial R_0}{\partial \alpha_1} \frac{\alpha_1}{R_0} = \frac{1}{\gamma_2} \frac{\alpha_1}{R_0} = \frac{\alpha_1 (\beta_1 + \beta_2)}{\alpha_2 \beta_1 + \alpha_1 (\beta_1 + \beta_2)} > 0, \\ \Upsilon_{\alpha_2} &= \frac{\partial R_0}{\partial \alpha_2} \frac{\alpha_2}{R_0} = \frac{\beta_1}{\gamma_2 (\beta_1 + \beta_2)} \frac{\alpha_2}{R_0} = \frac{\alpha_2 \beta_1}{\alpha_2 \beta_1 + \alpha_1 (\beta_1 + \beta_2)} > 0, \\ \Upsilon_{\beta_1} &= \frac{\partial R_0}{\partial \beta_1} \frac{\beta_1}{R_0} = \frac{\alpha_2 \beta_2}{\gamma_2 (\beta_1 + \beta_2)^2} \frac{\beta_1}{R_0} = \frac{\alpha_2 \beta_1 \beta_2}{(\beta_1 + \beta_2) (\alpha_2 \beta_1 + \alpha_1 (\beta_1 + \beta_2))} > 0, \\ \Upsilon_{\beta_2} &= \frac{\partial R_0}{\partial \beta_2} \frac{\beta_2}{R_0} = \left(-\frac{\alpha_2 \beta_1}{\gamma_2 (\beta_1 + \beta_2)^2} \right) \frac{\beta_2}{R_0} = -\frac{\alpha_2 \beta_1 \beta_2}{(\beta_1 + \beta_2) (\alpha_2 \beta_1 + \alpha_1 (\beta_1 + \beta_2))} < 0, \\ \Upsilon_{\gamma_2} &= \frac{\partial R_0}{\partial \gamma_2} \frac{\gamma_2}{R_0} = \frac{\gamma_2}{R_0} \left(-\frac{R_0}{\gamma_2} \right) = -1 < 0.\end{aligned}$$

The analytical results reveal that R_0 is positively correlated with α_1 , α_2 , and β_1 , meaning that an increase in any of these parameters will exacerbate the spread of the rumor. Conversely, R_0 is negatively correlated with β_2 and γ_2 ; increasing these parameters effectively suppresses rumor transmission. Notably, the sensitivity index of R_0 with respect to γ_2 is exactly -1 , indicating that a proportional increase in the spontaneous forgetting rate of Forwarders leads to an equivalent proportional decrease in the basic reproduction number.

Based on this sensitivity analysis, the following targeted intervention strategies can be implemented to control the spread of online rumors:

- **Enhancing user information literacy:** Educational initiatives should aim to improve users' critical thinking abilities, thereby reducing the transition rates α_1 and α_2 . By training users to identify misinformation upon initial exposure, the probability of accepting and forwarding rumors is minimized. Furthermore, platforms can optimize recommendation algorithms to limit the exposure of susceptible and neutral users to persuasive rumor content.
- **Strengthening rumor-refutation mechanisms:** Authorities and platforms should increase the value of β_2 by promoting fact-checking tags and encouraging user verification. Increasing the visibility of authoritative debunking information can effectively guide neutral individuals to transition into the Refuter state, converting potential spreaders into active suppressors of the rumor.
- **Disrupting user attention cycles:** Platform administrators can artificially increase the spontaneous forgetting rate γ_2 by suppressing the algorithmic recommendation traffic for the rumor topic or diverting user attention with other high-quality content. This accelerates the rate at which Forwarders lose interest and revert to the Unaware state, effectively shortening the propagation cycle and mitigating the rumor outbreak.

7. Conclusions

This paper systematically investigates the complex dynamics induced by time delays in rumor propagation using a delayed UNFR compartmental framework. Unlike traditional models that assume instantaneous state transitions, our model incorporates specific delays representing memory decay and decision-making processes. We analyzed three critical delays: Refuter-to-Unaware (R-U), Neutral-to-Forwarder (N-F), and Neutral-to-Refuter (N-R). Theoretically, we derived the Hopf bifurcation thresholds and analyzed the stability of bifurcating periodic solutions via center manifold theory. Numerical simulations validate these findings, demonstrating that exceeding the critical delay leads to sustained periodic oscillations (limit cycles). This provides a rigorous mathematical explanation for the “echo” effect and the periodic resurgence of rumors.

Remark 7.1. While reference [9] introduces the UNFR model and analyzes rumor propagation dynamics via user classification (unknown, neutral, repeater, refuter), our study extends this framework by incorporating discrete time delays into critical transitions ($R \rightarrow U$, $N \rightarrow R$, $N \rightarrow F$) to model memory decay and decision hesitation. Through characteristic equation analysis, Hopf bifurcation theory, and center manifold/normal form techniques, we rigorously examine the stability dynamics: Equilibria remain locally asymptotically stable below a critical delay threshold, while exceeding it induces a Hopf bifurcation and sustained periodic oscillations—offering a theoretical explanation for the cyclical resurgence of rumors.

Remark 7.2. The choice to investigate the R-U, N-F, and N-R delays specifically is physically grounded in realistic cognitive processes within social networks. Although delays could theoretically be introduced for any transition between the U, N, F, and R states, we restricted our analysis to these cases to ensure both analytical tractability and clear physical interpretation. Furthermore, to simplify the calculations, delays were added only to the linear terms of the model, and the effect of a single delay parameter was considered at a time. Future research could explore more complex scenarios involving multiple coupled delays or delays in nonlinear interaction terms.

Remark 7.3. Although mathematically feasible, a Hopf bifurcation at the rumor-free equilibrium E_0 lacks practical significance in the context of rumor propagation, as oscillations around the extinction state do not reflect the cyclical resurgence of rumors observed in reality. Therefore, our study primarily focuses on the dynamics of the endemic equilibrium E^* . Nevertheless, for mathematical completeness, we briefly address the stability of E_0 .

Linearizing the system in Eq (3.1) at $E_0(0, 0, 0)$ yields a characteristic equation that separates into a quadratic term and a transcendental term. Under the condition $R_0 < 1$, the quadratic term is stable. Consequently, the stability of E_0 is governed solely by the transcendental equation $\lambda + \delta e^{-\lambda\tau} = 0$. Solving this equation reveals that pure imaginary roots exist only when the delay reaches the critical value $\tau_0 = \frac{\pi}{2\delta}$, establishing the bifurcation threshold.

Remark 7.4. Sensitivity analysis of the basic reproduction number R_0 identifies key leverage points within the system, such as reducing user susceptibility (α_i) or accelerating memory decay (γ_2). In the context of the delayed model, adjusting these parameters serves a dual purpose: It lowers R_0 and directly modulates the system’s stability conditions. When the time delay exceeds a critical threshold, the endemic equilibrium loses stability and the system enters a limit cycle. Mathematically,

this sustained periodic oscillation precisely manifests as the phenomenon of “periodic resurgence of rumors”—a scenario where a rumor repeatedly transitions from a low-activity state back to a widespread outbreak instead of eventually dying out. Consequently, implementing time-sensitive strategies—such as enhancing user information literacy to curb transmission or optimizing algorithms to shorten attention spans—can effectively suppress these oscillatory outbreak cycles, thereby driving the system back to a stable state.

Use of AI tools declaration

During the preparation of this work, the authors used GLM-5.1 in order to improve readability and language. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

Conflicts of interest

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Acknowledgments

This research was funded by Research Start-up Fund of Suqian University, the interdisciplinary Integration and Innovation Project of Suqian University (Grant No. 2025XKTD02), the Key Project of the 2025 Annual Higher Education Scientific Research Planning Project (Grant No. 25LK0203).

Author contributions

Wamei Hou: Modelling, formal analysis, writing—original draft preparation, visualization; Xiangqing Zhao: Conceptualization, methodology, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

References

1. F. Zollo, A. Bessi, M. Del Vicario, A. Scala, G. Caldarelli, L. Shekhtman, et al. Debunking in a world of tribes, *PLoS One*, **12** (2017), e0181821. <https://doi.org/10.1371/journal.pone.0181821>
2. S. Vosoughi, D. Roy, S. Aral, The spread of true and false news online, *Science*, **359** (2018), 1146–1151. <https://doi.org/10.1126/science.aap9559>
3. Y. Ma, Z. Chen, F. Ren, Research on Weibo rumor propagation mechanism based on big data, *Highlights Sci. Eng. Technol.*, **136** (2025), 148–158. <https://doi.org/10.54097/kxt0dg89>
4. D. J. Daley, D. G. Kendall, Epidemics and rumours, *Nature*, **204** (1964), 1118. <https://doi.org/10.1038/2041118a0>

5. Z. Yu, S. Lu, D. Wang, Z. Li, Modeling and analysis of rumor propagation in social networks, *Inf. Sci.*, **580** (2021), 857–873. <https://doi.org/10.1016/j.ins.2021.09.012>
6. L. Fakih, A. Halanay, F. Avram, On a model of rumors spreading through social media, *Entropy*, **27** (2025), 903. <https://doi.org/10.3390/e27090903>
7. Q. Li, F. Jiang, H. Sun, R. Wang, C. Jia, T. Li, A rumor dissemination control model based on evolutionary game and multiple user states, *IEEE Trans. Netw. Sci. Eng.*, **12** (2025), 3625–3240. <https://doi.org/10.1109/TNSE.2025.3563360>
8. S. A. Polin, M. N. Hasan, S. Islam, C. N. Podder, Behavioral influences on rumor dynamics: A compartmental model with hesitation, forgetting, and self-remembering mechanisms in complex heterogeneous social networks, *Chaos, Solitons Fractals*, **201** (2025), 117317. <https://doi.org/10.1016/j.chaos.2025.117317>
9. H. Wang, W. Liu, L. Lu, Rumor propagation model of Microblog network with attenuation effect and forgetting mechanism, *Comput. Sci.*, **50** (2023), 762–768.
10. X. Sun, Y. Wang, Y. Chai, Y. Liu, Dynamic analysis and control strategies of the SEIHR rumor diffusion model in online social networks, *Appl. Math. Modell.*, **134** (2024), 611–634. <https://doi.org/10.1016/j.apm.2024.06.005>
11. L. Zhu, B. Wang, Stability analysis of a SAIR rumor spreading model with control strategies in online social networks, *Inf. Sci.*, **526** (2020), 1–19. <https://doi.org/10.1016/j.ins.2020.03.076>
12. Y. Ding, L. Zhu, Turing instability analysis of a rumor propagation model with time delay on non-network and complex networks, *Inf. Sci.*, **667** (2024), 120402. <https://doi.org/10.1016/j.ins.2024.120402>
13. H. Zhu, X. Zhang, Q. An, Global stability of a rumor spreading model with discontinuous control strategies, *Phys. A*, **606** (2022), 128157. <https://doi.org/10.1016/j.physa.2022.128157>
14. S. Wu, Y. Lu, C. Chen, S. Liu, Dynamic analysis of SICDR rumor propagation model with time delay and psychological factors, in *2024 7th International Conference on Computer Information Science and Application Technology (CISAT)*, Hangzhou, China, 746–755, 2024. <https://doi.org/10.1109/CISAT62382.2024.10695275>
15. R. Yin, A. Muhammadhaji, Dynamic analysis of rumor propagation model with distributed delay and saturation cure rate in multilingual environment, *Chaos, Solitons Fractals*, **200** (2025), 117055. <https://doi.org/10.1016/j.chaos.2025.117055>
16. Y. Cheng, L. Huo, L. Zhao, Stability analysis and optimal control of rumor spreading model under media coverage considering time delay and pulse vaccination, *Chaos, Solitons Fractals*, **157** (2022), 111931. <https://doi.org/10.1016/j.chaos.2022.111931>
17. M. Ye, H. Jiang, Bifurcation control for a fractional-order delayed SEIR rumor spreading model with incommensurate orders, *Acta Math. Sci.*, **43** (2023), 2662–2682. <https://doi.org/10.1007/s10473-023-0621-6>

18. J. Wang, H. Jiang, C. Hu, Z. Yu, J. Li, Stability and Hopf bifurcation analysis of multi-lingual rumor spreading model with nonlinear inhibition mechanism, *Chaos, Solitons Fractals*, **153** (2021), 111464. <https://doi.org/10.1016/j.chaos.2021.111464>
19. L. Zhu, X. Chen, Modeling the bifurcation dynamics of rumor propagation in the spatial environment, *Int. J. Bifurc. Chaos*, **34** (2024), 2450005. <https://doi.org/10.1142/S0218127424500056>
20. G. Li, *Optimal Control Theory and Parameter Optimization*, National Defense Industry Press, Beijing, China, 2005.



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)