



Research article

Bifurcation behavior and a control scheme in a fractional-order chemostat model incorporating delay

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Abstract: In this article, a new fractional-order chemostat model incorporating delay is formulated. Making use of the Lipschitz condition, we confirm the existence and uniqueness of the solution to the formulated model. By defining a reasonable function, we prove that the solution of the proposed model is bounded. The criterion of the stability and Hopf bifurcation emerging from the developed model is established by taking advantage of the stability theory and the Hopf bifurcation theorem of fractional-order differential systems. By contriving two distinctive controllers, we can effectually adjust the stability region and the delay length of the onset of Hopf bifurcation of the developed model. Finally, computer simulation experiments are provided to support the acquired conclusions. The derived results play an important role in balancing the concentrations of the substrate and the cell mass.

Keywords: fractional-order chemostat model; stability; Hopf bifurcation; speed feedback controller; dislocated feedback controller

1. Introduction

A great deal of studies show that differential dynamical equations play a vital role in illustrating the laws of various changes in many fields such as chemistry, mechanics, electronics, neural network science, electrical engineering, the communication field, and so on [1]. In chemistry, in many cases, the change in the density of various chemical substances can be described by differential equations. By analyzing the established differential equation models of chemical substances, we can find inherent changing laws of the density of various chemical substances. During the past decades, many scholars have paid great attention to the exploration of differential systems of chemical substances and many excellent results on these aspects have been reported. For example, in 2015, Xu and Wu [1] considered the chaos control of a three-dimensional chemical model. By designing a suitable delayed feedback controller, the chaotic behaviors of this three-dimensional chaotic chemical model were suppressed.

The sufficient conditions on the stability and the onset of Hopf bifurcation of this three-dimensional controlled chaotic chemical model were established. The critical delay value to ensure the generation of Hopf bifurcation in the controlled chaotic chemical model was found. Through knowledge of the center manifold and the normal form theorem, the concrete formulae to determine the direction, stability, and the period of bifurcating periodic solutions of the controlled chaotic chemical model are acquired. In 1996, Ruan and Wolkowicz [2] formulated a delayed chemostat model of a single species. By virtue of the average time delay as a Hopf bifurcation parameter, the authors explored the Hopf bifurcation of the delayed chemostat model. The effect of the average time delay on the Hopf bifurcation of this model was analyzed. The results of [2] play a critical role in balancing the concentration of the nutrient and the populations of microorganisms. In [3], Li et al. established a new fractional-order delayed Lengyel-Epstein model. By applying the stability and bifurcation of fractional-order differential equations, a new delay-dependent condition on the stability and the onset of Hopf bifurcation of the formulated fractional delayed Lengyel-Epstein model was acquired. Making use of a nonlinear delayed feedback controller and a hybrid controller, the stability region and the delay length of the generation of Hopf bifurcation of this controlled Lengyel-Epstein model were adjusted. For more detailed studies in this area, one can refer to [4, 5].

In 2025, Sharma et al. [6] put forward the following chemostat model:

$$\begin{cases} \frac{du_1(t)}{dt} = bu_{10} - bu_1(t) - \frac{A(u_1(t))u_2(t)}{B(u_1(t))}, \\ \frac{du_2(t)}{dt} = bu_{20} - bu_2(t) - A(u_1(t))u_2(t), \end{cases} \quad (1.1)$$

where $u_1(t)$ denotes the concentrations of the substrate, and $u_2(t)$ denotes the concentrations of the cell mass, $b > 0$ denotes the dilution rate, u_{10} denotes the concentration of the initial substrate, and u_{20} denotes the concentration of initial cell mass, u_{10} and u_{20} are positive constants. B denotes the yield parameter, $A(u_1(t))$ denotes the concrete growth rate that follows the Monod growth model typically relies on the yield parameter B and the substrate concentration. $A(u_1(t))$ and $B(u_1(t))$ take the following expression:

$$\begin{cases} A(u_1(t)) = \frac{\alpha u_1(t)}{c + u_1(t)}, \\ B(u_1(t)) = \gamma + \beta u_1(t), \end{cases} \quad (1.2)$$

where $\alpha > 0$ represents the maximum concrete growth rate, $c > 0$ denotes the substrate's concentration at which the concrete growth rate is half of its maximum value α and is the Monod constant, $\gamma > 0$ denotes the baseline yield coefficient, $\beta > 0$ denotes the rate at which the yield increases with the substrate's concentration, $\gamma > 0$ and $\beta > 0$ are real coefficients. From Eqs (1.2) and (1.1), we can get the following system:

$$\begin{cases} \frac{du_1(t)}{dt} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t)u_2(t)}{(c + u_1(t))(\gamma + \beta u_1(t))}, \\ \frac{du_2(t)}{dt} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t)u_2(t)}{c + u_1(t)}. \end{cases} \quad (1.3)$$

Here, we must mention that in most cases, delay often exists in dynamic systems, since the response of variables depends on the past time. In Model (8.3), we consider that the concrete growth rate $A(u_1(t))$ depends on the past history; that is to say, there is a self-feedback delay from u_1 to u_1 in $A(u_1(t))$, which

implies that

$$A(u_1(t)) = \frac{\alpha u_1(t - \tau)}{c + u_1(t - \tau)}. \quad (1.4)$$

Depending on this ideas, we modify Model (8.3) as follows:

$$\begin{cases} \frac{du_1(t)}{dt} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t - \tau)u_2(t)}{(c + u_1(t - \tau))(\gamma + \beta u_1(t))}, \\ \frac{du_2(t)}{dt} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t - \tau)u_2(t)}{c + u_1(t - \tau)}. \end{cases} \quad (1.5)$$

Recently, many scholars have argued that fractional-order dynamic equations have displayed widespread application in many disciplines such as biology, chemistry, quantum mechanics, control science, various light waves, image processing, network engineering, electric power system and so on [7, 8]. The extensive application of fractional-order dynamical equations depends on the their memory properties and hereditary of for the development of various substances [9]. Nowadays, a good deal of excellent works on fractional-order differential models have been published. For instance, Xu et al. [10] explored the impact of two inconsistent delays on the emergence of bifurcation and the stability region of fractional delayed neural networks. Mondal et al. [11] investigated the saddle-node bifurcation, the Hopf bifurcation, and the existence of multi-stable states and measurements of basin stability in a fractional-order predator-prey system. Pratap et al. [12] studied the stabilization and quasi-pinning synchronization of fractional-order delayed neural networks. For details, one can see [13–17]. It is a pity that all the abovementioned works are only concerned with the dynamical behavior of integer-order chemostat models. and they are not concerned with the dynamical behavior of fractional-order chemostat models. Motivated by this discussion, in this paper, we will explore the dynamic behavior of fractional-order chemostat models. On the basis of this idea, we develop the following fractional-order chemostat model:

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t - \tau)u_2(t)}{(c + u_1(t - \tau))(\gamma + \beta u_1(t))}, \\ \frac{d^q u_2(t)}{dt^q} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t - \tau)u_2(t)}{c + u_1(t - \tau)}, \end{cases} \quad (1.6)$$

where $q \in (0, 1]$ is a constant. All other coefficients and functions have the same meanings as those in Model (8.3). We know that delay-induced Hopf bifurcation is a very important dynamic phenomenon in fractional-order differential systems and it can describe the balance of different densities of chemical substances. Thus, it is of importance to investigate the delay-induced Hopf bifurcation of Model (1.6). We now give the initial condition of System (1.6) as follows:

$$\begin{cases} u_1(\varphi) = u_1(\varphi), \varphi \in [-\tau, t_0], \\ u_2(\varphi) = u_2(\varphi), \varphi \in [-\tau, t_0], \end{cases} \quad (1.7)$$

where $t_0 > 0$ represents a real constant.

The fundamental frame of this article is listed as follows. Section 2 provides several necessary lemmas and definitions on the fractional-order differential equation. Section 3 analyzes the existence and uniqueness of the solution to Model (1.6). Section 4 explores the boundedness of the solution to Model (1.6). Section 5 provides the delay-independent sufficient condition on the stability and

bifurcation generation of Model (1.6). Section 6 probes into the Hopf bifurcation control design of Model (1.6) by adopting speed feedback controller. Section 7 deals with Hopf bifurcation control design of Model (1.6) by applying dislocated feedback controller. Section 8 performs numerical simulations to confirm the primary results of this manuscript. Section 9 provides the conclusions of this manuscript.

2. Essential definitions and lemmas

This part will provide the necessary definitions and lemmas of fractional-order dynamic systems. Assume $R_+ = \{u | u \geq 0, u \in R\}$.

Definition 2.1 [18] *The Caputo-type fractional-order derivative is provided as follows:*

$$\mathcal{D}^q h(v) = \frac{1}{\Gamma(l-q)} \int_{v_0}^v \frac{h^{(l)}(s)}{(v-s)^{q-l+1}} ds, \quad (2.1)$$

where $h(v) \in ([v_0, \infty), R)$, $\Gamma(s) = \int_0^\infty v^{s-1} e^{-v} dv$, $v \geq v_0$ and $l \in Z^+$, $l-1 \leq q < l$, $h^{(l)}(t)$ stands for the l -th derivative of the function $h(t)$.

The Laplace transform of Caputo-type fractional-order derivative is provided as follows:

$$\mathcal{L}\{\mathcal{D}^q h(t); s\} = s^p \mathcal{H}(s) - \sum_{i=0}^{l-1} s^{q-i-1} h^{(i)}(0), l-1 \leq q < l \in Z^+,$$

where $\mathcal{H}(s) = \mathcal{L}\{h(t)\}$, and $\mathcal{L}\{h(t)\}$ stands for the laplace transform of the function $h(t)$. Particularly, if $h^{(i)}(0) = 0, i = 1, 2, \dots, n$, then $\mathcal{L}\{\mathcal{D}^q h(t); s\} = s^p \mathcal{H}(s)$.

Definition 2.2 [19] *Here, (u_{1*}, u_{2*}) is regarded as the equilibrium point of Model (1.6) if*

$$\begin{cases} bu_{10} - bu_{1*} - \frac{\alpha u_{1*} u_{2*}}{(c + u_{1*})(\gamma + \beta u_{1*})} = 0, \\ bu_{20} - bu_{2*} - \frac{\alpha u_{1*} u_{2*}}{c + u_{1*}} = 0. \end{cases} \quad (2.2)$$

Lemma 2.1 [20] *Assume the following fractional-order system:*

$$\frac{d^q u(t)}{dt^q} = f(t, u(t)), u(t_0) = u_{t_0}, \quad (2.3)$$

where t_0 stands for a positive real constant, $q \in (0, 1]$, and $f : [t_0, \infty) \times \chi \rightarrow R_+^n, \chi \subset R_+^n$. If $f(t, u)$ satisfies the local Lipschitz condition with regard to u , then System (2.3) has a unique solution defined on $[t_0, \infty)$.

Lemma 2.2 [21] *If $u(t) \in C[t_0, \infty]$ and*

$$\begin{cases} D^q u(t) \leq -\kappa_1 u(t) + \kappa_2, \\ u(t_0) = u_{t_0}, \end{cases} \quad (2.4)$$

where $0 < q < 1, \kappa_1, \kappa_2 \in R, \kappa_1 \neq 0, t_0 \geq 0$. Then

$$u(t) \leq \left(u(t_0) - \frac{\kappa_2}{\kappa_1}\right) E_q[-\kappa_1(t-t_0)^q] + \frac{\kappa_2}{\kappa_1}, \quad (2.5)$$

where $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}$ stands for the one parameter Mittag-Leffler function.

Lemma 2.3 [22,23] We take the following fractional-order differential system

$$\frac{d^q u(t)}{dt^q} = f(t, u(t)), u(0) = u_0, \quad (2.6)$$

where $q \in (0, 1]$ and $f(t, u(t)) : R^+ \times R^m \rightarrow R^m$. The equilibrium point of System (2.6) maintains locally asymptotically stable if every eigenvalue φ of the formula $\frac{\partial f(t, u)}{\partial u}$ evaluated at the equilibrium point satisfies $|\arg(\varphi)| > \frac{q\pi}{2}$.

Lemma 2.4 [24] Assume the following fractional-order differential system

$$\begin{cases} \frac{d^{q_1} \mathcal{J}_1(t)}{dt^{q_1}} = l_{11}\mathcal{J}_1(t - \tau_{11}) + l_{12}\mathcal{J}_2(t - \tau_{12}) + \cdots + l_{1n}\mathcal{J}_n(t - \tau_{1n}), \\ \frac{d^{q_2} \mathcal{J}_2(t)}{dt^{q_2}} = l_{21}\mathcal{J}_1(t - \tau_{21}) + l_{22}\mathcal{J}_2(t - \tau_{22}) + \cdots + l_{2n}\mathcal{J}_n(t - \tau_{2n}), \\ \vdots \\ \frac{d^{q_n} \mathcal{J}_n(t)}{dt^{q_n}} = l_{n1}\mathcal{J}_1(t - \tau_{n1}) + l_{n2}\mathcal{J}_2(t - \tau_{n2}) + \cdots + l_{nn}\mathcal{J}_n(t - \tau_{nn}), \end{cases} \quad (2.7)$$

where $0 < q_j < 1 (j = 1, 2, \dots, n)$, the initial values $\mathcal{J}_i(t) = \varpi_i(t) \in C[-\max_{i,l} \tau_{il}, 0]$, $t \in [-\max_{i,l} \tau_{il}, 0]$, $i, l = 1, 2, \dots, n$. If we set

$$\Lambda(\xi) = \begin{bmatrix} \xi^{q_1} - l_{11}e^{-\xi\tau_{11}} & -l_{12}e^{-\xi\tau_{12}} & \cdots & -l_{1n}e^{-\xi\tau_{1n}} \\ -l_{21}e^{-\xi\tau_{12}} & \xi^{q_2} - l_{22}e^{-\xi\tau_{22}} & \cdots & -l_{2n}e^{-\xi\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{n1}e^{-\xi\tau_{n1}} & -l_{n2}e^{-\xi\tau_{n2}} & \cdots & \xi^{q_n} - l_{nn}e^{-\xi\tau_{nn}} \end{bmatrix}, \quad (2.8)$$

then the zero solution of Eq (2.7) keeps Lyapunov asymptotically stability if all roots of $\det(\Lambda(\xi)) = 0$ have negative real part.

3. Existence and uniqueness

This part will discuss the existence and uniqueness of the solution of System (1.6) via the Lipschitz condition.

Theorem 3.1 Denote $\Phi = \{(u_1, u_2) \in R^2 : \max\{|u_1|, |u_2|\} < M\}$, where $M > 0$ denotes a constant. Then for each $\mathcal{U}_0 = (u_1(t_0), u_2(t_0)) \in \Phi$ and for all $t \geq t_0$, System (1.6) with the initial value $\mathcal{U}_0$ has a unique solution $\mathcal{U}(t) \in \Phi$.

Proof: We will analyze the existence and uniqueness of the solution of System (1.6) in the domain $\Phi \times [t_0, t_{01}]$, where $t_{01} < +\infty$. Let $\mathcal{U} = (u_1, u_2)$ and $\tilde{\mathcal{U}} = (\tilde{u}_1, \tilde{u}_2)$, and define a mapping $\Psi(\mathcal{U}) = (\Psi_1(\mathcal{U}), \Psi_2(\mathcal{U}))$, where

$$\begin{cases} \Psi_1(\mathcal{U}) = bu_{10} - bu_1(t) - \frac{\alpha u_1(t - \tau)u_2(t)}{(c + u_1(t - \tau))(\gamma + \beta u_1(t))}, \\ \Psi_2(\mathcal{U}) = bu_{20} - bu_2(t) - \frac{\alpha u_1(t - \tau)u_2(t)}{c + u_1(t - \tau)}. \end{cases} \quad (3.1)$$

For all $\mathcal{U}, \tilde{\mathcal{U}} \in \Phi$, $t_1, t_2 \in R$, $t_1, t_2 \geq t_0$, according to the work of Huo [25], we get

$$\begin{aligned}
& \|\Psi(\mathcal{U}) - \Psi(\tilde{\mathcal{U}})\| \\
&= |\Psi_1(\mathcal{U}) - \Psi_1(\tilde{\mathcal{U}})| + |\Psi_2(\mathcal{U}) - \Psi_2(\tilde{\mathcal{U}})| \\
&= \left\| \left[bu_{10} - bu_1(t_1) - \frac{\alpha u_1(t_1 - \tau)u_2(t_1)}{(c + u_1(t_1 - \tau))(\gamma + \beta u_1(t_1))} \right] \right. \\
&\quad \left. - \left[bu_{10} - bu_1(t_2) - \frac{\alpha u_1(t_2 - \tau)u_2(t_2)}{(c + u_1(t_2 - \tau))(\gamma + \beta u_1(t_2))} \right] \right\| \\
&\quad + \left\| \left[bu_{20} - bu_2(t_1) - \frac{\alpha u_1(t_1 - \tau)u_2(t_1)}{c + u_1(t_1 - \tau)} \right] \right. \\
&\quad \left. - \left[bu_{20} - bu_2(t_2) - \frac{\alpha u_1(t_2 - \tau)u_2(t_2)}{c + u_1(t_2 - \tau)} \right] \right\| \\
&\leq b|u_1(t_1) - u_1(t_2)| + b|u_2(t_1) - u_2(t_2)| \\
&\quad + \left| \frac{\alpha u_1(t_1 - \tau)u_2(t_1)}{(c + u_1(t_1 - \tau))(\gamma + \beta u_1(t_1))} - \frac{\alpha u_1(t_2 - \tau)u_2(t_2)}{(c + u_1(t_2 - \tau))(\gamma + \beta u_1(t_2))} \right| \\
&\quad + \left| \frac{\alpha u_1(t_1 - \tau)u_2(t_1)}{c + u_1(t_1 - \tau)} - \frac{\alpha u_1(t_2 - \tau)u_2(t_2)}{c + u_1(t_2 - \tau)} \right| \\
&\leq \alpha c \gamma M |u_1(t_1 - \tau) - u_1(t_2 - \tau)| + \alpha c \gamma M |u_2(t_1) - u_2(t_2)| \\
&\quad + \alpha \gamma M^2 |u_1(t_1 - \tau) - u_1(t_2 - \tau)| + \alpha \gamma M^2 |u_1(t_1 - \tau) - u_1(t_2 - \tau)| \\
&\quad + \alpha \gamma M^2 |u_2(t_1) - u_2(t_2)| + \alpha c \beta M^2 |u_1(t_1 - \tau) - u_1(t_2 - \tau)| \\
&\quad + \alpha c \beta M^2 |u_2(t_1) - u_2(t_2)| + \alpha c \beta M^2 |u_1(t_1) - u_1(t_2)| \\
&\quad + \alpha \beta^2 M |u_1(t_1) - u_1(t_2)| + \alpha \beta^2 M |u_2(t_1) - u_1(t_2)| \\
&\leq \alpha c \gamma M |u_1(t_1) - u_1(t_2)| + \alpha c \gamma M |u_2(t_1) - u_2(t_2)| \\
&\quad + \alpha \gamma M^2 |u_1(t_1) - u_1(t_2)| + \alpha \gamma M^2 |u_1(t_1) - u_1(t_2)| \\
&\quad + \alpha \gamma M^2 |u_2(t_1) - u_2(t_2)| + \alpha c \beta M^2 |u_1(t_1) - u_1(t_2)| \\
&\quad + \alpha c \beta M^2 |u_2(t_1) - u_2(t_2)| + \alpha c \beta M^2 |u_1(t_1) - u_1(t_2)| \\
&\quad + \alpha \beta^2 M |u_1(t_1) - u_1(t_2)| + \alpha \beta^2 M |u_2(t_1) - u_1(t_2)| \\
&= \eta_1 |u_1(t_1) - u_1(t_2)| + \eta_2 |u_2(t_1) - u_2(t_2)|, \tag{3.2}
\end{aligned}$$

where

$$\begin{cases} \eta_1 = \alpha c \gamma + 2\alpha \gamma M^2 + 2\alpha c \beta M^2 + \alpha \beta^2 M, \\ \eta_2 = \alpha c \gamma M + \alpha \gamma M^2 + \alpha c \gamma \beta M^2 + \alpha \beta^2 M. \end{cases} \tag{3.3}$$

It follows from Eq (3.2) that

$$\|\Psi(\mathcal{U}) - \Psi(\tilde{\mathcal{U}})\| \leq \eta \|\mathcal{U} - \tilde{\mathcal{U}}\|, \tag{3.4}$$

where

$$\eta = \max\{\eta_1, \eta_2\}. \tag{3.5}$$

Therefore, $\Psi(\mathcal{U})$ satisfies the Lipschitz condition with regard to $\mathcal{U}$. By Lemma 2.1, one concludes that Theorem 3.1 is valid. The proof of Theorem 3.1 finishes.

4. Boundedness

This part will discuss the boundedness of the solution to System (1.6) by constructing a reasonable function. Write $u_+ = \{(u_1, u_2) \in \Upsilon : u_1, u_2 \in R_+\}$.

Theorem 4.1 *All solutions to System (1.6) beginning with u_+ are uniformly bounded.*

Proof: Let

$$U(t) = u_1(t) + u_2(t). \quad (4.1)$$

From System (1.6), we have

$$\begin{aligned} \frac{d^q U(t)}{dt^q} &= \frac{d^q u_1(t)}{dt^q} + \frac{d^q u_2(t)}{dt^q} \\ &= bu_{10} - bu_1(t) - \frac{\alpha u_1(t-\tau)u_2(t)}{(c+u_1(t-\tau))(\gamma+\beta u_1(t))} \\ &\quad + bu_{20} - bu_2(t) - \frac{\alpha u_1(t-\tau)u_2(t)}{c+u_1(t-\tau)} \\ &\leq b(u_{10} + u_{20}) - b(u_1(t) + u_2(t)) \\ &\leq -bU(t) + b(u_{10} + u_{20}). \end{aligned} \quad (4.2)$$

By using Lemma 2.2, we have

$$U(t) \leq [u_{t_0} - (u_{10} + u_{20})] E_q[-b(t-t_0)^q] + u_{10} + u_{20} \rightarrow u_{10} + u_{20}, t \rightarrow \infty. \quad (4.3)$$

So one can know that each solution to System (1.6) beginning with u_+ remains uniformly bounded. The proof is complete.

5. Stability and existence of bifurcation

This part will explore the the onset of Hopf bifurcation and the stability of System (1.6) via applying the stability and bifurcation theory of fractional-order delayed dynamic systems. If we let $E(u_{1*}, u_{2*})$ be the positive equilibrium point of System (1.6), then u_{1*}, u_{2*} satisfy the following condition:

$$\begin{cases} bu_{10} - bu_{1*} - \frac{\alpha u_{1*}u_{2*}}{(c+u_{1*})(\gamma+\beta u_{1*})} = 0, \\ bu_{20} - bu_{2*} - \frac{\alpha u_{1*}u_{2*}}{c+u_{1*}} = 0. \end{cases} \quad (5.1)$$

The linear equation of System (5.1) at $E(u_{1*}, u_{2*})$ takes the following form:

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = a_1 u_1(t) + a_2 u_2(t) + a_3 u_1(t-\tau), \\ \frac{d^q u_2(t)}{dt^q} = a_4 u_2(t) + a_5 u_1(t-\tau), \end{cases} \quad (5.2)$$

where

$$\begin{cases} a_1 = \frac{\alpha u_{1*} u_{2*} \beta}{(c + u_{1*})^2 (\gamma + \beta u_{1*})^2} - b, \\ a_2 = -\frac{\alpha u_{1*}}{(c + u_{1*}) (\gamma + \beta u_{1*})}, \\ a_3 = \frac{\alpha u_{1*} u_{2*}}{(c + u_{1*})^2 (\gamma + \beta u_{1*})} - \frac{\alpha u_{2*}}{(c + u_{1*}) (\gamma + \beta u_{1*})}, \\ a_4 = \frac{c + u_{1*}}{\alpha u_{1*} u_{2*}} - b, \\ a_5 = \frac{c + u_{1*}}{\alpha u_{1*} u_{2*}} - \frac{\alpha u_{1*} u_{2*}}{(c + u_{1*})^2}. \end{cases} \quad (5.3)$$

The corresponding characteristic equation of Eq (5.2) reads as

$$\det \begin{bmatrix} s^q - a_1 - a_3 e^{-s\tau} & -a_2 \\ -a_5 e^{-s\tau} & s^q - a_4 \end{bmatrix} = 0. \quad (5.4)$$

By Eq (5.4), one has

$$s^{2q} + \bar{a}_1 s^q + \bar{a}_2 + (\bar{a}_3 s^q + \bar{a}_4) e^{-s\tau} = 0, \quad (5.5)$$

where

$$\begin{cases} \bar{a}_1 = -(a_1 + a_4), \\ \bar{a}_2 = a_1 a_4, \\ \bar{a}_3 = -a_3, \\ \bar{a}_4 = a_3 a_4 - a_2 a_5. \end{cases} \quad (5.6)$$

Assume that $s = i\omega = \omega \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$ is the root of Eq (5.5), then by Eq (5.5), we derive

$$\begin{cases} (k_1 \omega^q + k_2) \cos \omega \tau + k_3 \omega^q \sin \omega \tau = k_4 \omega^{2q} + k_5 \omega^q, \\ k_3 \omega^q \cos \omega \tau - (k_1 \omega^q + k_2) \sin \omega \tau = k_6 \omega^{2q} + k_7 \omega^q, \end{cases} \quad (5.7)$$

where

$$\begin{cases} k_1 = \bar{a}_3 \cos \frac{q\pi}{2}, \\ k_2 = \bar{a}_4, \\ k_3 = \bar{a}_3 \sin \frac{q\pi}{2}, \\ k_4 = -\cos q\pi, \\ k_5 = \bar{a}_1 \cos \frac{q\pi}{2}, \\ k_6 = -\sin q\pi, \\ k_7 = \bar{a}_1 \sin \frac{q\pi}{2}. \end{cases} \quad (5.8)$$

In view of Eq (5.7), one has

$$(k_1 \omega^q + k_2)^2 + (k_3 \omega^q)^2 = (k_4 \omega^{2q} + k_5 \omega^q)^2 + (k_6 \omega^{2q} + k_7 \omega^q)^2 \quad (5.9)$$

which results in

$$l_1 \omega^{4q} + l_2 \omega^{3q} + l_3 \omega^{2q} + l_4 \omega^q + l_5 = 0, \quad (5.10)$$

where

$$\begin{cases} l_1 = k_4^2 + k_6^2, \\ l_2 = 2(k_4k_5 + k_6k_7), \\ l_3 = k_5^2 + k_7^2 - k_1^2 - k_3^2, \\ l_4 = -2k_1k_2, \\ l_5 = -k_2^2. \end{cases} \quad (5.11)$$

If we let $z = \omega^q$, then Eq (5.10) takes the form:

$$l_1 z^4 + l_2 z^3 + l_3 z^2 + l_4 z + l_5 = 0. \quad (5.12)$$

Note that $l_1 > 0$ and $l_5 < 0$, and then Eq (5.12) has at least one positive real root. Thus Eq (5.5) has at least one pair of purely imaginary roots. Making use of the work of Sun et al. [26], the following results can be easily derived.

Lemma 5.1 Assume that $l_i > 0 (i = 2, 3, 4)$. Let ω_0 be a root of Eq (5.10). Then Eq (5.5) has a pair of purely imaginary roots $\pm i\omega_0$ if $\tau = \tau_i (i = 1, 2, 3, \dots)$ where

$$\tau_i = \frac{1}{\omega_0} \left[\arccos \left(\frac{(k_1\omega_0^{2q} + k_5\omega_0^q)(k_1\omega_0^q + k_2) + (k_6\omega_0^{2q} + k_7\omega_0^q)k_3\omega_0^q}{(k_1\omega_0^q + k_2)^2 + k_3^2\omega_0^{2q}} \right) + 2i\pi \right], \quad (5.13)$$

where $i = 0, 1, 2, \dots$.

Here we omit the proof of Lemma 5.1. One can refer to the work of Sun et al. [26].

We now give the necessary assumption as follows:

(A₁) $H_{11}H_{21} + H_{12}H_{22} > 0$, where

$$\begin{cases} H_{11} = 2q\omega_0^{2q-1} \cos \frac{(2q-1)\pi}{2} + q_1\bar{a}_1\omega_0^{q-1} \cos \frac{(q-1)\pi}{2} \\ \quad + q\bar{a}_3\omega_0^{q-1} \cos \frac{(q-1)\pi}{2} \cos \omega_0\tau_0 + q\bar{a}_3\omega_0^{q-1} \sin \frac{(q-1)\pi}{2} \sin \omega_0\tau_0, \\ H_{12} = 2q\omega_0^{2q-1} \sin \frac{(2q-1)\pi}{2} + q_1\bar{a}_1\omega_0^{q-1} \sin \frac{(q-1)\pi}{2} \\ \quad - q\bar{a}_3\omega_0^{q-1} \cos \frac{(q-1)\pi}{2} \sin \omega_0\tau_0 + q\bar{a}_3\omega_0^{q-1} \sin \frac{(q-1)\pi}{2} \cos \omega_0\tau_0, \\ H_{21} = \left(\bar{a}_3\omega_0^q \cos \frac{q\pi}{2} + \bar{a}_4 \right) \omega_0 \sin \omega_0\tau_0 - \left(\bar{a}_3\omega_0^q \sin \frac{q\pi}{2} \right) \omega_0 \cos \omega_0\tau_0, \\ H_{22} = \left(\bar{a}_3\omega_0^q \cos \frac{q\pi}{2} + \bar{a}_4 \right) \omega_0 \cos \omega_0\tau_0 - \left(\bar{a}_3\omega_0^q \sin \frac{q\pi}{2} \right) \omega_0 \sin \omega_0\tau_0. \end{cases} \quad (5.14)$$

Lemma 5.2 If we let $s(\tau) = \xi_1(\tau) + i\xi_2(\tau)$ be a root of Eq (5.5) around $\tau = \tau_0$ such that $\xi_1(\tau_0) = 0$, $\xi_2(\tau_0) = \omega_0$, then $\text{Re} \left[\frac{ds}{d\tau} \right] \Big|_{\tau=\tau_0, \omega=\omega_0} > 0$.

Proof: In view of Eq (5.5), we obtain

$$\begin{aligned} & (2qs^{2q-1} + q\bar{a}_1s^{q-1}) \frac{ds}{d\tau} + q\bar{a}_3s^{q-1}e^{-s\tau} \frac{ds}{d\tau} \\ & - e^{-s\tau} \left(\frac{ds}{d\tau} \tau + s \right) (\bar{a}_3s^q + \bar{a}_4) = 0. \end{aligned} \quad (5.15)$$

It follows from Eq (5.15) that

$$\left[\frac{ds}{d\tau} \right]^{-1} = \frac{H_1(s)}{H_2(s)} - \frac{\tau}{s}, \quad (5.16)$$

where

$$\begin{cases} H_1(s) = 2qs^{2q-1} + q\bar{a}_1s^{q-1} + q\bar{a}_3s^{q-1}e^{-s\tau}, \\ H_2(s) = se^{-s\tau}(\bar{a}_3s^q + \bar{a}_4). \end{cases} \quad (5.17)$$

Then

$$\operatorname{Re} \left\{ \frac{ds}{d\tau} \right\} \bigg|_{\tau=\tau_0, \omega=\omega_0} = \operatorname{Re} \left\{ \frac{H_1(s)}{H_2(s)} \right\} \bigg|_{\tau=\tau_0, \omega=\omega_0} = \frac{H_{11}H_{21} + H_{12}H_{22}}{H_{21}^2 + H_{22}^2}. \quad (5.18)$$

By (A_1) , we derive

$$\operatorname{Re} \left\{ \left[\frac{ds}{d\tau} \right]^{-1} \right\} \bigg|_{\tau=\tau_0, \omega=\omega_0} > 0. \quad (5.19)$$

The proof is complete.

Next, we present the assumption as follows:

(A_2) $\bar{a}_1 + \bar{a}_3 > 0, \bar{a}_2 + \bar{a}_4 > 0$.

Lemma 5.3 Assume that $\tau = 0$ and (A_2) is fulfilled, then System (1.6) keeps locally asymptotically stable state.

Proof: When $\tau = 0$, then Eq (5.5) can be rewritten as follows:

$$\lambda^2 + (\bar{a}_1 + \bar{a}_3)\lambda + \bar{a}_2 + \bar{a}_4 = 0, \quad (5.20)$$

where $\lambda = s^q$. In view of (A_2) , we conclude that the two roots $\lambda_i (i = 1, 2)$ of (5.20) satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2} (i = 1, 2)$. Thus Lemma 5.3 is true. The proof is complete.

On the basis of the analysis above, one can establish the following assertion.

Theorem 5.1 If (A_1) and (A_2) hold, then the equilibrium point $E(u_{1*}, u_{2*})$ of System (1.6) remains locally asymptotically stable if $\tau \in [0, \tau_0)$ and System (1.6) will generate a cluster of Hopf bifurcations around the equilibrium point $E(u_{1*}, u_{2*})$ when $\tau = \tau_0$.

Remark 5.1 In view of Eq (5.1), we get the following expression:

$$\tilde{a}_1 u_{1*}^3 + \tilde{a}_2 u_{1*}^2 + \tilde{a}_3 u_{1*} + \tilde{a}_4 = 0,$$

where $\tilde{a}_1 = b^2\beta, \tilde{a}_2 = b\beta - b^2\beta u_{10} - b^2\gamma + b^2\gamma + b^2\beta, \tilde{a}_3 = abu_{20} - b^2\gamma u_{10} - b^2\beta u_{10} + b^2c\gamma, \tilde{a}_4 = -(b^2c\gamma u_{10} + b\gamma u_{10})$. If the equation $\tilde{a}_1 u_{1*}^3 + \tilde{a}_2 u_{1*}^2 + \tilde{a}_3 u_{1*} + \tilde{a}_4 = 0$ has at least positive root (say u_{1*}) and $(bu_{10} - bu_{1*})(\gamma + \beta u_{1*}) - bu_{20} > 0$, then System (1.6) has a positive equilibrium point

$$E \left(u_{1*}, \frac{(bu_{10} - bu_{1*})(\gamma + \beta u_{1*}) - bu_{20}}{b} \right).$$

6. Bifurcation control utilizing speed feedback controller

This part will focus on the controller design of Hopf bifurcation of System (1.6) by constructing a suitable speed feedback controller. Depending on the ideas of [15], we design the controller $\mu(t) = -\varrho_1 \frac{d^q u_1(t)}{dt^q}$ where ϱ_1 represents a feedback coefficient. Adding this controller $\mu(t) = -\varrho_1 \frac{d^q u_1(t)}{dt^q}$ to the second equation of System (1.6), we get the following fractional-order controlled chemostat model:

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t - \tau) u_2(t)}{(c + u_1(t - \tau))(\gamma + \beta u_1(t))}, \\ \frac{d^q u_2(t)}{dt^q} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t - \tau) u_2(t)}{c + u_1(t - \tau)} + \mu(t). \end{cases} \quad (6.1)$$

Specifically,

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t-\tau)u_2(t)}{(c+u_1(t-\tau))(\gamma+\beta u_1(t))}, \\ \frac{d^q u_2(t)}{dt^q} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t-\tau)u_2(t)}{c+u_1(t-\tau)} \\ \quad - \varrho_1 \left[bu_{10} - bu_1(t) - \frac{\alpha u_1(t-\tau)u_2(t)}{(c+u_1(t-\tau))(\gamma+\beta u_1(t))} \right]. \end{cases} \quad (6.2)$$

System (6.2) has the equilibrium point $E(u_{1*}, u_{2*})$. The linear equation of System (6.2) at the equilibrium point $E(u_{1*}, u_{2*})$ is given by

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = b_1 u_1(t) + b_2 u_2(t) + b_3 u_1(t-\tau), \\ \frac{d^q u_2(t)}{dt^q} = b_4 u_1(t) + b_5 u_2(t) + b_6 u_1(t-\tau), \end{cases} \quad (6.3)$$

where

$$\begin{cases} b_1 = \frac{\alpha u_{1*} u_{2*} \beta}{(c+u_{1*})^2 (\gamma+\beta u_{1*})^2} - b, \\ b_2 = -\frac{\alpha u_{1*} u_{2*}}{(c+u_{1*})(\gamma+\beta u_{1*})}, \\ b_3 = \frac{\alpha u_{1*} u_{2*} \beta}{(c+u_{1*})^2 (\gamma+\beta u_{1*})^2} - \frac{\alpha u_{2*}}{(c+u_{1*})(\gamma+\beta u_{1*})}, \\ b_4 = -\varrho_1 \left[\frac{\alpha u_{1*} u_{2*} \beta}{(c+u_{1*})^2 (\gamma+\beta u_{1*})^2} - b \right], \\ b_5 = \frac{\alpha u_{1*}}{c+u_{1*}} - b + \frac{\varrho_1 \alpha u_{1*}}{(c+u_{1*})(\gamma+\beta u_{1*})}, \\ b_6 = \frac{\alpha u_{1*} u_{2*}}{c+u_{1*}} - \frac{\alpha u_{1*} u_{2*}}{(c+u_{1*})^2} - \varrho_1 \left[\frac{\alpha u_{1*} u_{2*}}{(c+u_{1*})^2 (\gamma+\beta u_{1*})} - \frac{\alpha u_{2*}}{(c+u_{1*})(\gamma+\beta u_{1*})} \right]. \end{cases} \quad (6.4)$$

The corresponding characteristic equation of Eq (6.3) reads as

$$\det \begin{bmatrix} s^q - b_1 - b_3 e^{-s\tau} & -b_2 \\ -b_4 - b_6 e^{-s\tau} & s^q - b_5 \end{bmatrix} = 0. \quad (6.5)$$

By Eq (6.5), one has

$$s^{2q} + \bar{b}_1 s^q + \bar{b}_2 + (\bar{b}_3 s^q + \bar{b}_4) e^{-s\tau} = 0, \quad (6.6)$$

where

$$\begin{cases} \bar{b}_1 = -(b_1 + b_5), \\ \bar{b}_2 = b_1 b_5 - b_2 b_4, \\ \bar{b}_3 = -b_3, \\ \bar{b}_4 = b_3 b_5 - b_2 b_6. \end{cases} \quad (6.7)$$

If we let $s = i\psi = \psi \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$ be the root of Eq (6.6), then it follows from Eq (6.6) that

$$\begin{cases} (m_1 \psi^q + m_2) \cos \psi \tau + m_3 \psi^q \sin \psi \tau = m_4 \psi^{2q} + m_5 \psi^q, \\ m_3 \psi^q \cos \psi \tau - (m_1 \psi^q + m_2) \sin \psi \tau = m_6 \psi^{2q} + m_7 \psi^q, \end{cases} \quad (6.8)$$

where

$$\begin{cases} m_1 = \bar{b}_3 \cos \frac{q\pi}{2}, \\ m_2 = \bar{b}_4, \\ m_3 = \bar{b}_3 \sin \frac{q\pi}{2}, \\ m_4 = -\cos q\pi, \\ m_5 = \bar{b}_1 \cos \frac{q\pi}{2}, \\ k_6 = -\sin q\pi, \\ m_7 = \bar{b}_1 \sin \frac{q\pi}{2}. \end{cases} \quad (6.9)$$

In view of Eq (6.8), one has

$$(m_1\psi^q + m_2)^2 + (m_3\psi^q)^2 = (m_4\psi^{2q} + m_5\psi^q)^2 + (m_6\psi^{2q} + m_7\psi^q)^2 \quad (6.10)$$

which results in

$$\varepsilon_1\psi^{4q} + \varepsilon_2\psi^{3q} + \varepsilon_3\psi^{2q} + \varepsilon_4\psi^q + \varepsilon_5 = 0, \quad (6.11)$$

where

$$\begin{cases} \varepsilon_1 = m_4^2 + m_6^2, \\ \varepsilon_2 = 2(m_4m_5 + m_6m_7), \\ \varepsilon_3 = m_5^2 + m_7^2 - m_1^2 - m_3^2, \\ \varepsilon_4 = -2m_1m_2, \\ \varepsilon_5 = -m_2^2. \end{cases} \quad (6.12)$$

If we let $\bar{y} = \psi^q$, then Eq (6.11) becomes

$$\varepsilon_1\bar{y}^4 + \varepsilon_2\bar{y}^3 + \varepsilon_3\bar{y}^2 + \varepsilon_4\bar{y} + \varepsilon_5 = 0. \quad (6.13)$$

Note that if $\varepsilon_1 > 0$ and $\varepsilon_5 < 0$, then Eq (6.13) has at least one positive real root. Thus Eq (6.6) has at least one pair of purely roots. Making use of the work of Sun et al. [26], the following results can be easily derived.

Lemma 6.1 Assume that $\varepsilon_i > 0 (i = 1, 2, 3, 4)$. Let ψ_0 be a root of Eq (6.11). Then Eq (6.6) has a pair of purely imaginary roots $\pm i\psi_0$ if $\tau = \tau_{0i} (i = 1, 2, 3, \dots)$ where

$$\tau_{0j} = \frac{1}{\psi_0} \left[\arccos \left(\frac{(m_1\psi_0^{2q} + m_5\psi_0^q)(m_1\psi_0^q + m_2) + (m_6\psi_0^{2q} + m_7\psi_0^q)m_3\psi_0^q}{(m_1\psi_0^q + m_2)^2 + m_3^2\psi_0^{2q}} \right) + 2j\pi \right], \quad (6.14)$$

where $j = 0, 1, 2, \dots$.

Here we omit the proof of Lemma 6.1. One can refer to the work of Sun et al. [26].

We now give the necessary assumption as follows.

(A₃) $I_{11}I_{21} + I_{12}I_{22} > 0$, where

$$\begin{cases} I_{11} = 2q\psi_0^{2q-1} \cos \frac{(2q-1)\pi}{2} + q_1\bar{b}_1\psi_0^{q-1} \cos \frac{(q-1)\pi}{2} \\ \quad + q\bar{b}_3\psi_0^{q-1} \cos \frac{(q-1)\pi}{2} \cos \psi_0\tau_{00} + q\bar{b}_3\psi_0^{q-1} \sin \frac{(q-1)\pi}{2} \sin \psi_0\tau_{00}, \\ I_{12} = 2q\psi_0^{2q-1} \sin \frac{(2q-1)\pi}{2} + q_1\bar{b}_1\psi_0^{q-1} \sin \frac{(q-1)\pi}{2} \\ \quad - q\bar{b}_3\psi_0^{q-1} \cos \frac{(q-1)\pi}{2} \sin \psi_0\tau_{00} + q\bar{b}_3\psi_0^{q-1} \sin \frac{(q-1)\pi}{2} \cos \psi_0\tau_{00}, \\ I_{21} = \left(\bar{b}_3\psi_0^q \cos \frac{q\pi}{2} + \bar{b}_4 \right) \psi_0 \sin \psi_0\tau_{00} - \left(\bar{b}_3\psi_0^q \sin \frac{q\pi}{2} \right) \psi_0 \cos \psi_0\tau_{00}, \\ I_{22} = \left(\bar{b}_3\psi_0^q \cos \frac{q\pi}{2} + \bar{b}_4 \right) \psi_0 \cos \psi_0\tau_{00} - \left(\bar{b}_3\psi_0^q \sin \frac{q\pi}{2} \right) \psi_0 \sin \psi_0\tau_{00}. \end{cases} \quad (6.15)$$

Lemma 6.2 *If we let $s(\tau) = \mu_1(\tau) + i\mu_2(\tau)$ be a root of Eq (6.6) around $\tau = \tau_{00}$ such that $\mu_1(\tau_{00}) = 0, \mu_2(\tau_{00}) = \psi_0$, then $\operatorname{Re} \left[\frac{ds}{d\tau} \right] \Big|_{\tau=\tau_{00}, \psi=\psi_0} > 0$.*

Proof: In view of Eq (6.6), we obtain

$$\begin{aligned} & (2qs^{2q-1} + q\bar{b}_1s^{q-1}) \frac{ds}{d\tau} + q\bar{b}_3s^{q-1}e^{-s\tau} \frac{ds}{d\tau} \\ & - e^{-s\tau} \left(\frac{ds}{d\tau} \tau + s \right) (\bar{b}_3s^q + \bar{b}_4) = 0. \end{aligned} \quad (6.16)$$

It follows from Eq (6.16) that

$$\left[\frac{ds}{d\tau} \right]^{-1} = \frac{I_1(s)}{I_2(s)} - \frac{\tau}{s}, \quad (6.17)$$

where

$$\begin{cases} I_1(s) = 2qs^{2q-1} + q\bar{b}_1s^{q-1} + q\bar{b}_3s^{q-1}e^{-s\tau}, \\ I_2(s) = se^{-s\tau} (\bar{b}_3s^q + \bar{b}_4). \end{cases} \quad (6.18)$$

Then

$$\operatorname{Re} \left\{ \frac{ds}{d\tau} \right\} \Big|_{\tau=\tau_{00}, \psi=\psi_0} = \operatorname{Re} \left\{ \frac{I_1(s)}{I_2(s)} \right\} \Big|_{\tau=\tau_{00}, \psi=\psi_0} = \frac{I_{11}I_{21} + I_{12}I_{22}}{I_{21}^2 + I_{22}^2}. \quad (6.19)$$

By (A₃), we derive

$$\operatorname{Re} \left\{ \left[\frac{ds}{d\tau} \right]^{-1} \right\} \Big|_{\tau=\tau_{00}, \psi=\psi_0} > 0. \quad (6.20)$$

The proof is complete.

Next, we present the assumption as follows:

(A₄) $\bar{b}_1 + \bar{b}_3 > 0, \bar{b}_2 + \bar{b}_4 > 0$.

Lemma 6.3 *Assume that $\tau = 0$ and (A₄) is fulfilled, then system (6.1) keeps locally asymptotically stable state.*

Proof: When $\tau = 0$, then Eq (6.6) can be rewritten as follows:

$$\lambda^2 + (\bar{b}_1 + \bar{b}_3)\lambda + \bar{b}_2 + \bar{b}_4 = 0, \quad (6.21)$$

where $\lambda = s^q$. In view of (A_4) , we conclude that the two roots $\lambda_i (i = 1, 2)$ of Eq (6.21) satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2} (i = 1, 2, \dots)$. Thus Lemma 6.3 is true. The proof completes.

On the basis of the analysis above, one can establish the following assertion.

Theorem 6.1 *If (A_3) and (A_4) hold, then the equilibrium point $E(u_{1*}, u_{2*})$ of System (6.1) remains locally asymptotically stable if $\tau \in [0, \tau_{00})$ and System (6.1) will generate a cluster of Hopf bifurcations around the equilibrium point $E(u_{1*}, u_{2*})$ when $\tau = \tau_{00}$.*

7. Bifurcation control utilizing dislocated feedback controller

This part will focus on the controller design for the Hopf bifurcation of System (1.6) by constructing a suitable dislocated feedback controller. Depending on the ideas of [15], we design the controller $v(t) = -\varrho_2 u_1(t)$ where ϱ_2 represents the feedback coefficient. Adding this controller $v(t) = -\varrho_2 u_1(t)$ to the second equation of System (1.6), we then get the following fractional-order controlled chemostat model:

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t - \tau) u_2(t)}{(c + u_1(t - \tau))(\gamma + \beta u_1(t))}, \\ \frac{d^q u_2(t)}{dt^q} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t - \tau) u_2(t)}{c + u_1(t - \tau)} + v(t). \end{cases} \quad (7.1)$$

Specifically,

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = bu_{10} - bu_1(t) - \frac{\alpha u_1(t - \tau) u_2(t)}{(c + u_1(t - \tau))(\gamma + \beta u_1(t))}, \\ \frac{d^q u_2(t)}{dt^q} = bu_{20} - bu_2(t) - \frac{\alpha u_1(t - \tau) u_2(t)}{c + u_1(t - \tau)} - \varrho_2 u_1(t). \end{cases} \quad (7.2)$$

If we let $E(u_{1*}, u_{2*})$ be the equilibrium point of System (7.2). Then u_{1*}, u_{2*} satisfy

$$\begin{cases} bu_{10} - bu_{1*} - \frac{\alpha u_{1*} u_{2*}}{(c + u_{1*})(\gamma + \beta u_{1*})} = 0, \\ bu_{20} - bu_{2*} - \frac{\alpha u_{1*} u_{2*}}{c + u_{1*}} - \varrho_2 u_{1*} = 0. \end{cases} \quad (7.3)$$

The linear equation of System (7.2) at the equilibrium point $E(u_{1*}, u_{2*})$ is given by

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = c_1 u_1(t) + c_2 u_2(t) + c_3 u_1(t - \tau), \\ \frac{d^q u_2(t)}{dt^q} = c_4 u_1(t) + c_5 u_2(t) + c_6 u_1(t - \tau), \end{cases} \quad (7.4)$$

where

$$\begin{cases} c_1 = \frac{\alpha u_{1*} u_{2*} \beta}{(c + u_{1*})^2 (\gamma + \beta u_{1*})^2} - b, \\ c_2 = -\frac{\alpha u_{1*}}{(c + u_{1*})(\gamma + \beta u_{1*})}, \\ c_3 = \frac{\alpha u_{1*} u_{2*}}{(c + u_{1*})^2 (\gamma + \beta u_{1*})} - \frac{\alpha u_{2*}}{(c + u_{1*})(\gamma + \beta u_{1*})}, \\ c_4 = -\varrho_2, \\ c_5 = \frac{\alpha u_{1*}}{c + u_{1*}} - b, \\ c_6 = \frac{c + u_{1*}}{\alpha u_{1*} u_{2*}} - \frac{\alpha u_{1*} u_{2*}}{(c + u_{1*})^2}. \end{cases} \quad (7.5)$$

The corresponding characteristic equation of Eq (7.4) reads as follows:

$$\det \begin{bmatrix} s^q - c_1 - b_3 e^{-s\tau} & -c_2 \\ -c_4 - c_6 e^{-s\tau} & s^q - c_5 \end{bmatrix} = 0. \quad (7.6)$$

By Eq (7.6), one has

$$s^{2q} + \bar{c}_1 s^q + \bar{c}_2 + (\bar{c}_3 s^q + \bar{c}_4) e^{-s\tau} = 0, \quad (7.7)$$

where

$$\begin{cases} \bar{c}_1 = -(c_1 + c_5), \\ \bar{c}_2 = c_1 c_5 - c_2 c_4, \\ \bar{c}_3 = -c_3, \\ \bar{c}_4 = c_3 c_5 - c_2 c_6. \end{cases} \quad (7.8)$$

If we assume that $s = i\chi = \chi \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$ is the root of Eq (7.7), then by Eq (7.7), we have

$$\begin{cases} (n_1 \chi^q + n_2) \cos \chi \tau + n_3 \chi^q \sin \chi \tau = n_4 \chi^{2q} + n_5 \chi^q, \\ n_3 \chi^q \cos \chi \tau - (n_1 \chi^q + n_2) \sin \chi \tau = n_6 \chi^{2q} + n_7 \chi^q, \end{cases} \quad (7.9)$$

where

$$\begin{cases} n_1 = \bar{c}_3 \cos \frac{q\pi}{2}, \\ n_2 = \bar{c}_4, \\ n_3 = \bar{c}_3 \sin \frac{q\pi}{2}, \\ n_4 = -\cos q\pi, \\ n_5 = \bar{c}_1 \cos \frac{q\pi}{2}, \\ n_6 = -\sin q\pi, \\ n_7 = \bar{c}_1 \sin \frac{q\pi}{2}. \end{cases} \quad (7.10)$$

In view of Eq (7.9), one has

$$(n_1 \chi^q + n_2)^2 + (n_3 \chi^q)^2 = (n_4 \chi^{2q} + n_5 \chi^q)^2 + (n_6 \chi^{2q} + n_7 \chi^q)^2 \quad (7.11)$$

which results in

$$\epsilon_1 \chi^{4q} + \epsilon_2 \chi^{3q} + \epsilon_3 \chi^{2q} + \epsilon_4 \chi^q + \epsilon_5 = 0, \quad (7.12)$$

where

$$\begin{cases} \epsilon_1 = n_4^2 + n_6^2, \\ \epsilon_2 = 2(n_4 n_5 + n_6 n_7), \\ \epsilon_3 = n_5^2 + n_7^2 - n_1^2 - n_3^2, \\ \epsilon_4 = -2n_1 n_2, \\ \epsilon_5 = -n_2^2. \end{cases} \quad (7.13)$$

If we let $\bar{z} = \chi^q$, then Eq (7.12) becomes

$$\epsilon_1 \bar{z}^4 + \epsilon_2 \bar{z}^3 + \epsilon_3 \bar{z}^2 + \epsilon_4 \bar{z} + \epsilon_5 = 0. \quad (7.14)$$

Note that $\epsilon_1 > 0$ and $\epsilon_5 < 0$, and thus Eq (7.14) has at least one positive real root. Thus Eq (7.7) has at least one pair of purely roots. Making use of the work of Sun et al. [26], the following results can be easily derived.

Lemma 7.1 Assume that $\epsilon_i > 0 (i = 2, 3, 4)$. Let χ_0 be a root of Eq (7.12). Then Eq (7.7) has a pair of purely imaginary roots $\pm i\chi_0$ if $\tau = \tau_{00i} (i = 1, 2, 3, \dots)$ where

$$\tau_{00i} = \frac{1}{\chi_0} \left[\arccos \left(\frac{(n_1\chi_0^{2q} + n_5\chi_0^q)(n_1\chi_0^q + n_2) + (n_6\chi_0^{2q} + n_7\chi_0^q)n_3\chi_0^q}{(n_1\chi_0^q + n_2)^2 + n_3^2\chi_0^{2q}} \right) + 2i\pi \right], \quad (7.15)$$

where $i = 0, 1, 2, \dots$.

Here we omit the proof of Lemma 7.1. One can refer to the work of Sun et al. [26].

We now give the necessary assumption as follows.

(A₅) $J_{11}J_{21} + J_{12}J_{22} > 0$, where

$$\begin{cases} J_{11} = 2q\chi_0^{2q-1} \cos \frac{(2q-1)\pi}{2} + q_1\bar{c}_1\chi_0^{q-1} \cos \frac{(q-1)\pi}{2} \\ \quad + q\bar{c}_3\chi_0^{q-1} \cos \frac{(q-1)\pi}{2} \cos \chi_0\tau_{000} + q\bar{c}_3\omega_0^{q-1} \sin \frac{(q-1)\pi}{2} \sin \chi_0\tau_{000}, \\ J_{12} = 2q\chi_0^{2q-1} \sin \frac{(2q-1)\pi}{2} + q_1\bar{c}_1\chi_0^{q-1} \sin \frac{(q-1)\pi}{2} \\ \quad - q\bar{c}_3\chi_0^{q-1} \cos \frac{(q-1)\pi}{2} \sin \chi_0\tau_{000} + q\bar{c}_3\chi_0^{q-1} \sin \frac{(q-1)\pi}{2} \cos \chi_0\tau_{000}, \\ J_{21} = \left(\bar{c}_3\chi_0^q \cos \frac{q\pi}{2} + \bar{c}_4 \right) \chi_0 \sin \chi_0\tau_{000} - \left(\bar{c}_3\chi_0^q \sin \frac{q\pi}{2} \right) \chi_0 \cos \chi_0\tau_{000}, \\ J_{22} = \left(\bar{c}_3\chi_0^q \cos \frac{q\pi}{2} + \bar{c}_4 \right) \chi_0 \cos \chi_0\tau_{000} - \left(\bar{c}_3\chi_0^q \sin \frac{q\pi}{2} \right) \chi_0 \sin \chi_0\tau_{000}. \end{cases} \quad (7.16)$$

Lemma 7.2 Let $s(\tau) = \rho_1(\tau) + i\rho_2(\tau)$ be a root of Eq (7.7) around $\tau = \tau_{000}$ such that $\rho_1(\tau_{000}) = 0, \rho_2(\tau_{000}) = \chi_0$, then $\text{Re} \left[\frac{ds}{d\tau} \right] \Big|_{\tau=\tau_{000}, \chi=\chi_0} > 0$.

Proof: In view of Eq (7.7), we obtain

$$\begin{aligned} & (2qs^{2q-1} + q\bar{c}_1s^{q-1}) \frac{ds}{d\tau} + q\bar{c}_3s^{q-1}e^{-s\tau} \frac{ds}{d\tau} \\ & - e^{-s\tau} \left(\frac{ds}{d\tau} \tau + s \right) (\bar{c}_3s^q + \bar{c}_4) = 0. \end{aligned} \quad (7.17)$$

It follows from Eq (7.17) that

$$\left[\frac{ds}{d\tau} \right]^{-1} = \frac{J_1(s)}{J_2(s)} - \frac{\tau}{s}, \quad (7.18)$$

where

$$\begin{cases} J_1(s) = 2qs^{2q-1} + q\bar{c}_1s^{q-1} + q\bar{c}_3s^{q-1}e^{-s\tau}, \\ J_2(s) = se^{-s\tau} (\bar{c}_3s^q + \bar{c}_4). \end{cases} \quad (7.19)$$

Then

$$\text{Re} \left\{ \frac{ds}{d\tau} \right\} \Big|_{\tau=\tau_{000}, \chi=\chi_0} = \text{Re} \left\{ \frac{J_1(s)}{J_2(s)} \right\} \Big|_{\tau=\tau_{000}, \chi=\chi_0} = \frac{J_{11}J_{21} + J_{12}J_{22}}{J_{21}^2 + J_{22}^2}. \quad (7.20)$$

By (A₅), we derive

$$\text{Re} \left\{ \left[\frac{ds}{d\tau} \right]^{-1} \right\} \Big|_{\tau=\tau_{000}, \chi=\chi_0} > 0. \quad (7.21)$$

The proof is complete.

Next, we present the assumption as follows:

$$(A_6) \quad \bar{c}_1 + \bar{c}_3 > 0, \bar{c}_2 + \bar{c}_4 > 0.$$

Lemma 7.3 *If we assume $\tau = 0$ and (A_6) is fulfilled, then System (7.1) keeps a locally asymptotically stable state.*

Proof: When $\tau = 0$, then Eq 7.7 can be rewritten as follows:

$$\lambda^2 + (\bar{c}_1 + \bar{c}_3)\lambda + \bar{c}_2 + \bar{c}_4 = 0, \quad (7.22)$$

where $\lambda = s^q$. In view of (A_6) , we conclude that the two roots $\lambda_i (i = 1, 2)$ of Eq (7.22) satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2} (i = 1, 2,)$. Thus Lemma 7.3 is true. The proof is complete.

On the basis of the analysis above, one can establish the following assertion.

Theorem 7.1 *If (A_5) and (A_6) hold, then the equilibrium point $E(u_{1*}, u_{2*})$ of System (7.1) remains locally asymptotically stable if $\tau \in [0, \tau_{000})$, and System (7.1) will generate a cluster of Hopf bifurcations around the equilibrium point $E(u_{1*}, u_{2*})$ when $\tau = \tau_{000}$.*

Remark 7.1 *In this section, we design a suitable dislocated feedback controller to control the Hopf bifurcation of System (1.6). The controlled System (7.2) and the uncontrolled System (1.6) have different positive equilibria. It is shown that we change the position of the equilibrium point to achieve control the local asymptotic stability of the equilibrium point and the critical value of Hopf bifurcation of System (1.6).*

Remark 7.2 *Mohd Aris and Jamaian [27] introduced a delay into Model (8.3) and investigated the dynamics of the proposed fractional delayed chemostat model. In this paper, we consider the stability and Hopf bifurcation and the new controller design issue of a new fractional delayed chemostat model (namely, Model (1.6)). The obtained results are novel.*

Remark 7.3 *In Section 6 and 7, we adopt a speed feedback controller and a dislocated feedback controller to adjust the stability domain of the system and the delay length of the onset of Hopf bifurcation. For the speed feedback controller, we add a perturbation term with speed variation and this controller does not change the position of the equilibrium point. For the dislocated feedback controller, we add a perturbation term with a state variable, and this controller changes the position of the equilibrium point.*

Remark 7.4 *By designing the dislocated feedback controller, the controlled System (7.2) and the uncontrolled System (1.6) have different equilibrium points. From view point of ecology, we know that the concentrations of the substrate and the concentrations of the cell mass will tend to different values, compared with the concentrations of the substrate and the concentrations of the cell mass in original system (1.6) when the time $t \rightarrow +\infty$. Moreover, the critical values of time delay in the controlled system (7.2) have a close connection with the control parameter q_2 and the positive equilibrium of the controlled system (7.2).*

Remark 7.5 *In [27], the authors adopted the time-delay feedback controller and the traditional feedback controller to adjust the stability and Hopf bifurcation of the proposed models. In this paper, we apply the speed feedback controller and the dislocated feedback controller to adjust the the stability and Hopf bifurcation of the proposed Model (1.6), which is different from the previous work [27]. According to this idea, our controllers are new.*

8. Computer simulation diagrams

Example 8.1 Consider the following fractional-order chemostat model incorporating delay:

$$\begin{cases} \frac{d^{0.87} u_1(t)}{dt^{0.87}} = 0.43 \times 0.15 - 0.43 u_1(t) - \frac{2.5 u_1(t - \tau) u_2(t)}{(0.85 + u_1(t - \tau))(0.52 + 3.25 u_1(t))}, \\ \frac{d^{0.87} u_2(t)}{dt^{0.87}} = 0.43 \times 0.35 - 0.43 u_2(t) - \frac{2.5 u_1(t - \tau) u_2(t)}{0.85 + u_1(t - \tau)}. \end{cases} \quad (8.1)$$

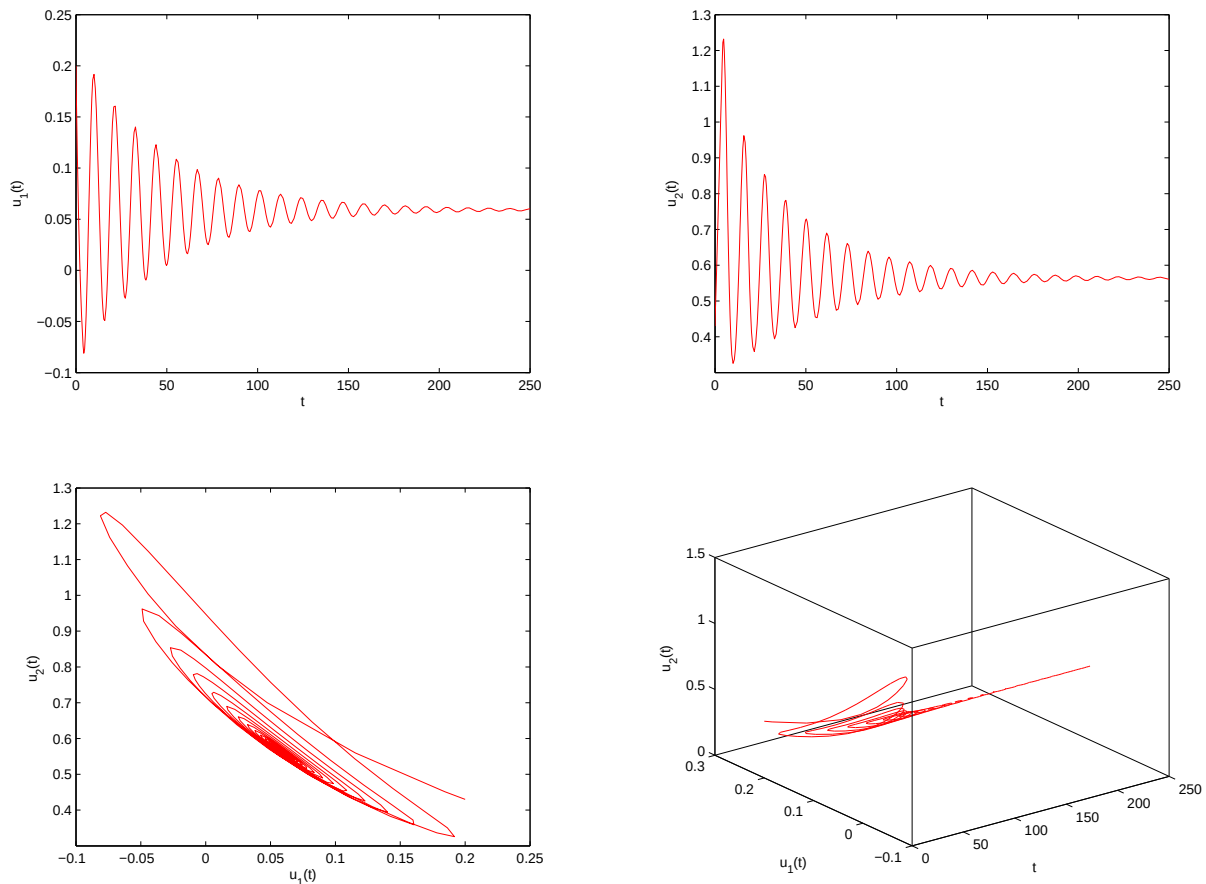


Figure 1. Matlab simulation figures of System (8.1) concerning the delay $\tau = 3.92 < \tau_0 = 4.25$. The positive equilibrium point $E(0.0592, 0.5634)$ maintains a locally asymptotically stable position.

It is easy to acquire that System (8.1) has a positive equilibrium point $E(0.0592, 0.5634)$. With Matlab software, we can easily obtain $\tau_0 = 4.25$. One can verify that the conditions (A_1) and (A_2) in Theorem 5.1 hold. To check the correctness of the conclusions of Theorem 5.1, we now choose both two delays. One is $\tau = 3.92$ and another is $\tau = 4.85$. For $\tau = 3.92 < \tau_0 = 4.25$, we get the simulation plots which are presented in Figure 1. In view of Figure 1, we know that $u_1 \rightarrow 0.0592, u_2 \rightarrow 0.5634$ as $t \rightarrow \infty$, which shows that the positive equilibrium point $E(0.0592, 0.5634)$ of System (8.1) will remain in a locally asymptotically stable state when the time t gradually approaches infinity. From the view

point of ecology, the concentrations of the substrate and the concentrations of the cell mass will tend to the values 0.0592 and 0.5634, respectively. For $\tau = 4.85 > \tau_0 = 4.25$, we get the simulation plots which are presented in Figure 2. In view of Figure 2, we know that u_1, u_2 are to remain in periodic motion as $t \rightarrow \infty$, which shows that the Hopf bifurcation of the system will arise near the positive equilibrium point $E(0.0592, 0.5634)$ when the time t gradually approaches infinity. From the view point of ecology, the concentrations of the substrate and the concentrations of the cell mass will keep periodic motion around the values 0.0592 and 0.5634, respectively. In addition, the bifurcation diagrams are provided to display the bifurcation value more intuitively (see Figures 3 and 4). Obviously, Figures 3 and 4 show that the bifurcation value is equal to 4.25. Moreover, we also give a table to show the relation of ω_0 and τ_0 (see Table 1).

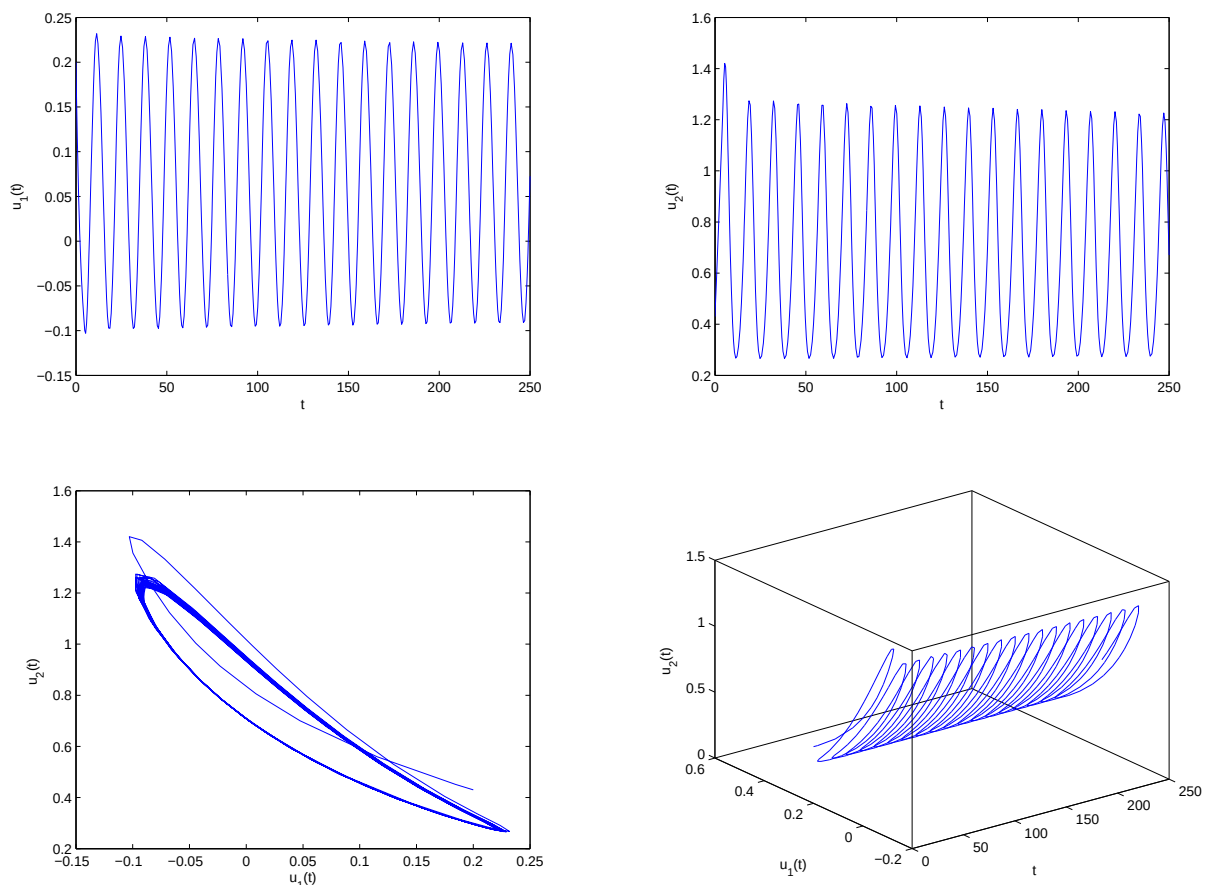


Figure 2. Matlab simulation figures of System (8.1) concerning the delay $\tau = 4.85 > \tau_0 = 4.25$. Two states (u_1, u_2) of System (8.1) will oscillate periodically around the positive equilibrium point $E(0.0592, 0.5634)$.

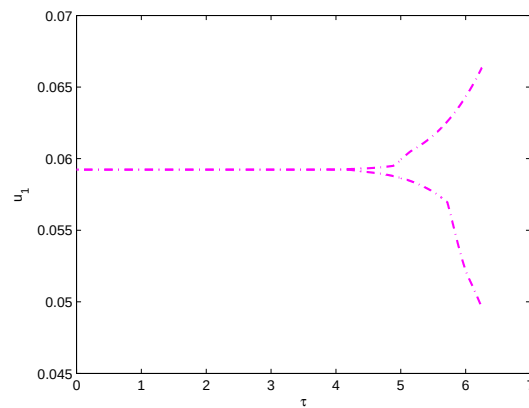


Figure 3. Bifurcation diagram of System (8.1): The delay τ versus the state variable u_1 . The bifurcation value of System (8.1) is 4.25.

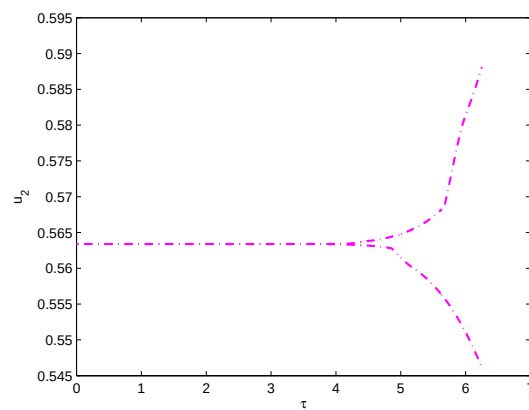


Figure 4. Bifurcation diagram of System (8.1): The delay τ versus the state variable u_2 . The bifurcation value of System (8.1) is 4.25.

Table 1. The interrelation of ω_0 and τ_0 of System (8.1).

ω_0	τ_0
0.9725	1.0235
1.5093	2.4784
2.2534	3.7731
2.8892	4.250
3.2587	4.9783
3.9987	5.4688
4.8734	6.6623
5.2908	7.8721
6.2148	8.6544

Example 8.2 Consider the following fractional-order controlled chemostat model incorporating delay:

$$\begin{cases} \frac{d^{0.87} u_1(t)}{dt^{0.87}} = 0.43 \times 0.15 - 0.43u_1(t) - \frac{2.5u_1(t-\tau)u_2(t)}{(0.85 + u_1(t-\tau))(0.52 + 3.25u_1(t))}, \\ \frac{d^{0.87} u_2(t)}{dt^{0.87}} = 0.43 \times 0.35 - 0.43u_2(t) - \frac{2.5u_1(t-\tau)u_2(t)}{0.85 + u_1(t-\tau)} \\ \quad - \varrho_1 \left[0.43 \times 0.15 - 0.43u_1(t) - \frac{2.5u_1(t-\tau)u_2(t)}{(0.85 + u_1(t-\tau))(0.52 + 3.25u_1(t))} \right], \end{cases} \quad (8.2)$$

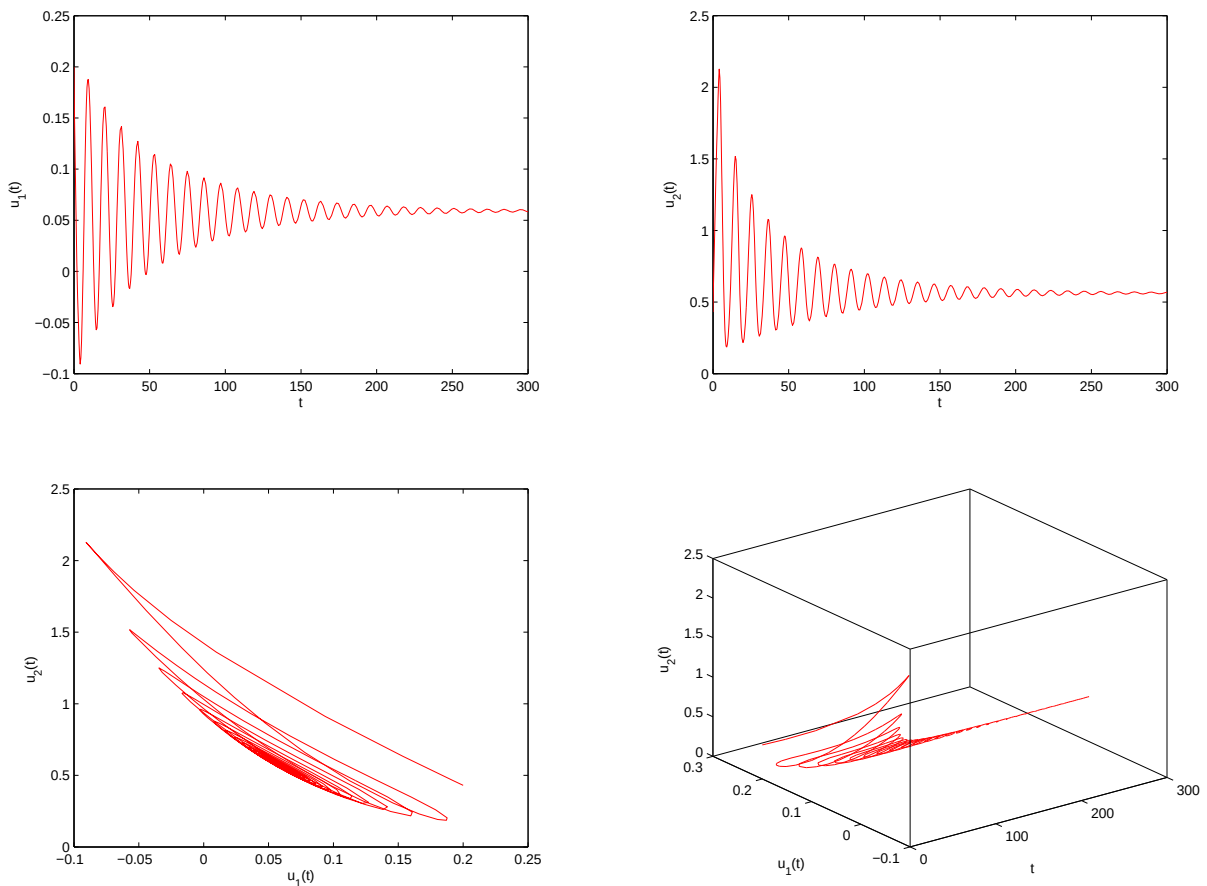


Figure 5. Matlab simulation figures of System (8.2) concerning the delay $\tau = 3.7 < \tau_{00} = 4.05$. The positive equilibrium point $E(0.0592, 0.5634)$ maintains a locally asymptotically stable position.

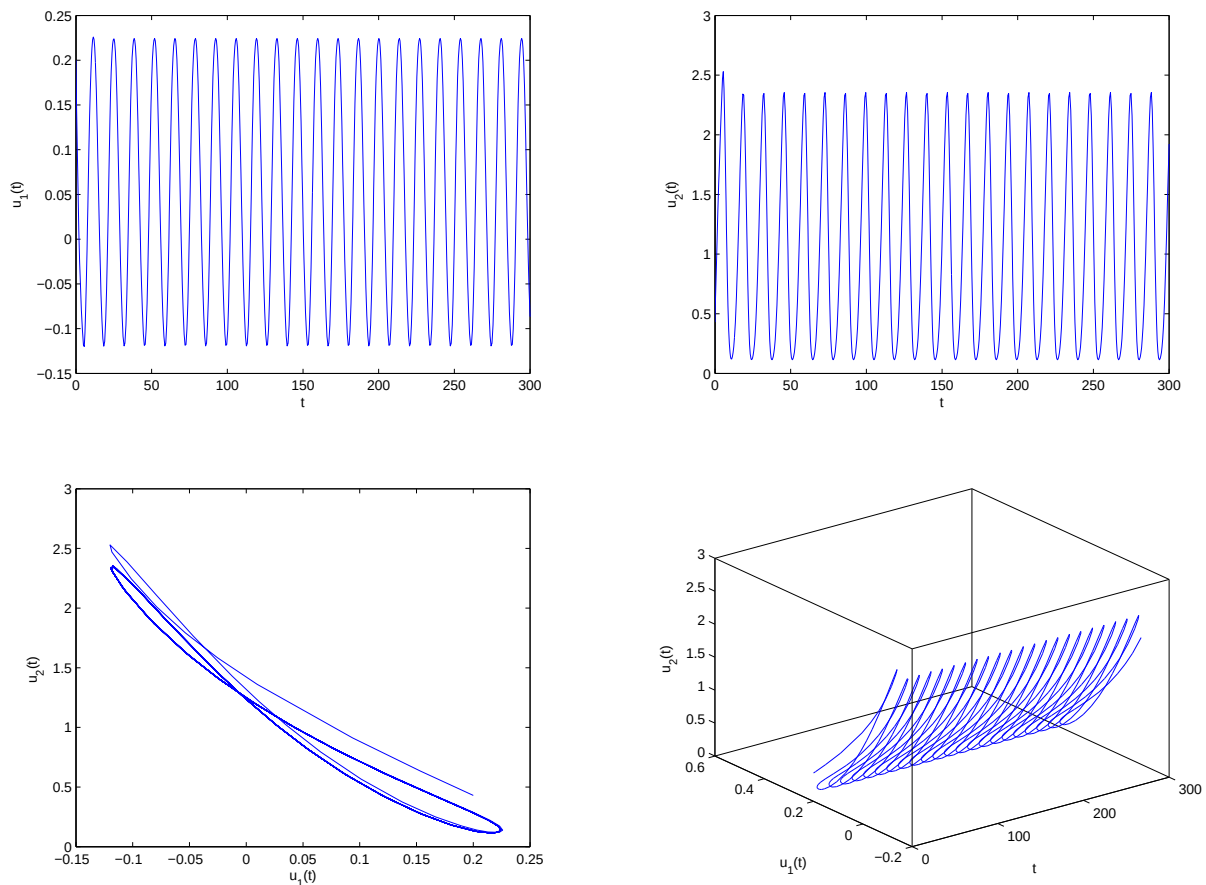


Figure 6. Matlab simulation figures of System (8.2) concerning the delay $\tau = 4.95 > \tau_{00} = 4.05$. Four state states (u_1, u_2) of System (8.2) will oscillate periodically near the positive equilibrium point $E(0.0592, 0.5634)$.

It is easy to see that System (8.2) has a positive equilibrium point $E(0.0592, 0.5634)$. Let $\varrho_1 = 2.96$. By virtue of Matlab software, we can easily obtain $\tau_{00} = 4.05$. One can verify that the conditions (A_3) and (A_4) in Theorem 6.1 hold. To check the correctness of the conclusions of Theorem 6.1, now we choose both two delays. One is $\tau = 3.7$ and another is $\tau = 4.95$. For $\tau = 3.7 < \tau_{00} = 4.05$, we get the simulation plots which are presented in Figure 5. In view of Figure 5, we know that $u_1 \rightarrow 0.0592$ and $u_2 \rightarrow 0.5634$ as $t \rightarrow \infty$, which shows that the positive equilibrium point $E(0.0592, 0.5634)$ of System (8.2) will keep locally asymptotically stable state when the time t gradually approaches infinity. From the view point of ecology, the concentrations of the substrate and the concentrations of the cell mass will tend to the values 0.0592 and 0.5634, respectively. For $\tau = 4.95 > \tau_{00} = 4.05$, we get the simulation plots which are presented in Figure 6. In view of Figure 6, we know that u_1, u_2 are to remain in periodic motion as $t \rightarrow \infty$, which shows that the Hopf bifurcation of system will arise near the positive equilibrium point $E(0.0592, 0.5634)$ when the time t gradually approaches infinity. From the view point of ecology, the concentrations of the substrate and the concentrations of the cell mass will keep periodic motion around the values 0.0592 and 0.5634, respectively. In addition, the bifurcation diagrams display the bifurcation value more intuitively (see Figures 7 and 8). Obviously, Figures 7 and 8 show that the bifurcation value is equal to 4.05. Moreover, we also give a table to show the relation of ψ_0 and τ_{00} (see Table 2).

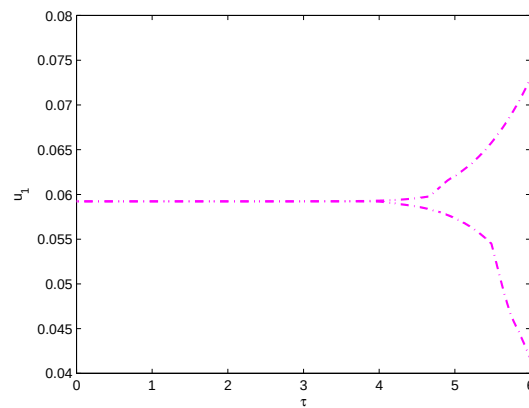


Figure 7. Bifurcation diagram of System (8.2): The delay τ versus the state variable u_1 . The bifurcation value of System (8.2) is 4.05.

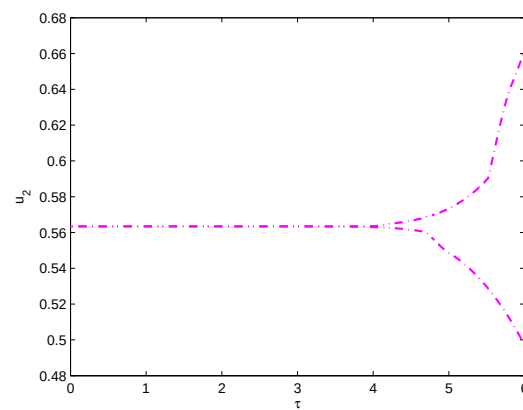


Figure 8. Bifurcation diagram of System (8.2): The delay τ versus the state variable u_2 . The bifurcation value of System (8.2) is 4.05.

Table 2. The interrelation of ψ_0 and τ_{00} of System (8.2).

ψ_0	τ_{00}
0.3352	1.1341
0.4098	2.1058
0.5542	3.4525
0.6021	4.050
0.7091	4.8509
0.8961	5.1875
1.1968	6.2875
1.5875	7.2768
1.9177	8.0939

Example 8.3 Consider the following fractional-order controlled chemostat model incorporating delay:

$$\begin{cases} \frac{d^{0.87} u_1(t)}{dt^{0.87}} = 0.43 \times 0.15 - 0.43 u_1(t) - \frac{2.5 u_1(t - \tau) u_2(t)}{(0.85 + u_1(t - \tau))(0.52 + 3.25 u_1(t))}, \\ \frac{d^{0.87} u_2(t)}{dt^{0.87}} = 0.43 \times 0.35 - 0.43 u_2(t) - \frac{2.5 u_1(t - \tau) u_2(t)}{0.85 + u_1(t - \tau)} - \varrho_2 u_1(t). \end{cases} \quad (8.3)$$

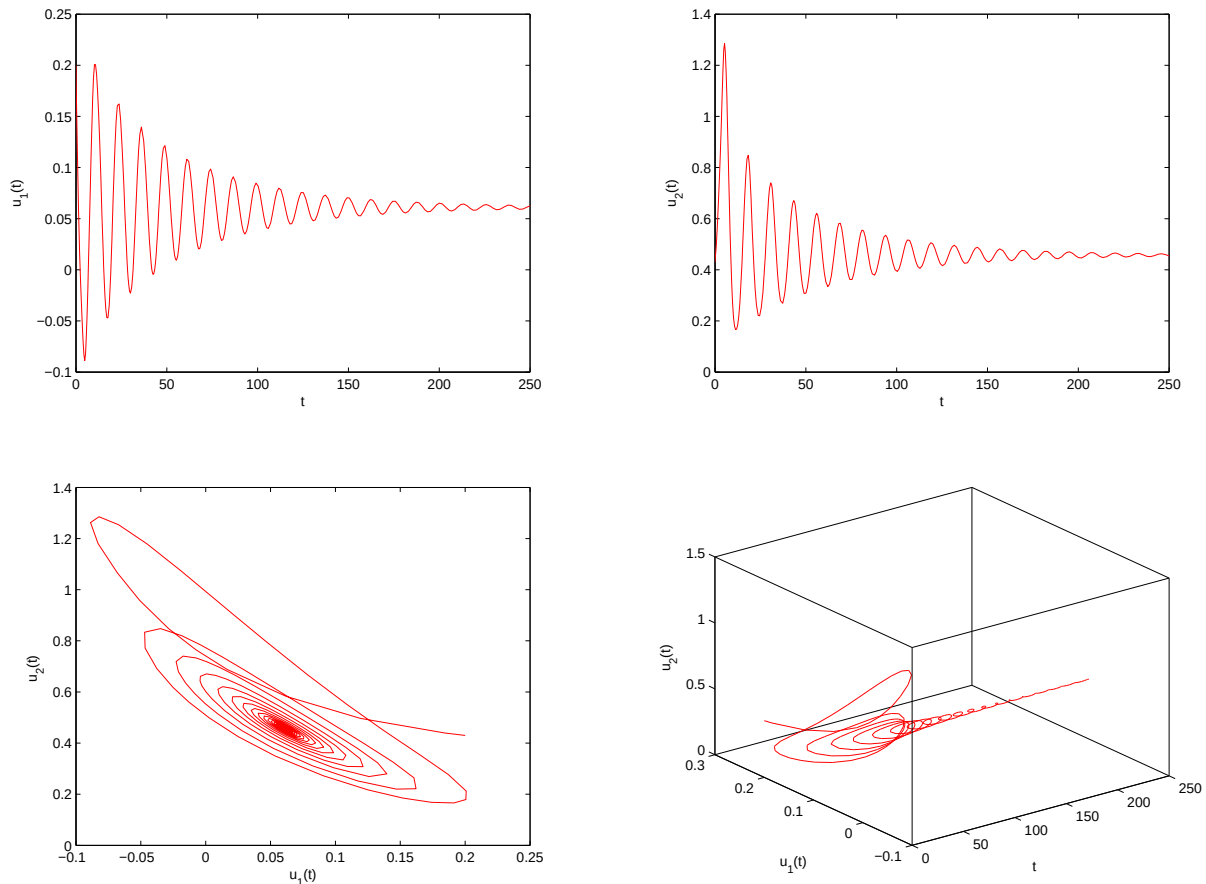


Figure 9. Matlab simulation figures of System (8.3) concerning the delay $\tau = 4.35 < \tau_{000} = 4.72$. The positive equilibrium point $E(0.0611, 0.4573)$ maintains a locally asymptotically stable position.

It is easy to see that System (8.3) has a positive equilibrium point $E(0.0611, 0.4573)$. Let $\varrho_2 = 0.5$. Using Matlab software, we can easily obtain $\tau_{000} = 4.72$. One can verify that the conditions (A_5) and (A_6) in Theorem 7.1 hold. To check the correctness of the conclusions of Theorem 7.1, we now choose both two delays. One is $\tau = 4.35$ and another is $\tau = 5.65$. For $\tau = 4.35 < \tau_{000} = 4.72$, we get the simulation plots which are presented in Figure 9. In view of Figure 9, we know that $u_1 \rightarrow 0.0611, u_2 \rightarrow 0.4573$ as $t \rightarrow \infty$, which shows that the positive equilibrium point $E(0.0611, 0.4573)$ of System (8.3) will remain in a locally asymptotically stable state when the time t gradually approaches infinity. From the view point of ecology, the concentrations of the substrate and the concentrations of the cell mass will tend to the values 0.0611 and 0.4573, respectively. For $\tau = 5.65 > \tau_{000} = 4.72$, we get the simulation

plots which are presented in Figure 10. In view of Figure 10, we know that u_1, u_2 are to remain in periodic motion as $t \rightarrow \infty$, which shows that the Hopf bifurcation of system will arise near the positive equilibrium point $E(0.0611, 0.4573)$ when the time t gradually approaches infinity. From the view point of ecology, the concentrations of the substrate and the concentrations of the cell mass will keep periodic motion around the values 0.0611 and 0.4573, respectively. In addition, the bifurcation diagrams display the bifurcation value more intuitively (see Figures 11 and 12). Obviously, Figures 11 and 12 show that the bifurcation value is equal to 4.72. Moreover, we also give a table to show the relation of χ_0 and τ_{000} (see Table 3).

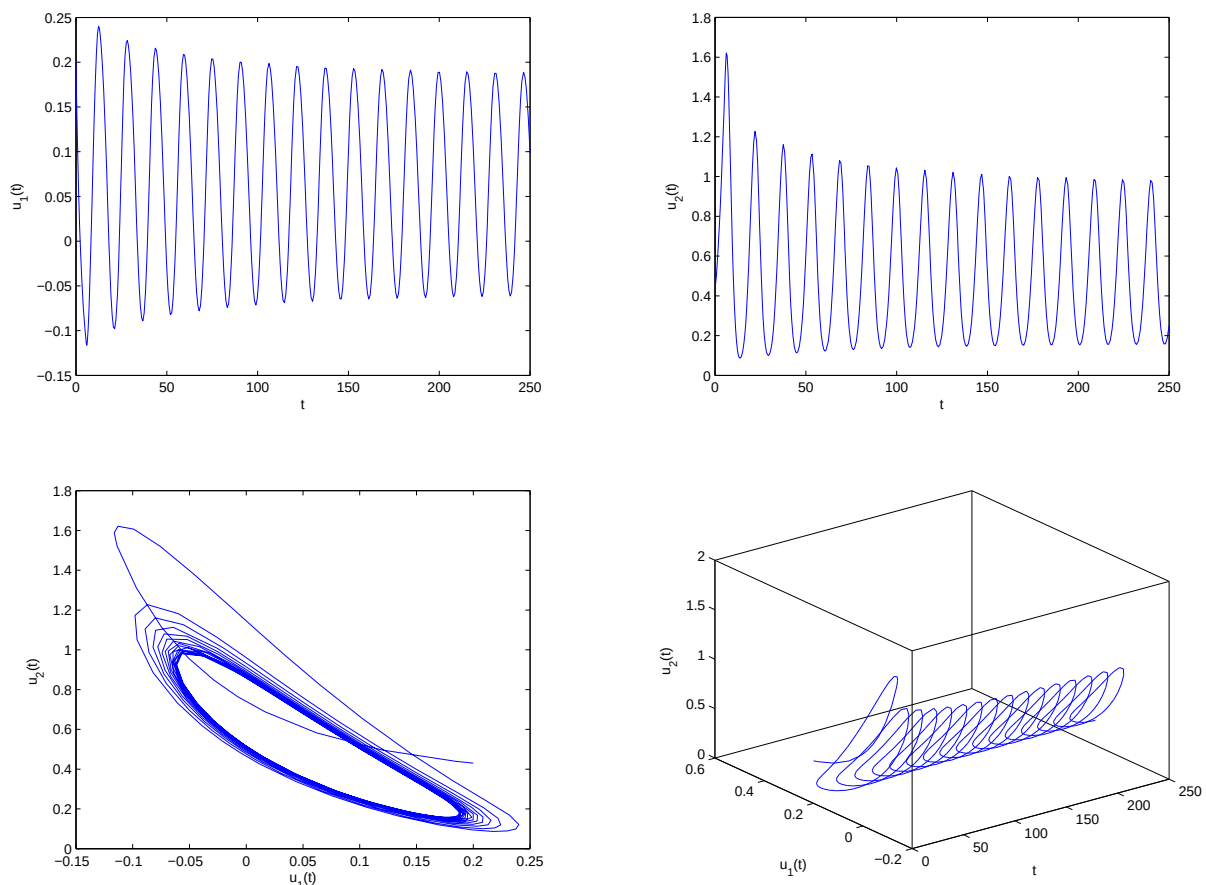


Figure 10. Matlab simulation figures of System (8.3) concerning the delay $\tau = 5.65 > \tau_{000} = 4.72$. Four state states (u_1, u_2) of System (8.3) will oscillate periodically near the positive equilibrium point $E(0.0611, 0.4573)$.

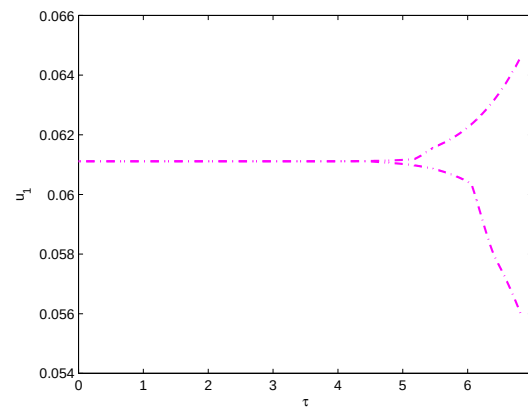


Figure 11. Bifurcation diagram of System (8.3): The delay τ versus the state variable u_1 . The bifurcation value of System (8.3) is 4.72.

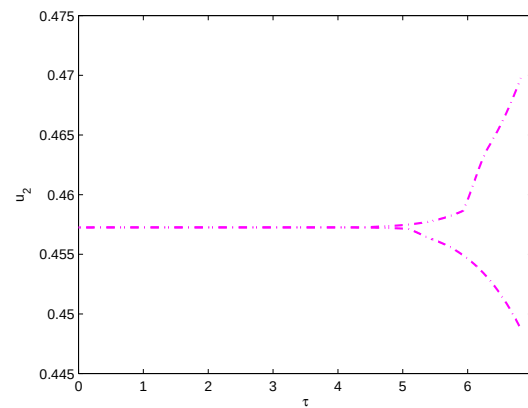


Figure 12. Bifurcation diagram of System (8.3): The delay τ versus the state variable u_2 . The bifurcation value of System (8.3) is 4.72.

Table 3. The interrelation of χ_0 and τ_{000} of System (8.3).

χ_0	τ_{000}
0.8312	1.2577
1.1452	2.0921
1.8025	3.3387
2.4418	4.720
3.6088	5.6032
4.2839	6.5875
4.6872	7.6783
5.2916	8.0832
5.8765	8.9877

Remark 8.1 In Tables 1–3, τ_0 , τ_{00} , and τ_{000} stand for the critical delay values, and ω_0 , ψ_0 , and χ_0 are

frequency values.

Remark 8.2 *In Examples 8.1–8.3, All the selected parameters have realistic biological significance and satisfy the conditions of Theorems 5.1, 6.1, and 7.1, respectively. Thus the quantitative results have practical meaning in biology.*

Remark 8.3 *We choose $\varrho_2 = 0.5$ by virtue of Theorem 7.1. In fact, we can choose other values which satisfy the assumptions of Theorem 7.1.*

Remark 8.4 *On the basis of Eqs (5.13), (6.14), and (7.15), we can obtain the interrelation of ω_0 and τ_0 of System (8.1), the interrelation of ψ_0 and τ_{00} of System (8.2), and the interrelation of χ_0 and τ_{000} of System (8.3).*

9. Conclusions

During the past decades, integer-order delayed differential equations have played a great role in depicting the interaction among variables of systems in the natural world [28–32]. Recently, with the development of fractional-order differential equations, many scholars have argued that fractional-order differential equation is more effective tool to portray the interaction among variables of the system than integer-order delayed differential equations. In this work, drawing on the previous works, we formulate a novel fractional-order chemostat model incorporating delay. We have discussed the well-posedness, involving the boundedness of solution, and the existence and uniqueness of the solution of the developed fractional-order chemostat model. The parameter conditions on the boundedness of the solution, the existence and uniqueness of the solution of the developed fractional-order chemostat model are obtained. The stability behavior and bifurcation phenomenon of this model are analyzed via the stability theorem and the Hopf bifurcation theory of fractional-order differential equations. The delay-independent parameter condition on the stability and the onset of Hopf bifurcation of the proposed model are acquired. The delay threshold value to ensure the onset of Hopf bifurcation of this model is found. In order to control the delay length of the emergence of Hopf bifurcation and the stability domain of this model, the reasonable speed feedback controller and the dislocated feedback controller are effectively designed. The impact of delay and control parameters of these controllers are displayed. In order to check the validity of the derived theoretical outcomes, three numerical examples with their Matlab simulations are provided. The gained conclusions have crucial theoretical guiding significance for stabilizing and balancing and the concentrations of the substrate and the cell mass in chemistry. Moreover, the research methodologies and control strategies can also be used to deal with the control of bifurcation and chaos in abundant integer-order and fractional-order dynamic models. In addition, we can obtain the same results on the existence and uniqueness of solution and the boundedness of solution of Model (1.6) and Model (1.6) with $\tau = 0$. As for the bifurcation issue, there is different results on Model (1.6) and Model (1.6) with $\tau = 0$, so we leave it for future work. In this paper, we only consider the single fractional order q . We will establish stability/bifurcation theory for incommensurate fractional-order systems in the near future. In addition, in this paper, we mainly explore the effect of delay on the stability and Hopf bifurcation of Model (1.6) and its controlled Systems [33–36]. In the near future, we will also investigate the influence of the fractional order q , compare the controlled and uncontrolled systems quantitatively, and provide sensitivity analyses with respect to the key parameters.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

Tiancheng Wang: Conceptualization, formal analysis, methodology, software, validation, methodology, investigation, writing-original draft, writing-review and editing. Xiang Li: Conceptualization, formal analysis, methodology, software, validation, methodology, investigation, writing-original draft, writing-review and editing.

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