



Research article

Semi-analytical valuation of an option on the maximum of two assets under a stochastic volatility model with stochastic long-term mean and stochastic liquidity

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Abstract: This paper proposes a semi-analytical valuation framework for a European call option on the maximum of two assets under a two-factor stochastic volatility model that incorporates both stochastic long-term means and stochastic liquidity risk. By applying the Feynman–Kac theorem, the joint characteristic function of the two liquidity-adjusted log-prices is explicitly derived. Using this joint characteristic function, a semi-analytical pricing formula for the option is obtained through measure changes and bivariate Fourier inversion techniques. The formula is validated against Monte Carlo simulation, and comparisons with nested benchmark models quantify the incremental pricing effects of stochastic liquidity and the stochastic long-term mean. Finally, through various numerical examples, a sensitivity analysis is conducted to illustrate how changes in significant parameters affect the option’s price.

Keywords: option on the maximum; stochastic volatility; stochastic liquidity risk; stochastic long-term mean

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1. Introduction

The valuation of multi-asset derivatives is important in mathematical finance as financial contracts depend on the behavior of several assets, rather than on a single asset. Extremum options, which are determined by the maximum or the minimum of several underlying asset prices at maturity, are an essential type of derivative. Options written on the maximum or the minimum of several underlying assets, often called rainbow or extremum options, are widely used to hedge correlated price risk across

two or more assets. Their payoffs depend jointly on the terminal values of the underlying assets, so their valuation requires a joint description of the underlying dynamics.

The classical pricing theory for extremum options was studied by Stulz [1], who derived formulas for options on the maximum and minimum of two risky assets in the Black–Scholes framework. Johnson [2] extended this line of research to options on the maximum or minimum of several assets. Since then, extremum options have become standard benchmark products in the study of multi-asset option pricing. For example, Zhang and Zhou [3] derived analytical pricing formulas for options on the maximum or minimum of several assets with counterparty default risk before maturity, and Wang and Deng [4] studied geometric Asian extremum options on the maximum or minimum of two assets under a mixed fractional Brownian motion with jumps.

Although the Black–Scholes model [5] provides analytical tractability, its assumptions are too restrictive for modern financial markets. In particular, the assumptions of constant volatility, perfect liquidity, and simple dependence structures between assets are often inconsistent with observed market behavior. The Heston stochastic volatility model [6] is one of the most widely used extensions, since it captures stochastic changes in variance while retaining a tractable characteristic function representation. However, the standard one-factor Heston model may not be sufficiently flexible to reproduce the observed term structure and shape of implied volatility surfaces. This has motivated multifactor stochastic volatility models, including the multifactor Heston-type models studied in [7]. In the multi-asset setting, such models are particularly useful because they allow asset returns to share a common market volatility factor while also retaining asset specific idiosyncratic volatility factors. In addition to stochastic volatility, the long-term mean of volatility is an important consideration in modeling. He and Chen [8] proposed a stochastic volatility model in which the long-term mean of the variance process is itself stochastic. Their stochastic long-term mean specification improves the flexibility of the Heston-type variance dynamics while preserving analytical tractability. More recently, stochastic long-term mean models have been incorporated into various option pricing frameworks [9, 10]. In particular, Cho and Kim [9] priced vulnerable options under a stochastic volatility model with a stochastic long-term mean, in which the second stochastic process is the asset value of the option writer that triggers default at maturity. In a parallel strand of research, a growing body of empirical evidence documents that volatility is rough; see Gatheral et al. [11], together with estimation methods based on the fractional Ornstein–Uhlenbeck process [12, 13]. Although rough volatility lies outside the affine framework adopted in this paper, it constitutes an important complementary direction for modeling volatility dynamics.

Another important market imperfection is liquidity risk. To incorporate this factor, Feng et al. [14] introduced a stochastic liquidity discount mechanism in which the traded price is adjusted by a liquidity factor driven by a mean-reverting market liquidity process. Subsequent studies extended this approach to various derivative pricing problems, particularly vulnerable options [15–17]. Recently, He and Lin [18] analytically priced European options under two-factor stochastic volatility with stochastic liquidity risks, and He et al. [19] developed a new analytical approach for pricing variance and volatility swaps that incorporates liquidity risk and self-exciting jumps. These studies show that liquidity risk can affect option values not only through price discounting, but also through the volatility and dependence structure of the underlying assets.

The present paper contributes to this literature by developing a semi-analytical valuation framework for a European option on the maximum of two assets under a two-factor stochastic volatility model with stochastic long-term means and stochastic liquidity risk. Specifically, we construct a two-asset pricing

model that simultaneously incorporates stochastic volatility, stochastic long-term means, and stochastic liquidity risk. In addition, we derive the joint characteristic function of the two log-prices by applying the Feynman–Kac theorem under the proposed framework, and obtain an explicit semi-analytical pricing formula for a European call option on the maximum of two assets using the joint characteristic function.

The contribution of this paper relative to the closest existing studies can be clarified as follows. Compared with Cho and Kim [9], who priced vulnerable options under a stochastic volatility model with a stochastic long-term mean, the present framework differs in three substantive respects: both stochastic processes are tradable underlying assets rather than the asset value of an option writer, stochastic liquidity discount factors are attached to both traded assets, and the extremum payoff depends on the joint law of the two log-prices over the intersection of two exercise regions rather than on a default event at maturity. Accordingly, the pricing methodology also differs: two share measures associated with the two assets are introduced, and the bivariate Shephard inversion is applied to suitably transformed pairs of log-price variables, which is technically more involved than the univariate inversions used in single-asset or vulnerable option settings. In addition, we make the market prices of risk explicit, validate the pricing formula against Monte Carlo simulation and nested special cases, and quantify the incremental pricing effects of stochastic liquidity and the stochastic long-term mean through comparisons with benchmark models.

The remainder of this paper is organized as follows. Section 2 introduces the two-asset stochastic volatility model with stochastic long-term means and stochastic liquidity discount factors, and derives the joint characteristic function of the two liquidity-adjusted log-prices. Section 3 presents the semi-analytical pricing formula for the European call option on the maximum of two assets using a change of measure and bivariate Fourier inversion. Section 4 provides several numerical examples to illustrate the sensitivity of option prices to key parameters. Section 5 concludes the paper.

2. The model

We work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ denotes the physical measure. Let $S_1^L(t)$ and $S_2^L(t)$ denote the perfectly liquid prices of the two underlying assets. Following Christoffersen et al. [7], their dynamics under $\mathbb{P}$ are driven by a two-factor stochastic volatility model:

$$\frac{dS_1^L(t)}{S_1^L(t)} = \mu_1 dt + \sigma_1 \sqrt{v_1(t)} dW_1(t) + \sqrt{v_2(t)} dW_2(t), \quad (2.1)$$

$$\frac{dS_2^L(t)}{S_2^L(t)} = \mu_2 dt + \sigma_2 \sqrt{v_1(t)} dW_1(t) + \sqrt{v_3(t)} dW_3(t). \quad (2.2)$$

Here μ_1 and μ_2 are drift terms, σ_1 and σ_2 are scaling coefficients controlling the sensitivity of the two assets to the common volatility factor $\sqrt{v_1(t)}$, and $v_2(t)$, $v_3(t)$ are the idiosyncratic variance factors of the two assets. The variance processes evolve with time-dependent long-term means,

$$dv_i(t) = k_i(\theta_i(t) - v_i(t))dt + \zeta_i \sqrt{v_i(t)} dW_i^v(t), \quad i = 1, 2, 3, \quad (2.3)$$

where k_i is the mean-reversion speed, $\theta_i(t)$ is the stochastic long-term mean, and ζ_i is the volatility of volatility. Following He and Chen [8], each long-term mean follows an arithmetic Brownian motion,

$$d\theta_i(t) = \lambda_i dt + \psi_i dW_i^\theta(t), \quad i = 1, 2, 3, \quad (2.4)$$

with drift λ_i and volatility ψ_i . The Wiener processes satisfy

$$d\langle W_i(t), W_i^v(t) \rangle = \rho_i^v dt, \quad i = 1, 2, 3, \quad (2.5)$$

with all other pairs uncorrelated.

Remark 1. *The arithmetic Brownian motion specification Eq (2.4) follows He and Chen [8] and the subsequent stochastic long-term mean literature [9, 10, 20], and is adopted here because it preserves the exponentially affine structure of the joint characteristic function. Replacing Eq (2.4) with a CIR-type square-root process for $\theta_i(t)$ would turn the coefficient equation (2.30) for C_i below into a Riccati equation with a time-dependent inhomogeneous term, which no longer admits a closed-form solution. Two theoretical caveats of the specification should nevertheless be acknowledged: $\theta_i(t)$ is not mean-reverting and can in principle become negative. Over the horizons relevant for option pricing, however, the probability of a negative long-term mean is negligible for empirically plausible parameter values; under the risk-neutral baseline parameters of Section 4, $\mathbb{Q}(\tilde{\theta}_i(T) < 0) = \Phi(-(\tilde{\theta}_i(0) + \tilde{\lambda}_i T)/(\tilde{\psi}_i \sqrt{T})) = \Phi(-14) < 10^{-44}$ for $T = 1$. The specification should therefore be understood as a locally valid approximation over option maturities, and replacing it with mean-reverting or nonnegative long-term mean dynamics at the cost of analytical tractability is left for future research.*

Following Feng et al. [14], the liquidity-adjusted prices $S_1(t)$ and $S_2(t)$ are defined by

$$S_1(t) = \frac{S_1^L(t)}{\xi_1(t)}, \quad S_2(t) = \frac{S_2^L(t)}{\xi_2(t)}, \quad (2.6)$$

where the liquidity discount factors $\xi_i(t)$ follow

$$\frac{d\xi_i(t)}{\xi_i(t)} = \left(-v_i m(t) + \frac{1}{2} v_i^2 m^2(t)\right) dt - v_i m(t) dW_i^\xi(t), \quad i = 1, 2. \quad (2.7)$$

Here $m(t)$ is the market liquidity level and v_i measures the sensitivity of $\xi_i(t)$ to $m(t)$. The liquidity level mean-reverts,

$$dm(t) = \alpha(\beta - m(t))dt + \gamma dW^m(t), \quad (2.8)$$

with speed α , long-term level β , and volatility γ . The remaining correlations are

$$d\langle W^m(t), W_i^\xi(t) \rangle = \rho_i^\xi dt, \quad i = 1, 2, \quad d\langle W_1^\xi(t), W_2^\xi(t) \rangle = \rho^\xi dt, \quad (2.9)$$

with all other unspecified pairs equal to zero.

Remark 2. *The dependence between the two assets is generated through three channels: the common variance factor $v_1(t)$ with loadings σ_1, σ_2 , the common market liquidity level $m(t)$, and the correlation ρ^ξ between the asset-specific liquidity shocks. In particular, the instantaneous correlation between the two liquidity-adjusted returns in Eqs (2.10) and (2.11) below,*

$$\frac{\sigma_1 \sigma_2 v_1(t) + \rho^\xi v_1 v_2 m^2(t)}{\sqrt{(\sigma_1^2 v_1(t) + v_2(t) + v_1^2 m^2(t))(\sigma_2^2 v_1(t) + v_3(t) + v_2^2 m^2(t))}},$$

is itself stochastic, driven jointly by v_1, v_2, v_3 and m , so the model generates a stochastic correlation structure even though the correlation parameters are constants. The independence of W_2 and W_3 is

precisely what defines v_2 and v_3 as idiosyncratic factors, following Christoffersen et al. [7]; allowing $d\langle W_2(t), W_3(t) \rangle = \rho_{23} dt$ with $\rho_{23} \neq 0$ would introduce a term proportional to $\sqrt{v_2(t)v_3(t)}$ into the pricing equation, which is not affine in the state variables and destroys the closed-form solvability of the characteristic function. Similarly, following Feng et al. [14], a single market-wide liquidity level $m(t)$ drives the discount factors of both assets, consistent with the interpretation of $m(t)$ as systematic market liquidity; asset-specific heterogeneity is nevertheless retained through the sensitivities v_1, v_2 and the imperfectly correlated liquidity shocks W_1^ξ, W_2^ξ with $\rho^\xi < 1$. Richer specifications, such as asset-specific liquidity levels $m_i(t)$, feedback from liquidity to volatility, or correlated idiosyncratic volatility shocks, are empirically appealing but lead to non-affine dynamics; these extensions are discussed in Section 5.

Applying Itô's lemma to $\xi_i^{-1}(t)$ with the dynamics Eq (2.7) yields

$$\frac{d(\xi_i^{-1}(t))}{\xi_i^{-1}(t)} = \left[-\left(-v_i m(t) + \frac{1}{2} v_i^2 m^2(t) \right) + v_i^2 m^2(t) \right] dt + v_i m(t) dW_i^\xi(t) = \left(v_i m(t) + \frac{1}{2} v_i^2 m^2(t) \right) dt + v_i m(t) dW_i^\xi(t),$$

where the reversed sign of the diffusion term and the additional Itô correction $v_i^2 m^2(t) dt$ arise from $d(\xi_i^{-1}) = -\xi_i^{-2} d\xi_i + \xi_i^{-3} d\langle \xi_i \rangle_t$. Since $S_i(t) = S_i^L(t) \xi_i^{-1}(t)$, the Itô product rule gives

$$dS_i(t) = \xi_i^{-1}(t) dS_i^L(t) + S_i^L(t) d(\xi_i^{-1}(t)) + d\langle S_i^L, \xi_i^{-1} \rangle_t,$$

in which the quadratic covariation term vanishes because the Brownian motions W_1, W_2, W_3 driving S_i^L are uncorrelated with W_i^ξ under the correlation structure Eqs (2.5) and (2.9). Combining the two displays, the liquidity-adjusted dynamics under $\mathbb{P}$ become

$$\frac{dS_1(t)}{S_1(t)} = \left(\mu_1 + v_1 m(t) + \frac{1}{2} v_1^2 m^2(t) \right) dt + \sigma_1 \sqrt{v_1(t)} dW_1(t) + \sqrt{v_2(t)} dW_2(t) + v_1 m(t) dW_1^\xi(t), \quad (2.10)$$

$$\frac{dS_2(t)}{S_2(t)} = \left(\mu_2 + v_2 m(t) + \frac{1}{2} v_2^2 m^2(t) \right) dt + \sigma_2 \sqrt{v_1(t)} dW_1(t) + \sqrt{v_3(t)} dW_3(t) + v_2 m(t) dW_2^\xi(t). \quad (2.11)$$

Because the market is incomplete, there are infinitely many equivalent martingale measures. We decompose the correlated processes using the independent standard Brownian motions $\widehat{B}_i(t)$, $i = 1, \dots, 12$.

$$\begin{aligned} dW_1 &= d\widehat{B}_1, & dW_1^\nu &= \rho_1^\nu d\widehat{B}_1 + \sqrt{1 - (\rho_1^\nu)^2} d\widehat{B}_2, \\ dW_2 &= d\widehat{B}_3, & dW_2^\nu &= \rho_2^\nu d\widehat{B}_3 + \sqrt{1 - (\rho_2^\nu)^2} d\widehat{B}_4, \\ dW_3 &= d\widehat{B}_5, & dW_3^\nu &= \rho_3^\nu d\widehat{B}_5 + \sqrt{1 - (\rho_3^\nu)^2} d\widehat{B}_6, \\ dW_1^\xi &= d\widehat{B}_7, & dW_2^\xi &= \rho^\xi d\widehat{B}_7 + \sqrt{1 - (\rho^\xi)^2} d\widehat{B}_8, \end{aligned}$$

$$dW^m = \rho_1^\xi d\widehat{B}_7 + \bar{\rho} d\widehat{B}_8 + \tilde{\rho} d\widehat{B}_9, \quad dW_i^\theta = d\widehat{B}_{9+i}, \quad i = 1, 2, 3,$$

where

$$\bar{\rho} = \frac{\rho_2^\xi - \rho_1^\xi \rho^\xi}{\sqrt{1 - (\rho^\xi)^2}}, \quad \tilde{\rho} = \sqrt{1 - (\rho_1^\xi)^2 - \bar{\rho}^2},$$

with $\rho_i^v \in [-1, 1]$, $i = 1, 2, 3$, and $1 - (\rho_1^\xi)^2 - \bar{\rho}^2 \geq 0$.

We now introduce a new measure $\mathbb{Q}$ using the following Radon–Nikodym derivative

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t = \exp\left(-\sum_{i=1}^{12} \int_0^t a_i(s) d\widehat{B}_i(s) - \frac{1}{2} \sum_{i=1}^{12} \int_0^t a_i(s)^2 ds\right),$$

where $a_i(t)$ are the market prices of risk.

By Girsanov's theorem, $\widetilde{B}_i = \widehat{B}_i(t) + \int_0^t a_i(s) ds$ become independent standard Brownian motions under the measure $\mathbb{Q}$. We let $\widetilde{W}$ represent the correlated processes under $\mathbb{Q}$, which are formulated using the same decomposition structure but constructed with $\widetilde{B}_i(t)$.

Since the market is incomplete, the pricing measure is not unique, and the specification of the $a_i(t)$ must be made explicit. Following the essentially affine specifications that are standard in the stochastic volatility and liquidity literature [6, 7, 14], we take the total prices of risk attached to the correlated Brownian motions to be

$$\begin{aligned} \Lambda_i^v(t) &:= \rho_i^v a_{2i-1}(t) + \sqrt{1 - (\rho_i^v)^2} a_{2i}(t) = \frac{b_i}{\zeta_i} \sqrt{v_i(t)}, \quad i = 1, 2, 3, \\ \Lambda^m(t) &:= \rho_1^\xi a_7(t) + \bar{\rho} a_8(t) + \tilde{\rho} a_9(t) = \frac{b_4}{\gamma} m(t), \quad a_{9+i}(t) = c_i, \quad i = 1, 2, 3, \end{aligned}$$

for constants $b_1, \dots, b_4$ and c_1, c_2, c_3 , so that the variance and liquidity risk premia are proportional to $\sqrt{v_i(t)}$ and $m(t)$, respectively, while the risk premia of the long-term mean drivers are constant. This family is chosen because it preserves the affine structure of the model under the measure change: the proportional variance premium changes the mean-reversion speed from k_i to $k_i + b_i$ while keeping square-root dynamics, exactly as in Heston [6]. The remaining prices of risk, namely those attached to W_1, W_2, W_3, W_1^ξ and W_2^ξ , are restricted only by the two martingale conditions imposed below on the discounted traded prices; any admissible choice satisfying these conditions leads to the same $\mathbb{Q}$ -dynamics Eqs (2.12)–(2.16) and hence to the same option prices. In practice, the risk-neutral parameters $(\widetilde{k}_i, \widetilde{\theta}_i(0), \widetilde{\lambda}_i, \widetilde{\psi}_i, \widetilde{\alpha}, \widetilde{\beta})$ are treated as primitives to be calibrated directly to observed option prices, so that market data effectively select the pricing measure within this parametric family [7, 14].

Under the measure $\mathbb{Q}$, variance, long-term mean, and liquidity dynamics are given by

$$dv_i(t) = \widetilde{k}_i(\widetilde{\theta}_i(t) - v_i(t))dt + \zeta_i \sqrt{v_i(t)} d\widetilde{W}_i^v(t), \quad i = 1, 2, 3, \quad (2.12)$$

$$d\widetilde{\theta}_i(t) = \widetilde{\lambda}_i dt + \widetilde{\psi}_i d\widetilde{W}_i^\theta(t), \quad i = 1, 2, 3, \quad (2.13)$$

$$dm(t) = \widetilde{\alpha}(\widetilde{\beta} - m(t))dt + \gamma d\widetilde{W}^m(t), \quad (2.14)$$

where, with constants b_i and c_i ,

$$\widetilde{k}_i = k_i + b_i, \quad \widetilde{\theta}_i(t) = \frac{k_i}{k_i + b_i} \theta_i(t), \quad \widetilde{\lambda}_i = \frac{k_i}{k_i + b_i} (\lambda_i - \psi_i c_i), \quad \widetilde{\psi}_i = \frac{k_i}{k_i + b_i} \psi_i,$$

$$\widetilde{\alpha} = \alpha + b_4, \quad \widetilde{\beta} = \frac{\alpha\beta}{\alpha + b_4}.$$

Indeed, substituting $dW_i^v(t) = d\widetilde{W}_i^v(t) - \Lambda_i^v(t) dt$ into Eq (2.3) gives $dv_i(t) = [k_i \theta_i(t) - (k_i + b_i) v_i(t)] dt + \zeta_i \sqrt{v_i(t)} d\widetilde{W}_i^v(t)$, which is Eq (2.12) under the reparametrization above, and the expressions for $\widetilde{\lambda}_i, \widetilde{\psi}_i, \widetilde{\alpha}$

and $\widetilde{\beta}$ follow in the same way from the constant premia c_i and the liquidity premium $\Lambda^m(t)$. Imposing that the discounted prices $e^{-rt}S_1(t)$ and $e^{-rt}S_2(t)$ be $\mathbb{Q}$ -martingales with constant risk-free rate r , the asset dynamics under $\mathbb{Q}$ become

$$\frac{dS_1(t)}{S_1(t)} = r dt + \sigma_1 \sqrt{v_1(t)} d\widetilde{W}_1(t) + \sqrt{v_2(t)} d\widetilde{W}_2(t) + \nu_1 m(t) d\widetilde{W}_1^\xi(t), \quad (2.15)$$

$$\frac{dS_2(t)}{S_2(t)} = r dt + \sigma_2 \sqrt{v_1(t)} d\widetilde{W}_1(t) + \sqrt{v_3(t)} d\widetilde{W}_3(t) + \nu_2 m(t) d\widetilde{W}_2^\xi(t). \quad (2.16)$$

Setting $x(t) = \ln S_1(t)$ and $y(t) = \ln S_2(t)$ and applying Itô's lemma gives the risk-neutral log-price dynamics

$$dx(t) = \left(r - \frac{1}{2}(\sigma_1^2 v_1(t) + v_2(t) + \nu_1^2 m^2(t)) \right) dt + \sigma_1 \sqrt{v_1(t)} d\widetilde{W}_1(t) + \sqrt{v_2(t)} d\widetilde{W}_2(t) + \nu_1 m(t) d\widetilde{W}_1^\xi(t), \quad (2.17)$$

$$dy(t) = \left(r - \frac{1}{2}(\sigma_2^2 v_1(t) + v_3(t) + \nu_2^2 m^2(t)) \right) dt + \sigma_2 \sqrt{v_1(t)} d\widetilde{W}_1(t) + \sqrt{v_3(t)} d\widetilde{W}_3(t) + \nu_2 m(t) d\widetilde{W}_2^\xi(t). \quad (2.18)$$

Theorem 1. Let $\tau = T - t$. Define the joint characteristic function of $x(T)$ and $y(T)$ under $\mathbb{Q}$ by

$$h(\eta_1, \eta_2; t, T) = \mathbb{E}^\mathbb{Q} \left[e^{i\eta_1 x(T) + i\eta_2 y(T)} \mid x, y, v_1, v_2, v_3, \theta_1, \theta_2, \theta_3, m \right]. \quad (2.19)$$

Then

$$h(\eta_1, \eta_2; \tau) = \exp \left(A(\tau) + \sum_{i=1}^3 B_i(\tau) v_i(t) + \sum_{i=1}^3 C_i(\tau) \widetilde{\theta}_i(t) + D(\tau) m(t) + E(\tau) m^2(t) + i\eta_1 x(t) + i\eta_2 y(t) \right), \quad (2.20)$$

where (suppressing the dependence on η_1, η_2) the coefficient functions are

$$\begin{aligned} A(\tau) = & \int_0^\tau \left\{ \frac{1}{2} \sum_{i=1}^3 (\widetilde{\psi}_i^2 C_i(s)^2 + 2\widetilde{\lambda}_i C_i(s)) \right\} ds + r(i\eta_1 + i\eta_2)\tau \\ & + \frac{1}{2} \left(\omega_E \tau - \ln(\cosh(\delta_E \tau) + \widetilde{\delta}_E \sinh(\delta_E \tau)) \right) + \frac{\widetilde{\alpha}^2 \widetilde{\beta}^2}{\gamma^2} \left[\frac{1}{2} (\widetilde{\delta}_E^2 - 1) \tau \right. \\ & - \frac{\widetilde{\delta}_E^2}{2\delta_E} \tanh(\delta_E \tau + \operatorname{arctanh}(\widetilde{\delta}_E)) - \frac{\widetilde{\delta}_E}{\delta_E} (1 - \widetilde{\delta}_E^2) \frac{1}{\cosh(\delta_E \tau) + \widetilde{\delta}_E \sinh(\delta_E \tau)} \\ & \left. + \frac{1 - \widetilde{\delta}_E^2}{2\delta_E} \frac{\sinh(\delta_E \tau) + \widetilde{\delta}_E \cosh(\delta_E \tau)}{\cosh(\delta_E \tau) + \widetilde{\delta}_E \sinh(\delta_E \tau)} + \frac{\widetilde{\delta}_E}{2\delta_E} \right], \end{aligned} \quad (2.21)$$

$$B_i(\tau) = \frac{2}{\zeta_i^2} \left(\omega_B^i - \delta_B^i \frac{\sinh(\delta_B^i \tau) + \widetilde{\delta}_B^i \cosh(\delta_B^i \tau)}{\cosh(\delta_B^i \tau) + \widetilde{\delta}_B^i \sinh(\delta_B^i \tau)} \right), \quad i = 1, 2, 3, \quad (2.22)$$

$$C_i(\tau) = \frac{2\widetilde{k}_i}{\zeta_i^2} \left(\omega_B^i \tau - \ln(\cosh(\delta_B^i \tau) + \widetilde{\delta}_B^i \sinh(\delta_B^i \tau)) \right), \quad i = 1, 2, 3, \quad (2.23)$$

$$D(\tau) = \frac{\widetilde{\alpha} \widetilde{\beta}}{\gamma^2} \cdot \frac{(\widetilde{\delta}_E^2 - 1)(\cosh(\delta_E \tau) - 1)}{\cosh(\delta_E \tau) + \widetilde{\delta}_E \sinh(\delta_E \tau)}, \quad (2.24)$$

$$E(\tau) = \frac{1}{2\gamma^2} \left(\omega_E - \delta_E \frac{\sinh(\delta_E \tau) + \tilde{\delta}_E \cosh(\delta_E \tau)}{\cosh(\delta_E \tau) + \tilde{\delta}_E \sinh(\delta_E \tau)} \right). \quad (2.25)$$

The auxiliary quantities, in which u_i denotes the Fourier argument attached to the idiosyncratic variance factor v_i ($u_2 = \eta_1$ and $u_3 = \eta_2$, since v_2 appears only in the dynamics of $x(t)$ and v_3 only in those of $y(t)$), are

$$\begin{aligned} \omega_B^1 &= \frac{1}{2}(\tilde{k}_1 - i\rho_1^v \zeta_1(\sigma_1 \eta_1 + \sigma_2 \eta_2)), \\ \omega_B^i &= \frac{1}{2}(\tilde{k}_i - i\rho_i^v \zeta_i u_i), \quad i = 2, 3, \\ \delta_B^1 &= \sqrt{(\omega_B^1)^2 + \frac{\zeta_1^2}{4}(\sigma_1^2 \eta_1(\eta_1 + i) + \sigma_2^2 \eta_2(\eta_2 + i) + 2\sigma_1 \sigma_2 \eta_1 \eta_2)}, \\ \delta_B^i &= \sqrt{(\omega_B^i)^2 + \frac{\zeta_i^2}{4} u_i(u_i + i)}, \quad i = 2, 3, \\ \tilde{\delta}_B^i &= \frac{\omega_B^i}{\delta_B^i}, \quad i = 1, 2, 3, \\ \omega_E &= \tilde{\alpha} - i \sum_{j=1}^2 \eta_j \gamma v_j \rho_j^\xi, \\ \delta_E &= \sqrt{\omega_E^2 + 2\gamma^2 \rho^\xi \eta_1 \eta_2 v_1 v_2 + \gamma^2 \sum_{j=1}^2 \eta_j v_j^2 (\eta_j + i)}, \\ \tilde{\delta}_E &= \frac{\omega_E}{\delta_E}. \end{aligned}$$

Proof. By the Feynman–Kac theorem, h satisfies the backward PDE

$$\begin{aligned} \frac{\partial h}{\partial \tau} &= \frac{1}{2}(\sigma_1^2 v_1 + v_2 + v_1^2 m^2) \frac{\partial^2 h}{\partial x^2} + \frac{1}{2}(\sigma_2^2 v_1 + v_3 + v_2^2 m^2) \frac{\partial^2 h}{\partial y^2} + \frac{1}{2} \sum_{i=1}^3 \zeta_i^2 v_i \frac{\partial^2 h}{\partial v_i^2} \\ &+ \frac{1}{2} \sum_{i=1}^3 \tilde{\psi}_i \frac{\partial^2 h}{\partial \tilde{\theta}_i^2} + \frac{1}{2} \gamma^2 \frac{\partial^2 h}{\partial m^2} + \rho_1^v \zeta_1 \sigma_1 v_1 \frac{\partial^2 h}{\partial x \partial v_1} + \rho_2^v \zeta_2 v_2 \frac{\partial^2 h}{\partial x \partial v_2} + \rho_1^\xi \gamma v_1 m \frac{\partial^2 h}{\partial x \partial m} \\ &+ \rho_1^v \zeta_1 \sigma_2 v_1 \frac{\partial^2 h}{\partial y \partial v_1} + \rho_3^v \zeta_3 v_3 \frac{\partial^2 h}{\partial y \partial v_3} + \rho_2^\xi \gamma v_2 m \frac{\partial^2 h}{\partial y \partial m} + (\sigma_1 \sigma_2 v_1 + \rho^\xi v_1 v_2 m^2) \frac{\partial^2 h}{\partial x \partial y} \\ &+ \left(r - \frac{1}{2}(\sigma_1^2 v_1 + v_2 + v_1^2 m^2) \right) \frac{\partial h}{\partial x} + \left(r - \frac{1}{2}(\sigma_2^2 v_1 + v_3 + v_2^2 m^2) \right) \frac{\partial h}{\partial y} \\ &+ \sum_{i=1}^3 \tilde{k}_i (\tilde{\theta}_i - v_i) \frac{\partial h}{\partial v_i} + \sum_{i=1}^3 \tilde{\lambda}_i \frac{\partial h}{\partial \tilde{\theta}_i} + \tilde{\alpha} (\tilde{\beta} - m) \frac{\partial h}{\partial m}, \end{aligned} \quad (2.26)$$

with terminal condition $h|_{\tau=0} = e^{i\eta_1 x(T) + i\eta_2 y(T)}$.

Moreover, the required derivatives are

$$\begin{aligned} \frac{\partial h}{\partial x} &= i\eta_1 h, & \frac{\partial h}{\partial y} &= i\eta_2 h, & \frac{\partial h}{\partial v_i} &= B_i h, & \frac{\partial h}{\partial \tilde{\theta}_i} &= C_i h, & \frac{\partial h}{\partial m} &= (D + 2Em)h, \\ \frac{\partial^2 h}{\partial x^2} &= -\eta_1^2 h, & \frac{\partial^2 h}{\partial y^2} &= -\eta_2^2 h, & \frac{\partial^2 h}{\partial x \partial y} &= -\eta_1 \eta_2 h, & \frac{\partial^2 h}{\partial v_i^2} &= B_i^2 h, & \frac{\partial^2 h}{\partial \tilde{\theta}_i^2} &= C_i^2 h, \end{aligned}$$

$$\frac{\partial^2 h}{\partial m^2} = (2E + (D + 2Em)^2)h, \quad \frac{\partial^2 h}{\partial x \partial m} = i\eta_1(D + 2Em)h, \quad \frac{\partial^2 h}{\partial y \partial m} = i\eta_2(D + 2Em)h,$$

and the remaining cross-derivatives are

$$\frac{\partial^2 h}{\partial x \partial v_i} = i\eta_1 B_i h, \quad \frac{\partial^2 h}{\partial y \partial v_i} = i\eta_2 B_i h, \quad i = 1, 2, 3.$$

Substituting these derivatives into Eq (2.26) and dividing by h , both sides of the equation become polynomials that are affine in $(v_1, v_2, v_3, \theta_1, \theta_2, \theta_3)$ and quadratic in m . Since the resulting identity must hold for all admissible values of the state variables, and the functions $1, v_i, \theta_i, m, m^2$ are linearly independent, the coefficients of v_i, θ_i, m, m^2 and the constant term on the two sides must coincide. For instance, collecting the coefficient of v_1 , the diffusion term $\frac{1}{2}\zeta_1^2 v_1 \frac{\partial^2 h}{\partial v_1^2}$ contributes $\frac{1}{2}\zeta_1^2 B_1^2$, the cross-derivatives $\frac{\partial^2 h}{\partial x \partial v_1}$ and $\frac{\partial^2 h}{\partial y \partial v_1}$ together with the drift term $-\tilde{k}_1 v_1 \frac{\partial h}{\partial v_1}$ contribute $-(\tilde{k}_1 - i\rho_1^v \zeta_1(\sigma_1 \eta_1 + \sigma_2 \eta_2))B_1$, and the second- and first-order derivatives with respect to x and y contribute the remaining B_1 -free terms, which produces Eq (2.28); the other equations are obtained in the same manner. We thus obtain the system of ordinary differential equations

$$\frac{\partial A}{\partial \tau} = \frac{1}{2} \sum_{i=1}^3 (\tilde{\psi}_i^2 C_i^2 + 2\tilde{\lambda}_i C_i) + \frac{1}{2}\gamma^2(D^2 + 2E) + \tilde{\alpha}\tilde{\beta}D + r(i\eta_1 + i\eta_2), \quad (2.27)$$

$$\frac{\partial B_1}{\partial \tau} = \frac{1}{2}\zeta_1^2 B_1^2 - (\tilde{k}_1 - i\rho_1^v \zeta_1(\sigma_1 \eta_1 + \sigma_2 \eta_2))B_1 - \frac{1}{2}\sigma_1^2 \eta_1(i + \eta_1) - \frac{1}{2}\sigma_2^2 \eta_2(i + \eta_2) - \sigma_1 \sigma_2 \eta_1 \eta_2, \quad (2.28)$$

$$\frac{\partial B_i}{\partial \tau} = \frac{1}{2}\zeta_i^2 B_i^2 - (\tilde{k}_i - i\rho_i^v \zeta_i u_i)B_i - \frac{1}{2}u_i(u_i + i), \quad i = 2, 3, \quad (2.29)$$

$$\frac{\partial C_i}{\partial \tau} = \tilde{k}_i B_i, \quad i = 1, 2, 3, \quad (2.30)$$

$$\frac{\partial D}{\partial \tau} = (2\gamma^2 E - \tilde{\alpha} + i \sum_{j=1}^2 \eta_j \gamma v_j \rho_j^\xi)D + 2\tilde{\alpha}\tilde{\beta}E, \quad (2.31)$$

$$\frac{\partial E}{\partial \tau} = 2\gamma^2 E^2 + 2\left(i \sum_{j=1}^2 \eta_j \gamma v_j \rho_j^\xi - \tilde{\alpha}\right)E - \rho^\xi \eta_1 \eta_2 v_1 v_2 - \sum_{j=1}^2 \frac{1}{2} \eta_j v_j^2 (\eta_j + i), \quad (2.32)$$

with zero initial conditions.

Equations (2.28), (2.29), and (2.32) are Riccati equations of the form

$$\frac{\partial H}{\partial \tau} = aH^2 - 2\omega H + c, \quad H(0) = 0.$$

The substitution $H(\tau) = -\frac{1}{a} u'(\tau)/u(\tau)$ linearizes them to $u'' + 2\omega u' + ac u = 0$, whose characteristic roots are $-\omega \pm \delta$ with $\delta = \sqrt{\omega^2 - ac}$. Using $H(0) = 0$, the solution is

$$H(\tau) = \frac{1}{a} \left(\omega - \delta \frac{\sinh(\delta\tau) + \tilde{\delta} \cosh(\delta\tau)}{\cosh(\delta\tau) + \tilde{\delta} \sinh(\delta\tau)} \right), \quad \tilde{\delta} = \frac{\omega}{\delta}.$$

Mapping $a = \frac{1}{2}\zeta_i^2$, $\omega = \omega_B^i$ gives $B_i(\tau)$ in Eq (2.22), and mapping $a = 2\gamma^2$, $\omega = \omega_E$ gives $E(\tau)$ in Eq (2.25). For C_i , integrating Eq (2.30) and writing B_i as a logarithmic derivative yields Eq (2.23). Given

E , Eq (2.31) is a first-order linear ODE; using the integrating factor $f_E(\tau) = \cosh(\delta_E \tau) + \tilde{\delta}_E \sinh(\delta_E \tau)$ and the identity $\omega_E = \delta_E \tilde{\delta}_E$ gives $D(\tau)$ in Eq (2.24). Finally, $A(\tau)$ in Eq (2.21) follows by direct integration of Eq (2.27); the term involving C_i cannot be integrated in closed form and is left in integral form. Consequently, all coefficient functions except this one-dimensional time integral are available in closed form; the resulting pricing formula in Section 3 is therefore referred to as semi-analytical, and the remaining integral is evaluated numerically at negligible cost (see Section 4). This completes the proof.

Remark 3. Two structural features of Theorem 1 deserve comment. The Brownian motions W_i^y and W_i^θ are mutually independent, following He and Chen [8]; a nonzero correlation $\rho_i^{y\theta}$ would introduce a term $\rho_i^{y\theta} \zeta_i \tilde{\psi}_i \sqrt{v_i} \partial^2 h / \partial v_i \partial \tilde{\theta}_i$ into Eq (2.26), which is not affine in the state variables, so this independence is a tractability assumption shared by the stochastic long-term mean literature [8–10]. Moreover, although the loading $C_i(\tau)$ on $\tilde{\theta}_i(t)$ does not involve $\tilde{\psi}_i$, the volatility of the long-term mean does affect option prices: $\tilde{\psi}_i$ enters the characteristic function through $A(\tau)$ via the term $\frac{1}{2} \tilde{\psi}_i^2 C_i(s)^2$ in Eq (2.21), so that its effect is of second order in $\tilde{\psi}_i$. At the baseline parameters of Section 4, increasing $\tilde{\psi}_i$ from 0 to 0.03 changes the option price by less than 0.01, which explains why the first-order drift effect of λ_i dominates in the numerical experiments.

For notational convenience we write $h(\eta_1, \eta_2; T)$ for $h(\eta_1, \eta_2; 0, T)$.

3. Pricing the option on the maximum of two assets

In this section, we consider the pricing of a call option on the maximum of the two liquidity-adjusted assets S_1 and S_2 , with strike K and maturity T . The option price at time 0 under $\mathbb{Q}$ is given by

$$C_{\max} = e^{-rT} \mathbb{E}^{\mathbb{Q}}[(\max(S_1(T), S_2(T)) - K)^+], \quad (3.1)$$

where r is the constant interest rate.

Using the joint characteristic function of Theorem 1, we obtain the following semi-analytical pricing formula.

Theorem 2. Under the proposed model, the price at time 0 of the European call option on the maximum of the two liquidity-adjusted assets, defined in Eq (3.1), is given by

$$C_{\max} = e^{-rT} [h(-i, 0; T) \Pi_1 - K \Pi_2 + h(0, -i; T) \Pi_3 - K \Pi_4], \quad (3.2)$$

where $h(\eta_1, \eta_2; T)$ is the joint characteristic function defined in Theorem 1, with $h(-i, 0; T) = S_1(0)e^{rT}$ and $h(0, -i; T) = S_2(0)e^{rT}$. The probabilities Π_j ($j = 1, 2, 3, 4$), with $k = \ln K$, are

$$\begin{aligned} \Pi_1 &= \frac{1}{4} + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi_1 k} f_1(\phi_1, 0)}{i\phi_1} \right] d\phi_1 + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{f_1(\phi_2, -\phi_2)}{i\phi_2} \right] d\phi_2 \\ &\quad - \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \Re \left[\frac{e^{-i\phi_1 k} f_1(\phi_1 + \phi_2, -\phi_2) - e^{-i\phi_1 k} f_1(\phi_1 - \phi_2, \phi_2)}{\phi_1 \phi_2} \right] d\phi_1 d\phi_2, \\ \Pi_2 &= \frac{1}{4} + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi_1 k} f_2(\phi_1, 0)}{i\phi_1} \right] d\phi_1 + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{f_2(\phi_2, -\phi_2)}{i\phi_2} \right] d\phi_2 \end{aligned} \quad (3.3)$$

$$-\frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \Re \left[\frac{e^{-i\phi_1 k} f_2(\phi_1 + \phi_2, -\phi_2) - e^{-i\phi_1 k} f_2(\phi_1 - \phi_2, \phi_2)}{\phi_1 \phi_2} \right] d\phi_1 d\phi_2, \quad (3.4)$$

$$\begin{aligned} \Pi_3 = & \frac{1}{4} + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi_2 k} f_3(0, \phi_2)}{i\phi_2} \right] d\phi_2 + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{f_3(-\phi_1, \phi_1)}{i\phi_1} \right] d\phi_1 \\ & - \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \Re \left[\frac{e^{-i\phi_2 k} f_3(-\phi_1, \phi_1 + \phi_2) - e^{-i\phi_2 k} f_3(\phi_1, \phi_2 - \phi_1)}{\phi_1 \phi_2} \right] d\phi_1 d\phi_2, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \Pi_4 = & \frac{1}{4} + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi_2 k} f_4(0, \phi_2)}{i\phi_2} \right] d\phi_2 + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{f_4(-\phi_1, \phi_1)}{i\phi_1} \right] d\phi_1 \\ & - \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \Re \left[\frac{e^{-i\phi_2 k} f_4(-\phi_1, \phi_1 + \phi_2) - e^{-i\phi_2 k} f_4(\phi_1, \phi_2 - \phi_1)}{\phi_1 \phi_2} \right] d\phi_1 d\phi_2, \end{aligned} \quad (3.6)$$

and the characteristic functions $f_j(\phi_1, \phi_2)$ are

$$f_1(\phi_1, \phi_2) = \frac{h(\phi_1 - i, \phi_2; T)}{h(-i, 0; T)}, \quad f_2(\phi_1, \phi_2) = h(\phi_1, \phi_2; T), \quad (3.7)$$

$$f_3(\phi_1, \phi_2) = \frac{h(\phi_1, \phi_2 - i; T)}{h(0, -i; T)}, \quad f_4(\phi_1, \phi_2) = h(\phi_1, \phi_2; T). \quad (3.8)$$

Proof. The payoff of the option on the maximum of the two assets can be decomposed as follows.

$$\begin{aligned} (\max(S_1(T), S_2(T)) - K)^+ = & (S_1(T) - K) \mathbf{1}_{\{S_1(T) \geq S_2(T), S_1(T) \geq K\}} \\ & + (S_2(T) - K) \mathbf{1}_{\{S_2(T) > S_1(T), S_2(T) \geq K\}}. \end{aligned} \quad (3.9)$$

Here and in what follows, $\mathbf{1}_A$ denotes the indicator function of an event A , which equals one if A occurs and zero otherwise. Substituting this decomposition into Eq (3.1) gives

$$C_{\max} = e^{-rT} (J_1 - KJ_2 + J_3 - KJ_4), \quad (3.10)$$

where

$$\begin{aligned} J_1 &= \mathbb{E}^{\mathbb{Q}}[e^{x(T)} \mathbf{1}_{\{x(T) \geq y(T), x(T) \geq k\}}], & J_2 &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{x(T) \geq y(T), x(T) \geq k\}}], \\ J_3 &= \mathbb{E}^{\mathbb{Q}}[e^{y(T)} \mathbf{1}_{\{y(T) > x(T), y(T) \geq k\}}], & J_4 &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{y(T) > x(T), y(T) \geq k\}}]. \end{aligned}$$

To evaluate J_1 and J_3 we introduce two share measures, while J_2 and J_4 are computed directly under $\mathbb{Q}$. Define

$$\frac{d\mathbb{Q}_1}{d\mathbb{Q}} = \frac{e^{x(T)}}{\mathbb{E}^{\mathbb{Q}}[e^{x(T)}]} = \frac{e^{x(T)}}{h(-i, 0; T)}, \quad \frac{d\mathbb{Q}_2}{d\mathbb{Q}} = \frac{e^{y(T)}}{\mathbb{E}^{\mathbb{Q}}[e^{y(T)}]} = \frac{e^{y(T)}}{h(0, -i; T)}. \quad (3.11)$$

Then $J_1 = h(-i, 0; T) \mathbb{Q}_1(x(T) \geq y(T), x(T) \geq k)$, $J_2 = \mathbb{Q}(x(T) \geq y(T), x(T) \geq k)$, $J_3 = h(0, -i; T) \mathbb{Q}_2(y(T) \geq x(T), y(T) \geq k)$, and $J_4 = \mathbb{Q}(y(T) \geq x(T), y(T) \geq k)$.

The characteristic functions under the measures are obtained by incorporating the Radon–Nikodym densities into the expectations.

$$f_1(\phi_1, \phi_2) = \mathbb{E}^{\mathbb{Q}_1}[e^{i\phi_1 x(T) + i\phi_2 y(T)}] = \frac{\mathbb{E}^{\mathbb{Q}}[e^{(i\phi_1 + 1)x(T) + i\phi_2 y(T)}]}{h(-i, 0; T)} = \frac{h(\phi_1 - i, \phi_2; T)}{h(-i, 0; T)},$$

$$f_3(\phi_1, \phi_2) = \mathbb{E}^{\mathbb{Q}_2}[e^{i\phi_1 x(T) + i\phi_2 y(T)}] = \frac{\mathbb{E}^{\mathbb{Q}}[e^{i\phi_1 x(T) + (i\phi_2 + 1)y(T)}]}{h(0, -i; T)} = \frac{h(\phi_1, \phi_2 - i; T)}{h(0, -i; T)},$$

while $f_2(\phi_1, \phi_2) = f_4(\phi_1, \phi_2) = \mathbb{E}^{\mathbb{Q}}[e^{i\phi_1 x(T) + i\phi_2 y(T)}] = h(\phi_1, \phi_2; T)$.

For a bivariate random vector (X_1, X_2) with characteristic function $f(\cdot, \cdot)$, the Shephard inversion formula [21] gives the joint distribution

$$\begin{aligned} \mathbb{P}(X_1 \geq a_1, X_2 \geq a_2) &= \frac{1}{4} + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi_1 a_1} f(\phi_1, 0)}{i\phi_1} \right] d\phi_1 + \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi_2 a_2} f(0, \phi_2)}{i\phi_2} \right] d\phi_2 \\ &\quad - \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \Re \left[\frac{e^{-i\phi_1 a_1 - i\phi_2 a_2} f(\phi_1, \phi_2) - e^{-i\phi_1 a_1 + i\phi_2 a_2} f(\phi_1, -\phi_2)}{\phi_1 \phi_2} \right] d\phi_1 d\phi_2. \end{aligned} \quad (3.12)$$

For J_1 , work under $\mathbb{Q}_1$ with the pair $(x(T), x(T) - y(T))$ and thresholds $(k, 0)$: the characteristic function of $x(T) - y(T)$ at argument ϕ equals $f_1(\phi, -\phi)$, and that of the pair $(x(T), x(T) - y(T))$ at (ϕ_1, ϕ_2) equals $f_1(\phi_1 + \phi_2, -\phi_2)$. Applying Eq (3.12) produces Π_1 in Eq (3.3), so that $J_1 = h(-i, 0; T) \Pi_1$. The same construction under $\mathbb{Q}$ (with $f_2 = h$) gives Π_2 in Eq (3.4) and $J_2 = \Pi_2$.

For J_3 , work under $\mathbb{Q}_2$ with the pair $(y(T), y(T) - x(T))$ and thresholds $(k, 0)$: the characteristic function of $y(T) - x(T)$ at ϕ equals $f_3(-\phi, \phi)$, and that of $(y(T), y(T) - x(T))$ at (ϕ_2, ϕ_1) equals $f_3(-\phi_1, \phi_1 + \phi_2)$, yielding Π_3 in Eq (3.5) and $J_3 = h(0, -i; T) \Pi_3$. Likewise under $\mathbb{Q}$ (with $f_4 = h$) one obtains Π_4 in Eq (3.6) and $J_4 = \Pi_4$.

Substituting J_1, J_2, J_3, J_4 into Eq (3.10) completes the proof.

Remark 4 (Option on the minimum). *The price of the call on the minimum follows from the put–call–type identity for extremum payoffs,*

$$C_{\max}(S_1, S_2; K) + C_{\min}(S_1, S_2; K) = C(S_1; K) + C(S_2; K),$$

where $C(S_j; K)$ is the single-asset European call. Each single-asset call is recovered from h by setting the unused argument to zero, e.g., $C(S_1; K) = e^{-rT} [S_1(0)e^{rT}\Pi_1^s - K\Pi_2^s]$ with one-dimensional inversions Π_1^s, Π_2^s built from $f_1(\phi_1, 0)$ and $f_2(\phi_1, 0)$.

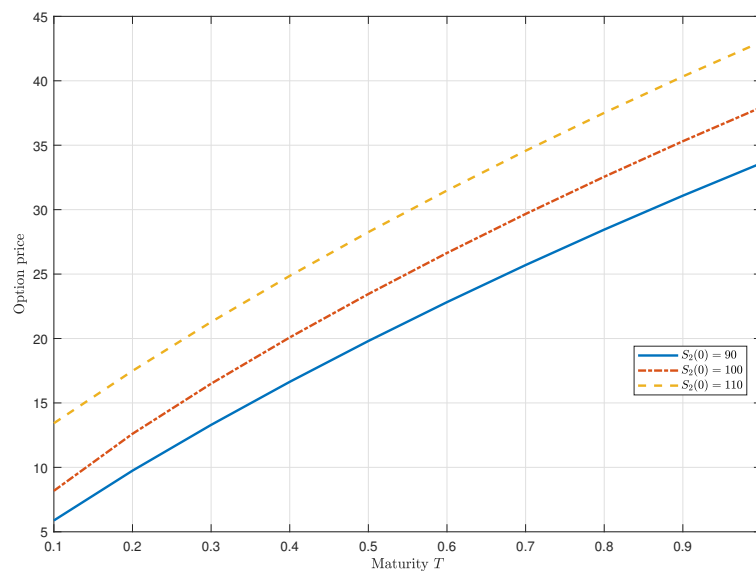
4. Numerical examples

In this section, we present numerical examples to illustrate the pricing behavior of the European call option on the maximum of two liquidity-adjusted assets under the proposed model. The option prices are computed from the semi-analytical formula in Theorem 2, validated against Monte Carlo simulation, and compared with nested benchmark models.

Unless otherwise stated, we use the baseline parameter values reported in Table 1. The baseline values are chosen for illustration but lie within the ranges employed in the related literature: the two-factor variance parameters are comparable to those used by Christoffersen et al. [7] and He and Chen [8], and the liquidity parameters $(\tilde{\alpha}, \tilde{\beta}, \gamma, \nu_i, m(0))$ follow the magnitudes adopted by Feng et al. [14] and subsequent liquidity studies [16, 22]. The values $\tilde{\lambda}_i = -0.01$ and $\tilde{\psi}_i = 0.01$ keep the probability of a negative long-term mean negligible over the maturities considered (below 10^{-44} ; see Remark 1). A full calibration of the model to market option data is left for future research.

Table 1. Baseline parameter values for the numerical experiments.

Parameter	Value	Parameter	Value
$S_1(0)$	100	$S_2(0)$	100
K	100	T	1
r	0.03	σ_1, σ_2	0.8, 0.8
$v_1(0), v_2(0), v_3(0)$	0.03, 0.03, 0.03	$\tilde{\theta}_1(0), \tilde{\theta}_2(0), \tilde{\theta}_3(0)$	0.15, 0.15, 0.15
$\tilde{k}_1, \tilde{k}_2, \tilde{k}_3$	2, 2, 2	$\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$	0.4, 0.1, 0.1
$\rho_1^v, \rho_2^v, \rho_3^v$	-0.5, -0.5, -0.5	$\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$	-0.01, -0.01, -0.01
ψ_1, ψ_2, ψ_3	0.01, 0.01, 0.01	$m(0)$	0.5
$\tilde{\alpha}$	2	$\tilde{\beta}$	0.8
γ	0.1	v_1, v_2	0.6, 0.6
ρ_1^ξ, ρ_2^ξ	0.5, 0.5	ρ^ξ	0.5
U	50	N	64
ε	10^{-8}	N_τ	32

**Figure 1.** Option price against the maturity T with different $S_2(0)$.

The improper integrals in Eqs (3.3)–(3.6) are truncated to $[\varepsilon, U]$ and evaluated by tensorized N -point Gauss–Legendre quadrature in each dimension, and the one-dimensional time integral appearing in $A(\tau)$ of Eq (2.21) is evaluated by N_τ -point Gauss–Legendre quadrature. The apparent singularities at the origin are removable because the numerators vanish at the same rate as the denominators, and the truncation at U is justified by the exponential decay of the characteristic function; in practice the integration starts from a small $\varepsilon > 0$. Under the baseline configuration $(U, N, N_\tau) = (50, 64, 32)$ with $\varepsilon = 10^{-8}$, the computed prices are numerically converged: enlarging the configuration to $(90, 256, 64)$ changes the reported prices by less than 10^{-9} , so all reported figures are unaffected by the quadrature settings. A single price evaluation takes about one second in our single-threaded implementation in an

interpreted language, whereas the Monte Carlo estimate described below requires roughly forty seconds, so the semi-analytical formula is more than an order of magnitude faster and, unlike simulation, free of statistical error. The Fourier inversion was numerically stable for all parameter configurations considered.

To validate the pricing formula, Table 2 compares the semi-analytical prices of Theorem 2 with Monte Carlo estimates for $T \in \{0.5, 1\}$ and $K \in \{90, 100, 110\}$. The Monte Carlo estimates are based on an Euler–Maruyama discretization of the risk-neutral dynamics with full truncation of the variance processes, using the Brownian decomposition of Section 2, 4×10^5 antithetic paths, and 500 time steps per year. All differences are below 0.05 in absolute value at price levels of 18–44 and lie within about one Monte Carlo standard error, which confirms the correctness of both the derived characteristic function and its numerical implementation. As additional checks, the martingale identities $h(-i, 0; T) = S_1(0)e^{rT}$ and $h(0, -i; T) = S_2(0)e^{rT}$ are reproduced to thirteen significant digits by the implemented characteristic function, and the closed-form expressions Eqs (2.22)–(2.25) agree with a direct Runge–Kutta integration of the system Eqs (2.27)–(2.32) to machine precision.

Table 2. Validation of the semi-analytical pricing formula against Monte Carlo simulation.

T	K	Formula	Monte Carlo	Std. error	Difference
0.5	90	30.1516	30.1175	0.0388	0.0341
0.5	100	23.4518	23.4118	0.0389	0.0400
0.5	110	17.9085	17.8682	0.0378	0.0403
1	90	43.9779	43.9629	0.0779	0.0150
1	100	37.9488	37.9321	0.0776	0.0167
1	110	32.6609	32.6432	0.0766	0.0177

Table 3. Comparison with nested benchmark models ($T = 1$).

Model	$K = 90$	$K = 100$	$K = 110$
Proposed model	43.9779	37.9488	32.6609
No liquidity risk ($\nu_i = 0$)	34.0500	27.2438	21.4356
Constant long-term mean ($\tilde{\psi}_i = \tilde{\lambda}_i = 0$)	44.1806	38.1597	32.8761
Two-factor Heston [7]	34.3236	27.5364	21.7378

Table 3 quantifies the incremental contribution of each model component by comparing the proposed model with three nested benchmarks at $T = 1$: the model without stochastic liquidity risk ($\nu_i = 0$), the model with a constant long-term mean ($\tilde{\psi}_i = \tilde{\lambda}_i = 0$), and the two-factor Heston model of Christoffersen et al. [7] obtained by imposing both restrictions. The prices of the nested models are computed from the same formula with the corresponding parameters set to zero, and the two-factor Heston benchmark was additionally verified by Monte Carlo simulation (27.5364 versus 27.4752 with standard error 0.0612 at $K = 100$). Ignoring stochastic liquidity understates the option value substantially at the baseline calibration, by 9.9 to 11.2 depending on the strike, because the liquidity factor both discounts the traded prices and adds an additional source of variance to the terminal maximum. The stochastic long-term mean has a more moderate effect at the baseline values of $\tilde{\lambda}_i$ and $\tilde{\psi}_i$ (about -0.21 at the money, driven mainly by the negative drift $\tilde{\lambda}_i = -0.01$), although this impact becomes substantial when $\tilde{\lambda}_i$ varies over a wider range, especially at long maturities. The pricing error incurred by ignoring liquidity risk is therefore first order in this calibration, while the stochastic long-term mean provides additional flexibility

for the term structure of option prices.

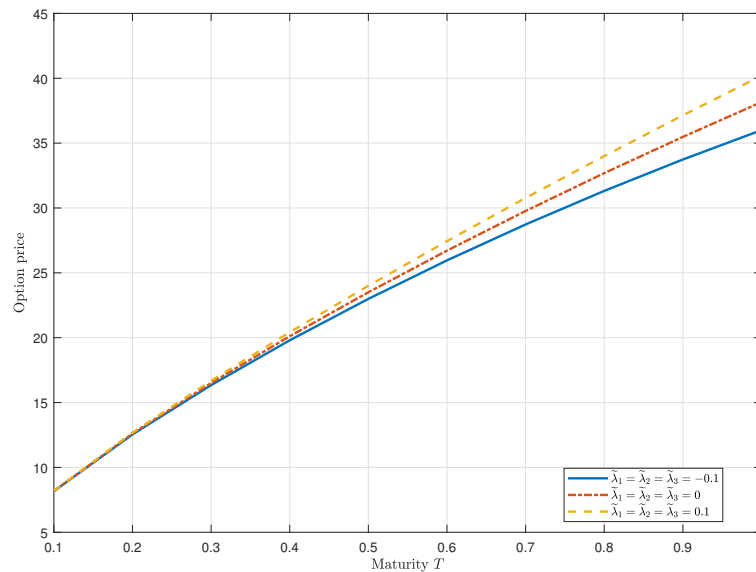


Figure 2. Option price against the maturity T with different $\tilde{\lambda}_i$ ($i = 1, 2, 3$).

Because the model contains several interacting risk factors, the one-parameter sensitivities reported in Figures 1–6 are complemented by a joint variation of the liquidity sensitivity $\nu := \nu_1 = \nu_2$ and the liquidity volatility γ in Table 4 ($T = 1$, $K = 100$). The two parameters interact strongly: when $\nu = 0$ the option price is completely insensitive to γ , whereas at $\nu = 0.6$ increasing γ from 0.1 to 0.9 raises the price by 2.98; conversely, the effect of ν becomes stronger as γ grows. Interactions of this type indicate that liquidity-related parameters should be calibrated jointly rather than one at a time.

Table 4. Interaction between the liquidity sensitivity $\nu = \nu_1 = \nu_2$ and the liquidity volatility γ ($T = 1$, $K = 100$).

	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.9$
$\nu = 0$	27.2438	27.2438	27.2438
$\nu = 0.3$	30.2540	30.5415	31.1716
$\nu = 0.6$	37.9488	38.9488	40.9288

Several examples are provided for the sensitivity analysis which is conducted based on the values in Table 1. In all figures, the horizontal axis is the maturity T , and the vertical axis is the option price.

Figure 1 shows the sensitivity of the option price to the initial price of the second asset. The price is increasing in $S_2(0)$. For fixed model parameters and fixed paths of the risk factors, a higher initial value $S_2(0)$ raises the terminal value $S_2(T)$ and hence cannot decrease the maximum payoff. As a result, the figure provides a numerical consistency verification for the pricing formula. Figure 2 illustrates the effect of the drift parameter $\tilde{\lambda}_i$ of the stochastic long-term mean process. In this experiment, we set $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 \in \{-0.1, 0, 0.1\}$. The option price increases as the common value of $\tilde{\lambda}_i$ increases. A larger $\tilde{\lambda}_i$ raises the expected future level of the stochastic long-term variance mean $\tilde{\theta}_i(t)$. This increases the expected future variance of the assets and therefore increases the time value of the maximum option.

The impact becomes more pronounced for longer maturities because the stochastic long-term mean has more time to affect the variance dynamics.

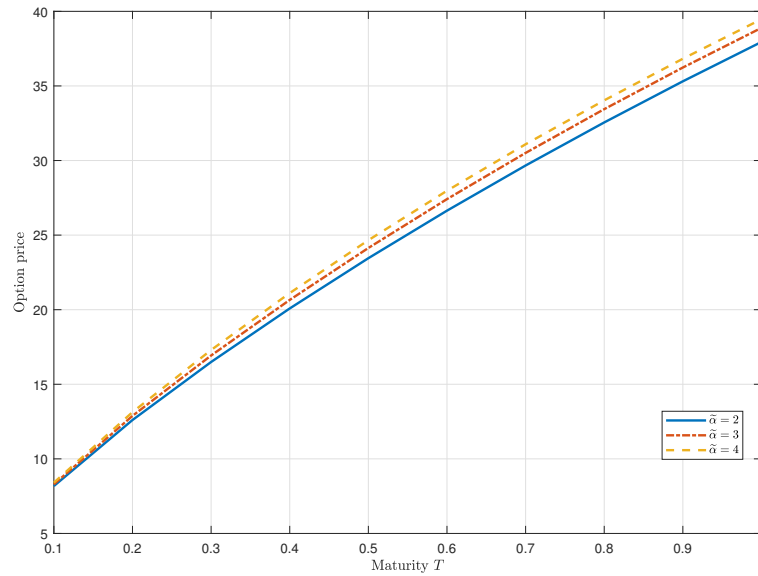


Figure 3. Option price against the maturity T with different $\tilde{\alpha}$.

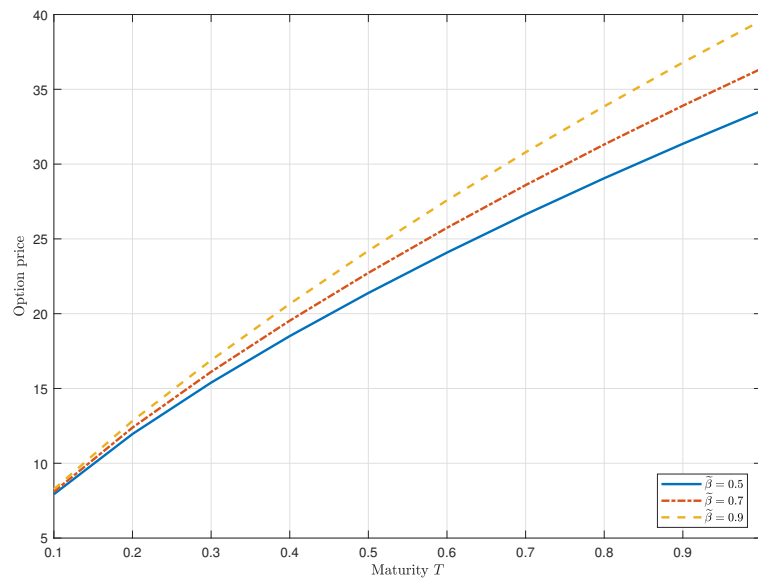


Figure 4. Option price against the maturity T with different $\tilde{\beta}$.

Figure 3 examines the role of the mean-reversion speed $\tilde{\alpha}$ of the market liquidity process $m(t)$. Under the baseline parameters, $m(0) = 0.5$ and $\tilde{\beta} = 0.8$, so the market liquidity process mean-reverts upward. Consequently, a larger $\tilde{\alpha}$ moves $m(t)$ more rapidly toward the higher long-term level $\tilde{\beta}$. Therefore, the

option price increases with $\tilde{\alpha}$ in this baseline setting. Figure 4 reports the sensitivity of the option price to the long-term mean $\tilde{\beta}$ of the liquidity process. The option price increases as $\tilde{\beta}$ increases. A larger $\tilde{\beta}$ indicates a higher long-run level of $m(t)$, enhancing the liquidity component in asset dynamics. As a result, the distribution of the terminal maximum becomes more dispersed, and the convexity of the call payoff leads to a higher option price.

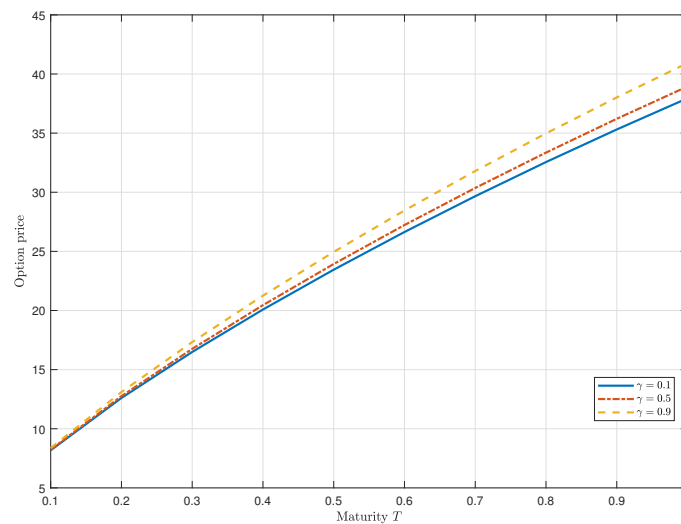


Figure 5. Option price against the maturity T with different γ .

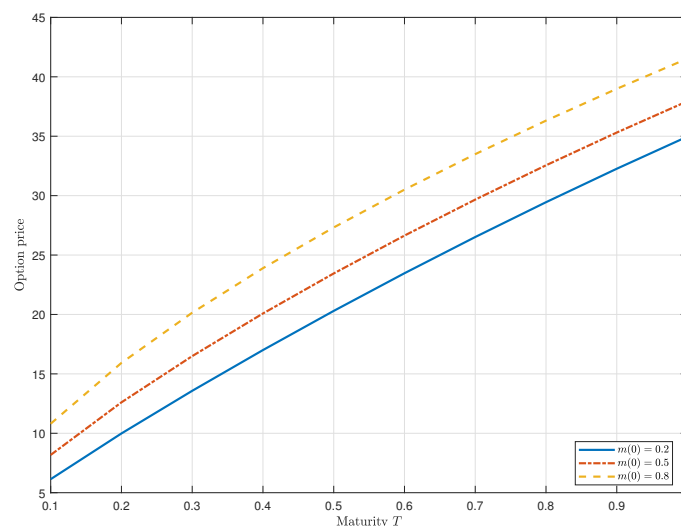


Figure 6. Option price against the maturity T with different $m(0)$.

Figure 5 displays the effect of the volatility parameter γ of the market liquidity process. The option price increases in γ . A larger γ introduces more uncertainty into the future liquidity level $m(t)$. Since $m(t)$ affects both the drift and diffusion components of the liquidity-adjusted asset prices, higher liquidity

volatility increases the variance of the terminal asset prices. Figure 6 investigates the sensitivity to the initial market liquidity level $m(0)$. The option price increases with $m(0)$. A higher initial value of $m(t)$ immediately amplifies the liquidity-induced terms in the asset dynamics and increases the uncertainty of the liquidity-adjusted terminal prices. The effect is observed across all maturities.

5. Conclusions

This paper develops a semi-analytical valuation framework for a European call option on the maximum of two assets. The two asset prices are driven by a common systematic variance factor and asset-specific idiosyncratic variance factors. To increase the flexibility of the variance dynamics, the long-term mean of each variance factor is modeled as a stochastic process. In addition, the traded asset prices incorporate stochastic liquidity risk through liquidity discount factors driven by a mean-reverting market liquidity process.

Despite the additional complexity generated by stochastic long-term means and stochastic liquidity, the proposed model preserves analytical tractability. Under the risk-neutral measure, the joint characteristic function of the two log-prices is derived explicitly. The associated coefficient equations are reduced to Riccati equations, which can be solved explicitly. As a result, we obtain a semi-analytical pricing formula for the European call option on the maximum of two assets by combining the characteristic function with a change of measure technique. Furthermore, we validate the formula against Monte Carlo simulation and nested benchmark models, and provide several numerical examples to illustrate important implications of the model.

Several limitations of the proposed framework suggest directions for future research. The arithmetic Brownian motion specification of the long-term means, the independence of the idiosyncratic volatility factors, and the single market-wide liquidity level with constant sensitivities are tractability-driven assumptions (see Remarks 1–3). Relaxing these assumptions would generally replace the present scalar coefficient system with more complex equations, but would not necessarily eliminate transform tractability. CIR-type long-term means and Gaussian asset-specific liquidity levels can lead to non-autonomous or higher-dimensional Riccati systems that are solvable numerically, whereas correlated idiosyncratic shocks or nonlinear liquidity–volatility feedback may require numerical PDE or simulation methods. Incorporating rough volatility dynamics [11–13] into multi-asset extremum option pricing is another promising but technically demanding direction. Finally, calibrating the model to market data on multi-asset options and assessing its empirical performance relative to simpler benchmarks remain important topics for future work.

Use of AI tools declaration

The authors declare they have used Large Language Models (LLMs) in the creation of this article. The AI tools were employed solely for language polishing, including grammar correction, improving sentence structure, enhancing clarity, and organizing the presentation of content. AI-assisted language refinements were applied across all sections of the manuscript. The authors reviewed and approved all modifications.

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Conflict of interest

Geonwoo Kim is a guest editor for *Networks and Heterogeneous Media*, and was not involved in the editorial review or the decision to publish this article. All authors declare that they have no competing interests.

Author contributions

Both authors contributed equally to the literature review, discussion, writing, and revision of the paper.

References

1. R. M. Stulz, Options on the minimum or the maximum of two risky assets: Analysis and applications, *J. Financ. Econ.*, **10** (1982), 161–185. [https://doi.org/10.1016/0304-405X\(82\)90011-3](https://doi.org/10.1016/0304-405X(82)90011-3)
2. H. Johnson, Options on the maximum or the minimum of several assets, *J. Financ. Quant. Anal.*, **22** (1987), 277–283. <https://doi.org/10.2307/2330963>
3. J. Zhang, K. Zhou, Pricing options on the maximum or the minimum of several assets with default risk, *N. Am. J. Econ. Finance*, **75** (2025), 102272. <https://doi.org/10.1016/j.najef.2024.102272>
4. R. Wang, G. Deng, Pricing geometric Asian extremum options under mixed fractional Brownian motion with jumps, *Jpn. J. Ind. Appl. Math.*, **42** (2025), 2233–2261. <https://doi.org/10.1007/s13160-025-00723-4>
5. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Political Econ.*, **81** (1973), 637–654. <https://doi.org/10.1086/260062>
6. S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.*, **6** (1993), 327–343. <https://doi.org/10.1093/rfs/6.2.327>
7. P. Christoffersen, S. Heston, K. Jacobs, The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well, *Manag. Sci.*, **55** (2009), 1914–1932. <https://doi.org/10.1287/mnsc.1090.1065>
8. X. J. He, W. Chen, A closed-form pricing formula for European options under a new stochastic volatility model with a stochastic long-term mean, *Math. Financ. Econ.*, **15** (2021), 381–396. <https://doi.org/10.1007/s11579-020-00281-y>
9. S. Y. Cho, G. Kim, Analytical valuation of vulnerable options under a stochastic volatility model with a stochastic long-term mean, *AIMS Math.*, **10** (2025), 20219–20234. <https://doi.org/10.3934/math.2025903>
10. Y. Wang, Y. Bai, European option pricing under the approximate fractional Heston jump–diffusion model with a stochastic long-term mean, *Alex. Eng. J.*, **123** (2025), 145–156. <https://doi.org/10.1016/j.aej.2025.03.045>

11. J. Gatheral, T. Jaisson, M. Rosenbaum, Volatility is rough, *Quant. Finance*, **18** (2018), 933–949. <https://doi.org/10.1080/14697688.2017.1393551>
12. X. Wang, W. Xiao, J. Yu, Modeling and forecasting realized volatility with the fractional Ornstein–Uhlenbeck process, *J. Econom.*, **232** (2023), 389–415. <https://doi.org/10.1016/j.jeconom.2021.08.001>
13. X. Wang, W. Xiao, J. Yu, C. Zhang, Maximum likelihood estimation of fractional Ornstein–Uhlenbeck process with discretely sampled data, Working Paper No. 202527, Faculty of Business Administration, University of Macau, 2025. Available from: <https://ideas.repec.org/p/boa/wpaper/202527.html>.
14. S. P. Feng, M. W. Hung, Y. H. Wang, Option pricing with stochastic liquidity risk: Theory and evidence, *J. Financ. Mark.*, **18** (2014), 77–95. <https://doi.org/10.1016/j.finmar.2013.05.002>
15. X. J. He, S. D. Huang, S. Lin, A closed-form solution for pricing European-style options under the Heston model with credit and liquidity risks, *Commun. Nonlinear Sci. Numer. Simul.*, **143** (2025), 108595. <https://doi.org/10.1016/j.cnsns.2025.108595>
16. J. Jeon, G. Kim, Analytically pricing a vulnerable option under a stochastic liquidity risk model with stochastic volatility, *Mathematics*, **12** (2024), 2642. <https://doi.org/10.3390/math12172642>
17. B. Wang, X. Wang, M. Zhao, Valuing vulnerable basket options with stochastic liquidity risk in reduced-form models, *Comput. Econ.*, **66** (2025), 2439–2455. <https://doi.org/10.1007/s10614-024-10794-z>
18. X. J. He, H. Chen, S. Lin, Analytically pricing European options under two-factor stochastic volatility with stochastic liquidity risks, *J. Futures Mark.*, **46** (2026), 1423–1439. <https://doi.org/10.1002/fut.70114>
19. X. J. He, D. Yan, S. Lin, A new analytical approach for pricing variance and volatility swaps: Incorporating liquidity and self-exciting jumps, *J. Futures Mark.*, (2026), e70120. <https://doi.org/10.1002/fut.70120>
20. S. Y. Cho, G. Kim, Analytically pricing vulnerable options under the stochastic volatility model with stochastic long-term mean and stochastic liquidity, *Commun. Nonlinear Sci. Numer. Simul.*, **161** (2026), 110130. <https://doi.org/10.1016/j.cnsns.2026.110130>
21. N. G. Shephard, From characteristic function to distribution function: A simple framework for the theory, *Econom. Theory*, **7** (1991), 519–529. <https://doi.org/10.1017/S0266466600004746>
22. S. Lin, X. J. He, Closed-form formulae for variance and volatility swaps under stochastic volatility with stochastic liquidity risks, *J. Futures Mark.*, **44** (2024), 1447–1461. <https://doi.org/10.1002/fut.22531>



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