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Research article

Two classes of coupled second order evolution equations with indirect damping: Lack of exponential decay

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Abstract: In this work, we consider two classes of coupled second-order abstract evolution equations with past history only acting in one of the equations of the systems, where the coupling terms are composed of fractional powers of operators. With the aid of the semigroup method and operator theory, we successfully establish that the coupled systems have no exponential decay rate for some fractional powers of coupled operators. Simultaneously, some applications of our abstract results in concrete models of the real world are given. It can be seen that our theorems cover and generalize the previous related results.

Keywords: coupled systems; indirect damping; exponential decay; past history

1. Introduction

We consider the following two classes of *abstract* systems of coupled equations with a past history term (memory term) in Hilbert space H.

$$\begin{cases} u_{tt} + bAu - \int_0^{+\infty} g(s)Au(t-s)ds + \alpha u + \beta A^\theta v = 0, \\ v_{tt} + Av + \beta A^\theta u = 0, \end{cases} \tag{1.1}$$

or

$$\begin{cases} u_{tt} + bAu - \int_0^{+\infty} g(s)Au(t-s)ds + \alpha u + \beta A^\theta v_t = 0, \\ v_{tt} + Av - \beta A^\theta u_t = 0, \end{cases} \tag{1.2}$$

with initial condition

$$u(t) = u^0(t), u_t(t) = \partial_t u^0(t),\ t \le 0,$$

and

$$v(0) = v_0, v_t(0) = v_1.$$

Here, A is a positive self-adjoint linear operator, $0 \le \theta \le 1$ (see [1] about the fractional operator), and $g(t) \in L^1(0, +\infty)$ is the kernel function. Furthermore, let A be an unbounded operator, which is very common in many special models, such as differential operators.

It is known that the term $\int_0^{+\infty} g(s)Au(t - s)ds$ in Eq (1.1) (or Eq (1.2)) will produce a damping mechanism, which turns out to be quite subtle. It can be seen that in the above two systems only the first equation has a damping term, so Eqs (1.1) and (1.2) are the indirectly damped systems. This damping mechanism indirectly converts mechanical vibration energy into thermal energy and dissipates it through the viscoelastic or hysteresis properties of the material. This type of indirect memory damping technology is widely used in fields that require both vibration reduction performance and durability, including transportation, building acoustics, and precision instruments. Its core lies in the memory effect of materials, such as viscoelastic hysteresis and deformation recovery, which can achieve vibration suppression and noise control. It has dynamic adaptability, long-term stability, and wide applicability. Therefore, it is a key technology to improve the reliability and comfort of mechanical, civil, and electronic systems.

In recent years, research on the stability of indirect damping systems has been a very hot topic (see, e.g., [2–6]). Specifically, the study of viscoelasticity with memory damping and coupled indirect damping has attracted significant attention, as seen in references [7–12]. In particular, the dependence of the stability of the whole system on the main operator coefficients, coupling terms, and kernel function in the system is the key problem in the research; see, e.g., [13–16] and the references therein. For more research on coupled systems, please refer to references [17–19]. For these two general forms of indirect damping systems (1.1) and (1.2) in the present paper, our concern is the interrelationship between coupling terms and exponential stability. In the previous literature, some authors have discussed this problem for some special forms of concrete systems, such as the following coupled wave equations:

$$\begin{cases} u_{tt} - b\Delta u + \displaystyle\int_0^{+\infty} g(s)\Delta u(t - s)ds + \alpha u + \beta v = 0,\ in\ \Omega \times (0, +\infty), \\ \hfill v_{tt} - \Delta v + \beta u = 0,\ in\ \Omega \times (0, +\infty), \\ \hfill u = v = 0,\ on\ \partial\Omega \times [0, +\infty). \end{cases} \tag{1.3}$$

Denote

$$\begin{aligned} &A : \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega), \\ &u \mapsto Au = -\Delta u. \end{aligned}$$

Then, the system (1.3) can be transferred to our abstract system (1.1) with $\theta = 0$. In [13], the authors verified that when $b = 1$ and $\alpha = 0$, system (1.3) has no exponential decay rate.

Moreover, the authors in [16] considered an isotropic porous and centrosymmetric viscoelastic solid with porous dissipation whose motion is described by the following coupled evolution system, which is also a special prototype of our abstract system (1.1) (see Section 3 for the details),

$$\begin{cases} J\varphi_{tt} = (\delta + g_1(0))\varphi_{xx} + \int_0^{+\infty} g_1'(s)\varphi_{xx}(t-s)ds - \xi\varphi - bu_x, \ in\ (0,\pi)\times(0,+\infty), \\ \rho u_{tt} = \mu u_{xx} + b\varphi_x, \ in\ (0,\pi)\times(0,+\infty), \\ \varphi(0,t) = \varphi(\pi,t) = u(0,t) = u(\pi,t) = 0, \quad t \geq 0, \end{cases} \tag{1.4}$$

and when $\frac{J\mu}{\rho} \neq \delta + g(0)$, the authors verified that the above system (1.4) lacked an exponential decay rate.

In [20], the authors investigated the stabilization of a locally coupled wave equations with local viscoelastic damping of past history type acting only in one equation via non smooth coefficients, which is represented as follows:

$$\begin{cases} u_{tt} - \left(au_x - b(x)\int_0^\infty g(s)u_x(x,t-s)ds\right)_x + c(x)y_t = 0, \ (x,s,t)\in(0,L)\times(0,\infty)\times(0,\infty), \\ y_{tt} - y_{xx} - c(x)u_t = 0, \ (x,t)\in(0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, \quad t > 0, \\ (u(x,-s), u_t(x,0)) = (u_0(x,s), u_1(x)), \ (x,s)\in(0,L)\times(0,\infty), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), \ x\in(0,L). \end{cases}$$

The authors proved that the energy decays exponentially when the two wave equations propagate at the same speed (i.e., $a = 1$), and polynomially with a rate of $\frac{1}{t}$ when the wave speeds differ.

In the present paper, with the help of the methods of semigroup and operator theory, for the different fractional powers θ in the coupling operators in systems (1.1) and (1.2), we give a sufficient condition that the systems do not exist with exponential stability. In other words, we prove a necessary condition for the exponential stability of systems (1.1) and (1.2). Furthermore, some specific applications of our abstract theories are given in Section 3, that is, we use our abstract results to verify whether some specific differential models can have exponential decay rates.

The relationship between the lack of exponential decay and system dynamic behavior is mainly reflected in the damping characteristics, stability, and energy dissipation mechanism of the system. The lack of exponential decay usually indicates insufficient system damping or missing energy dissipation mechanisms, directly affecting the persistence and stability of vibrations and dynamic performance. In engineering design, it is necessary to optimize the system response by adjusting the damping ratio or introducing new dissipation mechanisms. Due to the indirect damping system being considered, the system only has an infinite historical dissipation mechanism in one equation, which is transmitted to the second equation through coupling terms, thereby affecting the dissipation mechanism of the entire system. From our conclusion, it can be seen that when the system has different wave speeds, if the coupling mechanism is not strong enough, the system cannot transmit sufficiently strong damping dissipation to another part, resulting in us being unable to achieve exponential decay of the system by adjusting the infinite memory damping. Therefore, only when a new damping is introduced into another equation can the system achieve exponential decay, that is, by transforming the indirect damping coupling system into a direct damping coupling system. From these, it can be seen that our abstract theories not only cover some previous related results, but also generalize and improve related conclusions.

2. Main results and their proofs

The purpose of this section is to establish the main results of the abstract models considered in Eqs (1.1) and (1.2). To this aim, we are going to make the necessary assumptions and define the functional spaces.

Now we consider the assumptions on b, A, α, β, and g given below. Let H be a Hilbert space with scalar product $\langle \bullet, \bullet \rangle$ and the norm $\| \bullet \|$.

(A1) Assume that $\alpha \geq 0, \beta > 0$, and $0 \leq \theta \leq 1$ are three real constants. Let A be an unbounded positive self-adjoint linear operator in Hilbert space H satisfying that

$$\langle Au, u \rangle \geq M\|u\|^2, \quad u \in \mathcal{D}(A),$$

for some constant $M > 0$.

(A2) Let $b > 0$, and assume that $g(t) \in L^1(0, +\infty) \cap C^1[0, +\infty)$ is a nonnegative, non-increasing function satisfying for either $0 \leq \theta \leq \frac{1}{2}$,

$$b - \int_0^{+\infty} g(t)dt + M^{-1} \left(\alpha - \beta^2 \left\| A^{\theta - \frac{1}{2}} \right\|^2 \right) > 0,$$

or for $\frac{1}{2} < \theta \leq 1$,

$$b - \int_0^{+\infty} g(t)dt - \beta^2 \left\| A^{\theta - 1} \right\|^2 > 0.$$

(A2)′ Let $b > 0$, and assume that $g(t) \in L^1(0, +\infty) \cap C^1[0, +\infty)$ is a nonnegative, non-increasing function satisfying

$$b - \int_0^{+\infty} g(t)dt > 0.$$

Remark: We take the exponential decay kernel function $g(t) = ce^{-\delta t}$, $\delta > 0$, and the polynomial decay kernel function $g(t) = c\frac{1}{(1+t)^p}$, $p > 1$. As long as we choose the appropriate c, the kernel function g can satisfy assumptions $(A1), (A2)$, and $(A2')$. That is, $b - \int_0^{+\infty} g(t)dt = b - c\frac{1}{\delta} > 0$ and $b - \int_0^{+\infty} g(t)dt = b - c\frac{1}{p-1} > 0$.

We are interested to see whether the energy decays exponentially for systems (1.1) and (1.2). In order to apply some semigroup results to solve the problem, we first make some modifications to the original systems (1.1) and (1.2). Now let us introduce the following notation:

$$\eta^t(s) = u(t) - u(t - s), \quad t \geq 0, s \in (0, +\infty). \tag{2.1}$$

Then, proceeding formally, we obtain

$$\begin{cases} \eta_t + \eta_s = u_t, & (t, s) \in (0, +\infty) \times (0, +\infty), \\ \eta^0(s) = u^0(0) - u^0(s), & s \in (0, +\infty), \\ \eta^t(0) = \lim\limits_{s \to 0^+} \eta^t(s) = 0, & t \in [0, +\infty). \end{cases}$$

Therefore, systems (1.1) and (1.2) can be rewritten as the autonomous system

$$\begin{cases} u_{tt} + (b - \int_0^{+\infty} g(s)ds)Au + \int_0^{+\infty} g(s)A\eta^t(s)ds + \alpha u + \beta A^\theta v = 0, \\ v_{tt} + Av + \beta A^\theta u = 0, \\ \eta_t + \eta_s - u_t = 0, \end{cases} \tag{2.2}$$

and

$$\begin{cases} u_{tt} + (b - \int_0^{+\infty} g(s)ds)Au + \int_0^{+\infty} g(s)A\eta^t(s)ds + \alpha u + \beta A^\theta v_t = 0, \\ v_{tt} + Av - \beta A^\theta u_t = 0, \\ \eta_t + \eta_s - u_t = 0, \end{cases} \tag{2.3}$$

with initial condition

$$u(0) = u_0, u_t(0) = u_1, v(0) = v_0, v_t(0) = v_1,$$

and

$$\eta^0(s) = \eta_0(s), s \in (0, +\infty); \eta^t(0) = 0, t \in [0, +\infty),$$

where

$$u_0 = u^0(0), u_1 = \partial_t u^0(t)|_{t=0}, \eta_0(s) = u^0(0) - u^0(s), s \in (0, +\infty).$$

For our convenience, we transform system (2.2) into first-order equations. Therefore, defining

$$\Phi := (v, v_t(t), u, u_t(t), \eta)^T, \ \ b_1 = b - \int_0^{+\infty} g(s)ds,$$

we then have

$$\frac{d\Phi}{dt} = \mathfrak{U}\Phi, \tag{2.4}$$

where $\mathfrak{U}$ is given by

$$\mathfrak{U} = \begin{pmatrix} 0 & I(\cdot) & 0 & 0 & 0 \\ -A(\cdot) & 0 & -\beta A^\theta(\cdot) & 0 & 0 \\ 0 & 0 & 0 & I(\cdot) & 0 \\ -\beta A^\theta(\cdot) & 0 & (-\alpha - b_1 A)(\cdot) & 0 & -\int_0^{+\infty} g(s)A(\cdot)ds \\ 0 & 0 & 0 & I(\cdot) & -\frac{\partial}{\partial s}(\cdot) \end{pmatrix}. \tag{2.5}$$

We note $L_g^2(\mathbb{R}^+, D(A^{\frac{1}{2}}))$ as the Hilbert space of any function $\varphi \in D(A^{\frac{1}{2}})$, $\psi \in D(A^{\frac{1}{2}})$ on $\mathbb{R}^+$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L_g^2(\mathbb{R}^+, D(A^{\frac{1}{2}}))} = \int_0^{+\infty} g(s)\langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi \rangle ds. \tag{2.6}$$

Also, we introduce the product Hilbert spaces

$$\mathcal{H} = H \times H \times H \times H \times L_g^2(\mathbb{R}^+, D(A^{\frac{1}{2}})). \tag{2.7}$$

Furthermore, according to the hypothesis of (A2), $\mathcal{H}$ can be endowed with the following inner product:

$$\begin{aligned}\left\langle \Psi^1, \Psi^2 \right\rangle_{\mathcal{H}} &= \left\langle A^{\frac{1}{2}}\psi_1^1, A^{\frac{1}{2}}\psi_1^2 \right\rangle + \left\langle \psi_2^1, \psi_2^2 \right\rangle + b\left\langle A^{\frac{1}{2}}\psi_3^1, A^{\frac{1}{2}}\psi_3^2 \right\rangle + \left\langle \psi_4^1, \psi_4^2 \right\rangle + \alpha\left\langle \psi_3^1, \psi_3^2 \right\rangle \\ &\quad +\beta\left\langle A^{\theta}\psi_1^1, \psi_3^2 \right\rangle + \beta\left\langle \psi_3^2, A^{\theta}\psi_1^1 \right\rangle + \left\langle \psi_5^1, \psi_5^2 \right\rangle_{L_g^2(\mathbb{R}^+, D(A^{\frac{1}{2}}))},\end{aligned}$$

where $\Psi^1 = \left(\psi_1^1, \psi_2^1, \psi_3^1, \psi_4^1, \psi_5^1\right)^T$, $\Psi^2 = \left(\psi_1^2, \psi_2^2, \psi_3^2, \psi_4^2, \psi_5^2\right)^T$, and the associated norm $\|\bullet\|_{\mathcal{H}}$ is equivalent to the usual one.

Then, the above inner product and Eq (2.4) yield that

$$Re\left\langle \mathfrak{U}\Phi, \Phi \right\rangle = \frac{1}{2}\int_0^{+\infty} g'(s)\left\|A^{\frac{1}{2}}\eta\right\|^2 ds \le 0.$$

Thus, from the Lumer-Phillips theorem, $\mathfrak{U}$ is the infinitesimal generator of C_0 semigroup $S(t)$ of contractions in $\mathcal{H}$. Then, the Cauchy problem of Eq (2.4) is well-posed and the global solution exists and is unique.

Similarly, if we denote

$$\mathcal{U} = \begin{pmatrix} 0 & I(\cdot) & 0 & 0 & 0 \\ -A(\cdot) & 0 & 0 & \beta A^{\theta}(\cdot) & 0 \\ 0 & 0 & 0 & I(\cdot) & 0 \\ 0 & -\beta A^{\theta}(\cdot) & (-\alpha - b_1 A)(\cdot) & 0 & -\int_0^{+\infty} g(s)A(\cdot)ds \\ 0 & 0 & 0 & I(\cdot) & -\frac{\partial}{\partial s}(\cdot) \end{pmatrix}, \tag{2.8}$$

system (2.3) becomes the following first-order equations:

$$\frac{d\Phi}{dt} = \mathcal{U}\Phi. \tag{2.9}$$

Furthermore, according to the hypothesis (A2)′, $\mathcal{H}$ can be endowed with another equivalent inner product

$$\begin{aligned}\left\langle \Psi^1, \Psi^2 \right\rangle_{\mathcal{H}} &= \left\langle A^{\frac{1}{2}}\psi_1^1, A^{\frac{1}{2}}\psi_1^2 \right\rangle + \left\langle \psi_2^1, \psi_2^2 \right\rangle + b\left\langle A^{\frac{1}{2}}\psi_3^1, A^{\frac{1}{2}}\psi_3^2 \right\rangle + \left\langle \psi_4^1, \psi_4^2 \right\rangle + \alpha\left\langle \psi_3^1, \psi_3^2 \right\rangle \\ &\quad + \left\langle \psi_5^1, \psi_5^2 \right\rangle_{L_g^2(\mathbb{R}^+, D(A^{\frac{1}{2}}))}.\end{aligned}$$

It can be seen that the present associated norm $\| \bullet \|_{\mathcal{H}}$ is also equivalent to the usual one.

Meanwhile, using the above inner product and Eq (2.9) yields that

$$Re\left\langle \mathcal{U}\Phi, \Phi \right\rangle = \frac{1}{2}\int_0^{+\infty} g'(s)\left\|A^{\frac{1}{2}}\eta\right\|^2 ds \le 0.$$

Therefore, $\mathcal{U}$ is the infinitesimal generator of C_0 semigroup $\overline{S}(t)$ of contractions on $\mathcal{H}$. Then, the Cauchy problem of Eq (2.9) is well-posed and the global solution exists and is unique.

In the following, we illustrate that systems (2.2) and (2.3) are nonexponential decays. First, we recall some helpful conclusions.

Lemma 2.1. *[21, 22] The spectrum set of the self-adjoint operator A is nonempty, and all of the spectrum is an approximate point spectrum. Moreover, if $\lambda \in \mathbb{C}$ and $\mathrm{Im}\lambda \neq 0$, then $\lambda \in \rho(A)$, where $\rho(A)$ represents the resolvent set of A (see [22, page 159]).*

See details in [22, Corollarys 4.2.21 and 4.2.22 and Proposition 4.2.26].

Lemma 2.2. *[1, 23, 24] A C_0 semigroup of contractions in a Hilbert space, with generator G, is exponentially stable if and only if $\rho(G) \supset i\mathbb{R}$ and $\sup\left\{\left\|(i\gamma I - G)^{-1}\right\|; \gamma \in \mathbb{R}\right\} < \infty$, where $\rho(G)$ represents the resolvent set of G (see [22, page 159]).*

Theorem 2.3. *Assume that conditions (A1) and (A2) are satisfied. Then,*

- *if $0 \leq \theta < \frac{1}{2}$, the semigroup $S(t)$ associated to system (2.4) is not exponentially stable;*
- *if $\frac{1}{2} \leq \theta < \frac{3}{4}$ and $b \neq 1$, the semigroup $S(t)$ associated to system (2.4) is not exponentially stable.*

Proof. To prove this theorem, according to Lemma 2.2, we shall need to find a bounded sequence $F_n = (i\lambda_n I - \mathfrak{U})\Phi_n$ in $\mathcal{H}$, a sequence $\lambda_n \in \mathbb{R}$, and a sequence Φ_n satisfying that

$$|\lambda_n|,\ \|\Phi_n\|_{\mathcal{H}} \to +\infty,\ \ as\ \ n \to +\infty.$$

Denote $F = (f_1, f_2, f_3, f_4, f_5)^T$ and $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \eta)^T$. The resolvent equation $(i\lambda I - \mathfrak{U})\Phi = F$ of system (2.4) is given as

$$\begin{aligned}
&i\lambda\varphi_1 - \varphi_2 = f_1,\\
&i\lambda\varphi_2 + A\varphi_1 + \beta A^\theta\varphi_3 = f_2,\\
&i\lambda\varphi_3 - \varphi_4 = f_3,\\
&i\lambda\varphi_4 + \beta A^\theta\varphi_1 + b_1 A\varphi_3 + \alpha\varphi_3 + \int_0^{+\infty} g(s)A\eta ds = f_4,\\
&i\lambda\eta + \eta_s - \varphi_4 = f_5.
\end{aligned}$$

Let us choose $f_1 = f_3 = f_5 = 0$. Then, we get

$$\varphi_2 = i\lambda\varphi_1, \varphi_4 = i\lambda\varphi_3. \tag{2.10}$$

Putting them into the above equations, we can eliminate φ_2 and φ_4 as follows:

$$-\lambda^2\varphi_1 + A\varphi_1 + \beta A^\theta\varphi_3 = f_2, \tag{2.11}$$

$$-\lambda^2\varphi_3 + \beta A^\theta\varphi_1 + b_1 A\varphi_3 + \alpha\varphi_3 + \int_0^t g(s)A\eta ds = f_4, \tag{2.12}$$

$$i\lambda\eta + \eta_s - i\lambda\varphi_3 = 0. \tag{2.13}$$

From Lemma 2.1, we know the spectrum of the self-adjoint operator is an approximate point spectrum and the spectrum set is nonempty. Therefore, if we note $0 < \lambda_n \in \sigma(A)$, there exist sequences $\{\omega_n^m\}$, $\|\omega_n^m\| = 1$, such that $\lim\limits_{m\to+\infty} \|(A - \lambda_n I)\omega_n^m\| = 0$. Furthermore, we have $0 < \lambda_n^\theta \in \sigma(A^\theta)$ and $\lim\limits_{m\to+\infty} \|(A^\theta - \lambda_n^\theta I)\omega_n^m\| = 0$, where $\sigma(A^\theta)$ represents the spectral set of A^θ (see [22, page 159]).

Now we look for solutions of the form

$$\varphi_1 = a_n\omega_n^m, \qquad \varphi_3 = c_n\omega_n^m, \qquad \eta = \gamma(s)\omega_n^m, \tag{2.14}$$

where we choose $f_2 = f_4 = \omega_n^m$ and take the limits for Eqs (2.11)–(2.13) as $m \to +\infty$, so we have

$$-\lambda^2 a_n + a_n\lambda_n + \beta c_n\lambda_n^\theta = 1, \tag{2.15}$$

$$-\lambda^2 c_n + \beta a_n\lambda_n^\theta + b_1 c_n\lambda_n + \alpha c_n + \lambda_n \int_0^t g(s)\gamma(s)ds = 1, \tag{2.16}$$

$$i\lambda\gamma(s) + \gamma'(s) - i\lambda c_n = 0. \tag{2.17}$$

Since $\eta(0) = 0$, we get $\gamma(s) = c_n - c_n e^{-i\lambda s}$ according to Eq (2.17). We choose $\lambda = \lambda_n^{\frac{1}{2}}$ and substitute it into Eqs (2.15) and (2.16). Then, we have

$$c_n = \frac{1}{\beta\lambda_n^\theta}, \tag{2.18}$$

$$a_n = \frac{1}{\beta\lambda_n^\theta} - \frac{\alpha}{\beta^2\lambda_n^{2\theta}} - \frac{(b-1)\lambda_n}{\beta^2\lambda_n^{2\theta}} + \frac{\lambda_n}{\beta^2\lambda_n^{2\theta}} \int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds. \tag{2.19}$$

Then, substituting Eqs (2.18) and (2.19) into Eqs (2.10) and (2.14), we get

$$\varphi_1 = \left(\frac{1}{\beta\lambda_n^\theta} - \frac{\alpha}{\beta^2\lambda_n^{2\theta}} - \frac{(b-1)\lambda_n^{1-2\theta}}{\beta^2} + \frac{\lambda_n^{1-2\theta}}{\beta^2}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right)\omega_n,$$

$$\begin{aligned}\varphi_2 &= i\lambda_n^{\frac{1}{2}}\varphi_1 \\ &= i\lambda_n^{\frac{1}{2}}\left(\frac{1}{\beta\lambda_n^\theta} - \frac{\alpha}{\beta^2\lambda_n^{2\theta}} - \frac{(b-1)\lambda_n}{\beta^2\lambda^{2\theta}} + \frac{\lambda_n}{\beta^2\lambda_n^{2\theta}}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right)\omega_n \\ &= i\left(\frac{1}{\beta}\lambda_n^{\frac{1}{2}-\theta} - \frac{\alpha}{\beta^2}\lambda_n^{\frac{1}{2}-2\theta} - \frac{(b-1)\lambda_n^{\frac{3}{2}-2\theta}}{\beta^2} + \frac{\lambda_n^{\frac{3}{2}-2\theta}}{\beta^2}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right)\omega_n,\end{aligned}$$

$$\varphi_3 = \frac{1}{\beta\lambda_n^\theta}\omega_n,$$

$$\varphi_4 = i\lambda_n^{\frac{1}{2}}\varphi_3 = i\lambda_n^{\frac{1}{2}-\theta}\frac{1}{\beta}\omega_n.$$

Therefore, we have
Case 1: For $0 \le \theta < \frac{1}{2}$, it is easy to check that

$$\lim_{\lambda_n\to\infty} \|\varphi_4\| = \frac{1}{\beta}\lim_{\lambda_n\to\infty} \lambda_n^{\frac{1}{2}-\theta} = +\infty.$$

Thus,

$$\lim_{\lambda_n\longrightarrow\infty} \|\Phi\| \ge \lim_{\lambda_n\longrightarrow\infty} \|\varphi_4\| = +\infty. \tag{2.20}$$

The necessary condition (see Lemma 2.2) for the system to have exponential decay is $\overline{\lim_{\lambda_n\to\infty}}\|\Phi\| < \infty$. Thus, in this case the theorem holds.

Case 2: For $\frac{1}{2} \le \theta < \frac{3}{4}$, we find

$$\lim_{\lambda_n\to\infty}\|\varphi_1\|,\quad \lim_{\lambda_n\to\infty}\|\varphi_3\|,\quad \lim_{\lambda_n\to\infty}\|\varphi_4\| < +\infty.$$

Moreover, observe

$$\begin{aligned}
&\lim_{\lambda_n\to\infty}\left|\frac{\lambda_n^{\frac{3}{2}-2\theta}}{\beta^2}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right| \\
=&\lim_{\lambda_n\to\infty}\left|\frac{\lambda_n^{\frac{3}{2}-2\theta}}{-i\lambda_n^{\frac{1}{2}}\beta^2}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}d\left(-i\lambda_n^{\frac{1}{2}}s\right)\right| \\
\le&\frac{1}{\beta^2}g(0)\lim_{\lambda_n\to\infty}\lambda_n^{1-2\theta}\left|e^{-i\lambda_n^{\frac{1}{2}}t}-1\right| \le \frac{2}{\beta^2}g(0).
\end{aligned}$$

Then, if $b = 1$,

$$\lim_{\lambda_n\to\infty}\|\varphi_2\| < +\infty.$$

On the other hand, if $b \neq 1$,

$$\lim_{\lambda_n\to\infty}\left|\frac{(b-1)\lambda_n^{\frac{3}{2}-2\theta}}{\beta^2}\right| = +\infty.$$

Therefore, if $b \neq 1$,

$$\lim_{\lambda_n\longrightarrow\infty}\|\Phi\| \ge \lim_{\lambda_n\longrightarrow\infty}\|\varphi_2\| = +\infty. \tag{2.21}$$

Thus, case 2 in the theorem also holds, and the proof of the theorem is complete.

Remark 2.4. *In Theorem 2.3, we suppose that A is an unbounded operator, and we know the bounded operator's spectrum radius is finite [23]. Then, we can easily give an example when A is a bounded operator, and the system is exponentially stable when* $b \neq 1$. *For example, A is the identity operator.*

Theorem 2.5. *Assume that conditions (A1) and (A2)′ are satisfied. Then, if* $0 \le \theta < \frac{1}{4}$ *and* $b \neq 1$, *the semigroup* $\overline{S}(t)$ *associated with system* (2.9) *is not exponentially stable.*

Proof. **Step 1.** Simplification of the resolvent equation:

The proof of this theorem is similar to that of Theorem 2.3. We also apply Lemma 2.2, which means that if we can find a bounded sequence $F_n = (i\lambda_n I - \mathfrak{U})\Phi_n$ in $\mathcal{H}$, a sequence $\lambda_n \in \mathbb{R}$, and a sequence Φ_n that satisfies

$$|\lambda_n|,\ \|\Phi_n\|_{\mathcal{H}} \to +\infty,\quad as\ \ n \to +\infty,$$

we can conclude that system (2.9) lacks exponential decay.

Denote $F = (f_1, f_2, f_3, f_4, f_5)^T$ and $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \eta)^T$. The resolvent equation $(i\lambda I - \mathcal{U})\Phi = F$ of system (2.9) is given by

$$\begin{aligned} &i\lambda\varphi_1 - \varphi_2 = f_1,\\ &i\lambda\varphi_2 + A\varphi_1 - \beta A^\theta\varphi_4 = f_2,\\ &i\lambda\varphi_3 - \varphi_4 = f_3,\\ &i\lambda\varphi_4 + \beta A^\theta\varphi_2 + b_1 A\varphi_3 + \alpha\varphi_3 + \int_0^{+\infty} g(s)A\eta ds = f_4,\\ &i\lambda\eta + \eta_s - \varphi_4 = f_5. \end{aligned}$$

Let us choose $f_1 = f_3 = f_5 = 0$. Then, we get

$$\varphi_2 = i\lambda\varphi_1, \varphi_4 = i\lambda\varphi_3. \tag{2.22}$$

Putting these into the above equations, we can eliminate φ_2 and φ_4 as follows:

$$-\lambda^2\varphi_1 + A\varphi_1 - \beta A^\theta\varphi_4 = f_2, \tag{2.23}$$

$$-\lambda^2\varphi_3 + \beta A^\theta\varphi_2 + b_1 A\varphi_3 + \alpha\varphi_3 + \int_0^t g(s)A\eta ds = f_4, \tag{2.24}$$

$$i\lambda\eta + \eta_s - i\lambda\varphi_3 = 0. \tag{2.25}$$

Step 2. Construction of special solutions:

From Lemma 2.1, we know the spectrum of the self-adjoint operator is an approximate point spectrum and the spectrum set is nonempty. Therefore, if we note $0 < \lambda_n \in \sigma(A)$, there exist sequences $\{\omega_n^m\}$, $\|\omega_n^m\| = 1$, such that $\lim\limits_{m\to+\infty} \|(A - \lambda_n I)\omega_n^m\| = 0$. Furthermore, we have $0 < \lambda_n^\theta \in \sigma(A^\theta)$, and $\lim\limits_{m\to+\infty} \|(A^\theta - \lambda_n^\theta I)\omega_n^m\| = 0$.
Now we look for solutions of the form

$$\varphi_1 = a_n\omega_n^m, \qquad \varphi_3 = c_n\omega_n^m, \qquad \eta = \gamma(s)\omega_n^m, \tag{2.26}$$

where we choose $f_2 = f_4 = \omega_n^m$ and take the limits for Eqs (2.23)–(2.25) as $m \to +\infty$, so we have

$$-\lambda^2 a_n + a_n\lambda_n - i\lambda\beta c_n\lambda_n^\theta = 1, \tag{2.27}$$

$$-\lambda^2 c_n + i\lambda\beta a_n\lambda_n^\theta + b_1 c_n\lambda_n + \alpha c_n + \lambda_n\int_0^t g(s)\gamma(s)ds = 1, \tag{2.28}$$

$$i\lambda\gamma(s) + \gamma'(s) - i\lambda c_n = 0. \tag{2.29}$$

Since $\eta(0) = 0$, we get $\gamma(s) = c_n - c_n e^{-i\lambda s}$ from Eq (2.29). Choosing $\lambda = \lambda_n^{\frac{1}{2}}$ and substituting it into Eqs (2.27) and (2.28), we then have

$$ic_n = \frac{-1}{\beta\lambda_n^{\frac{1}{2}+\theta}}, \tag{2.30}$$

$$ia_n = \frac{1}{\beta\lambda_n^{\frac{1}{2}+\theta}} + \frac{\alpha}{\beta^2\lambda_n^{\frac{1}{2}+2\theta}} + \frac{(b-1)}{\beta^2\lambda_n^{2\theta}} - \frac{1}{\beta^2\lambda_n^{2\theta}}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds. \tag{2.31}$$

Then, substituting Eqs (2.30) and (2.31) into Eqs (2.22) and (2.26), we get

$$i\varphi_1 = \left(\frac{1}{\beta\lambda_n^{\frac{1}{2}+\theta}} + \frac{\alpha}{\beta^2\lambda_n^{\frac{1}{2}+2\theta}} + \frac{(b-1)}{\beta^2\lambda_n^{2\theta}} - \frac{1}{\beta^2\lambda_n^{2\theta}}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right)\omega_n,$$

$$\begin{aligned}\varphi_2 &= i\lambda_n^{\frac{1}{2}}\varphi_1\\ &= \lambda_n^{\frac{1}{2}}\left(\frac{1}{\beta\lambda_n^{\frac{1}{2}+\theta}} + \frac{\alpha}{\beta^2\lambda_n^{\frac{1}{2}+2\theta}} + \frac{(b-1)}{\beta^2\lambda_n^{2\theta}} - \frac{1}{\beta^2\lambda_n^{2\theta}}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right)\omega_n\\ &= \left(\frac{1}{\beta}\lambda_n^{-\theta} + \frac{\alpha}{\beta^2}\lambda_n^{-2\theta} + \frac{(b-1)\lambda_n^{\frac{1}{2}-2\theta}}{\beta^2} - \frac{\lambda_n^{\frac{1}{2}-2\theta}}{\beta^2}\int_0^t g(s)e^{-i\lambda_n^{\frac{1}{2}}s}ds\right)\omega_n,\end{aligned}$$

$$i\varphi_3 = \frac{1}{\beta\lambda_n^{\frac{1}{2}+\theta}}\omega_n,$$

$$\varphi_4 = i\lambda_n^{\frac{1}{2}}\varphi_3 = \lambda_n^{-\theta}\frac{1}{\beta}\omega_n.$$

Therefore, if $0 \le \theta < \frac{1}{4}$ and $b \neq 1$, it is easy to check that

$$\lim_{\lambda_n\to\infty} \|\varphi_2\| = +\infty.$$

Thus,

$$\lim_{\lambda_n\longrightarrow\infty} \|\Phi\| \ge \lim_{\lambda_n\longrightarrow\infty} \|\varphi_2\| = +\infty.$$

However, the necessary condition (see Lemma 2.2) for the system to have exponential decay is $\overline{\lim_{\lambda_n\to\infty}} \|\Phi\| < \infty$. Therefore, the proof of the theorem is complete.

3. Applications

In this section, let us present several examples showing how to apply our abstract theory (Theorems 2.3 and 2.5) to specific problems.

We first apply our abstract results to the two models (1.3) and (1.4) mentioned in Section 1. Denote

$$\begin{aligned}&A : \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega),\\ &u \mapsto Au = -\Delta u.\end{aligned}$$

Then, system (2.2) becomes Eq (1.3) with $\theta = 0$. It can be seen from Theorem 2.3 that our results here cover the related results in [13].

Moreover, if we take

$$\beta = \frac{b}{\sqrt{\rho J}}, \bar{u} = \varphi, \bar{v} = i\sqrt{\frac{\rho}{J}}u, g(s) = \frac{g_1'(s)}{J}, \alpha = \frac{\xi}{J}, \bar{b} = \frac{\rho(\delta + g_1(0))}{\mu J},$$

and

$$A : \mathcal{D}(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi),$$
$$u \mapsto A\bar{u} = -\frac{\mu}{\rho}\bar{u}_{xx},$$

then the system (1.4) becomes

$$\begin{cases} \bar{u}_{tt} + \bar{b}A\bar{u} - \int_0^{+\infty} g(s)A\bar{u}(t-s)ds + \alpha\bar{u} + \beta A^{\frac{1}{2}}\bar{v} = 0, \\ \bar{v}_{tt} + A\bar{v} + \beta A^{\frac{1}{2}}\bar{u} = 0. \end{cases} \tag{3.1}$$

This satisfies system (2.2) for $\bar{u}, \bar{v}$, and $\bar{b}$. So, using Theorem 2.3, we can obtain the above system (3.1), i.e., system (1.4) does not have exponential decay for $\bar{b} \neq 1$ (i.e., $\frac{\rho(\delta+g_1(0))}{\mu J} \neq 1$). In this way, we also establish that the necessary condition of system (1.4) for exponential stability is $\frac{\delta+g_1(0)}{J} = \frac{\mu}{\rho}$, which corresponds to the case of equal wave speeds.

Now let us give more applications of our abstract results.

Example 3.1. *Let us consider the Timoshenko system*

$$\begin{cases} \rho_2\psi_{tt} - k_2\psi_{xx} + \int_0^{+\infty} g_1(s)\psi_{xx}(t-s)ds + k_1\psi + k_2\phi_x = 0, \ in\ (0,l)\times(0,+\infty), \\ \rho_1\phi_{tt} - k_1\phi_{xx} - k_1\psi_x = 0, \ in\ (0,l)\times(0,+\infty), \\ \psi(0,t) = \psi(l,t) = \phi(0,t) = \phi(l,t) = 0, \quad t \geq 0. \end{cases} \tag{3.2}$$

Set

$$u = \psi, \ v = i\sqrt{\frac{\rho_1}{\rho_2}}\phi, \ g(s) = \frac{g_1(s)}{k_2}, \ b = \frac{\rho_1 k_2}{\rho_2 k_1}, \ \alpha = \frac{k_1}{\rho_2}, \ \beta = \sqrt{\frac{k_1}{\rho_2}},$$

and

$$A : \mathcal{D}(A) = H^2(0,l) \cap H_0^1(0,l) \to L^2(0,l),$$
$$u \mapsto Au = -\frac{k_1}{\rho_1}u_{xx}.$$

From [15, Example 3.4], we can transfer the above Timoshenko system (3.2) *into our abstract system* (1.1) *with* $\theta = \frac{1}{2}$.

Thus, the necessary condition of Timoshenko system (3.2) *for exponential stability is* $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$, *which corresponds to the case of equal wave speeds, which was proved in [25, 26].*

As can be seen from the above three examples, our abstract results cover the previous results. Now we apply our abstract results to several other coupled systems, which we have not considered in the past.

Example 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}^+$). Take $H = L^2(0, \Omega)$. Denote*

$$\begin{aligned} &A : \mathcal{D}(B) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega), \\ &u \mapsto Bu = -\Delta u, \end{aligned}$$

and $A = B^2$.

- *Consider the coupled system of Petrovsky type equations*

$$\begin{cases} u_{tt} + b\Delta^2 u - \displaystyle\int_0^{+\infty} g(s)\Delta^2 u(t-s)ds + \alpha u - \beta\Delta v = 0, \ \text{in } \Omega \times (0, +\infty), \\ v_{tt} + \Delta^2 v - \beta\Delta u = 0, \ \text{in } \Omega \times (0, +\infty), \\ u = v = \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \ \text{on } \partial\Omega \times [0, +\infty), \end{cases} \tag{3.3}$$

  where $B = A^{\frac{1}{2}}$ according to the definition of A and B. Then, Eq (3.3) *can be transformed into our abstract system* (2.2) *with $\theta = \frac{1}{2}$. From Theorem 2.3, if $b \neq 1$, system* (3.3) *does not have exponential decay. That is, the necessary condition of Eq* (3.3) *for exponential stability is $b = 1$.*
- *A coupled system*

$$\begin{cases} u_{tt} + b\Delta^2 u - \displaystyle\int_0^{+\infty} g(s)\Delta^2 u(t-s)ds + \alpha u + \beta v = 0, \ \text{in } \Omega \times (0, +\infty), \\ v_{tt} + \Delta^2 v + \beta u = 0, \ \text{in } \Omega \times (0, +\infty), \\ u = v = \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \ \text{on } \partial\Omega \times [0, +\infty), \end{cases} \tag{3.4}$$

  becomes our abstract system (2.2) *with $\theta = 0$. Then, for any $b > 0$, the above system* (3.4) *has no exponential decay.*

Example 3.3. *Consider the coupled wave equations*

$$\begin{cases} u_{tt} - b\Delta u + \displaystyle\int_0^{+\infty} g(s)\Delta u(t-s)ds + \alpha u + \beta v_t = 0, \ \text{in } \Omega \times (0, +\infty), \\ v_{tt} - \Delta v - \beta u_t = 0, \ \text{in } \Omega \times (0, +\infty), \\ u = v = 0, \ \text{on } \partial\Omega \times [0, +\infty). \end{cases} \tag{3.5}$$

Therefore, taking

$$\begin{aligned} &A : \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega), \\ &u \mapsto Au = -\Delta u, \end{aligned}$$

the above system (3.5) *becomes our abstract system* (2.3) *with $\theta = 0$. Then, Eq* (3.5) *has no exponential decay for $b \neq 1$ according to Theorem 2.5. That is, the necessary condition of Eq* (3.5) *for exponential stability is also $b = 1$.*

In conclusion, from the above examples it can be found that our abstract results (Theorems 2.3 and 2.5) not only cover the previous related results, but also generalize and improve the related results.

4. Conclusions

This paper investigates two classes of coupled second-order abstract evolution equations with indirect damping, where damping only acts on one equation in the system, and the coupling terms are composed of the fractional powers of the operator.

By using the semigroup method and operator theory, we systematically analyze and prove that under specific fractional power conditions, the system does not have exponential decay. Specifically, we establish sufficient conditions that systems (1.1) and (1.2) cannot achieve exponential stability within a specific parameter range, thus revealing the intrinsic relationship between the coupling term and the decay behavior of the system.

Based on theoretical analysis, we apply abstract results to several specific partial differential equation models, such as coupled wave equations, Timoshenko systems, and Petrovsky-type equations, etc., verifying that these models cannot achieve exponential decay under specific conditions. These applications not only demonstrate that our theoretical results encompass relevant conclusions from existing literature, but also extend and improve the understanding of the stability of complex coupled systems.

The significance of this study lies in providing a unified theoretical framework and a practical discriminant tool for the stability analysis of indirect damping coupled systems, enriching the stability theory of infinite-dimensional dynamical systems, and providing a theoretical basis for the design and optimization of damping mechanisms in related engineering and physical models.

Authors contribution

Both authors contributed equally to this work.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. T. J. Xiao, J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Springer, Berlin, Germany, 1998.

2. B. Feng, W. Youssef, T. El Arwadi, Polynomial and exponential decay rates of a laminated beam system with thermodiffusion effects, *J. Math. Anal. Appl.*, **517** (2023), 126633. https://doi.org/10.1016/j.jmaa.2022.126633

3. K. P. Jin, Stability of a class of coupled systems, *Abstr. Appl. Anal.*, **2014** (2014), 835765. https://doi.org/10.1155/2014/835765

4. F. Alabau-Boussouira, M. Lautaud, Indirect stabilization of locally coupled wave-type systems, *ESAIM Control Optim. Calc. Var.*, **18** (2012), 548–582. https://doi.org/10.1051/cocv/2011106

5. K. P. Jin, Stability for locally coupled wave-type systems with indirect mixed-type dampings, *Appl. Math. Optim.*, **85** (2022), 11. https://doi.org/10.1007/s00245-022-09830-x

6. K. P. Jin, Indirect stabilization of coupled abstract evolution equations with memory: Different propagation speeds, *Math. Methods Appl. Sci.*, **45** (2022), 4451–4467. https://doi.org/10.1002/mma.8048

7. K. P. Jin, J. Liang, T. J. Xiao, Asymptotic behavior for coupled systems of second order abstract evolution equations with one infinite memory, *J. Math. Anal. Appl.*, **475** (2019), 554–575. https://doi.org/10.1016/j.jmaa.2019.02.055

8. F. Ammar-Khodja, A. Benabdallah, J. E. Muñoz Rivera, R. Racke, Energy decay for Timoshenko systems of memory type, *J. Differ. Equ.*, **194** (2003), 82–115. https://doi.org/10.1016/S0022-0396(03)00185-2

9. M. I. Mustafa, Viscoelastic Timoshenko beams with variable-exponent nonlinearity, *J. Math. Anal. Appl.*, **516** (2022), 126520. https://doi.org/10.1016/j.jmaa.2022.126520

10. T. J. Xiao, J. Liang, Coupled second order semilinear evolution equations indirectly damped via memory effects, *J. Differ. Equ.*, **254** (2013), 2128–2157. https://doi.org/10.1016/j.jde.2012.11.019

11. A. Youkana, A. M. Al-Mahdi, S. A. Messaoudi, General energy decay result for a viscoelastic swelling porous-elastic system, *Z. Angew. Math. Phys.*, **73** (2022), 88. https://doi.org/10.1007/s00033-022-01696-x

12. I. Lasiecka, S. A. Messaoudi, M. Mustafa, Note on intrinsic decay rates for abstract wave equations with memory, *J. Math. Phys.*, **54** (2013), 031504. https://doi.org/10.1063/1.4793988

13. R. G. C. Almeida, M. L. Santos, Lack of exponential decay of a coupled system of wave equations with memory, *Nonlinear Anal. Real World Appl.*, **12** (2011), 1023–1032. https://doi.org/10.1016/j.nonrwa.2010.08.025

14. A. M. Al-Mahdi, M. M. Al-Gharabli, S. A. Messaoudi, New general decay of solutions in a porous-thermoelastic system with infinite memory, *J. Math. Anal. Appl.*, **500** (2021), 125136. https://doi.org/10.1016/j.jmaa.2021.125136

15. K. P. Jin, J. Liang, T. J. Xiao, Coupled second order evolution equations with fading memory: Optimal energy decay rate, *J. Differ. Equ.*, **257** (2014), 1501–1528. https://doi.org/10.1016/j.jde.2014.05.018

16. T. J. Xiao, Y. Zhang, Lack of exponential decay in viscoelastic materials with voids, *Syst. Control Lett.*, **63** (2014), 39–42. https://doi.org/10.1016/j.sysconle.2013.11.006

17. D. Chalishajar, D. Kasinathan, K. Ramkumar, K. Ravikumar, Viscoelastic Kelvin-Voigt model on Ulam-Hyer's stability and T-controllability for a coupled integro fractional stochastic systems with integral boundary conditions via integral contractors, *Chaos, Solitons Fractals*, **191** (2025), 115785. https://doi.org/10.1016/j.chaos.2024.115785

18. D. Chalishajar, D. Kasinathan, K. Ramkumar, K. Ravikumar, Exponential stability, T-controllability and optimal controllability of higher-order fractional neutral stochastic differential equation via integral contractor, *Chaos, Solitons Fractals*, **186** (2024), 115278. https://doi.org/10.1016/j.chaos.2024.115278

19. K. Ramkumar, K. Ravikumar, D. Chalishajar, Existence trajectory and optimal control of Clarke subdifferential stochastic integrodifferential inclusions suffered by non-instantaneous impulses and deviated arguments, *Results Control Optim.*, **13** (2023), 100295. https://doi.org/10.1016/j.rico.2023.100295

20. M. Akil, H. Badawi, S. Nicaise, A. Wehbe, Stability results of coupled wavw models with locally memory in a past history framework via nonsmooth coefficients on the interface, *Math. Methods Appl. Sci.*, **44** (2021), 6950–6981. https://doi.org/10.1002/mma.7235

21. W. Rudin, *Functional Analysis*, McGraw-Hill Science/Engineering/Math, 2nd edition, 1991.

22. J. Sun, Z. Wang, W. Wang, *Science Press*, in pectral Analysis of Linear Operators, 2 edition, Publishing House: Beijin, China, 2015.

23. L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, *Trans. Am. Math. Soc.*, **236** (1978), 385–394. https://doi.org/10.1090/S0002-9947-1978-0461206-1

24. F. L. Huang, Characteristic condition for exponential stability of linear dynamical systems in Hilbert space, *Ann. Differ. Equ.*, **1** (1985), 43–56.

25. S. A. Messaoudi, B. Said-Houari, Uniform decay in a Timoshenko-type system with past history, *J. Math. Anal. Appl.*, **360** (2009), 459–475. https://doi.org/10.1016/j.jmaa.2009.06.064

26. J. E. M. Rivera, H. D. F. Sare, Stability of Timoshenko systems with past history, *J. Math. Anal. Appl.*, **339** (2008), 482–502. https://doi.org/10.1016/j.jmaa.2007.07.012