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Research article

Three wave solution and lump-type solution to a (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation with some variable coefficients in inhomogeneous media

Wenjing Yang[1], **Kaili Liu**[1], **Bo Tang**[1,2,3,4,*], **Shijie Deng**[1] **and Fengyun Lv**[1]

[1] School of Mathematics and Statistics, Hubei University of Arts and Science, Xiangyang 441053, China

[2] Hubei Key Laboratory of Power System Design and Test for Electrical Vehicle, Hubei University of Arts and Science, Xiangyang 441053, China

[3] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China

[4] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shanxi 710049, China

* **Correspondence:** Email: tangbo0809@163.com

Abstract: This paper focuses on the new (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa (DJKM) equation with some variable coefficients and derives its new analytical solutions. Based on the (2+1)-dimensional DJKM equation, this equation adds an additional spatial dimension (z) and introduces time-dependent coefficients. Therefore, it is more suitable for describing dynamic wave behaviors in inhomogeneous media. Firstly, by virtue of the Hirota method, we derive the Hirota bilinear form of the equation. Secondly, based on the bilinear form, we present three types of solutions, including the three wave solution and lump-type solution. Finally, by choosing appropriate parameters, we plot some graphs to intuitively display the physical characteristics of these solutions.

Keywords: Date-Jimbo-Kashiwara-Miwa equation; three wave solution; lump-type solution

1. Introduction

Nonlinear evolution equations (NLEEs) have been used to describe complex nonlinear phenomena. Since the analytical solutions of the NLEEs have a major impact on investigating the physical aspects of the models, many scholars have been very interested in these analytical solutions [1–3]. Nowadays, a range of powerful methods have emerged to obtain analytical solutions, such as the Painlevé approach [4–6], Hirota's bilinear method [7–9] and the Darboux transformation method [10–12]. By applying those methods, some analytic solutions of the NLEEs have been obtained, such as N-soliton solutions [13], the breather solutions [14], the periodic wave solutions [15], and so on. However, traditional methods mainly rely on the locality assumption of integer-order derivatives. As a generalization of integer-order derivatives, fractional derivatives have opened up a broader space for the research on nonlinear evolution equations through their non-locality property [16, 17]. In the future, introducing fractional derivatives into the analytical solutions for NLEEs and exploring their new physical significance will be a greatly meaningful research direction.

Symbolic computation, with its powerful capabilities, has been widely applied across multiple domains including algebraic operations, equation solving, and matrix calculations. It can effectively make up for the deficiencies of manual calculation in terms of efficiency and accuracy, and has become an indispensable tool in related research. In practical research, many scholars have successfully solved a multitude of intricate mathematical and scientific problems by using symbolic computation [13, 18–20].

The Date-Jimbo-Kashiwara-Miwa (DJKM) equation is a significant class of nonlinear partial differential equations with extensive applications across multiple fields of physics. Initially, this equation was proposed to simulate the propagation laws of long-wavelength shallow water waves under conditions of low surface tension. Furthermore, in the field of plasma physics, the DJKM equation also played an important role, for example, it could effectively describe the propagation characteristics of nonlinear dispersive waves, such as the evolutionary processes of typical wave phenomena like ion acoustic waves and magnetohydrodynamic waves. In 1981, based on the Kadomtsev-Petviashvili (KP) equation, Date et al. derived the DJKM equation in [21]. In 2020, Wazwaz conducted further research on the DJKM equation; in [22], he extended the DJKM equation to a variable-coefficient form; in [23], he further generalized it to a (3+1)-dimensional DJKM equation. In 2021, Dong Wang et al. studied the bilinear form and N-soliton solutions of the DJKM equation with constant coefficients [24]. In 2022, Hengchun Hu et al. conducted the periodic wave and kink soliton solutions of the DJKM equation with constant coefficients [25]. To our knowledge, the three wave solution and lump-type solution for the new (3+1)-dimensional DJKM equation with variable coefficient have not been reported. In this paper, we focus on the study of the new (3+1)-dimensional DJKM equation with variable coefficient. By employing Mathematica software, we obtain a variety of analytical solutions for the equation.

In this paper, we concentrate on a new (3+1)-dimensional DJKM equation [23]:

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \alpha u_{yyy} - 2\beta g(t)u_{xxt} + h(t)(au_x + bu_y + cu_z)_{xx} = 0. \tag{1.1}$$

When $g(t) = 1$ and $h(t) = 1$, Eq (1.1) simplifies to the following equation [23]:

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \alpha u_{yyy} - 2\beta u_{xxt} + (au_x + bu_y + cu_z)_{xx} = 0. \tag{1.2}$$

This paper primarily investigates some analytical solutions of Eq (1.1), and the results obtained also encompass the solutions for Eq (1.1).

Using variable transformation:

$$u(x, y, z, t) = 2(\ln f(x, y, z, t))_x, \tag{1.3}$$

we can obtain the Hirota bilinear form of the (3+1)-dimensional DJKM equation:

$$D_x\left[\left(D_x^3D_y-3\beta g\left(t\right)D_xD_t+\frac{3}{2}h\left(t\right)\left(aD_x^2+bD_xD_y+cD_xD_z\right)\right)f\cdot f\right]\cdot f^2+$$
$$\frac{1}{2}D_y\left[\left(D_x^4-3\alpha D_y^2\right)f\cdot f\right]\cdot f^2=0, \tag{1.4}$$

where the definition of the general bilinear operator D can be expressed as [26]:

$$D_t^mD_x^nf\cdot g=(\partial_t-\partial_{t'})^m(\partial_x-\partial_{x'})^nf(x,t)g(x',t')\Big|_{x'=x,t'=t}.$$

In consequence, the (3+1)-dimensional DJKM equation can be converted to the following form:

$$(-2\alpha f_y^3-4\beta g(t)f_x^2f_t+2ch(t)f_zf_x^2+2bh(t)f_yf_x^2+2ah(t)f_x^3-4f_xf_{xy}f_{xx}-2f_yf_{xx}^2+4f_{xxy}f_x^2+4f_yf_xf_{xxx})+$$
$$f(3\alpha f_yf_{yy}+4\beta g(t)f_xf_{xt}-2ch(t)f_xf_{xz}-2bh(t)f_xf_{xy}+2\beta g(t)f_tf_{xx}-ch(t)f_zf_{xx}-bh(t)f_yf_{xx}-3ah(t)f_xf_{xx}$$
$$+2f_{xx}f_{xxy}-4f_xf_{xxxy}-f_yf_{xxxx})+f^2(-\alpha f_{yyy}-2\beta g(t)f_{xxt}+ch(t)f_{xxz}+bh(t)f_{xxy}+ah(t)f_{xxx}+f_{xxxxy})=0. \tag{1.5}$$

This article primarily seeks some new solutions of Eq (1.1) by utilizing the Hirota's bilinear method. This paper is divided into five sections. In the first section, we introduced the (3+1)-dimensional DJKM equations and their Hirota bilinear form. Using the Hirota's bilinear method and Mathematica software, we calculated the three wave solution and lump-type solution of Eq (1.1) respectively in Sections 2 and 3. We also used Matlab to create some graphs; these graphs provide us with different perspectives on the solutions, enabling us to understand the nature of the solutions more comprehensively. In the fifth section, some conclusions are given.

2. Three wave solution

In this section, we will acquire the three wave solution for the new (3+1)-dimensional DJKM equation. First, we can start with a hypothesis:

$$f=e^{\sigma_1}+k_1\cos\left(\sigma_2\right)+k_2e^{-\sigma_1}+k_3\sin\left(\sigma_3\right)$$
$$\sigma_i=a_ix+b_iy+c_iz+d_i(t), \tag{2.1}$$

where $i=1,2,3$, $d_1(t)$, $d_2(t)$, $d_3(t)$ are undetermined functions of t.

Substituting Eq (2.1) into Eq (1.5), we can obtain a set of equations. Solving these equations, we can get the three wave solution.

Theorem 2.1. The three wave solution for Eq (1.1) can be simplified as:

$$u=\frac{2\left(a_1\left(e^{\sigma_1}-k_2e^{-\sigma_1}\right)-a_2k_1\sin\left(\sigma_2\right)+a_3k_3\cos\left(\sigma_3\right)\right)}{k_2e^{-\sigma_1}+k_3\sin\left(\sigma_3\right)+k_1\cos\left(\sigma_2\right)+e^{\sigma_1}}, \tag{2.2}$$

under the following constraint conditions:

$$
\begin{aligned}
d_1(t) &= \int \frac{\left(a_1^2+a_2^2\right)^2 A_1 h(t) + \left(a_1^2+a_2^2\right)^2 m_2 + \alpha m_3}{m_1 g(t)}\,dt,\\
d_2(t) &= \int \frac{\left(a_1^2+a_2^2\right)^2 A_2 h(t) + \left(a_1^2+a_2^2\right)^2 m_4 + \alpha m_5}{m_1 g(t)}\,dt,\\
d_3(t) &= \int \frac{2a_2a_3\left(a_1^2+a_2^2\right)^2 A_3 h(t) - \left(a_1^2+a_2^2\right)^2 m_6 + \alpha m_7}{2a_2a_3 m_1 g(t)}\,dt,\\
k_2 &= -\frac{\left(a_2^2-a_3^2\right)\left(a_1^2k_1^2+a_2^2k_1^2-a_1^2k_3^2-a_3^2k_3^2\right)}{4\left(a_1^2+a_2^2\right)\left(a_1^2+a_3^2\right)},\\
b_1 &= \frac{\alpha a_1a_2b_2 - \sqrt{-\alpha a_3^2a_2^6 - 2\alpha a_1^2a_3^2a_2^4 - \alpha a_1^4a_3^2a_2^2}}{\alpha a_2^2},\\
b_3 &= \frac{-\frac{a_1a_3^2\sqrt{-\alpha a_2^2\left(a_1^2+a_2^2\right)^2a_3^2}}{\alpha a_2^2} + \frac{a_1\sqrt{-\alpha a_2^2\left(a_1^2+a_2^2\right)^2a_3^2}}{\alpha} + a_2a_3^2b_2 + \frac{a_1^2a_3^2b_2}{a_2}}{a_3a_1^2+a_2^2a_3},
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
m_1 &= 2\beta\left(a_1^2+a_2^2\right)^2, \qquad m_2 = \left(a_1^2-a_2^2\right)b_1 - 2a_1a_2b_2, \qquad A_i = a\,a_i + b\,b_i + c\,c_i,\\
m_3 &= b_1\left(a_2^2-a_1^2\right)\left(b_1^2-3b_2^2\right) + 2a_1a_2b_2\left(b_2^2-3b_1^2\right), \qquad m_4 = \left(a_1^2-a_2^2\right)b_2 + 2a_1a_2b_2,\\
m_5 &= b_2\left(a_1^2-a_2^2\right)\left(b_2^2-3b_1^2\right) + 2a_1a_2b_1\left(b_1^2-3b_2^2\right),\\
m_6 &= \left(a_2^2+a_3^2\right)\left(2a_1a_2b_1+4a_2a_3b_3\right) + b_2\left(a_1^2a_2^4 + a_3^4 + \left(a_1^2+5a_2^2\right)a_3^2\right),\\
m_7 &= b_2\left[a_3^2\left(a_2^2-a_1^2\right)\left(b_2^2-3b_1^2\right) + 3a_2^2\left(a_1^2-a_2^2\right)\left(b_1^2+b_3^2\right) - a_1^2\left(3a_1^2+a_2^2\right)\left(b_2^2+3b_3^2\right)\right]\\
&\quad - 2a_1a_2b_1\left(a_2^2+a_3^2\right)\left(b_1^2-3b_2^2\right),
\end{aligned}
$$

and $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$, i = 1, 2, 3, $\alpha < 0$.

Figures 1 and 2 illustrate the three-dimensional plot, density plot, and x-axis plot of the three wave solution. The selected parameter settings are $a_1 = 1, a_2 = 1, a_3 = 1, b_2 = 1, c_1 = 0, c_2 = 1, c_3 = 1, k_1 = 1, k_3 = 1, a = 1, b = 1, c = 1, \alpha = -4$ and $\beta = 1$. (a)–(c) present the three-dimensional spatial distribution characteristics of the variable coefficients g(t) and h(t) under different parameter combinations respectively. As shown in Figure 1, the solution exhibits multiple distinct peaks and troughs.

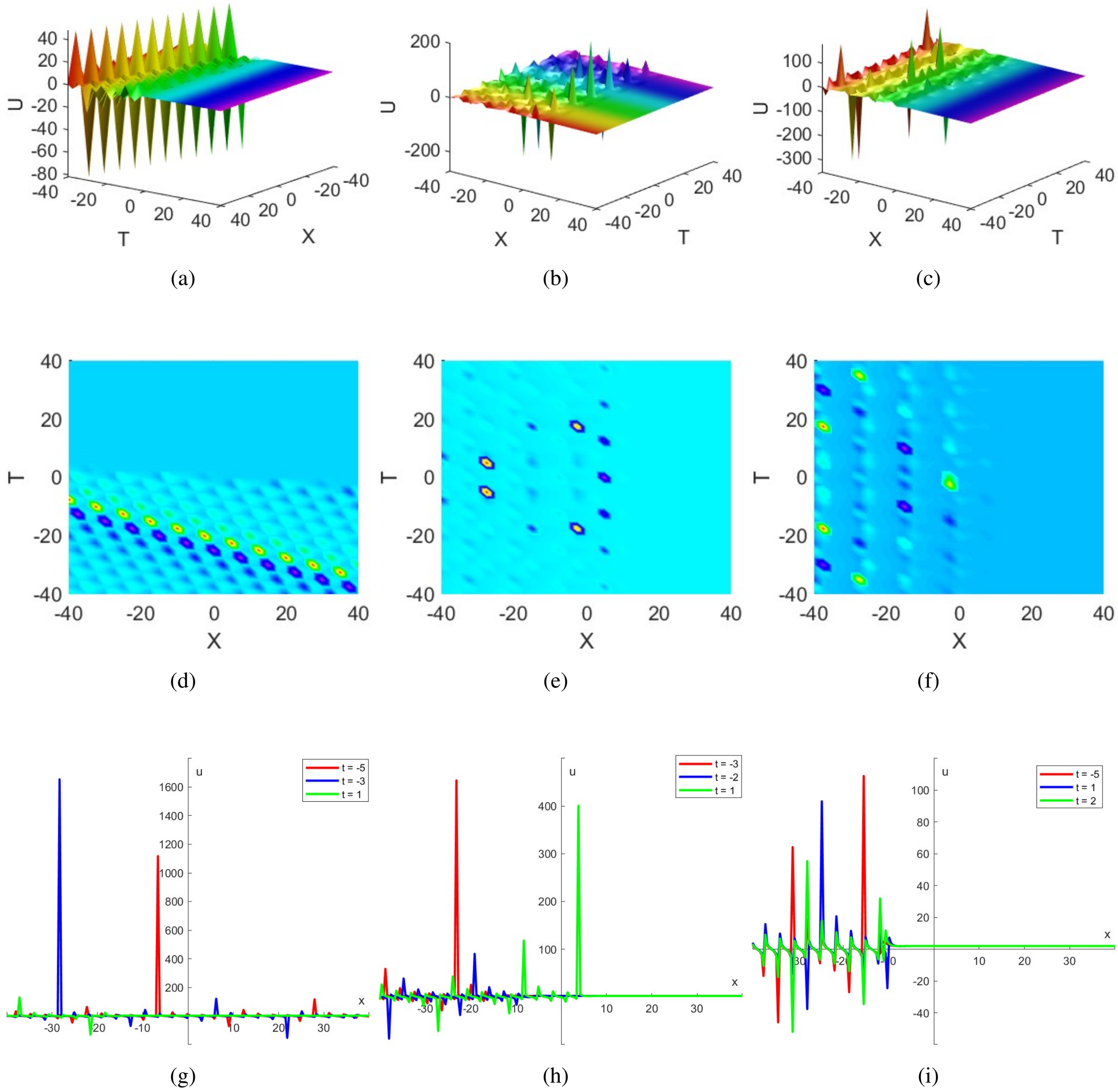

Figure 1. The three wave solution Eq (2.2) with $y = 1, z = 1$, (a)(d)(g): $h(t) = 2, g(t) = 1$; (b)(e)(h): $h(t) = 1, g(t) = \csc(t)$; (c)(f)(i): $h(t) = \sin(t), g(t) = \sec(t)$.

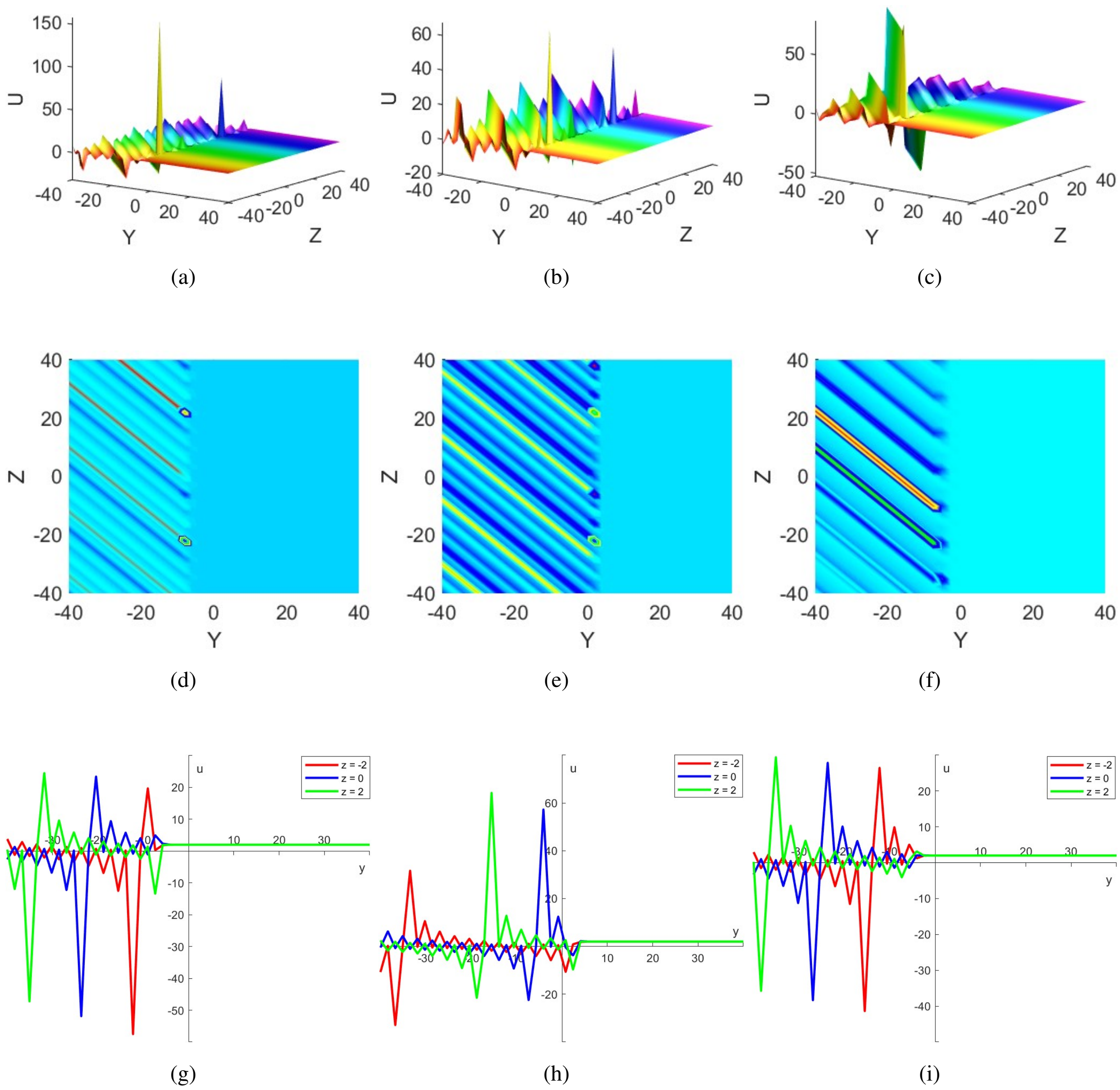

Figure 2. The three wave solution Eq (2.2) with $x = 1, t = 1$, (a)(d)(g): $h(t) = 2, g(t) = 1$; (b)(e)(h): $h(t) = 1, g(t) = \csc(t)$; (c)(f)(i): $h(t) = \sin(t), g(t) = \sec(t)$.

3. Lump-type solutions

3.1. Lump solution

To generate a lump solution, we can assume:

$$\begin{aligned} f &= {\sigma_1}^2 + {\sigma_2}^2 + k, \\ \sigma_i &= a_i x + b_i y + c_i z + d_i(t), \end{aligned} \tag{3.1}$$

where $i = 1, 2$, $d_1(t)$, $d_2(t)$, $d_3(t)$ are undetermined functions of t. Substituting Eq (3.1) into Eq (1.5), we can get the following solution.

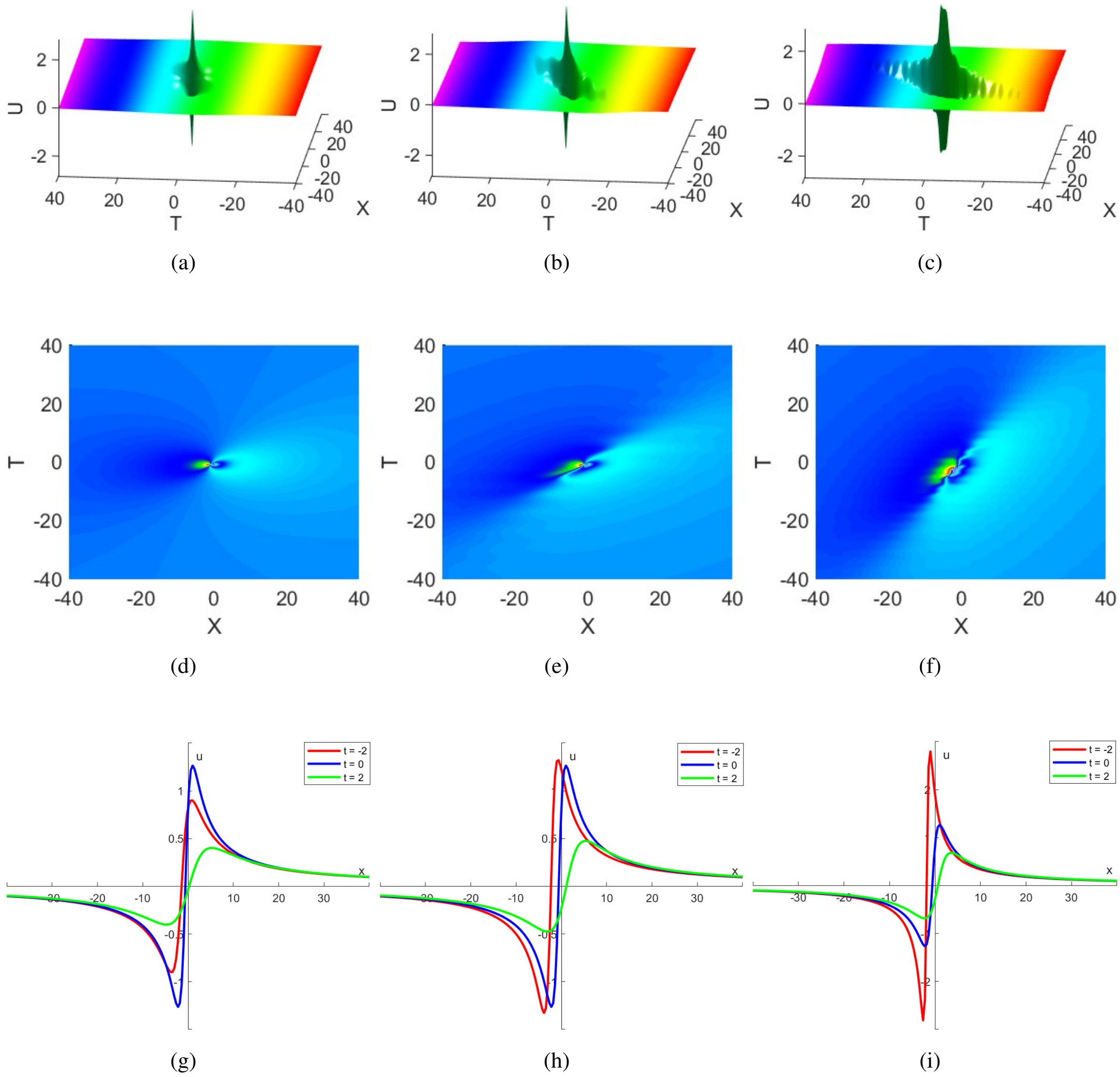

Figure 3. The lump solution Eq (3.2) with $y = 1, z = 1$, (a)(d)(g) $h(t) = 1, g(t) = 1$; (b)(e)(h) $h(t) = \cos(t), g(t) = 1$; (c)(f)(i) $h(t) = \cos(t), g(t) = \csc^2(t)$.

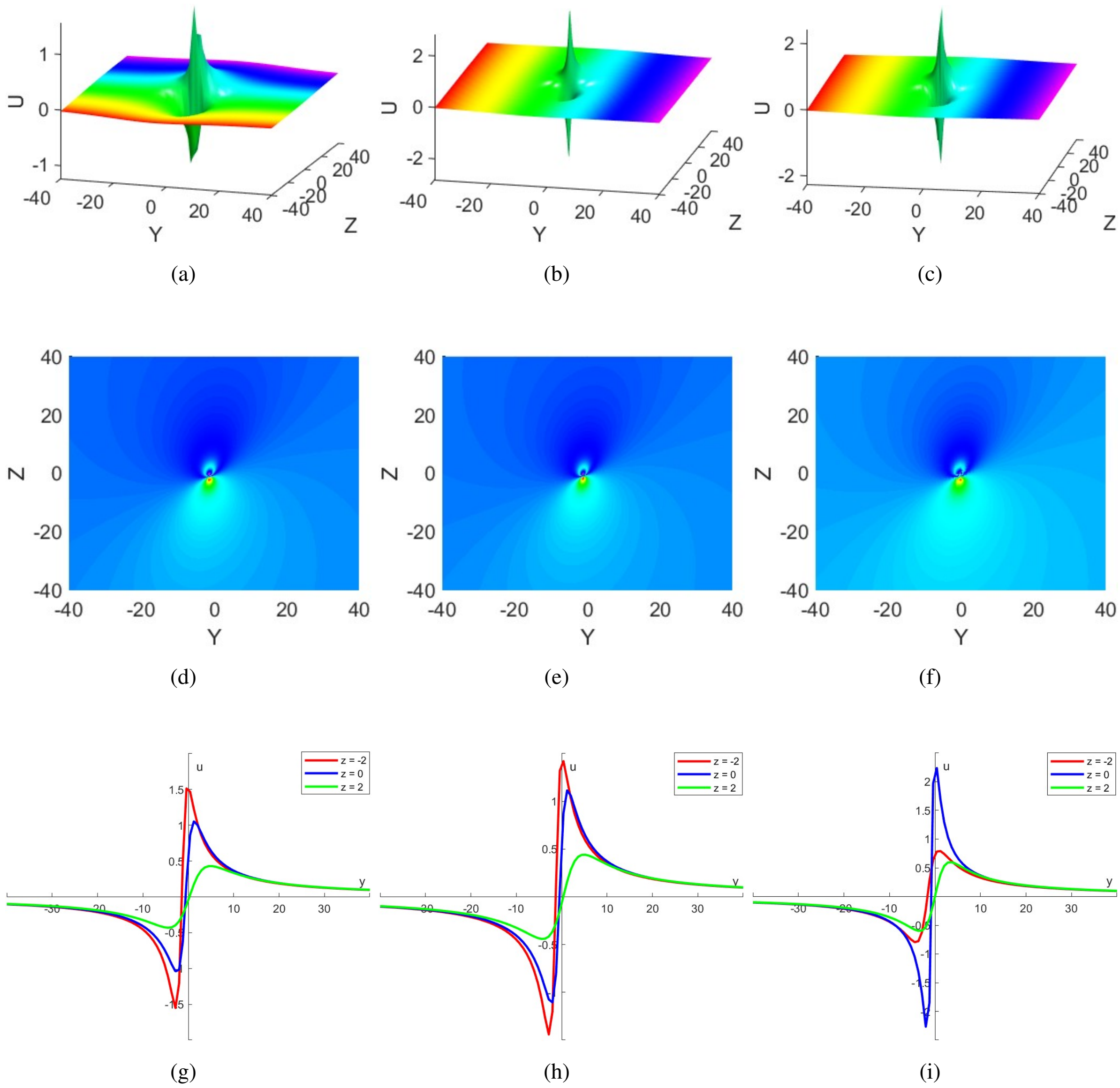

Figure 4. The lump solution Eq (3.2) with $x = 1, t = 1$, (a)(d)(g): $h(t) = 1,\ g(t) = 1$; (b)(e)(h): $h(t) = \cos(t),\ g(t) = 1$; (c)(f)(i): $h(t) = \cos(t),\ g(t) = \csc^2(t)$.

Theorem 3.1. The lump solution corresponding to Eq (1.1) can be represented as:

$$u = \frac{4(a_1\sigma_1 + a_2\sigma_2)}{{\sigma_1}^2 + {\sigma_2}^2 + k}, \tag{3.2}$$

under the following conditions:

$$\begin{aligned} &d_1(t) = \int \frac{n_2 + h(t)\left(a_1^2 + a_2^2\right)^2 B_1}{n_1 g(t)} dt, \qquad d_2(t) = \int \frac{n_3 + h(t)\left(a_1^2 + a_2^2\right)^2 B_2}{n_1 g(t)} dt, \\ &b_2 = \frac{-\sqrt{k\alpha a_1^8 + 3k\alpha a_1^6 a_2^2 + 3k\alpha a_1^4 a_2^4 + k\alpha a_1^2 a_2^6} + k\alpha a_1 a_2 b_1}{k\alpha a_1^2}, \end{aligned} \tag{3.3}$$

where

$$n_1 = 2\beta\left(a_1^2 + a_2^2\right)^2, \qquad n_2 = \alpha b_1\left(a_1^2 - a_2^2\right)\left(3b_2^2 - b_1^2\right) + 2\alpha a_1 a_2 b_2\left(b_2^2 - 3b_1^2\right),$$
$$n_3 = \alpha b_2\left(a_2^2 - a_1^2\right)\left(3b_1^2 - b_2^2\right) + 2\alpha a_1 a_2 b_1\left(b_1^2 - 3b_2^2\right), \qquad B_j = (aa_j + bb_j + cc_j),$$

and $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, j = 1, 2.$

Figures 3 and 4 display the lump solution of the DJKM equation. The parameters shared by Figure 3 and 4 are as follows: $a_1 = 1, a_2 = 1, b_1 = 1, c_1 = 1, c_2 = 1, k = 1, a = 1, b = 1, c = 1, \alpha = 1$ and $\beta = 1$. The lump solution presents a typical bimodal structure, with two wave peaks symmetrically distributed about the horizontal plane. As shown in Figure 3(h),(i), the peak amplitude gradually decays while the waveform progressively flattens with increasing time t.

3.2. Lump-type solution

In order to get the Lump-type solution, we assume:

$$\begin{aligned} f &= {\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2 + k, \\ \sigma_i &= a_i x + b_i y + c_i z + d_i(t), \end{aligned} \tag{3.4}$$

where $i = 1, 2, 3$, $d_1(t)$, $d_2(t)$, $d_3(t)$ are undetermined functions of t. Substituting Eq (3.4) into Eq (1.5) and solving the equations, we can get the Lump-type solution of Eq (1.1).

Theorem 3.2. The Lump-type solution can be written as:

$$u = \frac{4\left(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3\right)}{{\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2 + k}, \tag{3.5}$$

when

$$\begin{aligned} &d_1(t) = \int \frac{C_1 h(t) + \alpha r_2}{r_1 g(t)}\,dt, \qquad d_2(t) = \int \frac{C_2 h(t) + \alpha r_3}{r_1 g(t)}\,dt, \qquad d_3(t) = \int \frac{C_3 h(t) + \alpha r_4}{r_1 g(t)}\,dt, \\ &b_1 = \frac{a_2^2 b_3 c_1 - a_1 a_2 b_3 c_2 + a_3^2 b_3 c_1 - a_1 a_3 b_3 c_3}{a_1^2 c_3 - a_1 a_3 c_1 - a_2 a_3 c_2 + a_2^2 c_3}, \qquad b_2 = \frac{b_3\left(a_1^2 c_2 - a_1 a_2 c_1 + a_3^2 c_2 - a_2 a_3 c_3\right)}{a_1^2 c_3 - a_1 a_3 c_1 - a_2 a_3 c_2 + a_2^2 c_3}, \\ &k = \frac{\left(a_1^2 + a_2^2 + a_3^2\right) r_5 + b_3^2\left(-r_6^2 - r_7^2 - r_8^2 + 2r_7 r_6 + 2r_8 r_6 + 2r_7 r_8\right)}{\alpha b_3^2\left((a_1 c_2 - a_2 c_1)^2 + (a_1 c_3 - a_3 c_1)^2 + (a_2 c_3 - a_3 c_2)^2\right)}, \end{aligned} \tag{3.6}$$

where

$$
\begin{aligned}
r_1 &= 2\left(a_1^2 + a_2^2 + a_3^2\right)^2\beta, \\
r_2 &= \left(-a_1^2 + a_2^2 + a_3^2\right)b_1^3 + \left(3a_1^2 - 3a_2^2 + a_3^2\right)b_2^2 b_1 + \left(3a_1^2 + a_2^2 - 3a_3^2\right)b_3^2 b_1 \\
&\quad - 8a_2 a_3 b_2 b_3 b_1 + 2a_1\left(-3b_1^2 + b_2^2 + b_3^2\right)(a_2 b_2 + a_3 b_3), \\
r_3 &= \left(a_1^2 - a_2^2 + a_3^2\right)b_2^3 + \left(-3a_1^2 + 3a_2^2 + a_3^2\right)b_1^2 b_2 + \left(a_1^2 + 3a_2^2 - 3a_3^2\right)b_3^2 b_2 \\
&\quad - 8a_1 a_3 b_1 b_3 b_2 + 2a_2\left(b_1^2 - 3b_2^2 + b_3^2\right)(a_1 b_1 + a_3 b_3), \\
r_4 &= \left(a_1^2 + a_2^2 - a_3^2\right)b_3^3 + \left(-3a_1^2 + a_2^2 + 3a_3^2\right)b_1^2 b_3 + \left(a_1^2 - 3a_2^2 + 3a_3^2\right)b_2^2 b_3 \\
&\quad - 8a_1 a_2 b_1 b_2 b_3 + 2a_3\left(b_1^2 + b_2^2 - 3b_3^2\right)(a_1 b_1 + a_2 b_2), \\
r_5 &= \left(a_1^2 + a_2^2\right)^2 c_3^2 + \left(a_3^2(a_1 c_1 + a_2 c_2) - 2a_3\left(a_1^2 + a_2^2\right)c_3\right)(a_1 c_1 + a_2 c_2), \\
r_6 &= (a_2 c_3 - a_3 c_2)\, d_1(t), \qquad r_7 = (a_1 c_3 - a_3 c_1)\, d_2(t), \qquad r_8 = (a_1 c_2 - a_2 c_1)\, d_3(t), \\
C_i &= \left(\left(a_1^2 + a_2^2\right)^2 + a_3^2\left(2a_1^2 + 2a_2^2 + a_3^2\right)\right)(a a_i + b b_i + c c_i),
\end{aligned}
$$

and $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, i = 1, 2, 3$.

Figures 5 and 6 show the morphology of the Lump-type solution at $a_1 = 3, a_2 = 1, a_3 = 1, b_3 = 1, c_1 = 1, c_2 = 1, c_3 = 1, a = 1, b = 1, c = 1, \alpha = 1$ and $\beta = 1$. As can be seen from Figure 5(a)–(c), the Lump-type solution exhibits a distorted band-like morphology. Meanwhile, Figure 5(d)–(f) reveals the presence of peaks and troughs, which are uniformly spaced.

4. Conclusions

This paper has conducted an in-depth investigation of the new (3+1)-dimensional DJKM equation using the Hirota's bilinear method. Firstly, the bilinear form of the DJKM equation was derived through the D operator. On this basis, with the aid of Mathematica software for numerical calculations, the three wave solution (Eq (2.2)) and lump-type solutions (Eqs (3.2) and (3.5)) of Eq (1.1) were successfully obtained, enriching the solutions of the new (3+1)-dimensional DJKM equation. Finally, by adjusting the parameters, we plotted three-dimensional and density graphs to analyze the solutions' physical properties (Figures 1–6). In the future, we will continue to expand our research on nonlinear evolution equations (NLEE), including exploring new solution methods and constructing analytical solutions for fractional-order equations.

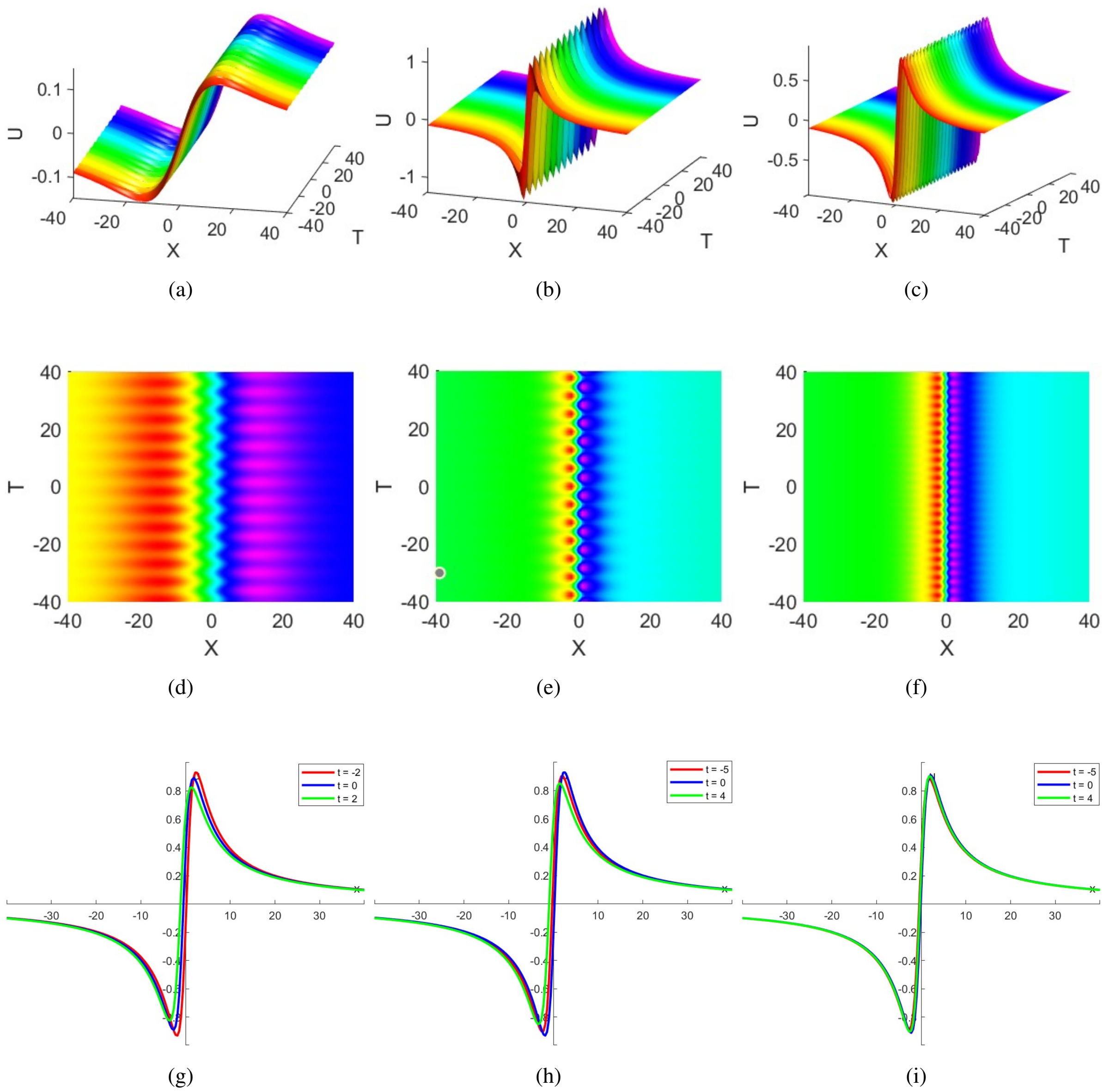

Figure 5. The lump-type solution Eq (3.5) with $y = 1, z = 1$, (a)(d)(g) $h(t) = \cos(t), g(t) = 1$; (b)(e)(h) $h(t) = 1, g(t) = \csc(t)$; (c)(f)(i) $h(t) = \sin(t), g(t) = \sec(t)$.

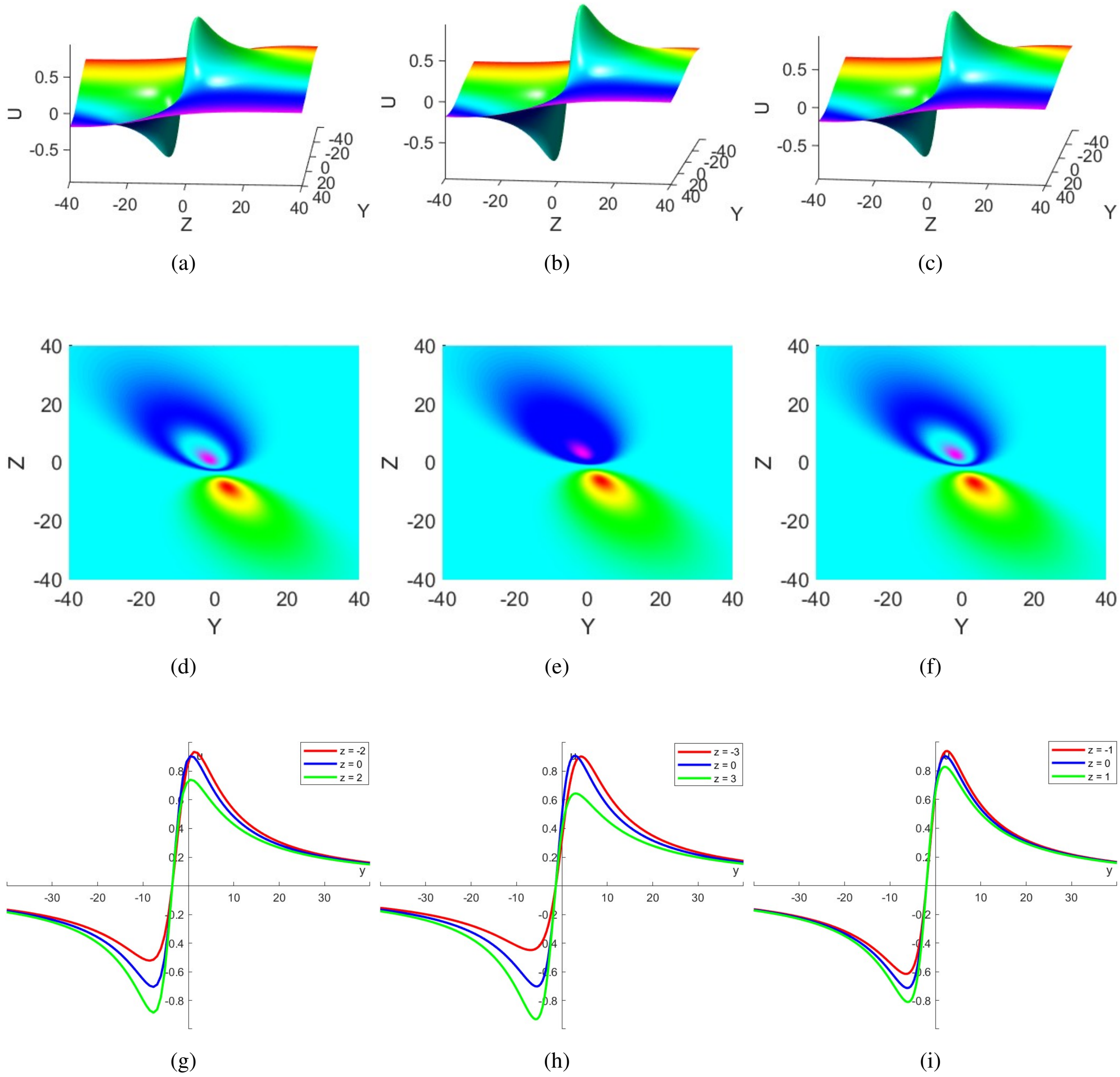

Figure 6. The lump-type solution Eq (3.5) with $x = 1, t = 1$, (a)(d)(g): $h(t) = \cos(t), g(t) = 1$; (b)(e)(h): $h(t) = 1, g(t) = \csc(t)$; (c)(f)(i): $h(t) = \sin(t), g(t) = \sec(t)$.

Author contributions

The first author was responsible for writing the original draft and revising the manuscript. The second author participated in reviewing and editing the writing. The third author was in charge of writing, reviewing and editing. The fourth author participated in reviewing and editing the writing. The fifth author conducted the software implementation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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