Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 20(3): 868–884.
DOI: 10.3934/nhm.2025037
Received: 07 April 2025
Revised: 29 May 2025
Accepted: 05 June 2025
Published: 18 July 2025

Research article

Generalized fractional derivatives and fourier transforms in tempered distributions with applications

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Abstract: The purpose of this paper is to define and prove that the Riemann–Liouville and Caputo fractional derivatives can be computed for tempered distributions, such that the fractional derivative of a tempered distribution remains a tempered distribution. The fact that the Fourier transform operator is an isomorphism in the dual of the Schwartz space is used, and we found that the fractional Riemann–Liouville and Caputo derivatives can be written as a Fourier transform composition and inverse. In this way, we are able to generalize both fractional Riemann–Liouville and Caputo derivatives for the tempered distributions. Moreover, certain examples of fractional derivatives for some tempered distributions are provided, such as the distribution of Dirac, the distribution of Heaviside, and the distribution of principal value.

Keywords: fractional derivatives; fourier transform; Schwartz space; tempered distributions

1. Introduction

Fractional analysis has a long history and extremely rich content. The ideas of fractional calculus have occupied many prominent scientists. The fractional operators are indispensable for describing and studying physical fractal problems and stochastic transfer processes. Work in this direction is only just beginning and is apparently restrained only by the exotic nature of fractional derivatives and fractional integrals. In a series of works by J. Liouville (1832–1835), using the expansion of functions into power series, he defined the fractional derivative. The author gave the first practical applications of the theory created for solving systems in mathematical physics. Later, B. Riemann (1847) considered another solution based on a definite integral, suitable for power series with non-integer exponents, which was completed by B. Riemann in 1876. The constructions of Liouville and Riemann are the main forms of fractional integration. Developing Liouville's idea, A. K. Grunwald (1867) introduced the concept of a fractional derivative as the limit of difference relations, and it has been strongly developed to the present day; see [1–3]. In recent years, considerable interest has been stimulated by its many applications in several fields of science, including physics, chemistry, aerodynamics, electrodynamics of complex media, signal processing, and optimal control [4–6]. At present, there is virtually no area of classical analysis that has not been touched on by fractional analysis. Fractional calculus, which generalizes the classical differentiation and integration to non-integer orders, was advanced from 2000 to 2025. New fractional derivative definitions were given in addressing previous limitations by introducing non-singular kernels and enhancing memory effect modeling in complex systems; see [7–9]. Tempered distributions have been extended to incorporate fractional and nonlocal operators, allowing the rigorous treatment of fractional derivatives within generalized function spaces; please see [10, 11]. This extension has opened new avenues for fractional calculus, providing analytical tools to handle functions and distributions with controlled growth at infinity. Today, significant advances in Fourier analysis of tempered distributions have enhanced the understanding of singularities and anisotropic phenomena, improving methods in harmonic and microlocal analysis; see [12, 13]. The theory of tempered distributions allows us to give a rigorous meaning to the Dirac delta function. It is defined by some special properties. Thus, tempered distributions are products of polynomials and derivatives of bounded continuous functions. In [14], the fractional Fourier transform of tempered distributions is considered, and generalized pseudo-differential operators involving two classes of symbols and fractional Fourier transforms are investigated. In [15], the theory of tempered fractional integrals and derivatives of a function with respect to another function is developed, and certain nonlinear fractional differential equations involving ψ-tempered derivatives are studied. Distributions are usually defined by duality, starting from a good choice of test functions.

It is well known that not all functions defined from $\mathbb{R}$ in $\mathbb{R}$ are differentiable on $\mathbb{R}$; on the other hand, the functions in $L^1_{loc}(\mathbb{R})$ are derivable in the sense of distribution $D'(\mathbb{R})$. That is to say, for $u \in L^1_{loc}(\mathbb{R})$, the derivative of u in the sense of $D'(\mathbb{R})$ is given by

$$\langle u', \varpi \rangle = -\langle u, \varpi' \rangle = \int_{\mathbb{R}} u(x)\,\varpi'(x)\,dx, \text{ for all } \varpi \in D(\mathbb{R}).$$

To compute the left Riemann-Liouville functional derivative for the functions defined from $\mathbb{R}$ to $\mathbb{R}$, we need a functional space that guarantees that the derivative exists and is well defined. This space being smaller, it will be shown that this space is the Sobolev space of order ν (see Lemma 3.3).

The primary aim of this paper is to answer the following question. How does the Riemann-Liouville fractional derivative work for functions that do not belong to the Sobolev space of order $\nu \in \mathbb{R}^+$? For example, let H be the Heaviside function given by

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

we have $I^{\nu}_{-\infty^+}\left[\frac{H}{x}\right]$ is diverged; that's to say, the Riemann-Liouville ν-fractional derivatives of $x \to \frac{H}{x}$ do not exist. Here we try to find a suitable space that contains the Sobolev space to calculate the fractional derivative (Riemann-Liouville or Caputo) in which we can calculate tempered distributions $S'(\mathbb{R})$, which means that we can obtain the fractional derivative of the distribution $x \to \dfrac{H}{x}$, see [16, 17].

It is known that the weak space properties are based on results in [18] and used for linear fractional partial differential equations in [19]. Indeed, fractional differential equations in the Riemann–Liouville type and Caputo type in Hilbert spaces are studied, and the existence and uniqueness of solutions are obtained in [20], where the main tools are extrapolation- and interpolation of fractional Sobolev spaces. Based on the exponentially weighted spaces of appropriate functions, fractional operators by means of a functional calculus using the Fourier transform are introduced. These results extend those in [21], in which a fractional derivative is defined as a derivative of a fractional integral, whereas in [22], the fractional derivative of C-valued functions on a bounded interval and linear fractional differential equations are also studied with a suitable functional analysis and fractional Sobolev spaces. Since the Sobolev space is of order ν, it is defined by a Fourier transformation (see Lemma 3.1). The Fourier transform is an isomorphism from $S'(\mathbb{R})$ into $S'(\mathbb{R})$, with $S'(\mathbb{R})$ a tempered distribution. By these properties, we generalize the Riemann–Liouville fractional derivative in tempered distributions and prove that this definition makes sense, so that we can calculate the fractional derivative of Riemann–Liouville for tempered distributions. To the best of our knowledge, this study represents the first contribution in this area by establishing a sufficient condition for the existence of fractional derivatives, where the function is required to belong to a specific subspace. Building upon this foundation, we extend and generalize the concept of fractional derivatives to the space of tempered distributions. Utilizing the Fourier transform technique

Motivated by the above works, in this article, we set an equivalent condition to obtain the fractional derivation, that is, a function in the Sobolev space of degree s (to obtain the derivative of degree s), and we see how to obtain the fractional derivation when it is not a function in the Sobolev space, i.e., we derive the fraction as a moderate distribution. We defined it with meaning and generalization and gave examples of the function not being granted to the Sobolev space, and therefore it is not possible to obtain the fractional derivation as a function, but we obtain it as a distribution which, actually arises in the study of differential and integral equations.

2. Preliminary results

A brief introduction to the fractional Fourier transform and tempered distributions is given.

2.1. Tempered distributions

Definition 2.1. *The Schwartz space $S(\mathbb{R})$ is the topological vector space of functions $u : \mathbb{R} \to \mathbb{C}$, such that $u \in C^{\infty}(\mathbb{R})$ and*

$$\lim_{|x|\to+\infty}\left(1+|x|^2\right)^{m/2}\left|u^{(n)}(x)\right| = 0, \ \forall n, m \in \mathbb{N}.$$

Introduce equivalent countable families of semi-norms on $S(\mathbb{R})$,

$$\|u\|_{m,n} = \sup_{x\in\mathbb{R}} \left|\left(1+|x|^2\right)^{m/2} u^{(n)}(x)\right|, \ \forall m,n \in \mathbb{R}.$$

Example 2.1. *We have the tempered distributions, Dirac, Heaviside, and principal value, respectively, for all $\varpi \in S(\mathbb{R})$,*

$$\langle \delta, \varpi \rangle = \varpi(0), \qquad \langle H, \varpi \rangle = \int_0^{+\infty} \varpi(x)\,dx,$$

$$\left\langle v_p(1/x), \varpi \right\rangle = \int_0^{+\infty} \frac{1}{x}\left(\varpi(x) - \varpi(-x)\right)dx.$$

Definition 2.2. *We define the space $O_M(\mathbb{R})$ as*

$$O_M(\mathbb{R}) = \left\{u \in C^\infty(\mathbb{R}) : \forall n \in \mathbb{N}, \exists C_n > 0, m_n \in \mathbb{N}, \ \left|u^{(n)}(x)\right| \le C_n \langle x \rangle^{m_n}\right\},$$

we call $O_M(\mathbb{R})$ the space of indefinitely differentiable functions with slow growth.

Definition 2.3. *A tempered distribution T on $\mathbb{R}$ is a continuous linear functional $T : S(\mathbb{R}) \to \mathbb{R}$. The topological vector space of tempered distributions is denoted by $S'(\mathbb{R})$.*

2.2. Fourier Transform

Definition 2.4. *The Fourier transform of a function $u : \mathbb{R} \to \mathbb{C}$ is the function $Fu := \hat{u} : \mathbb{R} \to \mathbb{C}$ defined by*

$$\hat{u}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\xi)\, e^{-i\xi x} d\xi.$$

The inverse Fourier transform of u is the function $F^{-1}u := \check{u} : \mathbb{R} \to \mathbb{C}$ given by

$$\check{u}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\xi)\, e^{i\xi x} d\xi.$$

Lemma 2.1. *The Fourier transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a one-to-one, onto bounded linear map. If $u, v \in L^2(\mathbb{R})$, then*

$$\int u(x)\,\overline{v}(x)\,dx = \int \hat{u}(\xi)\,\overline{\hat{v}}(\xi)\,d\xi.$$

Lemma 2.2. *The Fourier transform $F : S(\mathbb{R}) \to S(\mathbb{R})$ defined by $F : u \to \hat{u}$ is a continuous, one-to-one map of $S(\mathbb{R})$ onto itself. The inverse $F^{-1} : S(\mathbb{R}) \to S(\mathbb{R})$ is given by $F^{-1} : u \to \check{u}$. If $u \in S(\mathbb{R})$, then*

$$F\left(u^{(n)}\right) = (i\xi)^n F(u), \qquad F\left((-ix)^m u\right) = (Fu)^{(m)}.$$

Definition 2.5. *If $T \in S'(\mathbb{R})$, then the Fourier transform $\hat{T} \in S'(\mathbb{R})$ is the distribution defined by*

$$\left\langle \hat{T}, \varpi \right\rangle = \langle T, \hat{\varpi} \rangle, \qquad \textit{for all } \varpi \in S(\mathbb{R}).$$

The inverse Fourier transform $\check{T} \in S'(\mathbb{R})$ is the distribution defined by

$$\left\langle \check{T}, \varpi \right\rangle = \langle T, \check{\varpi} \rangle, \qquad \textit{for all } \varpi \in S(\mathbb{R}).$$

Remark 2.1. *For any function $u \in O_M(\mathbb{R})$, the operator defined by $\varpi \to u\varpi$ is continuous from $S(\mathbb{R})$ into itself.*

Definition 2.6. *For $\nu > 0$, the Sobolev space $H^\nu(\mathbb{R})$ consists of all tempered distributions $u \in S'(\mathbb{R})$ whose Fourier transform $\check{u}$ is a regular distribution such that*

$$\int_{\mathbb{R}} \langle \xi \rangle^{2\nu} |\hat{u}(\xi)|^2 \, d\xi < \infty.$$

The inner product and norm of $u, v \in H^\nu(\mathbb{R})$ are defined by

$$(u, v)_{H^\nu(\mathbb{R})} = \int_{\mathbb{R}} \langle \xi \rangle^{2\nu} \overline{\hat{u}(\xi)} \hat{v}(\xi) \, d\xi < \infty, \quad \|u\|_{H^\nu(\mathbb{R})} = \left(\int_{\mathbb{R}} \langle \xi \rangle^{2\nu} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

2.3. Fractional integrals-Fractional derivatives

Definition 2.7. *Let $u \in L^1(I)$ and $\nu > 0$. The left and right Riemann-Liouville ν-fractional integrals are defined by*

$$I_{a^+}^{\nu}[u](t) = \frac{1}{\Gamma(\nu)} \int_a^t (t - \xi)^{\nu - 1} u(\xi) \, d\xi,$$

and

$$I_{b^-}^{\nu}[u](t) = \frac{1}{\Gamma(\nu)} \int_t^b (\xi - t)^{\nu - 1} u(\xi) \, d\xi,$$

where $\Gamma(\nu)$ denotes the Euler's gamma function.

Definition 2.8. *Let $\nu \in (0, 1)$. For any $u : I \to \mathbb{R}$ sufficiently smooth, so that $I_{a^+}^{1-\nu}[u]$ and $I_{b^-}^{1-\nu}[u]$ are differentiable, the left and right Riemann-Liouville ν-fractional derivatives of u are defined by*

$$D_{a^+}^{\nu}[u](t) = \frac{d}{dt} I_{a^+}^{1-\nu}[u](t),$$

and

$$D_{b^-}^{\nu}[u](t) = \frac{d}{dt} I_{b^-}^{1-\nu}[u](t).$$

Definition 2.9. *Let $\nu \in (0, 1)$. For any $u : I \to \mathbb{R}$ sufficiently smooth, so that $I_{a^+}^{1-\nu}[u]$ and $I_{b^-}^{1-\nu}[u]$ are differentiable, the left and right Caputo ν-fractional derivatives of u are defined by*

$$D_{a^+}^{\nu}[u](t) = \frac{d}{dt} I_{a^+}^{1-\nu}[u'](t),$$

and

$$D_{b^-}^{\nu}[u](t) = \frac{d}{dt} I_{b^-}^{1-\nu}[u'](t).$$

Remark 2.2. *The left and right Riemann-Liouville derivative Riemann-Liouville derivatives with order $\nu > 0$ of the given function $u : I \to \mathbb{R}$ are respectively given as*

$$D_{a^+}^{\nu} u(t) = \frac{d^n}{dt} D_{a^+}^{\nu - n}[u](t),$$

and

$$D_{b^-}^{\nu}[u](t) = \frac{d^n}{dt} D_{b^-}^{\nu - n}[u](t),$$

where n is an integer that satisfies $n - 1 \leq \nu < n$.

3. Main results

3.1. Sobolev space of order $\nu \geq 0$

Lemma 3.1. *Let* $0 < \nu \leq 1$*, then*

$$F\left(D_{-\infty}^{\nu}[u]\right) = -\left(i\xi\right)^{\nu} F\left(u\right).$$

Proof. We have

$$F\left(D_{-\infty}^{\nu} u\right) = -i\xi F\left(I_{-\infty}^{1-\nu} u\right),$$

from Fubini's theorem, we obtain

$$F\left(I_{-\infty}^{1-\nu}[u]\right)(x) = \frac{1}{\Gamma(1-\nu)\sqrt{2\pi}} \int_{E} (\xi - y)^{-\nu} u(y) e^{-i\xi x} dy d\xi,$$

with

$$E = \left\{(y,\xi) \in \mathbb{R}^2 : y \leq \xi\right\}.$$

Then

$$\begin{aligned} F\left(I_{-\infty}^{1-\nu} u\right)(x) &= \frac{F(u)(x)}{\Gamma(1-\nu)} \int_0^{+\infty} \eta^{-\nu} e^{-i\xi x} d\xi \\ &= \frac{\Lambda(\nu, x)}{-i\xi\Gamma(1-\nu)} F(u)(x), \end{aligned}$$

where $\Lambda : (0,1] \times \mathbb{R} \to \mathbb{C}$ is given by

$$\Lambda(\nu,\xi) = -i\xi \int_0^{+\infty} t^{-\nu} e^{-it\xi} dt.$$

For all $\lambda > 0$, we have

$$\Lambda(\nu, \lambda\xi) = \lambda^{\nu}\Lambda(\nu,\xi),$$

and then

$$\Lambda(\nu,\xi) = \begin{cases} \xi^{\nu}\Lambda(\nu,1), & \text{if } \xi \geq 0, \\ (-\xi)^{\nu}\,\overline{\Lambda(\nu,1)}, & \text{if } \xi < 0. \end{cases}$$

Thus

$$\Lambda(\nu,1) = -i \int_0^{+\infty} t^{-\nu} e^{-it} dt = \Gamma(1-\nu)\, i^{\nu},$$

what is involved $\Lambda(\nu,\xi) = \Gamma(1-\nu)(\xi i)^{\nu}$. Hence,

$$\begin{aligned} F\left(D_{-\infty}^{\nu}[u]\right) &= -i\xi F\left(I_{-\infty}^{1-\nu} u\right) \\ &= \frac{\Lambda(\nu, x)}{\Gamma(1-\nu)} F(u)(x) \\ &= -(\xi i)^{\nu} F(u)(x). \end{aligned} \tag{3.1}$$

Lemma 3.2. *Let $\nu > 0$ and n are integers that satisfy $n - 1 \leq \nu < n$, then*

$$F\left(D^{\nu}_{-\infty}[u]\right) = (-1)^{n+1}(i\xi)^{\nu} F(u), \tag{3.2}$$

$$D^{\nu}_{-\infty}[u] = (-1)^{n+1} F^{-1}(i\xi)^{\nu} F(u), \tag{3.3}$$

$$F\left(I^{\nu}_{-\infty}[u]\right) = (\xi i)^{-\nu} F(u). \tag{3.4}$$

Proof. Let $\nu > 0$; we have

$$\begin{aligned} F\left(D^{\nu}_{-\infty}[u]\right) &= F\left(\frac{d^n}{dt}\left(D^{\nu-n}_{-\infty}[u]\right)\right) \\ &= (-i\xi)^n F\left(D^{\nu-n}_{-\infty}[u]\right), \end{aligned}$$

as $0 < \nu - n \leq 1$, by Lemma 3.2, we have

$$F\left(D^{\nu}_{-\infty}[u]\right) = (-1)^{n+1}(i\xi)^{\nu} F(u).$$

Then

$$F^{-1}F\left(D^{\nu}_{-\infty}[u]\right) = (-1)^{n+1} F^{-1}(i\xi)^{\nu} F(u).$$

As $F^{-1}F = Id$, we deduce equality (3.3).

By Eq (3.1), we have

$$F\left(I^{1-\nu}_{-\infty}u\right) = (\xi i)^{1-\nu} F(u)(x).$$

The previous lemma allows us to give another description of the Sobolev distances $H^{\nu}(\mathbb{R})$ to include the real positive exponents $\nu \geq 0$.

Lemma 3.3 provides an alternative definition of the scalar product and norm in a Sobolev space, proving their equivalence to the classical inner product and norm of that space. This equivalence helps us find a similar way to calculate the fractional derivative of functions in the Sobolev space, making it easier to study and analyze their differential properties using Sobolev spaces.

Lemma 3.3. *For $\nu > 0$, the Sobolev space $H^{\nu}(\mathbb{R})$ consists of all $u \in L^2(\mathbb{R})$ where u is ν-fractional derivative in the Riemann-Liouville sense, such that $D^{\nu}_{-\infty}[u] \in L^2(\mathbb{R})$.*

The inner product and norm of $u, v \in H^{\nu}(\mathbb{R})$ are defined by

$$(u, v)_{H^{\nu}(\mathbb{R})} = \int_{\mathbb{R}} D^{\nu}_{-\infty}[u](x)\, D^{\nu}_{-\infty}[v](x)\, dx,$$

$$\|u\|_{H^{\nu}(\mathbb{R})} = \left(\int_{\mathbb{R}} \left|D^{\nu}_{-\infty}[u](x)\right|^2 d\xi\right)^{1/2}.$$

Proof. For all $\nu > 0$, let

$$\mathcal{A}^{\nu}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : D^{\nu}_{-\infty}[u] \in L^2(\mathbb{R})\right\}.$$

Let $u \in H^{\nu}(\mathbb{R})$; then $u \in L^2(\mathbb{R})$ and $\xi^{\nu} F(u) \in L^2(\mathbb{R})$. By Lemma 3.2, $F\left(D^{\nu}_{-\infty,t}[u]\right) \in L^2(\mathbb{R})$. Since the operator $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is isometry, it means

$$F^{-1}\left(F\left(D^{\nu}_{-\infty}[u]\right)\right) = D^{\nu}_{-\infty}[u] \in L^2(\mathbb{R}),$$

then $u \in \mathcal{A}^{\nu}(\mathbb{R})$.

3.2. Tempered distribution fractional derivative

Lemma 3.4. *Let $\nu, \varepsilon \in (0,1)$; we consider the operator $D_{\nu,\varepsilon}$ from $S(\mathbb{R})$ to $S(\mathbb{R})$ given by:*

$$D_{\nu,\varepsilon} u = F^{-1}(ix+\varepsilon)^{\nu} F(u),$$

then the operator $D_{\nu,\varepsilon}$ is a bijection of $S(\mathbb{R})$ and

$$\lim_{\varepsilon\to 0} D_{\nu,\varepsilon} = -D^{\nu}_{-\infty}.$$

Proof. Let $u \in S(\mathbb{R})$, $\varepsilon > 0$, and the function $Q^{\nu}_{\varepsilon}(x) : \mathbb{R} \to \mathbb{C}$ given by: $Q^{\nu}_{\varepsilon}(x) = (ix+\varepsilon)^{\nu}$, then $Q^{\nu}_{\varepsilon} \in C^{\infty}(\mathbb{R},\mathbb{C})$. As $F : S(\mathbb{R}) \to S(\mathbb{R})$ is a bijection, then $F(u) \in S(\mathbb{R})$, that means $Q^{\nu}_{\varepsilon} F(u) \in C^{\infty}(\mathbb{R})$. On the other hand, we have

$$\lim_{|x|\to\infty} |x|^n \left(Q^{\nu}_{\varepsilon} u\right)^{(n)}(x) = \sum_{k=0}^{k=n} L_k \lim_{|x|\to\infty} |x|^n Q^{\nu-k}_{\varepsilon} u^{(n-k)}(x) = 0,$$

with

$$L_k = (i)^n C^k_n \prod_{k=0}^{k=n-1} (\nu - k),$$

then $\left(Q^{\nu}_{\varepsilon} u\right)^{(n)}$ is rapidly decreasing, $\forall n \in \mathbb{N}$. We conclude $Q^{\nu}_{\varepsilon} F(u) \in S(\mathbb{R})$ and further $F^{-1} Q_{\varepsilon} F(u) \in S(\mathbb{R})$. As $F : S(\mathbb{R}) \to S(\mathbb{R})$ is bijective, then $D_{\nu,\varepsilon}$ is continuous if and only if $F D_{\nu,\varepsilon}$ is continuous. Let $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\left(F D_{\nu,\varepsilon} u\right)^{(n)}\right| &\leq \widetilde{L}_n \sum_{k=0}^{k=n} \left(1+|x|^2\right)^{\frac{\nu-k}{2}} |x|^{n-k} |Fu| \\
&\leq \widetilde{L}_n \sum_{k=0}^{k=n} \left(1+|x|^2\right)^{\frac{n+\nu-2k}{2}} |Fu| \\
&\leq C_n \widetilde{L}_n \left(1+|x|^2\right)^{\frac{n+\nu}{2}} |Fu|,
\end{aligned} \tag{3.5}$$

with

$$\widetilde{L}_n = \max_{k\in\{1,2,..,n\}} |L_k| \quad \text{and} \quad C_n = \sup_{x\in\mathbb{R}} \sum_{k=0}^{k=n} \left(1+|x|^2\right)^{\frac{-k}{2}}.$$

By the last inequality, we get

$$\left\| FD_{\nu,\varepsilon}u \right\|_{m,n} \leq C_n \widetilde{L}_n \left\| Fu \right\|_{m+n+1,n}.$$

As F is continuous, deduce that $FD_{\nu,\varepsilon}$ is continuous. The operator $D_{\nu,\varepsilon}$ is bijective and $(D_{\nu,\varepsilon})^{-1} = D_{-\nu,\varepsilon}$, similarly, we find that $D_{-\nu,\varepsilon}$ is continuous. By (3.5), we have

$$\left| (FD_{\nu,\varepsilon}u - FD_{\nu,0}u)^{(n)} \right| \leq \varepsilon^2 \widetilde{L}_n C_n \left(1 + |x|^2\right)^{\frac{n}{2}} |Fu|,$$

which is equivalent to

$$\left\| FD_{\nu,\varepsilon}u - FD_{\nu,0}u \right\|_{m,n} \leq \varepsilon^2 C_n \widetilde{L}_n \left\| Fu \right\|_{m+n,n},$$

where $D_{\nu,\varepsilon}$ converges in $D_{\nu,0} = -D^{\nu}_{-\infty}$.

Lemma 3.5. *Let $\nu, \varepsilon > 0$, we deduce $D^*_{\nu,\varepsilon} : S'(\mathbb{R}) \to S'(\mathbb{R})$ is a bijection, where $D^*_{\nu,\varepsilon}(T) = T \circ D_{\nu,\varepsilon}$, that is to say*

$$\left\langle D^*_{\nu,\varepsilon}(T), \varpi \right\rangle = \left\langle T, D_{\nu,\varepsilon}[\varpi] \right\rangle, \textit{ for all } T \in S'(\mathbb{R}), \varpi \in S(\mathbb{R}).$$

By the continuity of T, we have

$$\lim_{\varepsilon \to 0} \left\langle D^*_{\nu,\varepsilon}(T), \varpi \right\rangle = -\left\langle T, D^{\nu}_{-\infty}[\varpi] \right\rangle, \textit{ for all } \varpi \in S(\mathbb{R}).$$

Proof. By Lemma 3.4, we find that $\lim_{\varepsilon \to 0} D_{\nu,\varepsilon} = -D^{\nu}_{-\infty}$ in $S(\mathbb{R})$. By passing to duality, we obtain the new concept we give in the following remark.

Remark 3.1. *From Lemma 3.4, we see that to calculate the Riemann-Liouville ν-derivative of a function, which must be at least in the Sobolev space $H^{\nu}(\mathbb{R})$, we would like to generalize the fractional derivative to tempered distributions. From Lemma 3.5, we can introduce the new concept of the fractional derivative of the Riemann-Liouville distribution $S'(\mathbb{R})$.*

Definition 3.1. *Let $\nu \in (0, 1)$, the right Riemann-Liouville ν -fractional derivatives of $T \in S'(\mathbb{R})$ are defined by*

$$\left\langle D^{\nu}_{-\infty}T, \varpi \right\rangle = -\left\langle T, D^{\nu}_{-\infty}[\varpi] \right\rangle, \textit{ for all } \varpi \in S(\mathbb{R}). \tag{3.6}$$

We give the following example.

Example 3.1. *Let $\nu \in (0, 1)$ and $u \in H^{\nu}(\mathbb{R})$, we have $T_u \in S'(\mathbb{R})$, by Definition 3.1, we have*

$$\left\langle D^{\nu}_{-\infty}T_u, \varpi \right\rangle = -\int_{\mathbb{R}} u(x)\, \overline{D^{\nu}_{-\infty}\varpi(x)} dx,$$

since $u \in H^{\nu}(\mathbb{R})$, by Lemma 2.1, we give

$$\begin{aligned} \left\langle D^{\nu}_{-\infty}T_u, \varpi \right\rangle &= \int_{\mathbb{R}} u(x)\, \overline{F^{-1}(ix)^{\nu} F(\varpi)(x)} dx \\ &= \int_{\mathbb{R}} u(x)\, F\overline{(ix)^{\nu}} F^{-1}(\varpi)(x)\, dx \\ &= \int_{\mathbb{R}} F(-ix)^{\nu} F^{-1}u(x)\, \varpi(x)\, dx \end{aligned}$$

$$= \int_{\mathbb{R}} D^{\nu}_{-\infty} u(x)\, \varpi(x)\, dx$$
$$= \left\langle T_{D^{\nu}_{-\infty} u}, \varpi \right\rangle, \textit{ for all } \varpi \in S(\mathbb{R}).$$

Then

$$D^{\nu}_{-\infty} T_u = T_{D^{\nu}_{-\infty} u}.$$

Remark 3.2. *By example 3.1, we conclude that the Riemann-Liouville fractional derivative of order ν for the functions u in the sense of distribution (by Definition 3.1), is the tempered distribution associated with the function $D^{\nu}_{-\infty}$, and it gives us the validity of the generalization presented in Definition 3.1.*

Example 3.2. *Let* $0 \in (0, 1)$*, we have*

$$D^{\nu}_{-\infty} \delta = \frac{\nu}{\Gamma(1-\nu)} (-x)^{(\nu+1)} H(-x),$$

$$D^{\nu}_{-\infty} v_p = \sqrt{2\pi}\Gamma(1+\nu)\cos(\pi(\nu+1)) \frac{1}{|x|^{\nu+1}},$$

With δ, H, and v_p defined in Example 2.1.

Remark 3.3. *For all $\nu > 0$, the integral Riemann-Liouville $I^{\nu}_{-\infty^+}\left[\frac{1}{x}\right]$ does not exist. Therefore, we cannot calculate the fractional Riemann-Liouville derivative of order ν, as a function, but we can calculate it as a tempered distribution.*

Remark 3.4. *Let $\nu > 0$, and $n \in \mathbb{N}$, such that $n - 1 \leq \nu < n$; then deducing the Riemann-Liouville derivative of order ν for the $T \in S'(\mathbb{R})$ is given by:*

$$\langle D^{\nu}_{-\infty} T, \varpi \rangle = (-1)^n \langle T, D^{\nu}_{-\infty}[\varpi] \rangle, \textit{ for all } \varpi \in S(\mathbb{R}). \tag{3.7}$$

Lemma 3.6. *Let* $0 < \nu \leq 1$ *and* $T \in S'(\mathbb{R})$*, then*

$$D^{\nu}_{-\infty} T = -\lim_{\varepsilon \to 0} F(i\xi + \varepsilon)^{\nu} F^{-1} T.$$

Proof. Let $T \in S'(\mathbb{R})$ and $\varpi \in S(\mathbb{R})$, we have

$$\begin{aligned} \langle D^{\nu}_{-\infty} T, \varpi \rangle &= -\langle T, D^{\nu}_{-\infty}[\varpi] \rangle \\ &= \lim_{\varepsilon \to 0} \langle T, D_{\nu,\varepsilon}[\varpi] \rangle \\ &= -\lim_{\varepsilon \to 0} \left\langle F(i\xi + \varepsilon)^{\nu} F^{-1} T, \varpi \right\rangle. \end{aligned}$$

The following lemma is the generalization of the derivative of the convolution product of the distribution.

Lemma 3.7. *Let* $\nu > 0$ *and* $S, T \in S'(\mathbb{R})$*, then*

$$D^{\nu}_{-\infty}(S * T) = (D^{\nu}_{-\infty} S) * T = S * D^{\nu}_{-\infty} T.$$

Proof. Let $n \in \mathbb{N}$, such that $n-1 \leq \nu < n$, for all $\varpi \in S(\mathbb{R})$, we have

$$\begin{aligned} \langle D^{\nu}_{-\infty}(S*T), \varpi\rangle &= (-1)^n \langle S*T, D^{\nu}_{-\infty}[\varpi]\rangle \\ &= (-1)^n \langle S_t, \langle T_x, D^{\nu}_{-\infty}[\varpi](x+t)\rangle\rangle \\ &= \langle S_t, \langle D^{\nu}_{-\infty}T_x, \varpi(x+t)\rangle\rangle \\ &= \langle S*D^{\nu}_{-\infty}T, \varpi\rangle. \end{aligned}$$

As $S*T = T*S$, we find $D^{\nu}_{-\infty}(S*T) = S*D^{\nu}_{-\infty}T$.

The following lemma is the generalization of the two-derived fractional composition.

Lemma 3.8. *Let $\nu, \beta > 0$ and $T \in S'(\mathbb{R})$, then*

$$D^{\beta}_{-\infty}(D^{\nu}_{-\infty}T) = (-1)^{j_{\nu,\beta}} D^{\nu+\beta}_{-\infty}T.$$

where $j_{\nu,\beta} = [\nu+\beta] - [\nu] - [\beta] \in \{0,1\}$.

Proof. Let $n, m \in \mathbb{N}$, such that $n-1 \leq \nu < n$, $m-1 \leq \beta < m$, for all $\varpi \in S(\mathbb{R})$, we have

$$\begin{aligned} \left\langle D^{\beta}_{-\infty}(D^{\nu}_{-\infty}T), \varpi\right\rangle &= (-1)^m \left\langle D^{\nu}_{-\infty}T, D^{\beta}_{-\infty}[\varpi]\right\rangle \\ &= (-1)^{m+n} \left\langle T, D^{\nu}_{-\infty}D^{\beta}_{-\infty}[\varpi]\right\rangle \\ &= (-1)^{m+n} \left\langle T, D^{\nu+\beta}_{-\infty}[\varpi]\right\rangle \\ &= (-1)^{m+n-k} \left\langle D^{\nu+\beta}_{-\infty}T, [\varpi]\right\rangle, \end{aligned}$$

with $k-1 \leq \nu+\beta < k$, as $k-(m+n) = j_{\nu,\beta}$, then

$$\left\langle D^{\beta}_{-\infty}(D^{\nu}_{-\infty}T), \varpi\right\rangle = (-1)^{j_{\nu,\beta}} \left\langle D^{\nu+\beta}_{-\infty}T, \varpi\right\rangle.$$

Remark 3.5. *The Riemann-Liouville fractional derivative of a tempered distribution is commutative, i.e., let $\nu > 0$, $\beta > 0$, $T \in S'(\mathbb{R})$, then*

$$D^{\beta}_{-\infty}(D^{\nu}_{-\infty}T) = D^{\nu}_{-\infty}\left(D^{\beta}_{-\infty}T\right).$$

3.3. Fractional integral of tempered distribution

The Fourier transform by the fractional derivative presented in Lemma 3.2 indicates that the function $(ix)^{\nu}$ does not belong to $O_M(\mathbb{R})$; for this end, we will introduce a sequence of functions $(ix+\epsilon)^{\nu}$ that belongs to $O_M(\mathbb{R})$, such that $(ix+\epsilon)^{\nu} \to (ix)^{\nu}$.

Lemma 3.9. *Let $\nu, \varepsilon \in (0,1)$; we consider the operator $I_{\nu,\varepsilon}$ from $S(\mathbb{R})$ to $S(\mathbb{R})$ given by:*

$$I_{\nu,\varepsilon}u = F^{-1}(ix+\varepsilon)^{-\nu} F(u),$$

then the operator $I_{\nu,\varepsilon}$ is a bijection of $S(\mathbb{R})$ and

$$\lim_{\varepsilon\to 0} I_{\nu,\varepsilon} = I^{\nu}_{-\infty}.$$

Proof. The proof of this lemma is simulated using the same steps as the proof of Lemma 3.4.

Remark 3.6. *By lemma 3.9, we deduce* $I^*_{\nu,\varepsilon} : S'(\mathbb{R}) \to S'(\mathbb{R})$ *is a bijection, where* $I^*_{\nu,\varepsilon}(T) = T \circ I_{\nu,\varepsilon}$, *that is to say*

$$\left\langle I^*_{\nu,\varepsilon}(T), \varpi \right\rangle = \langle T, I_{\nu,\varepsilon}[\varpi] \rangle, \text{ for all } T \in S'(\mathbb{R}), \varpi \in S(\mathbb{R}).$$

By the continuity of T, *we find*

$$\lim_{\varepsilon \to 0} \left\langle I^*_{\nu,\varepsilon}(T), \varpi \right\rangle = \langle T, I^{\nu}_{-\infty}[\varpi] \rangle, \text{ for all } \varpi \in S(\mathbb{R}).$$

From the last equation, we conclude that the Riemann-Liouville fractional integral can be generalized in a space $S'(\mathbb{R})$.

From Remark 3.6 and Lemma 3.9, we can give the new concept that presents the fractional integral of the Riemann-Liouville distribution $S'(\mathbb{R})$.

Definition 3.2. *Let* $\nu \in (0, 1)$; *the right Riemann-Liouville* ν*-fractional integral of* $T \in S'(\mathbb{R})$ *is defined by*

$$\langle I^{\nu}_{-\infty} T, \varpi \rangle = \langle T, I^{\nu}_{-\infty}[\varpi] \rangle, \text{ for all } \varpi \in S(\mathbb{R}). \tag{3.8}$$

Example 3.3. *Let* $0 \in (0, 1)$, *we have*

$$\langle I^{\nu}_{-\infty} \delta, \varpi \rangle = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{0} (-x)^{\nu-1} \varpi(x) \, d\xi,$$

then

$$I^{\nu}_{-\infty} \delta = \frac{H(-x)}{\Gamma(\nu)} (-x)^{\nu-1}.$$

with δ *and* H *defined in example 2.1.*

Example 3.4. *Let* $\nu \in (0, 1)$; *we consider the following problem:*

$$I^{\nu}_{-\infty}[u](x) = \frac{H(x)}{x}, \ \forall x \in \mathbb{R}.$$

Since $D^{\nu}_{-\infty}\left(\frac{H}{x}\right)$ *does not exist, we are looking for the solution in the distribution sense, then*

$$D^{\nu}_{-\infty} I^{\nu}_{-\infty}[u](x) = D^{\nu}_{-\infty}\left(\frac{H}{x}\right),$$

by example 3.2, we have

$$u(x) = \sqrt{2\pi}\,\Gamma(1+\nu)\cos(\pi(\nu+1)) \frac{1}{|x|^{\nu+1}}, \ \forall x \in \mathbb{R}. \tag{3.9}$$

Lemma 3.10. *Let* $\nu \in (0, 1)$ *and* $T \in S'(\mathbb{R})$, *then*

$$D^{\nu}_{-\infty} I^{\nu}_{-\infty} T = T.$$

Proof. Let $\varpi \in S(\mathbb{R})$, we have

$$\langle D^{\nu}_{-\infty} I^{\nu}_{-\infty} T, \varpi \rangle = \langle T, I^{\nu}_{-\infty} D^{\nu}_{-\infty}[\varpi] \rangle = \langle T, \varpi \rangle,$$

then $D^{\nu}_{-\infty} I^{\nu}_{-\infty} T = T$.

3.4. An application and discussions

For all $\nu > 0$, we present the following problem:

$$D^{\nu}_{-\infty} T_{\nu} = \delta, \quad T_{\nu} \in S'(\mathbb{R}). \tag{3.10}$$

Theorem 3.1. *Let $\nu > 0$; we define T_{ν} by*

$$T_{\nu} = \begin{cases} \dfrac{H(-x)}{\Gamma(\nu)} (-x)^{\nu-1}, & \text{if } \nu < 1 \\ \dfrac{H(x)}{\Gamma(\nu - n + 1)} x^{\nu-n}, & \text{if } \nu \geq 1, \end{cases} \tag{3.11}$$

with $n = [\nu]$, then T_{ν} is the solution of the problem (3.10).

Proof. Let $\nu < 1$ and $T_0 \in S'(\mathbb{R})$, such that $D^{\nu}_{-\infty} T_0 = \delta$, from Lemma 3.10, we will find

$$T_0 = \frac{H(-x)}{\Gamma(\nu)} (-x)^{\nu-1}.$$

Let $T_0 \in S'(\mathbb{R})$, such that $D^{\nu+1}_{-\infty} T_1 = \delta$, since $\delta = H'$, then $D^{\nu}_{-\infty} T = H$, hence

$$\left\langle D^{\nu+1}_{-\infty} T_1, \varpi \right\rangle = -\left\langle D^{\nu}_{-\infty} T_1, \varpi' \right\rangle = \langle H', \varpi \rangle = -\langle H, \varpi' \rangle,$$

On the other hand, we have

$$\langle I^{\nu}_{-\infty} H, \varpi \rangle = \langle H, I^{\nu}_{-\infty} \varpi \rangle = \frac{1}{\Gamma(\nu)} \int_0^{\infty} x^{\nu-1} \int_0^{\infty} \varpi(\xi + x)\, d\xi dx.$$

Then

$$\left\langle (I^{\nu}_{-\infty} H)', \varpi \right\rangle = \frac{1}{\Gamma(\nu)} \int_0^{\infty} x^{\nu-1} \varpi(x)\, dx,$$

that is to say, $(I^{\nu}_{-\infty} H)' = \frac{H(x)}{\Gamma(\nu)} x^{\nu-1}$, any $I^{\nu}_{-\infty} H = \frac{H(x)}{\Gamma(\nu+1)} x^{\nu}$. Consequently, we obtain $D^{\nu}_{-\infty} T_1 = H$, which means

$$T_1 = I^{\nu}_{-\infty} H = \frac{H(x)}{\Gamma(\nu + 1)} x^{\nu}.$$

Let $n = [\nu] \geq 2$ and $T_n \in S'(\mathbb{R})$, such that $D^{\nu}_{-\infty} T_n = \delta$, we pose $H_n = \frac{x^n}{n!} H(x)$, then $H_n^{(n+1)} = \delta$. For all $\varpi \in S(\mathbb{R})$, we have

$$\langle D^{\nu}_{-\infty} T_n, \varpi \rangle = \left\langle (D^{\nu-n}_{-\infty} T_n)^{(n)}, \varpi \right\rangle = \left\langle H^{(n)}_{n-1}, \varpi \right\rangle,$$

then $(D^{\nu-n}_{-\infty} T_n)^{(n)} = H_n^{(n)}$, what does it mean

$$D^{\nu-n}_{-\infty} T_n = H,$$

since $\nu - n < 1$, as a result

$$T_n = I^{\nu-n}_{-\infty} H = \frac{H(x)}{\Gamma(\nu + 1 - n)} x^{\nu-n}.$$

For all $\nu > 0$ and $f \in L^1_{loc}(\mathbb{R})$, we present the following problem

$$D^{\nu}_{-\infty} T_{\nu} = f, \quad T_{\nu} \in S'(\mathbb{R}). \tag{3.12}$$

Corollary 3.1. *For all $\nu > 0$ and $f \in L^1_{loc}(\mathbb{R})$, we have then $T = T_{\nu} * f$ is the solution of the problem* (3.12).

Proof. From Theorem 3.1 and Lemma 3.7, we have

$$D^{\nu}_{-\infty}(T_{\nu} * f) = D^{\nu}_{-\infty}(T_{\nu}) * f = \delta * f = f.$$

3.4.1. Riemann-Liouville ν-fractional

We may define the fractional derivative and the integral Riemann-Liouville for tempered distributions at ∞ in the same manner.

Definition 3.3. *Let $\nu > 0$, $T \in S'(\mathbb{R})$; we will define*

a) The left Riemann-Liouville ν-fractional derivatives of $T \in S'(\mathbb{R})$ as

$$\langle D^{\nu}_{+\infty} T, \varpi \rangle = (-1)^n \langle T, D^{\nu}_{+\infty}[\varpi] \rangle, \textit{ for all } \varpi \in S(\mathbb{R}).$$

b) The left Riemann-Liouville ν-fractional integral of $T \in S'(\mathbb{R})$ as

$$\langle I^{\nu}_{+\infty} T, \varpi \rangle = \langle T, I^{\nu}_{+\infty}[\varpi] \rangle, \textit{ for all } \varpi \in S(\mathbb{R}).$$

where $n = [\nu]$.

Lemma 3.11. *Let $\nu, \beta > 0$ and $T \in S'(\mathbb{R})$, we have*

1. $D^{\beta}_{-\infty}(D^{\nu}_{-\infty} T) = (-1)^{j_{\nu\beta}} D^{\nu+\beta}_{-\infty} T.$
2. $D^{\nu}_{+\infty} I^{\nu}_{+\infty} T = T.$
3. $D^{\nu}_{+\infty}(S * T) = (D^{\nu}_{+\infty} S) * T = S * D^{\nu}_{+\infty} T.$

where

$$j_{\nu,\beta} = [\nu + \beta] - [\nu] - [\beta] \in \{0, 1\}.$$

3.4.2. Caputo ν-fractional

For tempered distributions at $+\infty$ or $-\infty$, we may define the Caputo derivative and integral in the same manner.

Definition 3.4. *Let $\nu > 0$, $T \in S'(\mathbb{R})$, we will define*

1. The right Caputo ν-fractional derivatives of $T \in S'(\mathbb{R})$ as

$$\left\langle {}^{C}D^{\nu}_{-\infty} T, \varpi \right\rangle = (-1)^n \left\langle T, {}^{C}D^{\nu}_{-\infty}[\varpi] \right\rangle, \textit{ for all } \varpi \in S(\mathbb{R}).$$

2. *The left Caputo ν-fractional derivatives of $T \in S'(\mathbb{R})$ as*

$$\left\langle {}^{C}D^{\nu}_{+\infty}T, \varpi \right\rangle = (-1)^{n} \left\langle T, {}^{C}D^{\nu}_{+\infty}[\varpi] \right\rangle, \text{ for all } \varpi \in S(\mathbb{R}).$$

3. *The right Caputo ν-fractional integral of $T \in S'(\mathbb{R})$ as*

$$\left\langle {}^{C}I^{\nu}_{-\infty}T, \varpi \right\rangle = \left\langle T, {}^{C}I^{\nu}_{-\infty}[\varpi] \right\rangle, \text{ for all } \varpi \in S(\mathbb{R}).$$

4. *The left Caputo ν-fractional integral of $T \in S'(\mathbb{R})$ as*

$$\left\langle {}^{C}I^{\nu}_{+\infty}T, \varpi \right\rangle = \left\langle T, {}^{C}I^{\nu}_{+\infty}[\varpi] \right\rangle, \text{ for all } \varpi \in S(\mathbb{R}).$$

Lemma 3.12. *Let $\nu, \beta > 0$ and $T \in S'(\mathbb{R})$, we have*

1. ${}^{C}D^{\beta}_{-\infty}(D^{\nu}_{-\infty}T) = (-1)^{j_{\nu,\beta}} D^{\nu+\beta}_{-\infty}T.$
2. ${}^{C}D^{\beta}_{+\infty}(D^{\nu}_{+\infty}T) = (-1)^{j_{\nu,\beta}} D^{\nu+\beta}_{+\infty}T.$
3. ${}^{C}D^{\nu}_{-\infty}\left({}^{C}I^{\nu}_{-\infty}T\right) = T.$
4. ${}^{C}D^{\nu}_{+\infty}\left({}^{C}I^{\nu}_{+\infty}T\right) = T.$
5. ${}^{C}D^{\nu}_{+\infty}(S * T) = \left({}^{C}D^{\nu}_{+\infty}S\right) * T = S * {}^{C}D^{\nu}_{+\infty}T.$
6. ${}^{C}D^{\nu}_{-\infty}(S * T) = \left({}^{C}D^{\nu}_{-\infty}S\right) * T = S * {}^{C}D^{\nu}_{-\infty}T.$

where $j_{\nu,\beta} = [\nu + \beta] - [\nu] - [\beta] \in \{0, 1\}$.

4. Conclusions

In this paper, we define the fractional derivatives of Riemann–Liouville and Caputo for temperate distributions using the Fourier transform and inverse. The most important contributions we presented were:

The sufficient and necessary condition for calculating the fractional derivative (see Lemma 3.3).

Provided instruction for calculating the fractional derivative in the space $S'(\mathbb{R})$ (see Lemma 3.5 and Definition 3.1).

By definition 3.1, we can calculate the fractional derivative in the distribution sense for functions in $L^1(\mathbb{R})$, as an example, $\frac{H}{x} \notin H^{\nu}(\mathbb{R})$, for all $\nu > 0$, by Lemma 3.3, we have $D^{\nu}_{-\infty}\left(\frac{H}{x}\right)$ does not exist in any sense function, but in the distribution sense, see (3.9).

We can solve fractional differential equations in the space $S'(\mathbb{R})$; see problems (3.10) and (3.12).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

The authors declare there is no conflict of interest.

Authors contribution

A. Benaissa Cherif, F-Z. Ladrani: writing—original draft preparation. D. Alhwikem, A. Hammoudi and K. Bouhali: writing—review and editing, visualization. Kh. Zennir: visualization, supervision.

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