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Research article

Collisionless and decentralized formation control for strings

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Abstract: A decentralized feedback controller for multi-agent systems, inspired by vehicle platooning, is proposed. The closed loop resulting from the decentralized control action has three distinctive features: The generation of collision-free trajectories, flocking of the system towards a consensus state in velocity, and asymptotic convergence to a prescribed pattern of distances between agents. For each feature, a rigorous dynamical analysis is provided, yielding a characterization of the set of parameters and initial configurations where collision avoidance, flocking, and pattern formation are guaranteed. Numerical tests assess the theoretical results presented.

Keywords: multi-agent systems; decentralised feedback control; formation control; collision-avoidance control

1. Introduction

Multi-agent systems (MAS) are a versatile framework for studying diverse scalability problems in Science and Engineering, such as dynamic networks [1], autonomous vehicles [2, 3], collective behaviour of humans or animals [4,5], and many others [6,7]. Mathematically, MAS are often modelled as large-scale dynamical systems where each agent can be considered a subset of states, updated via interaction forces such as attraction, repulsion, alignment, etc., [8, 9] or through the optimization of a pay-off function in a control/game framework [10, 11].

In this work, we approach the study of MAS from a control viewpoint. We study a class of sparsely interconnected agents in one dimension, interacting through nonlinear couplings and a decentralized control law. The elementary building block of our approach is the celebrated Cucker-Smale model for consensus dynamics [9], which corresponds to a MAS where each agent is endowed with second-order nonlinear dynamics for velocity alignment, and where the influence of neighbouring agents decay with distance. The Cucker-Smale model and variants can represent the physical motion of agents on the real line, inspired by autonomous vehicle formations in platooning with a nearest-neighbour interaction scheme [12, 13]. The couplings to be studied are motivated by the more general setting of the Cucker-Smale dynamics in arbitrary dimension. The original Cucker-Smale dynamics consider full network connectivity in the agent interactions, generating flocking dynamics capable of exhibiting emergent consensus behavior, that is, agents that may reach a common velocity in steady state, without the action of external forces. This framework has been extended in several directions, including forcing terms and control [14–17], optimal control [18–21], formation control [22–25], leadership [26], graph topologies [27], stochasticity [28], unreliable communications [29, 30], short-range interactions [31, 32], and collision-avoidance capabilities [33–37], the latter being an increasingly sought-after property of formation control schemes for autonomous fleets of vehicles.

The main contribution of this paper is to propose a nearest neighbour interaction of agents on the real line that exhibits emergent consensus and collision avoidance under the action of a simple decentralised control law. The proposed feedback enforces a desired steady-state inter-agent distance and is inspired by formation control in vehicular platoons [38]. Such a model has the potential to achieve the goals of platooning applications, that is, the coordinated, scalable, secure, and efficient travel of automated vehicles [39], while offering flocking, pattern formation, and collision avoidance features from the non-linear dynamics. Moreover, we provide collision-avoidance guarantees that, from a safety viewpoint, do not rely on traditional concepts in platooning such as string stability. Here, we aim to provide alternative techniques for collision avoidance and formation control in platooning that avoid the need for long-range wireless networks while considering simple and interpretable control actions.

For the derivation of collision-avoidance results, we consider the framework developed in [40–43], which uses singular interaction kernels that blow up whenever two agents are at the same position. We modify this setup to also consider agents with a *volume* by including a threshold inter-agent distance where the kernel becomes singular.

Our main result is a rigorous characterization of flocking, collision-avoidance, and platooning behaviour for the proposed nonlinear model, in terms of the initial configuration of the system, interaction, and control law parameters. Recent works have studied different aspects of the one-dimensional case [44–49], relying on full connectivity of the agent network. Our contribution differentiates itself from these, showing that these emergent properties are present when a highly sparse nearest neighbour interaction is considered in the 1D case.

The remainder of the paper is structured as follows. In Section 2, we present the proposed model to be studied, a Cucker-Smale model with nearest neighbor singular interactions and a decentralized feedback control. We also define here a total energy functional $E(x, v)$ for the model and show that it is not increasing in time. In Section 3, we provide the results ensuring the collision-avoidance behaviour of the controlled system, and Section 4 includes a flocking estimate showing that the velocity alignment between individuals and the inter-agent distances are uniformly bounded in time. In Section 5, we present the main formation control result. We provide different numerical experiments that illustrate our theoretical results in Section 6, along with some concluding remarks.

2. Problem description and preliminary results

We consider a string of N agents, each characterized by a pair $(x_i(t), v_i(t))$ in $\mathbb{R}^2$ evolving in time t through second-order dynamics of the form

$$\begin{cases} \dfrac{dx_i(t)}{dt} = v_i(t), \quad i = 1, \ldots, N, \quad t > 0, \\ \dfrac{dv_i(t)}{dt} = I_i(t) + u_i(t), \end{cases} \tag{2.1}$$

subject to initial data

$$(x_i(0), v_i(0)) =: (x_i^0, v_i^0) \quad \text{for} \quad i = 1, \ldots, N. \tag{2.2}$$

Here, the term I_i describes nonlocal velocity interactions between individuals that are weighted by a singular communication function $\psi(r) : \mathbb{R}^+ \longrightarrow \mathbb{R}$,

$$\begin{cases} I_1 = \psi(|x_2 - x_1| - \delta_1)(v_2 - v_1), \\ I_k = \psi(|x_k - x_{k-1}| - \delta_{k-1})(v_{k-1} - v_k) \\ \qquad + \psi(|x_{k+1} - x_k| - \delta_k)(v_{k+1} - v_k), \\ I_N = \psi(|x_N - x_{N-1}| - \delta_{N-1})(v_{N-1} - v_N), \end{cases} \qquad k = 2, \ldots, N-1,$$

where parameters $\delta_i > 0$ are fixed. This interaction term induces consensus in the velocities of the agents while preventing collisions. The second term u_i serves as a decentralized feedback control depending on weight function $\phi(r) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and is given by

$$\begin{cases} u_1 = -\phi(|x_1 - x_2 - z_1|^2)(x_1 - x_2 - z_1), \\ u_k = \phi(|x_{k-1} - x_k - z_{k-1}|^2)(x_{k-1} - x_k - z_{k-1}) \\ \qquad - \phi(|x_k - x_{k+1} - z_k|^2)(x_k - x_{k+1} - z_k), \\ u_N = \phi(|x_{N-1} - x_N - z_{N-1}|^2)(x_{N-1} - x_N - z_{N-1}), \end{cases} \qquad k = 2, \ldots, N-1\,.$$

This feedback also depends on a vector of relative distances $z := (z_1, \ldots, z_{N-1}) \in \mathbb{R}^{N-1}$. The objective of this control law is to induce the formation of a string pattern characterized by z. The complete setting is depicted in Figure 1.

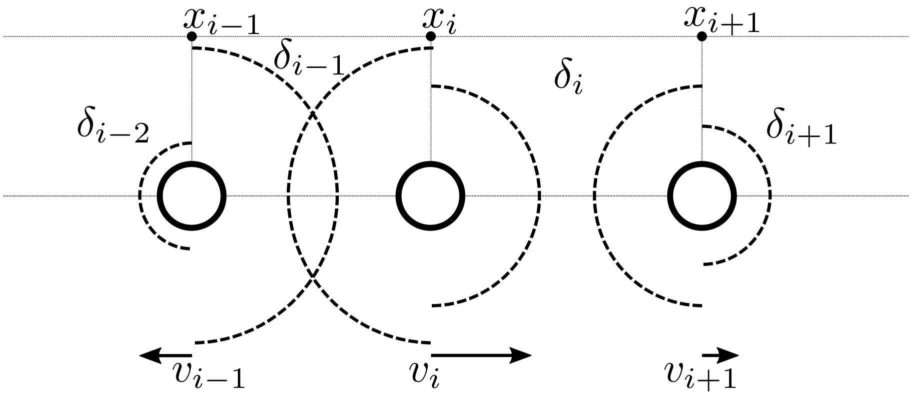

Figure 1. Diagram at a particular instant of three consecutive agents for the considered MAS. The singular interactions, providing *barriers* to the agents, are indicated with the radii δ_i of the semicircles. Note that with this nomenclature, in a teady state, the agents are not considered to have collided whenever $|x_{i-1} - x_i| - \delta_{i-1} > 0$. Moreover, note that the first and last agent have a barrier on only one of their sides.

For the sake of clarity, the weight functions ψ and ϕ are chosen as

$$\psi(r) = \frac{1}{r^\alpha}, \quad \text{and} \quad \phi(r) = \frac{1}{(1+r)^\beta}, \quad \alpha, \beta > 0\,. \tag{2.3}$$

The interaction kernel ψ is taken to be singular to enforce collision avoidance through velocity alignment. As two agents approach within a threshold δ_i, the singularity causes the interaction strength to diverge, inducing rapid alignment of their velocities before a collision occurs. On the other hand, the formation control kernel ϕ is chosen to be regular and bounded in order to ensure smooth convergence to the desired inter-agent distances. Using a nonsingular ϕ avoids unnecessary stiffness and enables robust decentralized control of spatial formations without interfering with the repulsive-alignment behavior induced by ψ. This design reflects a separation of roles: ψ handles safety, while ϕ drives pattern formation. We would like to point out that the results we will state in the forthcoming sections can extended to the case in which ϕ is bounded and Lipschitz continuous. We establish conditions under which the string (2.1) converge to a consensus state with a prescribed formation while avoiding collisions between agents. For this, we begin by stating an a priori energy estimate that is significantly used for estimating consensus emergence. We first define the total energy functional

$$E(x, v) := E_1(v) + E_2(x) = \frac{1}{4N} \sum_{i,j=1}^{N} |v_i - v_j|^2 + \frac{1}{2} \sum_{i=2}^{N} \int_0^{|x_{i-1}-x_i-z_{i-1}|^2} \phi(r)\, dr,$$

and its dissipation rate

$$D(x, v) := \sum_{i=2}^{N} \psi(|x_i - x_{i-1}| - \delta_{i-1})(v_i - v_{i-1})^2.$$

Lemma 1. *Let* $\{(x_i, v_i)\}_{i=1}^N$ *be a smooth solution to the system* (2.1) *on the time interval* $[0, T]$. *Then:*

(i) The mean velocity is conserved in time:

$$v_c(t) := \frac{1}{N}\sum_{i=1}^{N} v_i(t) = v_c(0)\,.$$

(ii) The total energy is not increasing in time:

$$\frac{d}{dt}E(x(t), v(t)) + D(x(t), v(t)) = 0\,.$$

Proof. Even though the proof is almost the same with [24, Lemma 3.1], we provide the proof for the completeness of our work.

(i) A straightforward computation yields

$$\sum_{i=1}^{N} I_i = \sum_{i=1}^{N} u_i = 0.$$

This implies

$$\frac{d}{dt}v_c(t) = 0, \quad \text{i.e.,} \quad v_c(t) = v_c(0) \tag{2.4}$$

for $t \in [0, T]$.

(ii) Let us first begin with the estimate for the kinetic energy:

$$\begin{aligned}\frac{1}{4N}\frac{d}{dt}\sum_{i,j=1}^{N}(v_i - v_j)^2 &= \frac{1}{2N}\sum_{i,j=1}^{N}(v_i - v_j)\left(\frac{dv_i}{dt} - \frac{dv_j}{dt}\right) = \frac{1}{2N}\sum_{i,j=1}^{N}\left(v_i\frac{dv_i}{dt} + v_j\frac{dv_j}{dt}\right)\\ &= \sum_{i=1}^{N} v_i\frac{dv_i}{dt} = \sum_{i=1}^{N} v_i(I_i + u_i),\end{aligned} \tag{2.5}$$

where we use system (2.4). Here, we use the same idea of [24, Lemma 3.1] to obtain

$$\sum_{i=1}^{N} v_i u_i = -\frac{1}{2}\frac{d}{dt}\sum_{i=2}^{N}\int_0^{|x_{i-1}-x_i-z_{i-1}|^2}\phi(r)\,dr. \tag{2.6}$$

On the other hand, we estimate the term with I_i as

$$\begin{aligned}\sum_{i=2}^{N-1} v_i I_i &= \sum_{i=2}^{N-1} v_i\left(\psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)\ + \psi(|x_i - x_{i+1}| - \delta_i)(v_{i+1} - v_i)\right)\\ &= \sum_{i=2}^{N-1} v_i\left(\psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)\right) + \ldots\\ &\quad \sum_{i=3}^{N} v_{i-1}\left(\psi(|x_i - x_{i-1}| - \delta_{i-1})(v_i - v_{i-1})\right)\\ &= v_2\psi(|x_2 - x_1| - \delta_1)(v_1 - v_2) - \ldots\\ &\quad \sum_{i=3}^{N-1}\psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)^2 + v_{N-1}\psi(|x_N - x_{N-1}| - \delta_{N-1})(v_N - v_{N-1})\\ &= -\,v_2 I_1 - v_{N-1}I_N - \sum_{i=3}^{N-1}\psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)^2.\end{aligned}$$

This asserts

$$\begin{aligned}\sum_{i=1}^{N} v_i I_i &= v_1 I_1 + \sum_{i=2}^{N-1} v_i I_i + v_N I_N \\ &= -(v_2 - v_1) I_1 - \sum_{i=3}^{N-1} \psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)^2 - (v_{N-1} - v_N) I_N \\ &= -\sum_{i=2}^{N} \psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)^2.\end{aligned} \tag{2.7}$$

Combining Eqs (2.5)–(2.7), we conclude the desired result.

3. Global and local existence of solutions

In this section, we show that system (2.1), under certain parametric and initial conditions, exhibits a non-collisional behaviour, which together with the Cauchy-Lipschitz theory, subsequently provides global-in-time existence and uniqueness of smooth solutions to the systems (2.1) and (2.2). Inspired by [24], we present two results regarding the existence of non-collisional trajectories. The first theorem requires a prescribed ordering for the initial datum x_i^0 and the power α in system (2.3) to be $\alpha \geq 1$. The second result requires $\alpha \geq 2$, but the initial ordering assumption is removed from the string. A third result characterizes a pathological case, where a 2-agent string blows up in finite time.

Theorem 1. *Suppose that $\alpha \geq 1$ and the initial configuration x_0 satisfies $x_{i+1}^0 > x_i^0 + \delta_i$ for all $i = 1, \dots, N-1$. Then, there exists the global unique smooth solution to the systems* (2.1) *and* (2.2) *satisfying $x_{i+1}(t) > x_i(t) + \delta_i$ for all $i = 1, \dots, N-1$ and all $t > 0$.*

Proof. We first notice that $\psi(x_{i+1} - x_i - \delta_i)$ is regular as long as $x_{i+1} > x_i + \delta_i$ and, thus, there exists a unique smooth solution to the system (2.1). For a fixed $T \in (0, \infty)$, let us assume that there is $t_* \in (0, T]$ where the smoothness of solutions breaks down for the first time, i.e., there is an index ℓ such that

$$x_{\ell+1}(t) - x_\ell(t) > \delta_\ell \text{ for } t \in (0, t_*) \quad \text{and} \quad \lim_{t \to t_*-} x_{\ell+1}(t) - x_\ell(t) = \delta_\ell. \tag{3.1}$$

We denote by $[\ell]$ the set of such indices and set $i_* = \min[\ell]$. We first claim $i_* \geq 2$. If $i_* = 1$, then for $t \in (0, t_*)$, we estimate

$$\frac{d}{dt}\Psi(x_2 - x_1 - \delta_1) = \psi(x_2 - x_1 - \delta_1)(v_2 - v_1) = I_1 = \frac{d}{dt}(v_1 - v_c) - u_1\,,$$

where we use system (2.4) and Ψ is the primitive of ψ, i.e.,

$$\Psi(r) = \begin{cases} \ln(r) & \text{for } \alpha = 1, \\ \dfrac{1}{1-\alpha} r^{1-\alpha} & \text{for } \alpha > 1. \end{cases}$$

From this, we deduce that

$$\Psi(x_2(t) - x_1(t) - \delta_1) = \Psi(x_2^0 - x_1^0 - \delta_1) = (v_1(t) - v_c(t)) - (v_1^0 - v_c(0)) - \int_0^t u_1(s)\, ds \tag{3.2}$$

for $t \in [0, t_*)$. On the other hand, by Hölder's inequality we find

$$|v_1(t) - v_c(t)| = \left| \frac{1}{N} \sum_{k=1}^{N} (v_1(t) - v_k(t)) \right| \leq \sqrt{\frac{1}{N} \sum_{k=1}^{N} (v_1(t) - v_k(t))^2}$$

and

$$|u_1(s)| \leq \|\phi\|_{L^\infty} |x_2 - x_1 - z_1| \leq \|\phi\|_{L^\infty} \left(|z_1| + |x_2^0 - x_1^0| + \int_0^s |v_2(\tau) - v_1(\tau)| \, d\tau \right).$$

These observations, together with the energy estimate in Lemma 1, imply that the right hand side of Eq (3.2) is bounded on the time interval $(0, t_*)$, and subsequently $t \mapsto \Psi(x_2(t) - x_1(t) - \delta_1)$ is bounded on the time interval $[0, t_*)$. This is a contradiction to Eq (3.1) and, thus, the claim follows. By the definition of i_*, there exists a constant $c_{i_*} > 0$, such that

$$x_{i_*}(t) - x_{i_*-1}(t) - \delta_{i_*-1} > c_{i_*} \tag{3.3}$$

for all $t \in (0, t_*)$. Similarly as above, we now estimate

$$\begin{aligned} \frac{d}{dt} \Psi(x_{i_*+1} - x_{i_*} - \delta_{i_*}) &= \psi(x_{i_*+1} - x_{i_*} - \delta_{i_*})(v_{i_*+1} - v_{i_*}) \\ &= I_{i_*} + \psi(x_{i_*} - x_{i_*-1} - \delta_{i_*-1})(v_{i_*} - v_{i_*-1}) \\ &= \frac{d}{dt}(v_{i_*} - v_c) + \psi(x_{i_*} - x_{i_*-1} - \delta_{i_*-1})(v_{i_*} - v_{i_*-1}) - u_{i_*}, \end{aligned}$$

and thus

$$\begin{aligned} &\Psi(x_{i_*+1}(t) - x_{i_*}(t) - \delta_{i_*}) \\ &\quad = \Psi(x_{i_*+1}^0 - x_{i_*}^0 - \delta_{i_*}) + (v_{i_*}(t) - v_c(t)) - (v_{i_*}^0 - v_c(0)) \\ &\qquad + \int_0^t \psi(x_{i_*}(s) - x_{i_*-1}(s) - \delta_{i_*-1})(v_{i_*}(s) - v_{i_*-1}(s)) \, ds - \int_0^t u_{i_*}(s) \, ds \end{aligned} \tag{3.4}$$

for $t \in (0, t_*)$. Here, the boundedness of the second and fourth terms can be obtained using almost the same argument as above. We also use Eq (3.3) to obtain

$$|\psi(x_{i_*}(s) - x_{i_*-1}(s) - \delta_{i_*-1})(v_{i_*}(s) - v_{i_*-1}(s)| \leq c_{i_*}^{-\alpha} 4N E_1(v(t)) \leq c_{i_*}^{-\alpha} 4N E(x^0, v^0).$$

Hence, the right hand side of Eq (3.4) is bounded on the time interval $[0, t_*)$, and so is the left hand side. This leads to a contradiction, and, thus, the unique smooth solution can exist up to an arbitrary finite time $T > 0$. This completes the proof.

We next present the second existence theorem whose proof is based on the energy estimate. For this, we first introduce a function $L_\delta^{\alpha-2}$ with $\alpha \geq 2$ given by

$$L_\delta^{\alpha-2}(t) = \begin{cases} \displaystyle\sum_{i=1}^{N-1} (|x_i(t) - x_{i+1}(t)| - \delta_i)^{-(\alpha-2)} & \text{for } \alpha > 2, \\ \displaystyle\sum_{i=1}^{N-1} \log(|x_i(t) - x_{i+1}(t)| - \delta_i) & \text{for } \alpha = 2. \end{cases}$$

Note that $|L^{\alpha-2}(t)| < \infty$ for $t \in [0, T]$ for some $\alpha \geq 2$ if and only if the distances between agents $x_i(t)$ and $x_{i+1}(t)$ are strictly greater than δ_i for all $i = 1, \ldots, N-1$ and $t \in [0, T]$.

Theorem 2. *Suppose that $\alpha \geq 2$ and that the initial configuration x_0 satisfies*

$$|x_i^0 - x_{i+1}^0| > \delta_i$$

for all $i = 1, \dots, N-1$. Then, there exists the global unique smooth solution to the systems (2.1) *and* (2.2) *where the distances between agents satisfy $|x_i(t) - x_{i+1}(t)| > \delta_i$ for all $i = 1, \dots, N-1$ and all $t > 0$.*

Proof. For the proof, as observed above, we show the boundedness of the function $L_\delta^{\alpha-2}$. We first introduce the maximal life-span $T_0 = T(x^0)$ of the initial datum x^0:

$$T_0 := \sup\{s \in \mathbb{R}_+ : \exists \text{ solution } (x(t), v(t)) \text{ for systems (2.1) and (2.2) in a time-interval } [0, s)\}$$

By the assumption, it is clear that $T_0 > 0$. We then claim $T_0 = \infty$. First, note that it follows from Lemma 1 that

$$\sum_{i=1}^{N-1} \int_0^t \frac{(v_{i+1}(s) - v_i(s))^2}{(|x_i(s) - x_{i+1}(s)| - \delta_i)^\alpha}\, ds \leq E(x^0, v^0). \tag{3.5}$$

Let us prove the claim above by dealing with two cases separately: $\alpha = 2$ and $\alpha > 2$.

(i) $\alpha = 2$: A straightforward computation gives

$$\begin{aligned}\left|\frac{d}{dt} L_\delta^0(t)\right| &= \left|\frac{d}{dt} \sum_{i=1}^{N-1} \log(|x_i(t) - x_{i+1}(t)| - \delta_i)\right| \\ &= \left|\sum_{i=1}^{N-1} \frac{(x_i(t) - x_j(t)) \cdot (v_i(t) - v_j(t))}{|x_i(t) - x_j(t)|(|x_i(t) - x_{i+1}(t)| - \delta_i)}\right| \leq \sum_{i=1}^{N-1} \frac{|v_i(t) - v_j(t)|}{|x_i(t) - x_j(t)| - \delta_i}\end{aligned}$$

for $t \in [0, T_0)$. This yields

$$\left|\sum_{i=1}^{N-1} \log(|x_i(t) - x_{i+1}(t)| - \delta_i)\right| \leq \left|\sum_{i=1}^{N-1} \log(|x_i^0 - x_{i+1}^0| - \delta_i)\right| + \sum_{i=1}^{N-1} \int_0^t \frac{|v_i(s) - v_{i+1}(s)|}{|x_i(s) - x_{i+1}(s)| - \delta_i}\, ds.$$

On the other hand, by using the Hölder inequality and Eq (3.5), we estimate

$$\sum_{i=1}^{N-1} \int_0^t \frac{|v_i(s) - v_{i+1}(s)|}{|x_i(s) - x_{i+1}(s)| - \delta_i}\, ds \leq \sqrt{t} \sum_{i=1}^{N-1} \left(\int_0^t \frac{|v_i(s) - v_{i+1}(s)|^2}{(|x_i(s) - x_{i+1}(s)| - \delta_i)^2}\, ds\right)^{1/2} \leq \sqrt{t(N-1)E(x^0, v^0)}.$$

Thus, we obtain

$$\left|L_\delta^0(t)\right| \leq \left|L_\delta^0(0)\right| + \sqrt{t(N-1)E(x^0, v^0)}, \tag{3.6}$$

for $t \in [0, T_0)$.

(ii) $\alpha > 2$: Taking the time derivative to $L_\delta^{\alpha-2}$, we get (omitting the time arguments)

$$\frac{dL_\delta^{\alpha-2}}{dt} = -(\alpha - 2) \sum_{i=1}^{N-1} (|x_i - x_{i+1}| - \delta_i)^{-\alpha+1} \frac{\langle x_i - x_{i+1}, v_i - v_{i+1}\rangle}{|x_i - x_{i+1}|}$$

$$
\begin{aligned}
&\le C\sum_{i=1}^{N-1}(|x_i - x_{i+1}| - \delta_i)^{-\alpha+1}|v_i - v_{i+1}| \\
&\le C\sum_{i=1}^{N-1}\frac{1}{(|x_i - x_{i+1}| - \delta_i)^{\alpha-2}} + C\sum_{i=1}^{N-1}\frac{|v_i - v_{i+1}|^2}{(|x_i - x_{i+1}| - \delta_i)^{\alpha}} \\
&= CL^{\alpha-2}(t) + C\sum_{i=1}^{N-1}\frac{|v_i - v_{i+1}|^2}{(|x_i - x_{i+1}| - \delta_i)^{\alpha}},
\end{aligned}
$$

for $t \in [0, T_0)$, where we use Young's inequality. Applying Gronwall's inequality to the above, we have

$$
L_\delta^{\alpha-2}(t) \le L_\delta^{\alpha-2}(0)e^{Ct} + Ce^{Ct}\sum_{i=1}^{N-1}\int_0^t \frac{|v_i(s) - v_{i+1}(s)|^2}{(|x_i(s) - x_{i+1}(s)| - \delta_i)^{\alpha}}\,ds \le e^{Ct}\left(L_\delta^{\alpha-2}(0) + CE(x^0, v^0)\right), \quad (3.7)
$$

for $t \in [0, T_0)$, due to Eq (3.5). Since the right hand sides of systems (3.6) and (3.7) are uniformly bounded in the time interval $[0, T_0)$, the life-span T_0 should be infinity, i.e., $T_0 = \infty$. This completes the proof.

Remark 1. *In Theorem 1, it is crucially used the fact that the system is posed in one dimension. However, Theorem 2 can also deal with higher dimensional problems, (see [41]).*

We conclude this section with a negative result characterizing a pathological configuration with 2 agents where the system blows up in finite time.

Theorem 3. *Let $\alpha \in (0, 1)$ and $N = 2$. Furthermore, we assume that δ_1, z_1, and the initial data $\{(x_i^0, v_i^0)\}_{i=1}^2$ satisfy $\delta_1 + z_1 \ge 0$, $x_2^0 > x_1^0 + \delta_1$, and*

$$
v_1^0 - v_2^0 = \frac{2}{1-\alpha}(x_2^0 - x_1^0 - \delta_1)^{1-\alpha}. \quad (3.8)
$$

Then, the smoothness of solutions to the systems (2.1) *and* (2.2) *breaks down in finite time.*

Proof. For the proof, it suffices to show that there exists a finite time $t_* < \infty$ such that $x_1(t_*) + \delta_1 = x_2(t_*)$. For notational simplicity, we set $x := x_2 - x_1$ and $v := v_2 - v_1$. Then, we easily find that x and v satisfy

$$
\begin{cases}
\dfrac{dx(t)}{dt} = v(t), \\
\dfrac{dv(t)}{dt} = -2(I_1(t) + u_1(t)) = -2\psi(x(t) - \delta_1)v - 2\phi(|x(t) + z_1|^2)(x(t) + z_1).
\end{cases}
$$

Note that the smooth solutions exist as long as $x(t) > \delta_1$, and this and the assumption $\delta_1 + z_1 \ge 0$ imply $x(t) + z_1 \ge 0$. Since $\phi \ge 0$, this implies that

$$
\frac{dv(t)}{dt} \le -2\psi(x(t) - \delta_1)v = -2\frac{d}{dt}\Psi(x(t) - \delta_1).
$$

Here, Ψ is the primitive of ψ, i.e.,

$$
\Psi(r) = \frac{1}{1-\alpha}r^{1-\alpha}.
$$

We then solve the above differential inequality to get

$$\frac{d(x(t)-\delta_1)}{dt} = v(t) \le -2\Psi(x(t)-\delta_1) = -\frac{2}{1-\alpha}(x(t)-\delta_1)^{1-\alpha} \tag{3.9}$$

due to system (3.8). We notice that the above differential inequality is sub-linear, and thus there exists $t_* < \infty$ such that $x(t_*) - \delta_1 = 0$. Indeed, we obtain from system (3.9) that

$$(x(t)-\delta_1)^\alpha \le (x^0-\delta_1)^\alpha - \frac{2\alpha}{1-\alpha}t.$$

Hence, we have

$$t_* \le \frac{(x^0-\delta_1)^\alpha}{2\alpha}(1-\alpha),$$

thus completing the proof.

4. Time-asymptotic behavior

Having characterized the well-posedness of the trajectories of system (2.1), we now turn our attention to the study of flocking emergence within the controlled string. In a flocking configuration, all agents travel with the same constant velocity, and as a direct consequence, the distance between agents remain constant. Let us recall our energy functionals

$$E_1(v) = \frac{1}{4N}\sum_{i,j=1}^{N}|v_i - v_j|^2 \quad \text{and} \quad E_2(x) = \frac{1}{2}\sum_{i=2}^{N}\int_0^{|x_{i-1}-x_i-z_{i-1}|^2}\phi(r)\,dr.$$

We provide a rigorous asymptotic flocking estimate for the system (2.1).

Theorem 4. *Suppose that either assumptions of Theorems 1 or 2 hold. Furthermore, we assume*

$$\int_0^\infty \phi(r)\,dr > \frac{1}{2N}\sum_{i,j=1}^{N}|v_i^0 - v_j^0|^2 + \sum_{i=2}^{N}\int_0^{|x_{i-1}^0-x_i^0-z_{i-1}|^2}\phi(r)\,dr. \tag{4.1}$$

Then, the string converges asymptotically towards a flocking state, that is

$$\sup_{0\le t\le\infty}\max_{i,j=1,\dots,N}|x_i(t)-x_j(t)| < \infty \qquad \textit{and} \qquad \max_{i,j=1,\dots,N}|v_i(t)-v_j(t)| \to 0$$

as $t \to \infty$.

Proof. From Theorems 1 or 2, existence and uniqueness of a smooth solution is found globally in time.

(Uniform-in-time boundedness): It follows from the energy estimate in Lemma 1 that

$$E_2(x(t)) \le E(x^0, v^0),$$

i.e.,

$$\sum_{i=2}^{N}\int_{|x_{i-1}^0-x_i^0-z_{i-1}|^2}^{|x_{i-1}(t)-x_i(t)-z_{i-1}|^2}\phi(r)\,dr \le \frac{1}{2N}\sum_{i,j=1}^{N}|v_i^0-v_j^0|^2 \quad \text{for} \quad t\ge 0. \tag{4.2}$$

On the other hand, under our major assumptions, we can find some constant $\rho > 0$ such that

$$\frac{1}{2N}\sum_{i,j=1}^{N}|v_i^0 - v_j^0|^2 + \sum_{i=2}^{N}\int_0^{|x_{i-1}^0 - x_i^0 - z_{i-1}|^2}\phi(r)\,dr \le \int_0^{\rho^2}\phi(r)\,dr.$$

This, together with system (4.2), yields

$$0 \le \frac{1}{2N}\sum_{i,j=1}^{N}|v_i^0 - v_j^0|^2 \le \int_{|x_{i-1}^0 - x_i^0 - z_{i-1}|^2}^{\rho^2}\phi(r)\,dr$$

for all $i = 2, \dots, N$. This implies that

$$|x_{i-1}(t) - x_i(t) - z_{i-1}| \le \rho \quad \text{for} \quad i = 2, \dots, N. \tag{4.3}$$

For any $i < j$, by telescoping and the triangle inequality, we estimate

$$|x_i - x_j| = \left|\sum_{\ell=i}^{j-1}(x_\ell - x_{\ell+1})\right| \le \sum_{\ell=i}^{j-1}|x_\ell - x_{\ell+1}| \le \sum_{\ell=i}^{j-1}|x_\ell - x_{\ell+1} - z_\ell| + \sum_{\ell=i}^{j-1}|z_\ell|,$$

and thus

$$|x_i - x_j| \le |j - i|\rho + \sum_{\ell=i}^{j-1}|z_\ell| \le (N-1)\rho + \sum_{i=1}^{N-1}|z_i| < \infty\,,$$

given the boundedness of distances between agents at all times.

(Velocity alignment behavior): From the bound above, we find

$$|x_i - x_{i-1}| - \delta_{i-1} \le |x_i - x_{i-1} - z_{i-1}| + |z_{i-1}| - \delta_{i-1} \le \rho + |z_{i-1}| + \delta_{i-1} \le \rho + \max_{i=1,\dots,N-1}(|z_i| + \delta_i).$$

Since ψ is monotonically decreasing, we obtain

$$\psi_m := \min_{2\le i\le N}\psi(|x_i - x_{i-1}| - \delta_{i-1}) \ge \psi\left(\rho + \max_{i=1,\dots,N-1}(|z_i| + \delta_i)\right) > 0.$$

This implies that the dissipation rate D is bounded from below by

$$D(x(t), v(t)) = \sum_{i=2}^{N}\psi(|x_i - x_{i-1}| - \delta_{i-1})(v_{i-1} - v_i)^2 \ge \psi_m\sum_{i=2}^{N}(v_{i-1} - v_i)^2.$$

Then, by Lemma 1, we get

$$\sum_{i=2}^{N}\int_0^{\infty}(v_{i-1}(t) - v_i(t))^2\,dt < \infty,$$

and subsequently, this leads to

$$\int_0^{\infty}E_1(v(t))\,dt = \frac{1}{4N}\sum_{i,j=1}^{N}\int_0^{\infty}|v_i(t) - v_j(t)|^2\,dt < \infty.$$

Indeed, by telescoping, for any $i < j$

$$|v_i - v_j| \leq \sum_{\ell=i}^{j-1} |v_\ell - v_{\ell+1}| \leq \sqrt{|i-j|} \sqrt{\sum_{\ell=i}^{j-1} |v_\ell - v_{\ell+1}|^2},$$

and thus

$$\sum_{i,j=1}^{N} |v_i - v_j|^2 \leq c_N \sum_{i=2}^{N} |v_{i-1} - v_i|^2, \quad \text{where} \quad c_N := \sum_{i,j=1}^{N} |i-j|. \tag{4.4}$$

Moreover, we also find

$$\begin{aligned} \left| \sum_{i=1}^{N} v_i u_i \right| &\leq \left| \sum_{i=2}^{N} \phi(|x_{i-1} - x_i - z_{i-1}|^2) \langle x_{i-1} - x_i - z_{i-1}, v_{i-1} - v_i \rangle \right| \\ &\leq \rho \sum_{i=2}^{N} |v_{i-1} - v_i| \leq C \sqrt{E(x^0, v^0)}, \end{aligned} \tag{4.5}$$

where C is independent of t, and we use

$$\max_{1\leq i,j\leq N} |v_i(t) - v_j(t)| \leq \sqrt{\sum_{i,j=1}^{N} |v_i(t) - v_j(t)|^2} \leq 2\sqrt{NE(x^0, v^0)}.$$

Furthermore, note that

$$\frac{1}{4N} \sum_{i,j=1}^{N} |v_i - v_j|^2 = \int_0^t (-D(x(s), v(s)))\, ds + \sum_{i=1}^{N} \int_0^t v_i(s) u_i(s)\, ds + \frac{1}{4N} \sum_{i,j=1}^{N} |v_i^0 - v_j^0|^2.$$

The dissipation rate D is integrable and, thus, the first term on the right side of the above equality is absolutely continuous. Regarding the second term, its time-derivative is uniformly bounded in time, see system (4.5), from where it follows that it is Lipschitz continuous. This implies that the $E_1(v(t))$ is the sum of an absolutely continuous function and a Lipschitz continuous function. Thus, we obtain that $E_1(v(t))$ is uniformly continuous. Since $E_1(v(t))$ is also integrable, then $E_1(v(t)) \to 0$ as $t \to \infty$. This completes the proof.

Remark 2. *If $\beta \leq 1$, then ϕ is not integrable, thus the left hand side of system* (4.1) *becomes infinity. This implies that the assumption system* (4.1) *automatically holds. On the other hand, if $\beta > 1$, we obtain*

$$\int_0^\infty \phi(r)\, dr = \int_0^\infty \frac{1}{(1+r)^\beta}\, dr = \frac{1}{\beta - 1},$$

and, thus, system (4.1) *can be rewritten as*

$$E(x_0, v_0) < \frac{1}{2(\beta - 1)}. \tag{4.6}$$

From these equivalences, it becomes evident that the fulfilment of the flocking condition depends only on two parameters of the model, namely, the number of agents N in the string and the control interaction constant $\beta > 1$, which regulates the strength of the control action. The constant α does not play a role on the condition. Having fixed a number of agents and β, flocking solely depends on the cohesiveness of the initial configuration.

Remark 3. *If we define Φ by the primitive of ϕ, then it is clear that $r \mapsto \Phi(r)$ is strictly increasing, and, thus, the constant ρ appearing in system* (4.3) *can be expressed by*

$$\rho = \sqrt{\Phi^{-1}\left(\frac{1}{2N}\sum_{i,j=1}^{N}|v_i^0 - v_j^0|^2 + \Phi(|x_{i-1}^0 - x_i^0 - z_{i-1}|^2)\right)}. \tag{4.7}$$

5. Exponential emergence of pattern formation and velocity alignment

In this section, we conclude our characterization of the string trajectories by studying the exponential emergence of pattern formation and velocity alignment behavior under additional assumptions on the solutions. We first provide an auxiliary result, a modification of Young's inequality, which can be proved by a similar argument as in [24, Lemma 6.1]. We thus omit its proof here.

Lemma 2. *Let $a_1, \dots, a_{N-1}$ be a set of vectors in $\mathbb{R}^d$ and $b_1, \dots, b_{N-1}$ be a set of positive scalars. Then,*

$$-\sum_{i=1}^{N-1} b_i|a_i|^2 + \sum_{i=1}^{N-2} b_i\langle a_i, a_{i+1}\rangle \leq -\epsilon_0 \sum_{i=1}^{N-1} b_i|a_i|^2,$$

where $\epsilon_0 \in (0, 1)$ is a sufficiently small number.

We now state our main result, which provides non-collisional behavior, flocking, and an exponential decay estimate towards the string configuration encoded in the relative distance vector z.

Proposition 1. *Suppose that the assumptions of Theorem 4 are satisfied. Furthermore, we assume that*

$$\inf_{t\geq 0} \min_{1\leq i\leq N-1} (|x_i(t) - x_{i+1}(t)| - \delta_i) > 0\,. \tag{5.1}$$

Then, we have

$$\max_{i=2,\dots,N} |x_{i-1}(t) - x_i(t) - z_{i-1}| + \max_{i,j=1,\dots,N} |v_i(t) - v_j(t)| \to 0$$

exponentially fast as $t \to \infty$.

Proof. We first notice that the energy estimate in Lemma 1 provides only a dissipation rate for the velocity. In order to have a complete exponential decay estimate, it is required to obtain the dissipation rate associated to the positions. For this, we consider the following quantity:

$$\sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(v_i - v_{i+1}).$$

Note that the total energy is bounded from below and above by

$$\frac{1}{4N}\sum_{i,j=1}^{N}|v_i - v_j|^2 + \phi_m \sum_{i=1}^{N-1}|x_i - x_{i+1} - z_i|^2 \leq E(x, v) \leq \frac{1}{4N}\sum_{i,j=1}^{N}|v_i - v_j|^2 + \sum_{i=1}^{N-1}|x_i - x_{i+1} - z_i|^2,$$

where we use

$$\phi_m := \min_{s\in[0,\rho]} \phi(s) \leq \phi(r) \leq 1.$$

This shows that a modified total energy E_γ, defined as

$$E_\gamma(x,v) := \gamma E(x,v) + \sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(v_i - v_{i+1}),$$

has similar upper and lower bound estimates as the one for $E(x,v)$ when $\gamma > 0$ is large enough. Indeed, for $\gamma > \sqrt{2N/\phi_m}$, we readily find

$$\begin{aligned} E_\gamma(x,v) &\geq \frac{\gamma\phi_m}{2}\sum_{i=1}^{N-1}|x_i - x_{i+1} - z_i|^2 + \left(\frac{\gamma}{4N} - \frac{1}{2\gamma\phi_m}\right)\sum_{i,j=1}^{N}|v_i - v_j|^2 \\ &\geq c_\gamma\left(\sum_{i=1}^{N-1}|x_i - x_{i+1} - z_i|^2 + \sum_{i,j=1}^{N}|v_i - v_j|^2\right), \end{aligned} \tag{5.2}$$

where $c_\gamma > 0$ is given by

$$c_\gamma := \min\left\{\frac{\gamma\phi_m}{2}, \frac{\gamma}{4N} - \frac{1}{2\gamma\phi_m}\right\}.$$

The upper bound on E_γ can be easily obtained. On the other hand, it follows from system (2.1) that

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(v_i - v_{i+1}) = &\sum_{i=1}^{N-1}(v_i - v_{i+1})^2 + \sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(I_i - I_{i+1}) + \\ &\sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(u_i - u_{i+1}). \end{aligned} \tag{5.3}$$

Since

$$|I_1| \leq \psi_M|v_2 - v_1|, \quad |I_N| \leq \psi_M|v_{N-1} - v_N|, \quad \text{and} \quad |I_i| \leq \psi_M(|v_{i-1} - v_i| + |v_i - v_{i+1}|)$$

for $i = 2, \dots, N-1$, we easily find

$$\sum_{i=1}^{N-1}|I_i - I_{i+1}|^2 \leq 2\left(|I_1|^2 + 2\sum_{i=2}^{N-1}|I_i|^2 + |I_N|^2\right) \leq 16\,\psi_M^2\sum_{i=1}^{N-1}|v_i - v_{i+1}|^2,$$

where $\psi_M := \sup_{r\in[0,\infty)}\psi(r) < \infty$, which can be defined by the assumption (5.1). Thus, using Young's inequality, we have

$$\sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(I_i - I_{i+1}) \leq \epsilon_1\sum_{i=1}^{N-1}|x_i - x_{i+1} - z_i|^2 + \frac{4\psi_M^2}{\epsilon_1}\sum_{i=1}^{N-1}|v_i - v_{i+1}|^2, \tag{5.4}$$

where ϵ_1 will be determined later. Regarding the term with u_i, we estimate

$$\begin{aligned} \sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(u_i - u_{i+1}) = &-2\sum_{i=1}^{N-1}\phi(|x_i - x_{i+1} - z_i|^2)|x_i - x_{i+1} - z_i|^2 \\ &+ 2\sum_{i=1}^{N-2}\phi(|x_i - x_{i+1} - z_i|^2)(x_i - x_{i+1} - z_i)(x_{i+1} - x_{i+2} - z_{i+1}). \end{aligned}$$

We then use Lemma 2 with $b_i = 2\phi(|x_i - x_{i+1} - z_i|^2)$ and $a_i = x_i - x_{i+1} - z_i$ to get

$$\begin{aligned}\sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(u_i - u_{i+1}) &\le -2\epsilon_0 \sum_{i=1}^{N-1} \phi(|x_i - x_{i+1} - z_i|^2)|x_i - x_{i+1} - z_i|^2 \\ &\le -2\epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2,\end{aligned}$$

where ϵ_0 is given in Lemma 2. This, together with systems (5.3) and (5.4), and choosing $\epsilon_1 = \epsilon_0\phi_m$, implies

$$\frac{d}{dt}\sum_{i=1}^{N-1}(x_i - x_{i+1} - z_i)(v_i - v_{i+1}) \le -\epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 + \frac{4\psi_M^2}{\epsilon_0\phi_m}\sum_{i=1}^{N-1} |v_i - v_{i+1}|^2.$$

Thus, the modified total energy E_γ satisfies

$$\frac{d}{dt}E_\gamma(x, v) \le -\left(\gamma\psi_m - \frac{4\psi_M^2}{\epsilon_0\phi_m}\right)\sum_{i=1}^{N-1} |v_i - v_{i+1}|^2 - \epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2.$$

By taking $\gamma > \max\{(4\psi_M^2)/(\epsilon_0\phi_m\psi_m), \sqrt{2N/\phi_m}\}$ and using systems (4.4) and (5.2), we further estimate

$$\begin{aligned}\frac{d}{dt}E_\gamma(x, v) &\le -\frac{1}{c_N}\left(\gamma\psi_m - \frac{4\psi_M^2}{\epsilon_0\phi_m}\right)\sum_{i,j=1}^{N} |v_i - v_j|^2 - \epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 \\ &\le -\frac{1}{c_\gamma}\min\left\{\frac{1}{c_N}\left(\gamma\psi_m - \frac{4\psi_M^2}{\epsilon_0\phi_m}\right), \epsilon_0\phi_m\right\} E_\gamma(x, v).\end{aligned}$$

Applying Grönwall's lemma to the above gives the exponential decay of the modified total energy E_γ. Moreover, the relation (5.2) concludes the desired result.

Remark 4. *The a priori assumption* (5.1) *used in Proposition 1 requires that the distance between neighboring agents always exceeds the collision threshold δ_i. While this assumption may appear restrictive, it can in fact be ensured by a suitable choice of the desired formation parameters, z_i. For instance, if we fix the order for the initial positions as $x_i^0 + \delta_i < x_{i+1}^0$ for $i = 1, \dots, N-1$, then, by Theorem 2, we have $x_i(t) + \delta_i < x_{i+1}(t)$ for all $t \ge 0$. This implies that in order to have the time-asymptotic pattern formation, z_i and δ_i should satisfy $z_i > \delta_i$ for all $i = 1, \dots, N-1$. In this case, if we further assume that*

$$z_{i-1} > \rho + \delta_{i-1}, \quad i = 2, \dots, N,$$

where ρ is given as in system (4.7)*, then*

$$|x_i(t) - x_{i-1}(t)| - \delta_{i-1} \ge z_{i-1} - \rho - \delta_{i-1} > 0$$

for all $t \ge 0$ and all $i = 2, \dots, N$. Indeed, it follows from system (4.3) *that*

$$|x_i(t) - x_{i-1}(t)| = |x_i(t) - x_{i-1}(t) - z_{i-1} + z_{i_1}| \ge z_{i-1} - \rho,$$

thus subtracting δ_{i-1} *from both sides gives that assumption* (5.1) *holds. Thus, the exponential convergence result in Proposition 1 can be obtained under verifiable structural contraints on* (z_i, δ_i) *and the initial data.*

We note that it would be an interesting direction for future work to establish the validity of system (5.1) *without relying on such parameter restrictions, possibly by identifying additional dissipative mechanisms or exploiting the system's long-time behavior.*

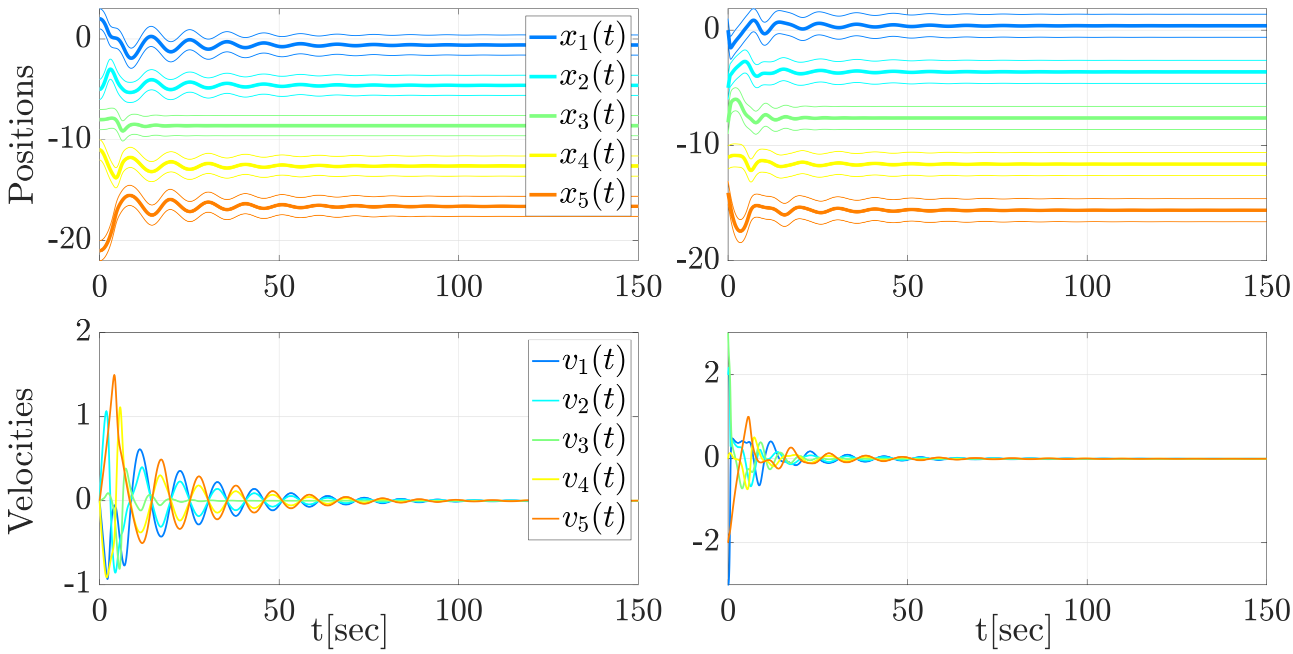

Figure 2. Positions over time for 5 particles on the line. Left: Case 1) Agents initially at rest and at non-collided positions. Right: Case 2) Agents with non-zero initial velocities and at close proximity but not collided. Thin lines represent the *volumes* of the agents. No collisions occur due to the singular interactions, and the desired formation is acquired in steady state. It can also be noted that the velocities seem to converge exponentially fast. Note that the *volumes* seem to change in time but this is only due to the chosen visualization style.

6. Numerical examples

In the following, we further elucidate the applicability of our results through two numerical experiments illustrating string flocking to a pattern formation, as well as energy evolution.

6.1. Regular collision-less behavior of the interconnected system

For simplicity in the visualization, we first consider a collection of $N = 5$ agents. We assume that the agents are in the desired order, that is, the final configuration of the agents does not require a collision to occur. For the model parameters, we select $\alpha = 2.1$, $\beta = 0.8$ and $\delta = 2$, equal for all inter-agent barriers. We consider two cases for the agents' initial conditions: 1) The agents are initially at rest and located at non-collided positions; and 2) the agents have different initial velocities with

$v_c(0) = 0$ and are close to each other but not colliding. In both cases, the desired inter-agent spacings are given by $z_i = \delta + 2$ (note that δ is added to avoid configurations that are collided in steady state). In such a scenario, the agents should reposition themselves to reach the consensus velocity of zero and the desired pattern. Figure 2 illustrates both cases. As predicted by Theorems 2 and 4, and given that $\alpha > 2$ and $\beta < 1$ with no initial collisions, we have that the agents do not collide as time progresses, even when they are initially almost touching. Moreover, the agents reach the desired inter-agent spacings and they reach flocking exponentially fast.

Now, for the same parameters as before, Figure 3 illustrates the behaviour of the multi-agent system when $v_c(0) = -0.2$. We can observe that the statement of Lemma 1 is satisfied over the time evolution of the dynamics, that is, the total energy of the system is non-increasing, and the dissipation rate is entirely determined by the interaction term (see Figure 3 bottom-left). Some agents are very close to the interaction limit determined by $\delta = 2$, at $t = 0[sec]$, which is highlighted by Figure 3 top-right. Then, agents 2 and 3 approach each other for a short period of time, and around $t = 10[sec]$ all the agents begin to spread out to reach the desired formation, with an average velocity of -0.2 in steady state.

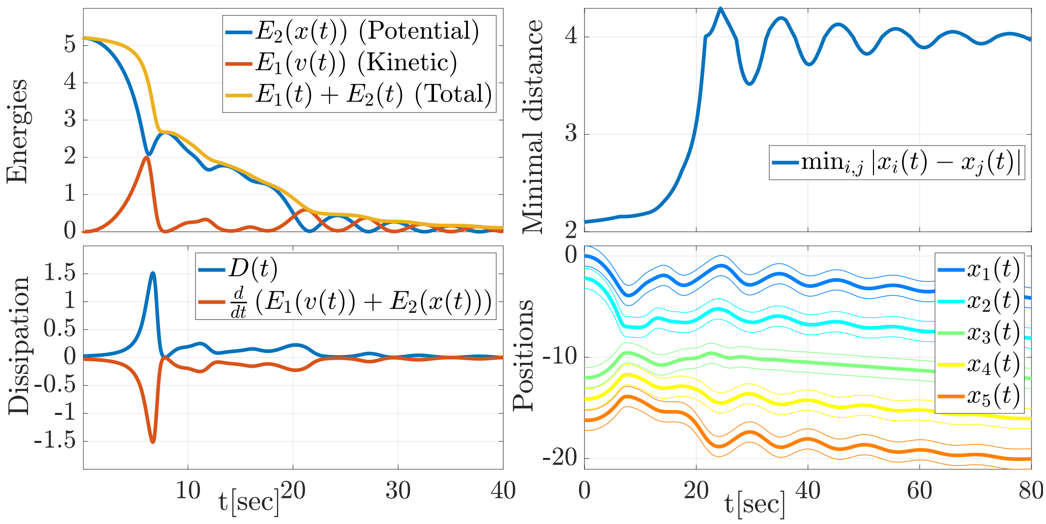

Figure 3. 5 particles on the line with non-zero average velocity. Left-Top: Energy decomposition; Left-Bottom: Dissipation; Right-Top: Minimal distance between agents; and Right-Bottom: Positions over time of the particles achieving a consensus speed and the desired spatial formation as the system evolves. Note that the energies satisfy statement (ii) in Lemma 1.

6.2. Varying barriers and small violation of the flocking conditions

We now select the system parameters in order to study what occurs when the flocking condition in Theorem 4 is violated. For $N = 5$, $\alpha = 2.1$, $\beta = 1.1$ and non-symmetric barriers given by $\delta_i = \{2.55, 2.15, 0.65, 1.05\}$. We consider the case where agents have non-zero initial velocities with $v_c(0) = 0$ and are initially positioned at non-colliding locations. The desired inter-agent spacings are given by $z_i = \{6, 6, 6, 3\}$. The time evolution for this case can be found in Figure 4.

According to system (4.6), since $\beta > 1$, flocking is guaranteed when $E(x_0, v_0) < \frac{1}{2(\beta-1)}$. For our chosen parameters, the theoretical threshold is $\frac{1}{2(1.1-1)} = 5$, but our initial configuration yields $E(x_0, v_0) \approx 5.2$, slightly exceeding this bound.

Despite violating the energy condition, the numerical results show that agents achieve velocity consensus and converge to the desired formation. This does not contradict Theorem 4, as the theorem provides a sufficient but not necessary condition for flocking. Collisions are avoided through the singular interactions while the system successfully reaches the desired pattern formation in steady state.

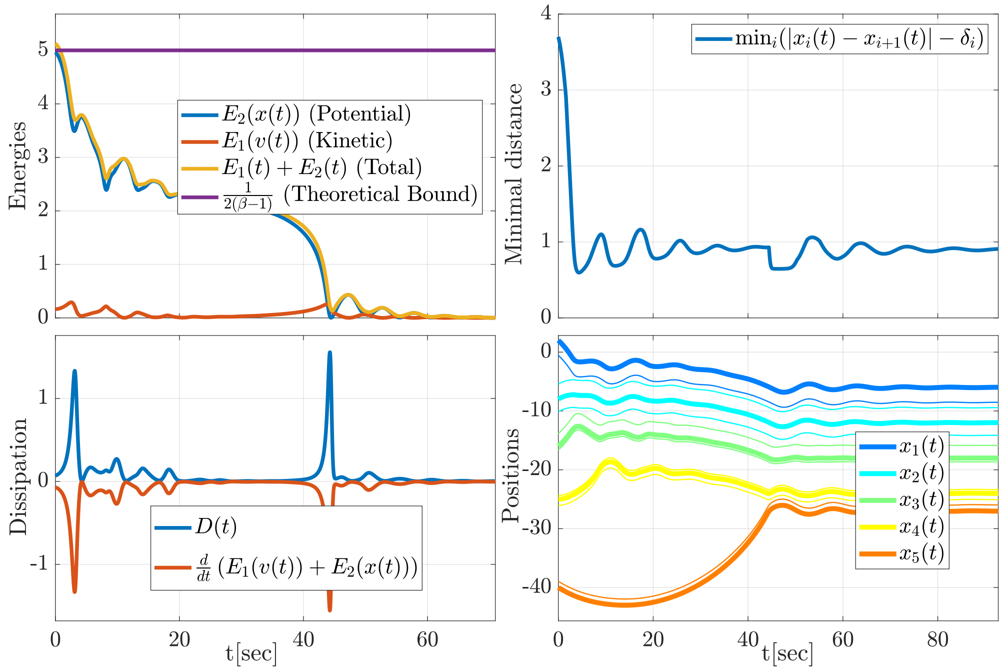

Figure 4. Five particles on the line with zero average velocity. Left-Top: Energy decomposition; Left-Bottom: Dissipation; Right-Top: Minimal distance between agents; and Right-Bottom: Positions over time of the particles achieving a consensus speed of zero and the desired spatial formation as the system evolves. Note that the initial configuration violates system (4.6).

6.3. *Flocking and formation acquisition for large violations of the flocking condition*

We now consider $N = 10$ agents with $\alpha = 2.2$ and $\delta_i = 0$ for all i. For the same initial conditions with $v_c(0) = 0$, we consider two cases: 1) $\beta = 4.1$ and 2) $\beta = 1.025$, which are presented in Figures 5 and 6, respectively. It can be seen that the parameter β influences the value of the initial energies, as noted earlier. However, the kinetic energy is positive and the same in both cases, that is, the agents are not initially moving with the same velocity. For $\beta = 4.1$, condition (4.1) of Theorem 4 is not satisfied, as the value $0.5/(4.1 - 1) \approx 0.1613$ is less than the initial energy of the system. We can see in Figure 5 top-right that the system does not achieve flocking nor the desired formation. Some errors $x_i(t) - x_{i+1}(t) - z_i$ diverge and we can appreciate clustering. On the other hand, for $\beta = 1.025$, condition (4.1) is met, and we have that the system does achieve the desired formation with all the errors $|x_i(t) - x_{i+1}(t) - z_i| \to 0$ as the system evolves. Note that the uncontrolled system is plotted at the bottom-right corner in both cases for comparison, that is, with $u_i(t) = 0$ for all t, i.

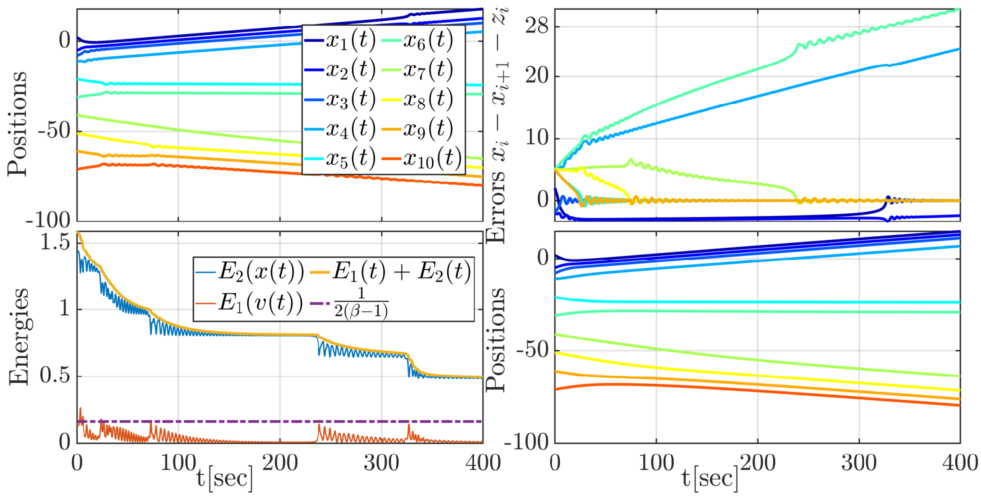

Figure 5. Ten particles on the line with zero average velocity when $\beta = 4.1$. Left-Top: Positions over time of the particles when the control is used; Left-Bottom: Energy decomposition and flocking condition for $\beta > 1$; and Right-Top: Errors from the desired formation $x_i - x_{i+1} - z_i$; Right-Bottom: Positions over time of the particles when the control is not used. Flocking does not occur, although collisions are avoided.

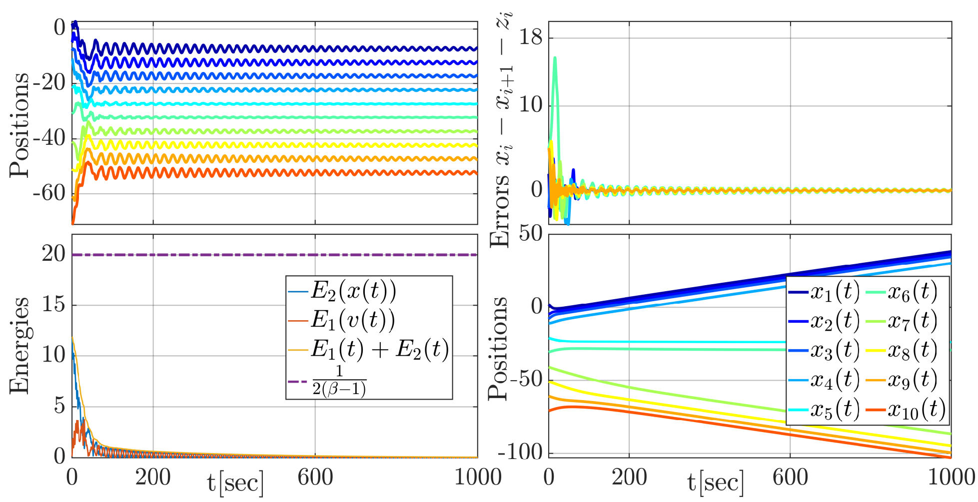

Figure 6. Ten particles on the line with zero average velocity when $\beta = 1.025$. Left-Top: Positions over time of the particles when the control is used; Left-Bottom: Energy decomposition and flocking condition for $\beta > 1$; and Right-Top: Errors from the desired formation $x_i - x_{i+1} - z_i$; Right-Bottom: Positions over time of the particles when the control is not used. As predicted by Theorem 4, given that the initial total energy satisfies (4.6), no collisions occur, and the desired formation is achieved in steady state.

7. Conclusions

We have presented a control system for platooning composed by a string of agents interacting under nonlinear singular dynamics and a decentralized feedback law. The resulting closed-loop exhibits important features for platooning control, namely, collision-avoidance, velocity flocking, and asymptotic pattern formation. The derivation of rigorous energy estimates enable the characterization of conditions under which the aforementioned features are guaranteed. Energy estimates are governed by: The number of agents in the string, the strength of the control interaction term expressed through parameter β in system (2.3), and the cohesiveness of the initial configuration. In particular, the dependence with respect to the number of agents is a relevant topic of interest for future research. Although our results are asymptotic, we have observed the transient behaviour of the control system and it exhibits similarities to linear-time invariant platooning, namely slow transients, as the number of agents increases. The energy analysis we presented can be extended to study mean field dynamics arising when $N \to \infty$, and the system is characterized by an agent density function [3]. Although the applicability of the mean field framework seems inadequate from a safety viewpoint as collision-avoidance is an eminently microscopic phenomenon, it can be a powerful mathematical method to further understand the large-scale structure of the control system.

Author contributions

All authors contributed equally to conceptualisation, formal analysis, computational experiments, and writing of this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Young-Pil Choi and Dante Kalise are Editors of the Editorial Board for [Networks and Heterogeneous Media] and were not involved in the editorial review or the decision to publish this article. The authors declare there are no conflict of interest.

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