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Research article

Homogenization of attractors to reaction–diffusion equations in domains with rapidly oscillating boundary: Critical case

Gaziz F. Azhmoldaev[1], **Kuanysh A. Bekmaganbetov**[2,3], **Gregory A. Chechkin**[4,5,*] **and Vladimir V. Chepyzhov**[6,7]

[1] Department of Fundamental Mathematics, L.N. Gumilyov Eurasian National University, Astana, Kazakhstan

[2] Department of Fundamental and Applied Mathematics, Kazakhstan Branch of M.V. Lomonosov Moscow State University, Astana, Kazakhstan

[3] Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

[4] Department of Differential Equations, Faculty of Mechanics and Mathematics, M.V. Lomonosov Moscow State University, Moscow, Russia

[5] Institute of Mathematics with Computing Center Subdivision of the Ufa Federal Research Center of Russian Academy of Science, Ufa, Russia

[6] Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia

[7] Higher School of Economics (NRU), Nizhny Novgorog, Russia

* **Correspondence:** Email: chechkin@mech.math.msu.su; Tel: +7 (910) 4341159.

Abstract: In the present paper, reaction–diffusion systems (RD-systems) with rapidly oscillating coefficients and righthand sides in equations and in boundary conditions were considered in domains with locally periodic oscillating (wavering) boundary. We proved a weak convergence of the trajectory attractors of the given systems to the trajectory attractors of the limit (homogenized) RD-systems in domain independent of the small parameter, characterizing the oscillation rate. We consider the critical case in which the type of boundary condition was preserved. For this aim, we used the approach of Chepyzhov and Vishik concerning trajectory attractors of evolutionary equations. Also, we applied the homogenization (averaging) method and asymptotic analysis to derive the limit (averaged) system and to prove the convergence. Defining the appropriate axillary functional spaces with weak topology, we proved the existence of trajectory attractors for these systems. Then, we formulated the main theorem and proved it with the help of auxiliary lemmata.

Keywords: attractors; homogenization; reaction–diffusion equations; nonlinear PDE, convergence in a weak sense; rapidly oscillating (wavering) boundary

1. Introduction

In the paper, one can find the homogenization problem for reaction–diffusion (RD) equations in domains with very rapidly wavering boundary (for detailed geometric settings [1]). We prove the existence of trajectory attractors and also obtain the convergence of the attractors as the small parameter, characterizing the oscillations, goes to zero, i.e., we prove the Hausdorff convergence of attractors as the small parameter goes to zero. Thus, we construct the limit attractor and prove the convergence of the attractors of the given problem to the attractor of the limit problem. In many pure mathematical papers, one can find the asymptotic methods applying to problems in domains with wavering (rough) boundaries (see, for example, rapidly oscillating boundaries in [1–5], fractal boundaries in [6], diffusivity through rough boundaries in [7], rapidly oscillating type of boundary conditions on oscillating (wavering) boundaries in [8, 9], boundaries with many thin rods in [10–13]). We want to mention here the basic frameworks [14–18] where one can find the detailed bibliography.

Concerning attractors, see, for instance, [19–21] and the references in these monographs. Homogenization of attractors were studied in [21–24] and applications of this theory were investigated in [25–28].

In this paper, we proved the weak convergence of the trajectory attractor $\mathfrak{A}_\epsilon$ to the RD-systems in domains with wavering boundary, as $\epsilon \to 0$, to the trajectory attractors $\overline{\mathfrak{A}}$ of homogenized systems in some natural functional space. Here, the small parameter ϵ characterizes the period and the amplitude of the oscillations. The parameter ϵ is included also in a Fourier condition on a part of the boundary, and we consider the case when the type of this condition is preserved (critical case).

Note that the subcritical case (the case of the Neumann homogenized condition) and supercritical (the case of the Dirichlet homogenized condition) are also interesting, but we suppose to study them in independent papers.

Section 2 is devoted to basic settings. In Section 3, one can find the framework of the theory of attractors. In Section 4, we describe the limiting (homogenized) RD-system and its trajectory attractor. Section 5 contains auxiliary results, and Section 6 is connected with the proof of the main result.

2. Settings of the problem

Suppose that D is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2$, where D lies in a semi-space $\{x_d > 0\}$ and $\Gamma_1 \subset \{x : x_d = 0\}$. Given a smooth nonpositive 1-periodic in the $\tilde{y}$ function $F(\tilde{x}, \tilde{y})$, $\tilde{x} = (x_1, ..., x_{d-1})$, $\tilde{y} = (y_1, ..., y_{d-1})$, we define the domain D_ϵ as follows: $\partial D_\epsilon = \Gamma_1^\epsilon \cup \Gamma_2$, where we set $\Gamma_1^\epsilon = \{x = (\tilde{x}, x_d) : (\tilde{x}, 0) \in \Gamma_1, x_d = \epsilon^\alpha F(\tilde{x}, \tilde{x}/\epsilon)\}$, $0 < \alpha < 1$, i.e., we add the thin oscillating layer $\Pi_\epsilon = \{x = (\tilde{x}, x_d) : (\tilde{x}, 0) \in \Gamma_1, x_d \in [0, \epsilon^\alpha F(\tilde{x}, \tilde{x}/\epsilon))\}$ to the domain D. Usually, we assume that $F(\tilde{x}, \tilde{y})$ is compactly supported on Γ_1 uniformly in $\tilde{y}$. Consider the problem

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} = \mathcal{A}\Delta u_\epsilon - a\left(x, \frac{x}{\epsilon}\right) f(u_\epsilon) + h\left(x, \frac{x}{\epsilon}\right), & x \in D_\epsilon, t > 0, \\ \frac{\partial u_\epsilon}{\partial \nu} + \epsilon^\beta p(\tilde{x}, \frac{\tilde{x}}{\epsilon}) u_\epsilon = \epsilon^{1-\alpha} g(\tilde{x}, \frac{\tilde{x}}{\epsilon}), & x = (\tilde{x}, x_d) \in \Gamma_1^\epsilon, t > 0, \\ u_\epsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\epsilon = U(x), & x \in D_\epsilon, t = 0, \end{cases} \tag{2.1}$$

where $u_\epsilon = u_\epsilon(x, t) = (u^1, \ldots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \ldots, f^n)^\top$ is given, $h = (h^1, \ldots, h^n)^\top$ is the known righthand side function, and $\mathcal{A}$ is an

$n \times d$-matrix with constant coefficients having a positive symmetrical part: $\frac{1}{2}(\mathcal{A}+\mathcal{A}^\top) \geq \varpi I,\ \varpi > 0$ (where I is the unit matrix with dimension n). We assume that $p(\tilde{x},\tilde{y}) = \operatorname{diag}\{p^1,\dots,p^n\}$, $g(\tilde{x},\tilde{y}) = (g^1,\dots,g^n)^\top$ are continuous, 1-periodic in $\tilde{y}$, and $p^i(\tilde{x},\tilde{y})$, $i = 1,\dots n$, are positive. Here, $\frac{\partial}{\partial\nu}$ is the co-normal derivative of the function, i.e., $\frac{\partial}{\partial\nu} := \sum\limits_{k,j=1}^{d} \mathcal{A}_{kj}\frac{\partial}{\partial x_k} N_j$ and $N = (N_1,\dots,N_d)$ is the outward normal vector to the boundary of the domain with unit length. We denote the maximum of p on Γ_1 by $p_{\max}$.

The function $a(x,y) \in C(\overline{D}_\epsilon \times \mathbb{R}^d)$ is such that $0 < a_1 \leq a(x,y) \leq a_2$ with some coefficient a_1, a_2. We assume that function $a_\epsilon(x) = a\left(x,\frac{x}{\epsilon}\right)$ has an average $\overline{a}(x)$ when $\epsilon \to 0+$ in space $L_{\infty,*w}(D)$, that is,

$$\int\limits_D a\left(x,\frac{x}{\epsilon}\right)\varphi(x)dx \to \int\limits_D \overline{a}(x)\varphi(x)dx \quad (\epsilon \to 0+), \tag{2.2}$$

for each $\varphi \in L_1(D)$.

Denote by V (respectively, V_ϵ) the Sobolev space $H^1(D,\Gamma_2)$ (respectively, $H^1(D_\epsilon,\Gamma_2)$), i.e., the space of functions from the Sobolev space $H^1(D)$ (respectively, $H^1(D_\epsilon)$) with zero trace on Γ_2. We also denote by V' (respectively, V'_ϵ) the dual space for V (respectively, V_ϵ), i.e., the space of linear bounded functionals on V (respectively V_ϵ). For vector function $h(x,y)$, assume that for any $\epsilon > 0$, function $h^i_\epsilon(x) = h^i\left(x,\frac{x}{\epsilon}\right) \in L_2(D_\epsilon)$ and it has an average $\overline{h^i}(x)$ in space $L_2(D_\epsilon)$ for $\epsilon \to 0+$, that is,

$$h^i\left(x,\frac{x}{\epsilon}\right) \rightharpoonup \overline{h^i}(x) \quad (\epsilon \to 0+) \ \text{ weakly in } \ L_2(D_\epsilon),$$

or

$$\int\limits_D h^i\left(x,\frac{x}{\epsilon}\right)\varphi(x)dx \to \int\limits_D \overline{h^i}(x)\varphi(x)dx \quad (\epsilon \to 0+), \tag{2.3}$$

for each function $\varphi \in L_2(D)$ and $i = 1,\dots,\ n$.

From the condition (2.3), it follows that the norms of functions $h^i_\epsilon(x)$ are bounded uniformly in ϵ, in the space $L_2(D_\epsilon)$, i.e.,

$$\|h^i_\epsilon(x)\|_{L_2(D_\epsilon)} \leq M_0, \quad \forall\epsilon \in (0,1]. \tag{2.4}$$

We suppose that the nonlinearity $f(w)$ is continuous, i.e., $f(w) \in C(\mathbb{R}^n;\mathbb{R}^n)$, and this function satisfies

$$\sum_{k=1}^{n} |f^k(w)|^{\frac{p_k}{(p_k-1)}} \leq M_0\left(\sum_{k=1}^{n} |w^k|^{p_k} + 1\right), \quad 2 \leq p_1 \leq \ldots \leq p_{n-1} \leq p_n, \tag{2.5}$$

$$\sum_{k=1}^{n} \gamma_k |w^k|^{p_k} - M_1 \leq \sum_{k=1}^{n} f^k(w)w^k, \quad \forall w \in \mathbb{R}^n, \tag{2.6}$$

for $\gamma_k > 0$ for any $k = 1,\dots,n$. The inequality (2.5) is due to the fact that in real RD-systems, the functions $f^k(w)$ are polynomials with possibly different degrees. The inequality (2.6) is called the *dissipativity condition* for the RD-system (2.1). In a simple model case $p_k \equiv p$ for each $k = 1,\dots,n$, bounds (2.5) and (2.6) are reduced to the following:

$$|f(w)| \leq M_0\left(|w|^{p-1} + 1\right), \quad \gamma|w|^p - M_1 \leq f(w)w, \quad \forall w \in \mathbb{R}^n. \tag{2.7}$$

Note that the fulfillment of the Lipschitz condition for the function $f(w)$ in the variable w is *not supposed*.

Remark 2.1. *Using the presented methods, it is also possible to study systems in which nonlinear terms look as follows:* $\sum_{k=1}^{m} a_k\left(x, \frac{x}{\epsilon}\right) f_k(w)$, *where* a_k *are matrices of the elements of which admit averaging and* $f_k(w)$ *are polynomial vectors of* w, *which satisfy conditions of the form* (2.5) *and* (2.6). *For brevity, we study the case* $m = 1$ *and* $a_1\left(x, \frac{x}{\epsilon}\right) = a\left(x, \frac{x}{\epsilon}\right) I$, *where* I *is the identity matrix.*

Denote

$$G(\tilde{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\tilde{y}} F(\tilde{x}, \tilde{y})|^2}\, g(\tilde{x}, \tilde{y})\, d\tilde{y}, \tag{2.8}$$

$$P(\tilde{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\tilde{y}} F(\tilde{x}, \tilde{y})|^2}\, p(\tilde{x}, \tilde{y})\, d\tilde{y}. \tag{2.9}$$

Note that $P(\tilde{x})$ is positive due to the positiveness of p. We have the convergences (see [1] and Section 5 of this paper)

$$\epsilon^{1-\alpha} \int\limits_{\Gamma_1^\epsilon} g^i\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right) \cdot \upsilon\left(\tilde{x}, \epsilon^\alpha F\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right)\right) ds \to \int\limits_{\Gamma_1} G^i(\tilde{x}) \cdot \upsilon(x)\, ds, \tag{2.10}$$

and

$$\epsilon^{1-\alpha} \int\limits_{\Gamma_1^\epsilon} p^i\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right) \upsilon\left(\tilde{x}, \epsilon^\alpha F\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right)\right) ds \to \int\limits_{\Gamma_1} P^i(\tilde{x})\, \upsilon(x)\, ds, \tag{2.11}$$

for each $\upsilon \in H^1(D_\epsilon)$ by $\epsilon \to 0$. Here, ds is the element of $(d-1)$-dimensional measure on the hypersurface.

In the further analysis we use the following notation for the spaces $\mathbf{U} := [L_2(D)]^n$, $\mathbf{U}_\epsilon := [L_2(D_\epsilon)]^n$, $\mathbf{W} := [H^1(D, \Gamma_2)]^n$, $\mathbf{W}_\epsilon := [H^1(D_\epsilon; \Gamma_2)]^n$. The norms in our spaces are defined in the following way:

$$\|v\|^2 := \int\limits_D \sum_{i=1}^n |v^i(x)|^2 dx, \qquad \|v\|_\epsilon^2 := \int\limits_{D_\epsilon} \sum_{i=1}^n |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int\limits_D \sum_{i=1}^n |\nabla v^i(x)|^2 dx, \qquad \|v\|_{1,\epsilon}^2 := \int\limits_{D_\epsilon} \sum_{i=1}^n |\nabla v^i(x)|^2 dx.$$

Denote by $\mathbf{W}'$ the dual space to the space $\mathbf{W}$, and by $\mathbf{W}'_\epsilon$ the dual space to the space $\mathbf{W}_\epsilon$.

Let $q_k = \frac{p_k}{(p_k-1)}$ for any $k = 1, \ldots, n$. We use the notation $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$, and define spaces

$$\mathbf{V_p} := L_{p_1}(D) \times \ldots \times L_{p_n}(D), \;\; \mathbf{V}_{\mathbf{p},\epsilon} := L_{p_1}(D_\epsilon) \times \ldots \times L_{p_n}(D_\epsilon),$$

$$\mathbf{V_p}(\mathbb{R}_+; \mathbf{V_p}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(D)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(D)),$$

$$\mathbf{V_p}(\mathbb{R}_+; \mathbf{V}_{\mathbf{p},\epsilon}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(D_\epsilon)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(D_\epsilon)).$$

As in [21, 29], we investigate weak (generalized) solutions of the problem (2.1), that is, functions

$$u_\epsilon(x,t) \in \mathbf{V}_\infty^{loc}(\mathbb{R}_+; \mathbf{U}_\epsilon) \cap \mathbf{V}_2^{loc}(\mathbb{R}_+; \mathbf{W}_\epsilon) \cap \mathbf{V}_\mathbf{p}^{loc}\left(\mathbb{R}_+; \mathbf{V}_{\mathbf{p},\epsilon}\right),$$

which satisfy the Eq (2.1) in the distributional sense (the sense of generalized functions), that is, the integral identity

$$-\int\limits_{D_\epsilon\times\mathbb{R}_+} u_\epsilon\cdot\frac{\partial\psi}{\partial t}\,dxdt+\int\limits_{D_\epsilon\times\mathbb{R}_+}\mathcal{A}\nabla u_\epsilon\cdot\nabla\psi\,dxdt+\int\limits_{D_\epsilon\times\mathbb{R}_+}a_\epsilon(x)f(u_\epsilon)\cdot\psi\,dxdt+$$
$$\epsilon^\beta\int\limits_{\Gamma_1^\epsilon\times\mathbb{R}_+}p\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)u_\epsilon\cdot\psi\,dsdt=\int\limits_{D_\epsilon\times\mathbb{R}_+}h_\epsilon(x)\cdot\psi\,dxdt+\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon\times\mathbb{R}_+}g\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)\cdot\psi\,dsdt, \tag{2.12}$$

for each function $\psi\in\mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{W}_\epsilon\cap\mathbf{V}_{\mathbf{p},\epsilon})$. Here, $z_1\cdot z_2$ denotes the scalar product of vectors $z_1,z_2\in\mathbb{R}^n$.

If $u_\epsilon(x,t)\in\mathbf{V}_\mathbf{p}(0,M;\mathbf{V}_{\mathbf{p},\epsilon})$, then from the condition (2.5) it follows that $f(u(x,t))\in\mathbf{V}_\mathbf{q}(0,M;\mathbf{V}_{\mathbf{q},\epsilon})$. At the same time, if $u_\epsilon(x,t)\in\mathbf{V}_2(0,M;\mathbf{W}_\epsilon)$, then $\mathcal{A}\Delta u_\epsilon(x,t)+h_\epsilon(x)\in\mathbf{V}_2(0,M;\mathbf{W}'_\epsilon)$. Therefore, for an arbitrary generalized solution $u_\epsilon(x,s)$ to problem (2.1), it satisfies

$$\frac{\partial u_\epsilon(x,t)}{\partial t}\in\mathbf{V}_\mathbf{q}(0,M;\mathbf{V}_{\mathbf{q},\epsilon})+\mathbf{V}_2(0,M;\mathbf{W}'_\epsilon).$$

Now, applying the Sobolev theorems, we get the following:

$$\mathbf{V}_\mathbf{q}(0,M;\mathbf{V}_{\mathbf{q},\epsilon})+\mathbf{V}_2(0,M;\mathbf{W}'_\epsilon)\subset\mathbf{V}_\mathbf{q}\left(0,M;\mathbf{U}_\epsilon^{-\mathbf{r}}\right).$$

Here $\mathbf{U}_\epsilon^{-\mathbf{r}}:=H^{-r_1}(D_\epsilon)\times\ldots\times H^{-r_n}(D_\epsilon)$, $\mathbf{r}=(r_1,\ldots,r_n)$ and $r_i=\max\{1,d(1/q_i-1/2)\}$ for $i=1,\ldots,n$, where $H^{-r}(D_\epsilon)$ denotes the space dual to the Sobolev space $H^r(D_\epsilon)$ with superscript $r>0$ in the domain D_ϵ.

Therefore, for all generalized (weak) solution $u_\epsilon(x,t)$ to problem (2.1), time derivative $\frac{\partial u_\epsilon(x,t)}{\partial t}$ belongs to $\mathbf{V}_\mathbf{q}\left(0,M;\mathbf{U}_\epsilon^{-\mathbf{r}}\right)$.

Remark 2.2. *Existence of a generalized solution $u(x,t)$ to problem* (2.1) *for any initial data $U\in\mathbf{U}_\epsilon$ and fixed ϵ, can be proved in the standard way (see, for instance,* [20], [29]*). This solution may not be unique, since the function $f(v)$ satisfies only the conditions* (2.5) *and* (2.6) *and it is not assumed that the Lipschitz condition is satisfied with respect to v.*

The next lemma is proved in a similar way to the proposition XV.3.1 from [21].

Lemma 2.1. *Let $u_\epsilon(x,t)\in\mathbf{V}_2^{loc}(\mathbb{R}_+;\mathbf{W}_\epsilon)\cap\mathbf{V}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{V}_{\mathbf{p},\epsilon})$ be the generalized solution of problem* (2.1)*. Then,*

(i) $u_\epsilon\in\mathbf{C}(\mathbb{R}_+;\mathbf{U}_\epsilon)$;

(ii) *function $\|u_\epsilon(\cdot,t)\|^2$ is absolutely continuous on $\mathbb{R}_+$, and moreover*

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon(\cdot,t)\|^2+\int\limits_{D_\epsilon}\mathcal{A}\nabla u_\epsilon(x,t)\cdot\nabla u_\epsilon(x,t)dx+$$
$$\int\limits_{D_\epsilon}a_\epsilon(x)f(u_\epsilon(x,t))\cdot u_\epsilon(x,t)dx+\epsilon^\beta\int\limits_{\Gamma_1^\epsilon}p\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)u_\epsilon(x,t)\cdot u_\epsilon(x,t)\,ds= \tag{2.13}$$
$$\int\limits_{D_\epsilon}h_\epsilon(x)\cdot u_\epsilon(x,t)dx+\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}g\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)\cdot u_\epsilon(x,t)\,ds,$$

for a. a. $t\in\mathbb{R}_+$.

To define the trajectory space $\mathcal{T}_\epsilon^+$ for Eq (2.1), we use the general approaches of Section 3, and for every $[t_0, t_1] \in \mathbb{R}$, we have the Banach spaces

$$\mathfrak{G}_{t_0,t_1} := \mathbf{V}_2(t_0, t_1; \mathbf{W}) \cap \mathbf{V}_\infty(t_0, t_1; \mathbf{U}) \cap \mathbf{V}_\mathbf{p}(t_0, t_1; \mathbf{V}_\mathbf{p}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{V}_\mathbf{q}\left(t_0, t_1; \mathbf{U}^{-r}\right) \right\},$$

(sometimes we omit the parameter ϵ for brevity) with the following norm:

$$\|w\|_{\mathfrak{G}_{t_0,t_1}} := \|w\|_{\mathbf{V}_2(t_0,t_1;\mathbf{W})} + \|w\|_{\mathbf{V}_\mathbf{p}(t_0,t_1;\mathbf{V}_\mathbf{p})} + \|w\|_{\mathbf{V}_\infty(0,M;\mathbf{U})} + \left\| \frac{\partial w}{\partial t} \right\|_{\mathbf{V}_\mathbf{q}(t_0,t_1;\mathbf{U}^{-r})}.$$

Letting $\mathcal{D}_{t_0,t_1} = \mathbf{V}_\mathbf{q}\left(t_0, t_1; \mathbf{U}^{-r}\right)$, we obtain $\mathfrak{G}_{t_0,t_1} \subseteq \mathcal{D}_{t_0,t_1}$, and for $u(t) \in \mathfrak{G}_{t_0,t_1}$, we have $\mathcal{L}(u(t)) \in \mathcal{D}_{t_0,t_1}$. One considers now the generalized solutions to Eq (2.1) as solutions of the equation in the general scheme of Section 3.

Consider the following spaces:

$$\mathfrak{G}_+^{loc} = \mathbf{V}_2^{loc}(\mathbb{R}_+; \mathbf{W}) \cap \mathbf{V}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{V}_\mathbf{p}) \cap \mathbf{V}_\infty^{loc}(\mathbb{R}_+; \mathbf{U}) \cap \left\{ w \;\middle|\; \frac{\partial w}{\partial t} \in \mathbf{V}_\mathbf{q}^{loc}(\mathbb{R}_+; \mathbf{U}^{-r}) \right\},$$

$$\mathfrak{G}_{\epsilon,+}^{loc} = \mathbf{V}_2^{loc}(\mathbb{R}_+; \mathbf{W}_\epsilon) \cap \mathbf{V}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{V}_{\mathbf{p},\epsilon}) \cap \mathbf{V}_\infty^{loc}(\mathbb{R}_+; \mathbf{U}_\epsilon) \cap \left\{ w \;\middle|\; \frac{\partial w}{\partial t} \in \mathbf{V}_\mathbf{q}^{loc}(\mathbb{R}_+; \mathbf{U}_\epsilon^{-r}) \right\}.$$

We introduce the following notation. Let $\mathcal{K}_\epsilon^+$ be the set of all generalized solutions to Eq (2.1). For any $U \in \mathbf{U}$, there exists at least one trajectory $u(\cdot) \in \mathcal{T}_\epsilon^+$ such that $u(0) = U(x)$. Hence, the space $\mathcal{T}_\epsilon^+$ to Eq (2.1) is not empty.

It is easy to see that $\mathcal{T}_\epsilon^+ \subset \mathfrak{G}_{\epsilon,+}^{loc}$ and the space $\mathcal{T}_\epsilon^+$ is translation invariant, i.e., if $u(t) \in \mathcal{T}_\epsilon^+$, then $u(\tau + t) \in \mathcal{T}_\epsilon^+$ for all $\tau \geq 0$. Hence, $S(\tau)\mathcal{T}_\epsilon^+ \subseteq \mathcal{T}_\epsilon^+$ for all $\tau \geq 0$.

In the set $\mathfrak{G}_{t_0,t_1}$ we can introduce metrics $\rho_{t_0,t_1}(\cdot,\cdot)$ in $\mathfrak{G}_{t_0,t_1}$ by means of $\mathbf{V}_2(t_0, t_1; \mathbf{U})$–norms. Hence, we obtain the following definition of this metric:

$$\rho_{t_0,t_1}(v, w) = \left(\int_{t_0}^{t_1} \|v(t) - w(t)\|_\mathbf{U}^2 dt \right)^{1/2} \qquad \forall v(\cdot), w(\cdot) \in \mathcal{F}_{t_0,t_1}.$$

The topology Θ_+^{loc} in $\mathfrak{G}_+^{loc}$ is generated by these metrics. Let us recall that $\{v_k\} \subset \mathfrak{G}_+^{loc}$ converges to $v \in \mathfrak{G}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{V}_2(t_0,t_1;\mathbf{U})} \to 0$ $(k \to \infty)$ for all $[t_0, t_1] \subset \mathbb{R}_+$. Bearing in mind Eq (3.2), we conclude that the topology Θ_+^{loc} is metrizable. We consider this topology in $\mathcal{T}_\epsilon^+$ of Eq (2.1). Similarly, we define the topology $\Theta_{\epsilon,+}^{loc}$ in $\mathfrak{G}_{\epsilon,+}^{loc}$.

Consider the semigroup of translation $\{S(\tau)\}$ on $\mathcal{T}_\epsilon^+$, $S(\tau) : \mathcal{T}_\epsilon^+ \to \mathcal{T}_\epsilon^+$, $\tau \geq 0$. This semigroup $\{S(\tau)\}$ acting on $\mathcal{T}_\epsilon^+$, is continuous in the topology $\Theta_{\epsilon,+}^{loc}$.

Using the scheme from Section 3, one can define bounded sets in $\mathcal{T}_\epsilon^+$ by means of the Banach space $\mathfrak{G}_{\epsilon,+}^b$. We naturally get

$$\mathfrak{G}_{\epsilon,+}^b = \mathbf{V}_2^b(\mathbb{R}_+; \mathbf{W}_\epsilon) \cap \mathbf{V}_\mathbf{p}^b(\mathbb{R}_+; \mathbf{V}_{\mathbf{p},\epsilon}) \cap \mathbf{V}_\infty(\mathbb{R}_+; \mathbf{U}_\epsilon) \cap \left\{ w \;\middle|\; \frac{\partial w}{\partial t} \in \mathbf{V}_\mathbf{q}^b(\mathbb{R}_+; \mathbf{U}_\epsilon^{-r}) \right\},$$

and the space $\mathfrak{G}_{\epsilon,+}^b$ is a subspace of $\mathfrak{G}_{\epsilon,+}^{loc}$.

Suppose that $\mathcal{T}_\epsilon$ is the kernel to Eq (2.1), i.e., we have the set of all generalized complete bounded solutions $u(t), t \in \mathbb{R}$, to our RD-system. We consider solutions bounded in

$$\mathfrak{F}_\epsilon^b = \mathbf{V}_2^b(\mathbb{R}; \mathbf{W}_\epsilon) \cap \mathbf{V}_\mathbf{p}^b(\mathbb{R}; \mathbf{V}_{\mathbf{p},\epsilon}) \cap \mathbf{V}_\infty(\mathbb{R}; \mathbf{U}_\epsilon) \cap \left\{ w \;\middle|\; \frac{\partial w}{\partial t} \in \mathbf{V}_\mathbf{q}^b(\mathbb{R}; \mathbf{U}_\epsilon^{-r}) \right\}.$$

Proposition 2.1. *Problem* (2.1) *has the trajectory attractors $\mathfrak{A}_\epsilon$ in the topological space $\Theta_{\epsilon,+}^{loc}$. The set $\mathfrak{A}_\epsilon$ is bounded in $\mathfrak{F}_{\epsilon,+}^b$ and compact in $\Theta_{\epsilon,+}^{loc}$. In addition, $\mathfrak{A}_\epsilon = \Pi_+ \mathcal{K}_\epsilon$, and the kernel $\mathcal{K}_\epsilon$ is nonempty and bounded in $\mathfrak{F}_\epsilon^b$. Recall that the spaces $\mathfrak{F}_{\epsilon,+}^b$ and $\Theta_{\epsilon,+}^{loc}$ depend on ϵ.*

To prove this proposition, we use the approach of the proof from [21]. To prove the existence of an absorbing set (bounded in $\mathcal{F}_{\epsilon,+}^b$ and compact in $\Theta_{\epsilon,+}^{loc}$), one can use Lemma 2.1 similar to [21].

It is easy to verify that $\mathfrak{A}_\epsilon \subset \mathcal{B}_0(R)$ for all $\epsilon \in (0, 1)$. Here, $\mathcal{B}_0(R)$ is a ball in $\mathfrak{F}_{\epsilon,+}^b$ with a sufficiently large radius R. Due to Lemma 3.1, we have

$$\mathcal{B}_0(R) \Subset \mathbf{V}_2^{loc}(\mathbb{R}_+; \mathbf{U}_\epsilon^{1-\eta}), \tag{2.14}$$

$$\mathcal{B}_0(R) \Subset \mathbf{C}^{loc}(\mathbb{R}_+; \mathbf{U}_\epsilon^{-\eta}), \quad 0 < \eta \le 1. \tag{2.15}$$

Bearing in mind Eqs (2.14) and (2.15), the attraction to the constructed trajectory attractor can be strengthened.

Corollary 2.1. *For any bounded in $\mathfrak{F}_{\epsilon,+}^b$ set $\mathcal{B} \subset \mathcal{T}_\epsilon^+$ we get*

$$\mathrm{dist}_{\mathbf{V}_2(0,M;\mathbf{U}_\epsilon^{1-\eta})} (\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{T}_\epsilon) \to 0,$$
$$\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{U}_\epsilon^{-\eta})} (\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{T}_\epsilon) \to 0 \ (\tau \to \infty),$$

where M is a positive constant.

Recall that $D \subset D_\epsilon$ and D lies in the positive half-space $\{x_d > 0\}$. Therefore, for each function $u(x, t)$ of the variable $x \in D_\epsilon$ that belongs to the space $\mathcal{F}_{\epsilon,+}^b$, its restriction to the domain D belongs to the space $\mathcal{F}_+^b$ and, moreover,

$$\|u\|_{\mathfrak{F}_+^b} \le \|u\|_{\mathfrak{F}_{\epsilon,+}^b}.$$

Using this observation, we have:

Corollary 2.2. *The trajectory attractors $\mathfrak{A}_\epsilon$ are uniformly (with respect to $\epsilon \in (0, 1)$) bounded in $\mathcal{F}_+^b$. It should be noted that the kernels $\mathcal{K}_\epsilon$ are uniformly bounded in the space $\mathfrak{F}^b$. We mean that they are uniformly bounded with respect to $\epsilon \in (0, 1)$.*

3. Attractors to evolutionary equations: General approach

The section is devoted to the trajectory attractors to autonomous evolutionary equations (see details in [21]).

Consider an autonomous equation of the form

$$\frac{\partial u}{\partial t} = \mathcal{L}(u), \quad t \ge 0. \tag{3.1}$$

Here, $\mathcal{L}(\cdot) : \Upsilon_1 \to \Upsilon_0$ is a nonlinear mapping, Υ_1, Υ_0 are Banach spaces, and $\Upsilon_1 \subseteq \Upsilon_0$. As an example, one can consider $\mathcal{L}(u) = \mathcal{A}\Delta u - a(\cdot)f(u) + h(\cdot)$.

We study generalized solutions $u(t)$ to Eq (3.1) as functions of $t \in \mathbb{R}_+$ as an object. The set of solutions of Eq (3.1) is said to be *a trajectory space* $\mathcal{T}^+$ of Eq (3.1). Now, we give a detailed description of $\mathcal{T}^+$.

Consider solutions $u(t)$ of Eq (3.1) on $[t_0, t_1] \subset \mathbb{R}$. We consider solutions to problem (3.1) in a Banach space $\mathfrak{G}_{t_0,t_1}$. The space $\mathfrak{G}_{t_0,t_1}$ is a set $f(s)$, $s \in [t_0, t_1]$ satisfying $f(t) \in \Upsilon$ for a.a. $t \in [t_0, t_1]$, where Υ is a Banach space, satisfying $\Upsilon_1 \subseteq \Upsilon \subseteq \Upsilon_0$.

We consider $\mathfrak{G}_{t_0,t_1}$ as the intersection of spaces $C([t_0, t_1]; E)$, or $L_p(t_0, t_1; E)$, for $p \in [1, \infty]$. Suppose that $R_{t_0,t_1}\mathfrak{G}_{\tau_0,\tau_1} \subseteq \mathfrak{G}_{t_0,t_1}$ and

$$\|R_{t_0,t_1} f\|_{\mathfrak{G}_{t_0,t_1}} \leq C(t_0, t_1, \tau_0, \tau_1)\|f\|_{\mathfrak{G}_{\tau_0,\tau_1}} \quad \forall f \in \mathfrak{G}_{\tau_1,\tau_2}.$$

Here, $[t_0, t_1] \subseteq [\tau_0, \tau_1]$ and we denote by R_{t_0,t_1} the restriction operator onto $[t_0, t_1]$, where $C(t_0, t_1, \tau_0, \tau_1)$ is independent of f.

Denote by $S(\tau)$ for $\tau \in \mathbb{R}$ the translation $S(\tau)f(t) = f(\tau + t)$. If the variable t of $f(\cdot)$ belongs to the segment $[t_0, t_1]$, then the variable t of $S(\tau)f(\cdot)$ belongs to $[t_0 - \tau, t_1 - \tau]$ for $\tau \in \mathbb{R}$. Suppose that $S(\tau)$ is an isomorphism from F_{t_0,t_1} to $F_{t_0-\tau,t_1-\tau}$ and $\|S(\tau)f\|_{\mathfrak{G}_{t_0-\tau,t_1-\tau}} = \|f\|_{\mathfrak{G}t_0,t_1}$ $\forall f \in \mathfrak{G}_{t_0,t_1}$.

Suppose that if $f(t) \in \mathfrak{G}_{t_0,t_1}$, then $\mathcal{L}(f(t)) \in \mathcal{D}_{t_0,t_1}$, where $\mathcal{D}_{t_0,t_1}$ is a Banach space, which is larger, $\mathfrak{G}_{t_0,t_1} \subseteq \mathcal{D}_{t_0,t_1}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a distribution with values in Υ_0, $\frac{\partial f}{\partial t} \in D'((t_0, t_1); \Upsilon_0)$, and we suppose that $\mathcal{D}_{t_0,t_1} \subseteq D'((t_0, t_1); \Upsilon_0)$ for all $(t_0, t_1) \subset \mathbb{R}$. A function $u(t) \in \mathfrak{G}_{t_0,t_1}$ is a *solution* of Eq (3.1) if $\frac{\partial u}{\partial t}(t) = \mathcal{L}(u(t))$ in the sense of $D'((t_0, t_1); \Upsilon_0)$.

Consider the space

$$\mathfrak{G}_+^{loc} = \{f(t),\ t \in \mathbb{R}_+ \mid R_{t_0,t_1} f(t) \in \mathfrak{G}_{t_0,t_1}, \quad \forall\ [t_0, t_1] \subset \mathbb{R}_+\}.$$

For instance, if $\mathfrak{G}_{t_0,t_1} = C([t_0, t_1]; E)$, then $\mathfrak{G}_+^{loc} = C(\mathbb{R}_+; E)$, and if $\mathfrak{G}_{t_0,t_1} = L_p(t_0, t_1; E)$, then $\mathfrak{G}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(t) \in \mathfrak{G}_+^{loc}$ is a solution of Eq (3.1) if $R_{t_0,t_1}u(t) \in \mathfrak{G}_{t_0,t_1}$, and $u(t)$ is a solution to Eq (3.1) for any $[t_0, t_1] \subset \mathbb{R}_+$.

Let $\mathcal{T}^+$ be a set of solutions to Eq (3.1) from $\mathfrak{G}_+^{loc}$. Note that $\mathcal{T}^+$ in general is not the set of *all* solutions from $\mathfrak{G}_+^{loc}$. The set $\mathcal{T}^+$ consists on elements, which are *trajectories*, and the set $\mathcal{T}^+$ is the *trajectory space* of the Eq (3.1).

Suppose that the trajectory space $\mathcal{T}^+$ is *translation invariant*, i.e., if $u(t) \in \mathcal{T}^+$, then $u(\tau + t) \in \mathcal{T}^+$ for every $\tau \geq 0$.

Consider the translations $S(\tau)$ in $\mathfrak{G}_+^{loc}$: $S(\tau)f(t) = f(\tau + t)$, $\tau \geq 0$. It is easy to see that the map $\{S(\tau), \tau \geq 0\}$ forms a semigroup in $\mathfrak{G}_+^{loc}$: $S(\tau_1)S(\tau_2) = S(\tau_1 + \tau_2)$ for $\tau_1, \tau_2 \geq 0$, and, in addition, $S(0)$ is the identity operator. The *semigroup* $\{S(\tau), \tau \geq 0\}$ maps the trajectory space $\mathcal{T}^+$ to itself: $S(\tau)\mathcal{T}^+ \subseteq \mathcal{T}^+$ for all $\tau \geq 0$.

We investigate attracting properties of the translation semigroup $\{S(\tau)\}$ acting on the trajectory space $\mathcal{T}^+ \subset \mathfrak{G}_+^{loc}$. Next step is to get a topology in $\mathfrak{G}_+^{loc}$.

Assume that some metrics $\rho_{t_0,t_1}(\cdot, \cdot)$ are defined on $\mathfrak{G}_{t_0,t_1}$ for any $[t_0, t_1] \subset \mathbb{R}$. Suppose that

$$\rho_{t_0,t_1}(R_{t_0,t_1} f, R_{t_0,t_1} g) \leq D(t_0, t_1, \tau_0, \tau_1)\rho_{\tau_0,\tau_1}(f, g) \quad \text{for all } f, g \in \mathfrak{G}_{\tau_0,\tau_1},\ [t_0, t_1] \subseteq [\tau_0, \tau_1],$$

$$\rho_{t_0-\tau,t_1-\tau}(S(\tau)f, S(\tau)g) = \rho_{t_0,t_1}(f,g) \quad \forall f,g \in \mathfrak{G}_{t_0,t_1},\ [t_0,t_1] \subset \mathbb{R},\ \tau \in \mathbb{R}.$$

Now, we denote by Θ_{t_0,t_1} the metric spaces on $\mathfrak{G}_{t_0,t_1}$. For instance, ρ_{t_0,t_1} is the metric defined by the norm $\|\cdot\|_{\mathfrak{G}_{t_0,t_1}}$ of $\mathfrak{G}_{t_0,t_1}$.

The *projective limit* of the spaces Θ_{t_0,t_1} defines the topology Θ_+^{loc} in $\mathfrak{G}_+^{loc}$, that is, by definition, a sequence $\{f_k(t)\} \subset \mathfrak{G}_+^{loc}$ goes to $f(t) \in \mathfrak{G}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\rho_{t_0,t_1}(R_{t_0,t_1}f_k, R_{t_0,t_1}f) \to 0$ as $k \to \infty$ for all $[t_0,t_1] \subset \mathbb{R}_+$. It is possible to show that the topology Θ_+^{loc} is metrizable. For this aim we use, for instance, the Fréchet metric

$$\rho_+(f_1,f_2) := \sum_{k\in\mathbb{N}} 2^{-k} \frac{\rho_{0,k}(f_1,f_2)}{1+\rho_{0,k}(f_1,f_2)}. \tag{3.2}$$

We define the Banach space

$$\mathfrak{G}_+^b := \{f(t) \in \mathfrak{G}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\},$$

with

$$\|f\|_{\mathfrak{G}_+^b} := \sup_{\tau\geq 0} \|R_{0,1}f(\tau+t)\|_{\mathfrak{G}_{0,1}}.$$

We recall that $\mathfrak{G}_+^b \subseteq \Theta_+^{loc}$. For our Banach space $\mathfrak{G}_+^b$, we need only the fact that it should define bounded subsets in the trajectory space $\mathcal{T}^+$.

Assume that $\mathcal{T}^+ \subseteq \mathfrak{G}_+^b$.

Definition 3.1. *A set $\Xi \subseteq \Theta_+^{loc}$ is said to be the* attracting set *of $\{S(\tau)\}$ acting on $\mathcal{T}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{T}^+$, the set Ξ attracts $S(\tau)\mathcal{B}$ as $\tau \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ϵ-neighbourhood $O_\epsilon(\Xi)$ in Θ_+^{loc} there is $\tau_1 \geq 0$ such that $S(\tau)\mathcal{B} \subseteq O_\epsilon(\Xi)$ for all $\tau \geq \tau_1$.*

It is easy to see that the attracting property of Ξ can be reformulated equivalently: we have

$$\mathrm{dist}_{\Theta_{0,M}}(R_{0,M}S(\tau)\mathcal{B}, R_{0,M}\Xi) \longrightarrow 0 \quad (\tau \to +\infty).$$

Here,

$$\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x,y),$$

is the Hausdorff semi-distance from a set X to a set Y in a metric space $\mathcal{M}$. We recall that the Hausdorff semi-distance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{T}^+$ bounded in $\mathfrak{G}_+^b$ and for all $M > 0$.

Definition 3.2. *([21]). A set $\mathfrak{A} \subseteq \mathcal{T}^+$ is said to be the* trajectory attractor *of the semigroup $\{S(\tau)\}$ on $\mathcal{T}^+$ in the topology Θ_+^{loc} if*

(i) $\mathfrak{A}$ is compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^b$,

(ii) the set $\mathfrak{A}$ is invariant: $S(\tau)\mathfrak{A} = \mathfrak{A}$ for all $\tau \geq 0$,

(iii) the set $\mathfrak{A}$ is an attracting for $\{S(\tau)\}$ on $\mathcal{T}^+$ in the topology Θ_+^{loc}, i.e., for every $M > 0$, we have

$$\mathrm{dist}_{\Theta_{0,M}}(R_{0,M}S(\tau)\mathcal{B}, R_{0,M}\mathfrak{A}) \to 0 \quad (\tau \to +\infty).$$

Let us give the main assertion on the trajectory attractor for Eq (3.1).

Theorem 3.1. *([20,21]). Let the trajectory space $\mathcal{T}^+$ corresponding to Eq (3.1) be contained in $\mathfrak{G}_+^b$. We also assume that our semigroup $\{S(t)\}$ has an attracting set $\Xi \subseteq \mathcal{T}^+$ which is bounded in $\mathfrak{G}_+^b$ and compact in Θ_+^{loc}. Then, the semigroup $\{S(\tau), \tau \geq 0\}$ acting on $\mathcal{T}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \Xi$. The set $\mathfrak{A}$ is compact in Θ_+^{loc} and bounded in $\mathfrak{G}_+^b$.*

Let us describe in detail, i.e., in terms of complete trajectories of the equation, the structure of the trajectory attractor $\mathfrak{A}$ to Eq (3.1). We study Eq (3.1) on the time axis

$$\frac{\partial u}{\partial t} = \mathcal{L}(u), \ t \in \mathbb{R}. \tag{3.3}$$

Note that the trajectory space $\mathcal{T}^+$ of Eq (3.3) on $\mathbb{R}_+$ has been defined. We need this notion on the entire $\mathbb{R}$. If a function $f(t), \ t \in \mathbb{R}$, is defined on the entire axis, then $S(\tau)f(t) = f(\tau + t)$ are also defined for $\tau < 0$. A function $u(t), t \in \mathbb{R}$, is a *complete trajectory* of Eq (3.3) if $R_+u(\tau + t) \in \mathcal{T}^+$ for all $\tau \in \mathbb{R}$. Here, $R_+ = R_{0,\infty}$ denotes the restriction operator to $\mathbb{R}_+$.

We have $\mathfrak{G}^{loc}_+$, $\mathfrak{G}^b_+$, and Θ^{loc}_+. Let us define spaces $\mathfrak{G}^{loc}$, $\mathfrak{G}^b$, and Θ^{loc} in the same way

$$\mathfrak{G}^{loc} := \{f(t), t \in \mathbb{R} \mid R_{t_0,t_1}f(s) \in \mathfrak{G}_{t_0,t_1} \ \forall \ [t_0, t_1] \subseteq \mathbb{R}\},$$

$$\mathfrak{G}^b := \{f(t) \in \mathfrak{G}^{loc} \mid \|f\|_{\mathfrak{G}^b} < +\infty\},$$

where

$$\|f\|_{\mathfrak{G}^b} := \sup_{h\in\mathbb{R}} \|R_{0,1}f(\tau + t)\|_{\mathfrak{G}_{0,1}}. \tag{3.4}$$

Note that our topological space Θ^{loc} coincides (the coincidence as a set) with $\mathfrak{G}^{loc}$ and, by definition, $f_k(t) \to f(t)$ $(k \to \infty)$ in Θ^{loc} if $R_{t_0,t_1}f_k(t) \to R_{t_0,t_1}f(t)$ $(k \to \infty)$ in Θ_{t_0,t_1} for each $[t_0, t_1] \subseteq \mathbb{R}$.

Definition 3.3. *The* kernel $\mathcal{T}$ *in* $\mathfrak{G}^b$ *of Eq (3.3) is the collection of all complete trajectories* $u(t), t \in \mathbb{R}$, *of Eq (3.3), bounded in* $\mathfrak{G}^b$ *w.r.t. the norm Eq (3.4), i.e.,*

$$\|R_{0,1}u(\tau + t)\|_{\mathfrak{G}_{0,1}} \le C_u \quad \forall \tau \in \mathbb{R}.$$

Theorem 3.2. *Suppose the assumptions of the previous theorem hold. Then,* $\mathfrak{A} = R_+\mathcal{T}$ *and the set* $\mathcal{T}$ *is bounded in* $\mathfrak{G}^b$ *and compact in* Θ^{loc}.

To prove this assertion, one can use the approach from [21].

Now we are going to prove that a ball in the space $\mathfrak{G}^b_+$ is compact in our topological space Θ^{loc}_+. For this aim we use the next lemma. Assume that Υ_0 and Υ_1 are Banach spaces and $\Upsilon_1 \subset \Upsilon_0$. We consider the spaces

$$W_{p_1,p_0}(0, M; \Upsilon_1, \Upsilon_0) = \left\{\xi(t), t \in 0, M \mid \xi(\cdot) \in L_{p_1}(0, M; \Upsilon_1), \ \xi'(\cdot) \in L_{p_0}(0, M; \Upsilon_0)\right\},$$

$$W_{\infty,p_0}(0, M; \Upsilon_1, \Upsilon_0) = \left\{\xi(t), t \in 0, M \mid \xi(\cdot) \in L_{\infty}(0, M; \Upsilon_1), \ \xi'(\cdot) \in L_{p_0}(0, M; \Upsilon_0)\right\},$$

$(p_1 \ge 1, \, p_0 > 1)$ with the norms

$$\|\xi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\xi(t)\|^{p_1}_{\Upsilon_1} dt\right)^{1/p_1} + \left(\int_0^M \|\xi'(t)\|^{p_0}_{\Upsilon_0} dt\right)^{1/p_0},$$

$$\|\xi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\xi(t)\|_{\Upsilon_1} \mid t \in [0, M]\} + \left(\int_0^M \|\xi'(t)\|^{p_0}_{\Upsilon_0} dt\right)^{1/p_0}.$$

Lemma 3.1. *([30]). Suppose that $\Upsilon_1 \Subset \Upsilon \subset \Upsilon_0$. Then, we have compact embeddings*

$$W_{p_1,p_0}(0,T;\Upsilon_1,\Upsilon_0) \Subset L_{p_1}(0,T;\Upsilon), \qquad W_{\infty,p_0}(0,T;\Upsilon_1,\Upsilon_0) \Subset C([0,T];\Upsilon).$$

In this paper we investigate evolutionary equations and their attractors that depend on a small parameter $\epsilon > 0$.

Definition 3.4. *The trajectory attractors $\mathfrak{A}_\epsilon$ tend to the trajectory attractor $\overline{\mathfrak{A}}$ as $\epsilon \to 0$ in the topology Θ_+^{loc} if for every vicinity $O(\overline{\mathfrak{A}})$ in Θ_+^{loc} there exists an $\epsilon_1 \geq 0$ such that $\mathfrak{A}_\epsilon \subseteq O(\overline{\mathfrak{A}})$ for all $\epsilon < \epsilon_1$, i.e., for every $M > 0$, we have*

$$\mathrm{dist}_{\Theta_{0,M}}(R_{0,M}\mathfrak{A}_\epsilon, R_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\epsilon \to 0).$$

4. Homogenized RD-system and its trajectory attractor (the case $\beta = 1 - \alpha$)

In the next sections, we study the behaviour of the problem (2.1) as $\epsilon \to 0$ in the critical case $\beta = 1-\alpha$. We have the following "formal" limit problem with an inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \mathcal{A}\Delta u_0 - \overline{a}(x) f(u_0) + \overline{h}(x), & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} + P(\tilde{x})u_0 = G(\tilde{x}), & x = (\tilde{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0. \end{cases} \tag{4.1}$$

Here, $\overline{a}(x)$ and $\overline{h}(x)$ are defined in Eqs (2.2) and (2.3), respectively, and $G(\tilde{x})$ and $P(\tilde{x})$ were defined in Eqs (2.8) and (2.9).

As before, we consider generalized solutions of the problem (4.1), that is, functions

$$u_0(x,t) \in \mathbf{V}_\infty^{loc}(\mathbb{R}_+;\mathbf{U}) \cap \mathbf{V}_2^{loc}(\mathbb{R}_+;\mathbf{W}) \cap \mathbf{V}_{\mathbf{p}}^{loc}\left(\mathbb{R}_+;\mathbf{V}_{\mathbf{p}}\right),$$

which obey the integral identity

$$\begin{gathered} -\int\limits_{D\times\mathbb{R}_+} u_0 \cdot \frac{\partial \xi}{\partial t}\, dxdt + \int\limits_{D\times\mathbb{R}_+} \mathcal{A}\nabla u_0 \cdot \nabla\xi \, dxdt + \int\limits_{D\times\mathbb{R}_+} \bar{a}(x) f(u_0)\cdot \xi \, dxdt + \\ \int\limits_{\Gamma_1\times\mathbb{R}_+} P(\tilde{x})\, u_0 \cdot \xi \, dsdt = \int\limits_{D\times\mathbb{R}_+} \bar{h}(x)\cdot\xi\, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} G(\tilde{x})\cdot\xi\, dsdt, \end{gathered} \tag{4.2}$$

for each function $\xi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{W}\cap\mathbf{V}_{\mathbf{p}})$. For each $u(x,t)$ to Eq (4.1), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{V}_{\mathbf{q}}(0,M;\mathbf{U}^{-\mathbf{r}})$ (see Section 2). Recall that the "limit" domain D in Eqs (4.1) and (4.2) is independent of ϵ and its boundary contains the plain part Γ_1.

Similar to Eq (2.1), for any initial data $U \in \mathbf{U}$, the problem (4.1) has at least one generalized solution (see Remark 2.2). Lemma 2.1 also holds true for the problem (4.1) with replacing the ϵ-depending coefficients $a, h, p,$ and g by the corresponding averaged coefficients $\overline{a}(x), \overline{h}(x), P(\tilde{x})$, and $G(\tilde{x})$.

As usual, let $\overline{\mathcal{T}}^+$ be the the trajectory space for Eq (4.1) (the set of all generalized solutions) that belongs to the corresponding spaces $\mathfrak{G}_+^{loc}$ and $\mathfrak{G}_+^b$ (see Section 3). Recall that $\overline{\mathcal{T}}^+ \subset \mathfrak{G}_+^{loc}$ and the space $\overline{\mathcal{T}}^+$ is translation invariant with respect to translation semigroup $\{S(\tau)\}$, that is, $S(\tau)\overline{\mathcal{T}}^+ \subseteq \overline{\mathcal{T}}^+$ for all $\tau \geq 0$. We now construct the trajectory attractor in the topology Θ_+^{loc} for the problem (4.1) (see Sections 2 and 3).

Similar to Proposition 2.1, we have:

Proposition 4.1. *Problem* (4.1) *has the trajectory attractor* $\overline{\mathfrak{A}}$ *in the topological space* Θ_+^{loc}. *The set* $\overline{\mathfrak{A}}$ *is bounded in* $\mathfrak{G}_+^b$ *and compact in* Θ_+^{loc}. *Moreover,*

$$\overline{\mathfrak{A}} = R_+\overline{\mathcal{K}},$$

and the kernel $\overline{\mathcal{K}}$ *of the problem* (4.1) *is nonempty and bounded in* $\mathfrak{G}^b$.

We also have $\overline{\mathfrak{A}} \subset \mathcal{B}_0(R)$, where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. Finally, the analog of Corollary 2.1 holds for the trajectory attractor $\overline{\mathfrak{A}}$.

Corollary 4.1. *For any bounded in* $\mathfrak{G}_+^b$ *set* $\mathcal{B} \subset \overline{\mathcal{T}}^+$, *we have*

$$\mathrm{dist}_{\mathbf{V}_2(0,M;\mathbf{U}^{1-\eta})}\left(R_{0,M}S(\tau)\mathcal{B}, R_{0,M}\overline{\mathcal{T}}\right) \to 0,$$
$$\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{U}_\epsilon^{-\eta})}\left(R_{0,M}S(\tau)\mathcal{B}, R_{0,M}\overline{\mathcal{T}}\right) \to 0 \ (\tau \to \infty), \ \forall M > 0.$$

5. Preliminary Lemmata (the case $\beta = 1 - \alpha$)

Next lemmata are proved in [1].

Lemma 5.1. *The convergence*

$$v\left(\tilde{x}, \epsilon^\alpha F\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right)\right) \to v(\tilde{x}, 0) \quad as\ \epsilon \to 0, \tag{5.1}$$

is strongly in $[L_2(\Gamma_1)]^n$ *and the inequality*

$$\|v\|_{[L_2(R_\epsilon)]^n} \le C_1 \sqrt{\epsilon^\alpha}\|v\|_{\mathbf{W}_\epsilon}, \tag{5.2}$$

is true for any $v \in \mathbf{W}_\epsilon$.

Let us consider auxiliary elliptic problems

$$\begin{cases} \mathcal{A}\Delta v_\epsilon + h\left(x, \frac{x}{\epsilon}\right) = 0, & x \in D_\epsilon, \\ \frac{\partial v_\epsilon}{\partial \nu} + \epsilon^\beta p(\tilde{x}, \frac{\tilde{x}}{\epsilon})v_\epsilon = \epsilon^{1-\alpha} g(\tilde{x}, \frac{\tilde{x}}{\epsilon}), & x = (\tilde{x}, x_d) \in \Gamma_1^\epsilon, \\ v_\epsilon = 0, & x \in \Gamma_2, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \mathcal{A}\Delta v_0 + \overline{h}(x) = 0, & x \in D, \\ \frac{\partial v_0}{\partial \nu} + P(\tilde{x})v_0 = G(\tilde{x}), & x = (\tilde{x}, 0) \in \Gamma_1, \\ v_0 = 0, & x \in \Gamma_2, \end{cases} \tag{5.4}$$

where $\overline{h}(x)$ is defined in Eq (2.3), and $G(\tilde{x})$ and $P(\tilde{x})$ are defined in Eqs (2.8) and (2.9).

Lemma 5.2. *For all* $v \in \mathbf{W}_\epsilon$, *the convergence*

$$\left|\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon} g\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right) \cdot v\left(\tilde{x}, \epsilon^\alpha F\left(\tilde{x}, \frac{\tilde{x}}{\epsilon}\right)\right) ds - \int\limits_{\Gamma_1} G(\tilde{x}) \cdot v(x)\, ds\right| \to 0, \tag{5.5}$$

is valid as $\epsilon \to 0$.

Lemma 5.3. *The convergence*

$$\left|\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon} p\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)v_0\left(\tilde{x},\epsilon^\alpha F\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)\right)\cdot v\left(\tilde{x},\epsilon^\alpha F\left(\tilde{x},\frac{\tilde{x}}{\epsilon}\right)\right)ds-\int\limits_{\Gamma_1} P(\tilde{x})v_0(x)\cdot v(x)\,ds\right|\to 0, \tag{5.6}$$

takes place as $\epsilon\to 0$. *Here,* v_0 *is a solution to Eq* (5.4) *and* $v\in\mathbf{W}_\epsilon$.

Remark 5.1. *Due to the smoothness of the boundary* ∂D, *the solution* v_0 *belongs to* $H^2(D)$ [31], *and, hence, can be continued on* R_ϵ *to belong to* $H^2(D_\epsilon)$ [32].

Lemma 5.4. *Let* $\beta=1-\alpha$ *and* $F(\tilde{x},\tilde{y})$, $g(\tilde{x},\tilde{y})$, $p(\tilde{x},\tilde{y})$ *be periodic in y smooth functions,* $\mathcal{A}$ *be a given matrix, and* $h(x,\frac{x}{\epsilon})$ *be a righthand function which satisfies the conditions* (2.3) *and* (2.4). *Suppose that* $F(\tilde{x},\tilde{y})$ *is compactly supported in* $x\in\Gamma_1$ *uniformly in y. Then, for all* $\epsilon>0$, *the existence and uniqueness of the solution to problem* (5.3) *follows, and the strong convergence*

$$v_\epsilon\to v_0, \tag{5.7}$$

in $\mathbf{W}$ *as* $\epsilon\to 0$ *is valid.*

Proof. The existence and uniqueness of v_ϵ (v_0) are due to the positiveness of $p(\tilde{x},\frac{\tilde{x}}{\epsilon})$ ($P(\tilde{x})$) and the Lax-Milgram lemma (see [33]). Then, according to Eqs (2.1) and (4.1),

$$\begin{aligned}
&\int\limits_{D_\epsilon}\mathcal{A}\nabla(v_0-v_\epsilon)\cdot\nabla w dx+\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}p(v_0-v_\epsilon)\cdot w ds=\int\limits_{D}\mathcal{A}\nabla v_0\cdot\nabla w dx-\\
&\int\limits_{D_\epsilon}h\cdot w dx-\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}g\cdot w ds+\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}pv_0\cdot w ds=\int\limits_{D}\mathcal{A}\nabla v_0\cdot\nabla w dx-\\
&\int\limits_{D_\epsilon}h\cdot w dx-\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}g\cdot w ds+\int\limits_{R_\epsilon}\mathcal{A}\nabla v_0\cdot\nabla w dx+\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}pv_0\cdot w ds=\\
&\int\limits_{R_\epsilon}\mathcal{A}\nabla v_0\cdot\nabla w dx-\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}g\cdot w ds+\int\limits_{\Gamma_1}G(\tilde{x})\cdot w ds-\\
&\int\limits_{R_\epsilon}h\cdot w dx+\epsilon^{1-\alpha}\int\limits_{\Gamma_1^\epsilon}pv_0\cdot w ds-\int\limits_{\Gamma_1}P(\tilde{x})v_0\cdot w ds.
\end{aligned}$$

Let us estimate all the terms in the righthand side of the last relation. By Eq (5.2), considering the smoothness of v_0, we have

$$\left|\int\limits_{R_\epsilon}\mathcal{A}\nabla v_0\cdot\nabla w dx\right|\le\|\mathcal{A}\|\|\nabla v_0\|_{[L_2(R_\epsilon)]^n}\|w\|_{\mathbf{W}_\epsilon}\le C_2\sqrt{\epsilon^\alpha}\|v_0\|_{\mathbf{W}_\epsilon}\|w\|_{\mathbf{W}_\epsilon},$$

and

$$\left|\int\limits_{R_\epsilon}h\cdot w dx\right|\le\|h\|_{[L_2(R_\epsilon)]^n}\|w\|_{[L_2(R_\epsilon)]^n}\le C_3\sqrt{\epsilon^\alpha}\|h\|_{\mathbf{U}_\epsilon}\|w\|_{\mathbf{W}_\epsilon}.$$

Then, according to Lemmas 5.2 and 5.3, the inequalities

$$\left| \epsilon^{1-\alpha} \int\limits_{\Gamma_1^\epsilon} g \cdot w ds - \int\limits_{\Gamma_1} G \cdot w ds \right| \leq C_4(\epsilon^{1-\alpha} + \sqrt{\epsilon^\alpha}) \|w\|_{\mathbf{W}_\epsilon},$$

and

$$\left| \epsilon^{1-\alpha} \int\limits_{\Gamma_1^\epsilon} p v_0 \cdot w ds - \int\limits_{\Gamma_1} P v_0 \cdot w\, ds \right| \leq C_5 \left(\epsilon^{1-\alpha} + \sqrt{\epsilon^\alpha} \right) \|v_0\|_{\mathbf{W}_\epsilon} \|w\|_{\mathbf{W}_\epsilon},$$

hold. With the help of these inequalities, we obtain

$$\left| \int\limits_{D_\epsilon} \mathcal{A} \nabla (v_0 - v_\epsilon) \cdot \nabla w dx + \int\limits_{\Gamma_1^\epsilon} p(v_0 - v_\epsilon) \cdot w ds \right| \leq C_6 \left(\epsilon^{1-\alpha} + \sqrt{\epsilon^\alpha} \right) \|w\|_{\mathbf{W}_\epsilon}.$$

Substituting $w = v_0 - v_\epsilon$ and using Lemma 5.3 and the Friedrichs type inequality (see [34–36]), we obtain Eq (5.7). The lemma is proved.

Lemma 5.5. *1) All solutions $u_\epsilon(t)$ to Eq* (2.1) *satisfy*

$$\|u_\epsilon(t)\|_\epsilon^2 \leq \|u_\epsilon(0)\|_\epsilon^2 e^{-\varkappa_1 t} + R_1^2, \tag{5.8}$$

$$\varpi \int_t^{t+1} \|u_\epsilon(s)\|_{\epsilon,1}^2 ds + 2a_0 \sum_{i=1}^N \gamma_i \int\limits_t^{t+1} \|u_\epsilon^i(s)\|_{L_{p_i}(D_\epsilon)}^{p_i} ds + \\ + 2p_{\max} \epsilon^{1-\alpha} \int\limits_t^{t+1} \|u_\epsilon(s)\|_{\mathbf{V}_2(\Gamma_1^\epsilon)}^2 ds \leq \|u_\epsilon(t)\|_\epsilon^2 + R_2^2, \tag{5.9}$$

where $\varkappa_1 > 0$ is a constant independent of ϵ. Positive values R_1 and R_2 depend on M_0 (see Eq (2.4)*) and are independent of $u_\epsilon(0)$ and ϵ.*

2) All solutions $u(t)$ to Eq (4.1) *satisfy the same inequalities* (5.8) *and* (5.9) *with the norms in the function spaces over the domain D instead D_ϵ.*

Proof. We give a brief outline of the proof (see the details in [21]).

In the righthand side of Eq (2.13), the integral over the part of the boundary Γ_1^ϵ is nonnegative because of the positiveness of the matrix p. We integrate Eq (2.13) with respect to t. Then, to estimate the terms

$$\epsilon^{1-\alpha} \int\limits_{\Gamma_1^\epsilon} g \cdot w ds \qquad \text{and} \qquad \epsilon^{1-\alpha} \int\limits_{\Gamma_1^\epsilon} p u_\epsilon \cdot w ds,$$

we use the Cauchy inequality and the compactness of embedding $\mathbf{V}_2(\Gamma_1^\epsilon) \Subset \mathbf{W}_\epsilon$. For other terms, we use a standard procedure (see [21]). The lemma is proved.

6. Main assertion

Here, we formulate the main result concerning the limiting behavior of the trajectory attractors $\mathfrak{A}_\epsilon$ of the systems (2.1) as $\epsilon \to 0$ in the critical case $\beta = 1 - \alpha$.

Theorem 6.1. *The following limit holds in the topological space* Θ_+^{loc}

$$\mathfrak{A}_\epsilon \to \overline{\mathfrak{A}} \ \ as \ \epsilon \to 0+. \tag{6.1}$$

Moreover,

$$\mathcal{K}_\epsilon \to \overline{\mathcal{K}} \ as \ \epsilon \to 0+ \ \ in \ \ \Theta^{loc}. \tag{6.2}$$

Proof. It is easy to see that Eq (6.2) implies Eq (6.1). Hence, it is sufficient to prove Eq (6.2), i.e., for every neighborhood $O(\overline{\mathcal{K}})$ in Θ^{loc}, there exists $\epsilon_1 = \epsilon_1(O) > 0$ such that

$$\mathcal{K}_\epsilon \subset O(\overline{\mathcal{K}}) \quad \text{for} \quad \epsilon < \epsilon_1. \tag{6.3}$$

Assume that Eq (6.3) is not true. Then, there exists a neighborhood $O'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\epsilon_k \to 0+ \ (k \to \infty)$, and a sequence $u_{\epsilon_k}(\cdot) = u_{\epsilon_k}(t) \in \mathcal{K}_{\epsilon_k}$, such that

$$u_{\epsilon_k} \notin O'(\overline{\mathcal{K}}) \quad \text{for all} \quad k \in \mathbb{N}.$$

The function $u_{\epsilon_k}(x,t), t \in \mathbb{R}$ is a solution to

$$\begin{cases} \frac{\partial u_{\epsilon_k}}{\partial t} = \mathcal{A}\Delta u_{\epsilon_k} - a\left(x, \frac{x}{\epsilon_k}\right) f(u_{\epsilon_k}) + h\left(x, \frac{x}{\epsilon_k}\right), & x \in D_{\epsilon_k}, \\ \frac{\partial u_{\epsilon_k}}{\partial \nu} + \epsilon_k^\beta p(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}) u_{\epsilon_k} = \epsilon_k^{1-\alpha} g(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}), & x \in \Gamma_1^{\epsilon_k}, \\ u_{\epsilon_k} = 0, & x \in \Gamma_2, \end{cases} \tag{6.4}$$

where $\beta = 1 - \alpha$. To get the uniform in ϵ estimate of the solution, we use Lemma 5.5 (see also Corollary 4.1). By means of Eqs (5.8) and (5.9), we obtain that the sequence $\{u_{\epsilon_k}(x,t)\}$ is bounded in $\mathfrak{F}^b$, i.e.,

$$\|u_{\epsilon_k}\|_{\mathfrak{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\epsilon_k}(t)\| + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\epsilon_k}(\vartheta)\|_1^2 \, d\vartheta \right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\epsilon_k}(\vartheta)\|_{\mathbf{V_p}}^p \, d\vartheta \right)^{1/p} +$$
$$\epsilon^\beta \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{\Gamma_1^\epsilon} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon}\Big) u_\epsilon(x,\vartheta) \cdot u_\epsilon(x,\vartheta) \, ds \, d\vartheta + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \frac{\partial u_{\epsilon_k}}{\partial t}(\vartheta) \right\|_{\mathbf{U}^{-r}}^q d\vartheta \right)^{1/q} \leq C \ \forall\, k \in \mathbb{N}. \tag{6.5}$$

Note that here, $\beta = 1 - \alpha$. The constant C is independent of ϵ. Consequently, there exists a subsequence $\{u_{\epsilon_k'}(x,t)\} \subset \{u_{\epsilon_k}(x,t)\}$, such that

$$u_{\epsilon_k'}(x,t) \to \overline{u}(x,t) \text{ as } k \to \infty \ \text{ in } \Theta^{loc}.$$

Here, $\overline{u}(x,t) \in \mathfrak{F}^b$ and $\overline{u}(t)$ satisfies Eq (6.5) with the same constant C. Because of Eq (6.5), we get

$$u_{\epsilon_k'}(x,t) \rightharpoonup \overline{u}(x,t) \ (k \to \infty),$$

weakly in $\mathbf{V}_2^{loc}(\mathbb{R};\mathbf{W})$, weakly in $\mathbf{V}_{\mathbf{p}}^{loc}\left(\mathbb{R};\mathbf{V}_{\mathbf{p}}\right)$, star-weakly in $\mathbf{V}_\infty^{loc}(\mathbb{R}_+;\mathbf{U})$ and

$$\frac{\partial u_{\epsilon_k'}(x,t)}{\partial t} \rightharpoonup \frac{\partial \overline{u}(x,t)}{\partial t} \ (k \to \infty),$$

weakly in $\mathbf{V}_{\mathbf{q},w}^{loc}(\mathbb{R};\mathbf{U}^{-r})$. We claim that $\overline{u}(x,t) \in \overline{\mathcal{K}}$. We have $\|\overline{u}\|_{\mathfrak{G}^b} \leq C$. Hence, we have to verify that $\overline{u}(x,t) = u_0(x,t)$, i.e. it is a generalized solution to Eq (4.1).

Using Eqs (6.5) and (2.3), we find that

$$\frac{\partial u_{\epsilon_k}}{\partial t} - \mathcal{A}\Delta u_{\epsilon_k} - h_{\epsilon_k}(x) \longrightarrow \frac{\partial \bar{u}}{\partial t} - \mathcal{A}\Delta \bar{u} - \overline{h}(x) \quad \text{as } k \to \infty, \tag{6.6}$$

in the space $D'(\mathbb{R};\mathbf{U}^{-\mathbf{r}})$, since the derivative operators are continuous in the space of distributions.

Let us prove that

$$a\left(x, \frac{x}{\epsilon_k}\right) f(u_{\epsilon_k}) \rightharpoonup \bar{a}(x) f(\bar{u}) \ \text{ as } k \to \infty, \tag{6.7}$$

weakly in $\mathbf{V}_{\mathbf{q},w}^{loc}\left(\mathbb{R};\mathbf{V}_{\mathbf{q}}\right)$. We fix any number $M > 0$. The sequence $\{u_{\epsilon_k}(x,t)\}$ is bounded in $\mathbf{V}_{\mathbf{p}}\left(-M,M;\mathbf{V}_{\mathbf{p}}\right)$ (see Eq (6.5)). Then, due to Eq (2.5), the sequence $\{f(u_{\epsilon_k}(t))\}$ is bounded in $\mathbf{V}_{\mathbf{q}}\left(-M,M;\mathbf{V}_{\mathbf{q}}\right)$. Because $\{u_{\epsilon_k}(x,t)\}$ is bounded in $\mathbf{V}_2(-M,M;\mathbf{W})$ and $\left\{\frac{\partial u_{\epsilon_k}}{\partial t}(t)\right\}$ is bounded in $\mathbf{V}_{\mathbf{q}}(-M,M;\mathbf{U}^{-\mathbf{r}})$, we may assume that

$$u_{\epsilon_k}(x,t) \to \bar{u}(x,t) \text{ as } k \to \infty \ \text{ in } \mathbf{V}_2(-M,M;\mathbf{V}_2) = \mathbf{V}_2(D\times]-M,M[),$$

hence,

$$u_{\epsilon_k}(x,t) \to \bar{u}(x,t) \ \text{ as } k \to \infty \ \text{ for almost all } (x,t) \in D\times]-M,M[.$$

Because the function $f(w)$ is continuous in $w \in \mathbb{R}$, we conclude that

$$f(u_{\epsilon_k}(x,t)) \to f(\bar{u}(x,t)) \ \text{ as } k \to \infty \ \text{ for almost all } (x,t) \in D\times]-M,M[. \tag{6.8}$$

We have

$$a\left(x, \frac{x}{\epsilon_k}\right) f(u_{\epsilon_k}) - \bar{a}(x) f(\bar{u}) =$$
$$a\left(x, \frac{x}{\epsilon_k}\right)(f(u_{\epsilon_k}) - f(\bar{u})) + \left(a\left(x, \frac{x}{\epsilon_k}\right) - \bar{a}(x)\right) f(\bar{u}). \tag{6.9}$$

Let us show that both terms in the righthand side of Eq (6.9) tend to zero as $k \to \infty$ weakly in $\mathbf{V}_{\mathbf{q}}\left(-M,M;\mathbf{V}_{\mathbf{q}}\right) = \mathbf{V}_{\mathbf{q}}(D\times]-M,M[)$. First, the sequence $a\left(x, \frac{x}{\epsilon_k}\right)(f(u_{\epsilon_k}) - f(\bar{u}))$ goes to zero as $k \to \infty$ for almost all $(x,t) \in D\times]-M,M[$ (see Eq (6.8)). Applying Lemma 1.3 from [37], we conclude that

$$a\left(x, \frac{x}{\epsilon_k}\right)\left(f(u_{\epsilon_k}) - f(\bar{u})\right) \rightharpoonup 0 \ \text{ as } k \to \infty,$$

weakly in $\mathbf{V}_{\mathbf{q}}(D\times]-M,M[)$. Second, the sequence $\left(a\left(x, \frac{x}{\epsilon_k}\right) - \bar{a}(x)\right) f(\bar{u})$ also goes to zero a $k \to \infty$ weakly in $\mathbf{V}_{\mathbf{q}}(D\times]-M,M[)$, since $a\left(x, \frac{x}{\epsilon_k}\right) \rightharpoonup \bar{a}(x)$ as $k \to \infty$ star-weakly in $\mathbf{V}_{\infty,*w}(-M,M;\mathbf{V}_2)$ and $f(\bar{u}) \in \mathbf{V}_{\mathbf{q}}(D\times]-M,M[)$. Thus, Eq (6.7) is proved.

Now, let us show that

$$\epsilon_k^{1-\alpha} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big) u_{\epsilon_k} \rightharpoonup P(\tilde{x})\bar{u} \quad \text{as} \quad k \to +\infty, \tag{6.10}$$

weakly in $\mathbf{V}_2(\Gamma_1 \times]-M, M[)$. Indeed, we have

$$\epsilon_k^{1-\alpha} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big) u_{\epsilon_k}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\Big) - P(\tilde{x})\bar{u}(x) =$$
$$\epsilon_k^{1-\alpha} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\Big(u_{\epsilon_k}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\Big) - \bar{u}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\Big)\Big) +$$
$$\epsilon_k^{1-\alpha} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\bar{u}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\Big) - P(\tilde{x})\bar{u}(\tilde{x}, 0).$$

We have

$$\epsilon_k^{1-\alpha} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\Big(u_{\epsilon_k}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big), t\Big) - \bar{u}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big), t\Big)\Big) \rightharpoonup 0 \quad \text{as} \quad k \to +\infty,$$

weakly in $\mathbf{V}_2(\Gamma_1 \times]-M, M[)$ due to Lemma 5.1. We state that

$$\epsilon_k^{1-\alpha} p\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)\bar{u}\Big(\tilde{x}, \epsilon_k^{\alpha} F\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big), t\Big) - P(\tilde{x})\bar{u}(\tilde{x}, 0, t) \rightharpoonup 0 \quad \text{as} \quad k \to +\infty, \tag{6.11}$$

weakly in $\mathbf{V}_2(\Gamma_1 \times]-M, M[)$. Indeed, due to Lemma 5.5, both terms are bounded $\mathbf{V}_2(\Gamma_1 \times]-M, M[)$. Also, one can see that this convergence due to Eq (2.11) is almost everywhere in $]-M, M[$. Using Lemma 1.3 from [37], we get the weak convergence Eq (6.11) and, hence, we obtain Eq (6.10).

In an analogous way, we act with the terms with $g\Big(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\Big)$ and $G(\tilde{x})$, using Lemma 5.2.

Hence, for $\overline{u}(x, t) = u_0(x, t)$, we have

$$-\int_{-M}^{M}\int_{D_{\epsilon_k}} u_{\epsilon_k} \cdot \frac{\partial \xi}{\partial t}\, dxdt + \int_{-M}^{M}\int_{D_{\epsilon_k}} \mathcal{A}\nabla u_{\epsilon_k} \cdot \nabla \xi \, dxdt + \int_{-M}^{M}\int_{D_{\epsilon_k}} a_{\epsilon_k}(x) f(u_{\epsilon_k}) \cdot \xi \, dxdt +$$
$$\epsilon_k^{\beta} \int_{-M}^{M}\int_{\Gamma_1^{\epsilon_k}} p\left(\tilde{x}, \frac{\tilde{x}}{\epsilon_k}\right) u_{\epsilon_k} \cdot \xi \, dsdt \longrightarrow -\int_{-M}^{M}\int_{D} u_0 \cdot \frac{\partial \xi}{\partial t}\, dxdt +$$
$$\int_{-M}^{M}\int_{D} \mathcal{A}\nabla u_0 \cdot \nabla \xi \, dxdt + \int_{D} \overline{a}(x) f(u_0) \cdot \xi \, dxdt + \int_{-M}^{M}\int_{\Gamma_1} P(\tilde{x})\, u_0 \cdot \xi \, dsdt,$$

as $k \to \infty$.

Using Eq (6.8), we pass to the limit in the Eq (6.4) as $k \to \infty$ in the space $D'(\mathbb{R}; \mathbf{U}^{-r})$ and obtain that the function $u_0(x, t)$ satisfies the integral identity Eq (4.2) and, hence, it is a complete trajectory of the Eq (4.1).

Consequently, $u_0 \in \overline{\mathcal{K}}$. We have proved above that $u_{\epsilon_k} \underset{}{\rightarrow} u_0$ as $k \to \infty$ in Θ^{loc}. Assumption $u_{\epsilon_k} \notin O'(\overline{\mathcal{K}})$ (see [38]) implies $u_0 \notin O'(\overline{\mathcal{K}})$, and, hence, $u_0 \notin \overline{\mathcal{K}}$. We arrive to the contradiction that completes the proof of the theorem.

Using the compact imbedding Eqs (2.14) and (2.15), we improve the convergence Eq (6.1).

Corollary 6.1. *For any* $0 < \eta \leq 1$ *and for all* $M > 0$*,*

$$\text{dist}_{\mathbf{V}_2([0,M];\mathbf{U}^{1-\eta})}\left(R_{0,M}\mathfrak{A}_\epsilon, R_{0,M}\overline{\mathfrak{A}}\right) \to 0, \tag{6.12}$$

$$\text{dist}_{\mathbf{C}([0,M];\mathbf{U}^{-\eta})}\left(R_{0,M}\mathfrak{A}_\epsilon, R_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\epsilon \to 0+). \tag{6.13}$$

To prove Eqs (6.12) and (6.13), we use the reasoning in proof Theorem 6.1, changing the topological space Θ^{loc} by $\mathbf{V}_2^{loc}(\mathbb{R}_+;\mathbf{U}^{1-\eta})$ or $\mathbf{C}^{loc}(\mathbb{R}_+;\mathbf{U}^{-\eta})$.

In conclusion, we consider the case of uniqueness of the Cauchy problem for RD-systems. It is sufficient to suppose that the nonlinear function $f(u)$ in Eq (2.1) satisfies the inequality

$$-C|w_1 - w_2|^2 \leq (f(w_1) - f(w_2), w_1 - w_2) \quad \text{for any } w_1, w_2 \in \mathbb{R}^n, \tag{6.14}$$

(see [21,29]). In [29], it was proved that if Eq (6.14) is true, then Eqs (2.1) and (4.1) generate dynamical semigroups in $\mathbf{U}$, possessing that global attractors A_ϵ and $\overline{A}$ are bounded in $\mathbf{W}$ (see [19,20]). Moreover,

$$A_\epsilon = \{u(0) \mid u \in \mathfrak{A}_\epsilon\}, \ \overline{A} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

The convergence Eq (6.13) gives:

Corollary 6.2. *Under the assumption of Theorem 6.1, the limit formula takes place*

$$\text{dist}_{\mathbf{U}^{-\eta}}\left(A_\epsilon, \overline{A}\right) \to 0 \ (\epsilon \to 0+).$$

7. Conclusions

In the paper, we consider RD-systems with rapidly oscillating terms in equations and in boundary conditions in domains with locally periodic wavering boundary (rough surface) depending on a small parameter. We define the trajectory attractors of these systems and prove that they weakly converge to the trajectory attractors of the limit (averaged) RD-systems in domain independent of the small parameter.

In this paper we consider the critical case in which the type of boundary condition is preserved under the limit passage.

Defining the appropriate axillary functional spaces with weak topology, we prove the existence of trajectory attractors for these systems. Then, we formulate the main theorem and prove it with the help of auxiliary lemmata.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Methodology, G.A.C. and V.V.C.; Formal analysis, K.A.B., G.A.C., and V.V.C.; Investigation, G.F.A., K.A.B., G.A.C., and V.V.C.; Writing—original draft, G.F.A.;Writing—review & editing, G.F.A., K.A.B., G.A.C., and V.V.C. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare there is no conflict of interest.

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