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Research article

Iterative learning algorithms for boundary tracing problems of nonlinear fractional diffusion equations

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Abstract: In this paper, the iterative learning control technique is extended to distributed parameter systems governed by nonlinear fractional diffusion equations. Based on P-type and PI^{θ}-type iterative learning control methods, sufficient conditions for the convergences of systems are given. Finally, numerical examples are presented to illustrate the efficiency of the proposed iterative schemes. The numerical results show that the closed-loop iterative learning control scheme converges faster than the open-loop iterative learning control scheme and the PI^{θ}-type iterative learning control scheme converges faster than the P-type and the PI-type iterative learning control scheme.

Keywords: nonlinear fractional diffusion equations; iterative learning control; boundary control; open-loop; closed-loop

1. Introduction

In this paper,we consider boundary tracing problem of nonlinear fractional diffusion equations with Neumann boundary condition

$$\begin{cases} D_t^{\alpha}\varphi = \varphi_{xx} + F(x,t,\varphi,\varphi_x), & (x,t) \in \Omega_T, \\ \varphi_x(0,t) = u(t), & t \in (0,T], \\ \varphi_x(1,t) = g(t), & t \in (0,T], \\ \varphi(x,0) = \varphi_0(x), & x \in [0,1] \end{cases} \tag{1.1}$$

by iterative learning algorithms, where D_t^{α} is the Caputo fractional derivative of order α, $0 < \alpha < 1$, $(x,t) \in \Omega_T \triangleq [0,1] \times [0,T]$ and $F(x,t,\varphi,\varphi_x)$ is the nonlinear function.

The basic idea of iterative learning control (ILC) [1, 4, 16] can be traced back to Garden [8] in 1967 and Uchiyama [28] in 1978. ILC is a control method suitable for dealing with iterative systems, which uses information obtained from previous trial to improve the tracking performance of current trial. Owing to simplicity and effectiveness, ILC plays an important role in many fields and applications [9, 10, 14].

ILC schemes are widely used for ordinary differential equations (ODEs) [23, 25, 26, 29]. However, there are few studies on its application to partial differential equations (PDEs) and fractional partial differential equations (FPDEs) [11, 24]. Choi et al. [3] employed the characteristic line method and the Q-ILC method to study the ILC schemes of a first-order hyperbolic PDE system. Huang et al. [12] studied the P-type ILC scheme for boundary tracking of nonlinear hyperbolic parametric systems and evaluated the robustness of the scheme. Kang et al. [15] proposed a Newton-type ILC algorithm for nonlinear parametric equations and provided sufficient conditions for convergence of the Newton descent method using the λ-norm. Different from the convergence in the sense of the λ norm, Dai et al. [5] derived the P-type ILC for linear parabolic parametric equations and proved its convergence in the sense of the L^2-norm and the $W^{1,2}$-norm. Lan et al. [22] presented a second-order ILC method for a class of multi-agent systems (MAS) with time-delay distributed parameters and proved its convergence.

For the diffusion equation, Xu et al. [30] proposed P-type and D-type ILC methods for infinite-dimensional linear systems in Hilbert spaces. Huang et al. [13] extended ILC to solve the boundary tracking problem of inhomogeneous heat equations and provided evidence for the effectiveness of the D-type ILC scheme. Zhang et al. [32] presented a novel intermittent updating PD-type ILC algorithm for semi-linear distributed parameter systems with sensors or actuators, and provided convergence conditions for the output error. For the fractional diffusion equation, Lan et al. [20] discussed the P-type ILC of fractional order parameter exchange systems and demonstrated that the exchange system maintains traceability over two time periods. Lan et al. [21] proposed a second-order P-type ILC scheme for a class of linear fractional order distributed parameter systems and established a sufficient condition for convergence using λ-norm and generalized Gronwall inequality.

Overall, there have been relatively few studies on iterative learning control algorithms for fractional diffusion equations, which can describe a variety of memory materials and genetic processes [6, 18]. Applying the ILC algorithm to fractional diffusion equations can improve control of the system for nonlocal transport phenomena and long-range memory effects, leading to faster convergence and improved tracking accuracy [19]. We aim to extend ILC to the nonlinear fractional diffusion equation and study their convergence. However, this work is challenging, as the difficulty lies in proving the convergence of the iterative learning control algorithm for fractional diffusion equations, with added challenges posed by the fractional derivatives and nonlinear source terms. Therefore, we assume that source term is Lipschitz continuous and employ Sobolev imbedding theorem to overcome difficulties in the proof.

In this paper, we consider boundary tracing problem of one dimensional fractional diffusion equation with input, state and output functions at the k-th iteration,

$$\begin{cases} D_t^{\alpha}\varphi^k = \varphi_{xx}^k + F(x,t,\varphi^k,\varphi_x^k), & (x,t)\in\Omega_T,\\ \varphi_x^k(0,t) = u^k(t), & t\in(0,T],\\ \varphi_x^k(1,t) = g(t), & t\in(0,T],\\ \varphi^k(x,0) = \varphi_0(x), & x\in[0,1],\\ y^k(t) = c(t)\varphi^k(1,t) + d(t)u^k(t), \end{cases} \tag{1.2}$$

where k denotes the iterative number of the process and $u^k, \varphi^k, y^k(t)$ are the input, state and output of the system at the k-th iteration respectively. The main idea is to adjust the control input $u^k(t)$ iteratively in order that system output $y^k(t)$ can track the predefined target $y^d(t)$ when $k \to \infty$.

In addition, we make some assumptions about the functions in system (1.2). Suppose $c(t)$ and $d(t)$ are bounded and $F(x, t, \varphi^k, \varphi^k_x)$ satisfies Lipschitz condition.

Assumption 1: The functions $c(t)$ and $d(t)$ satisfy

$$|c(t)| \leq c_1, 0 < d_1 \leq d(t) \leq d_2,$$

where c_1, d_1, d_2 are positive constants.

Assumption 2: The nonlinear function $F^k \triangleq F(x, t, \varphi^k, \varphi^k_x)$ is Lipschitz continuous,

$$|F^{k+1} - F^k| \leq C_F(|\varphi^{k+1} - \varphi^k| + |\varphi^{k+1}_x - \varphi^k_x|), \tag{1.3}$$

where C_F is a constant.

This paper is organized as follows. Preliminaries are presented in Section 2. In Section 3, P-type ILC scheme, PI^θ-type ILC scheme and corresponding convergence conditions are proposed for the nonlinear system. Numerical examples are given in Section 4 to illustrate the effectiveness of the methods. Finally, conclusions are drawn in Section 5.

2. Preliminaries

To prepare for our subsequent analysis, it is essential to introduce some definitions, useful lemmas and theorems.

Definition 2.1. *[17] Let $z(t) \in AC[0, T]$, the Caputo fractional derivative of order α is defined by*

$$D^\alpha_t z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z'(\tau)}{(t-\tau)^\alpha} d\tau, 0 < \alpha < 1, 0 < t \leq T.$$

Definition 2.2. *[17] Let $z(t) \in L(0, T)$, the Riemann-Liouville fractional integral of order α is defined by*

$$I^\alpha_t z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} z(\tau) d\tau, 0 < \alpha < 1, 0 < t \leq T.$$

Definition 2.3. *[27] The two-parameter Mittag-Leffler function is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0.$$

Lemma 2.1. *[27] Suppose $0 < \alpha < 1$. Caputo fractional derivative and fractional integral of order α have the following relationship*

$$I^\alpha_t (D^\alpha_t (x(t))) = x(t) - x(0).$$

Lemma 2.2. *[7] Assume $x(t)$ be a differentiable function. The following relationship holds*

$$\frac{1}{2} D^\alpha_t x(t)^2 \leq x(t) D^\alpha_t x(t), \ \forall \alpha \in (0, 1].$$

Lemma 2.3. *(Gronwall inequality [31]) Suppose a(t) is a nonnegative, nondecreasing, locally integrable function over* $0 \le t_0 \le t \le T$ *and g(t) is a nonnegative, nondecreasing continuous function over* $0 \le t_0 \le t \le T$*, g(t)* $\le$ *M, where M is a postive constant. If u(t) is nonnegative and locally integrable function over* $0 \le t_0 \le t \le T$ *and satisfies*

$$u(t) \le a(t) + g(t)\int_{t_0}^{t}(t-s)^{\alpha-1}u(s)ds, \quad \alpha > 0,$$

then, we have

$$u(t) \le a(t)E_{\alpha,1}\left(g(t)\Gamma(\alpha)t^{\alpha}\right).$$

Theorem 2.1. *(Sobolev imbedding theorem [2]) Let* $\Omega \in R^d$ *be a bounded Lipschitz domain and* $1 \le p \le \infty$*. If* $0 \le m < k - \frac{d}{p} < m+1$*, the space* $W^{k,p}(\Omega)$ *is continuously imbedded in* $C^{m,\alpha}(\overline{\Omega})$ *for* $\alpha = k - \frac{d}{p} - m$ *and compactly imbedded in* $C^{m,\beta}(\overline{\Omega})$ *for all* $0 \le \beta < \alpha$*.*

Remark 2.1. *Using the Sobolbev imbedding theorem 2.1 in the case of d=1, we can get*

$$\max_{x\in[0,1]}|\varphi(x,t)|^2 \le C_1\|\varphi(x,t)\|^2_{H^1}, \tag{2.1}$$

where $\|\varphi(x,t)\|^2_{H_1} \triangleq \int_\Omega \varphi^2 + \varphi_x^2 dx$ *and* C_1 *is a positive constant.*

3. ILC design for nonlinear systems

We need to give some necessary lemmas to obtain the convergence conditions for the ILC scheme.

Lemma 3.1. *Suppose* $e(t) \in AC[0,T)$ *and* $0.5 < \theta \le 1$*, then, we have*

$$|I_t^\theta e|^2 \le \frac{\Gamma(2\theta-1)e^{\lambda t}T}{\Gamma(\theta)^2\lambda^{2\theta-1}}|e|^2_\lambda. \tag{3.1}$$

Proof. From the Definition 2.2 of fractional integral, we can get

$$\begin{aligned}
|I_t^\theta e|^2 &= \frac{1}{\Gamma(\theta)^2}\Big(\int_0^t (t-\tau)^{\theta-1}e(\tau)d\tau\Big)^2\\
&= \frac{1}{\Gamma(\theta)^2}\Big(\int_0^t (t-\tau)^{\theta-1}e^{\frac{\lambda}{2}\tau}e^{-\frac{\lambda}{2}\tau}e(\tau)d\tau\Big)^2\\
&\le \frac{1}{\Gamma(\theta)^2}\int_0^t (t-\tau)^{2\theta-2}e^{\lambda\tau}d\tau\int_0^t e^2(\tau)e^{-\lambda\tau}d\tau\\
&\le \frac{1}{\Gamma(\theta)^2}\int_0^t (t-\tau)^{2\theta-2}e^{\lambda\tau}d\tau|e|^2_\lambda t\\
&= \frac{e^{\lambda t}}{\Gamma(\theta)^2}\int_0^t (t-\tau)^{2\theta-2}e^{-\lambda(t-\tau)}d\tau|e|^2_\lambda t.
\end{aligned}$$

where $|e|^2_\lambda \triangleq \sup_{t\in[0,T]}\{e^{-\lambda t}|e(t)|^2, \lambda > 0\}$ and $|e(t)|$ represents absolute value of $e(t)$. Let $t - \tau = \omega$ and

$\lambda\omega = v$, we have

$$\begin{aligned}
&\frac{e^{\lambda t}}{\Gamma(\theta)^2}\int_0^t (t-\tau)^{2\theta-2}e^{-\lambda(t-\tau)}d\tau|e|_\lambda^2 t\\
=&\frac{e^{\lambda t}}{\Gamma(\theta)^2}\int_0^t \omega^{2\theta-2}e^{-\lambda\omega}d\omega|e|_\lambda^2 t\\
=&\frac{e^{\lambda t}}{\Gamma(\theta)^2}\int_0^{\lambda t} (\frac{v}{\lambda})^{2\theta-2}e^{-v}\frac{1}{\lambda}dv|e|_\lambda^2 t = \frac{e^{\lambda t}}{\Gamma(\theta)^2}\int_0^{\lambda t} v^{2\theta-2}e^{-v}dv\frac{|e|_\lambda^2 t}{\lambda^{2\theta-1}}.
\end{aligned} \tag{3.2}$$

From the definition of the Gamma function, we can get

$$\begin{aligned}
&\frac{1}{\Gamma(\theta)^2}\int_0^t (t-\tau)^{2\theta-2}e^{\lambda\tau}d\tau|e|_\lambda^2 t\\
\leq&\frac{e^{\lambda t}}{\Gamma(\theta)^2}\int_0^{\infty} v^{2\theta-2}e^{-v}dv\frac{|e|_\lambda^2 t}{\lambda^{2\theta-1}}\\
=&\frac{e^{\lambda t}}{\Gamma(\theta)^2}\Gamma(2\theta-1)\frac{|e|_\lambda^2 t}{\lambda^{2\theta-1}} = \frac{\Gamma(2\theta-1)e^{\lambda t}T}{\Gamma(\theta)^2\lambda^{2\theta-1}}|e|_\lambda^2.
\end{aligned} \tag{3.3}$$

This completes the proof.

Lemma 3.2. *If ψ satisfies the equation*

$$\begin{cases}
D_t^\alpha\psi = \psi_{xx} + \delta F, & (x,t)\in\Omega_T,\\
\psi_x(0,t) = e(t), & t\in[0,T],\\
\psi_x(1,t) = 0, & t\in[0,T],\\
\psi(x,0) = 0, & x\in[0,1],
\end{cases} \tag{3.4}$$

we have

$$||\psi||_{L^2,\lambda}^2 \leq \frac{|e|_\lambda^2}{\lambda^\alpha}E_{\alpha,1}((C_F^2+2C_F+1)T^\alpha), \tag{3.5}$$

$$||\psi_x||_{L^2,\lambda}^2 \leq (\frac{|e|_\lambda^2}{\lambda^\alpha}+\frac{Mc_1}{\lambda^\alpha}+\frac{C_F^2}{\lambda^\alpha}||\psi||_{L^2,\lambda}^2)E_{\alpha,1}(C_F^2T^\alpha), \tag{3.6}$$

where

$$||\psi(\cdot,t)||_{L_2,\lambda}^2 \triangleq \sup_{t\in[0,T]}\{e^{-\lambda t}||\psi(\cdot,t)||_{L_2}^2, \lambda>0\},$$

$$|e(t)|_\lambda^2 \triangleq \sup_{t\in[0,T]}\{e^{-\lambda t}|e(t)|^2, \lambda>0\},$$

$|e(t)|$ represents absolute value of $e(t)$, $M = \max_{t\in[0,T]}|D_t^\alpha\psi(0,t)|^2$, $c_1 = \frac{\alpha^\alpha}{\alpha\Gamma(\alpha)e^\alpha}$, $\delta F = F(x,t,\varphi^{k+1},\varphi_x^{k+1}) - F(x,t,\varphi^k,\varphi_x^k)$ and $\psi = \varphi^{k+1}-\varphi^k$.

Proof. (i) We firstly prove the formula (3.5). Multiplying both sides of the equation $D_t^\alpha\psi = \psi_{xx}+\delta F$ by ψ and integrating with respect to x, it yields

$$\int_0^1 \psi D_t^\alpha\psi dx = \int_0^1 \psi\psi_{xx} + \psi\delta F dx.$$

Based on Lemma 2.2, formula (1.3) and boundary condition, it is not hard to know

$$\begin{aligned}\frac{1}{2}D_t^{\alpha}\|\psi\|_{L^2}^2 \leq& -\int_0^1 |\nabla\psi|^2 dx + \int_{\partial\Omega} \psi\psi_x ds + C_F\int_0^1 |\psi|(|\psi| + |\psi_x|)dx \\ \leq& -\int_0^1 |\psi_x|^2 dx + |\psi(0,t)e(t)| + C_F\|\psi\|_{L^2}^2 + C_F\int_0^1 |\psi\psi_x| dx.\end{aligned}$$

Using Young inequality (weighted form) and taking the positive constant C_1 in formula (2.1), it leads to

$$\begin{aligned}D_t^{\alpha}\|\psi\|_{L^2}^2 \leq& -2\|\psi_x\|_{L^2}^2 + 2|\psi(0,t)e(t)| + 2C_F\|\psi\|_{L^2}^2 + 2C_F\int_0^1 |\psi\psi_x| dx \\ \leq& -2\|\psi_x\|_{L^2}^2 + C_1|e(t)|^2 + \frac{1}{C_1}|\psi(0,t)|^2 + 2C_F\|\psi\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + C_F^2\|\psi\|_{L^2}^2 \\ \leq& C_1|e(t)|^2 + \frac{1}{C_1}|\psi(0,t)|^2 + c_2\|\psi\|_{L^2}^2 - \|\psi_x\|_{L^2}^2,\end{aligned}$$

where $c_2 = C_F^2 + 2C_F$. It follows from Theorem 2.1 that

$$\begin{aligned}D_t^{\alpha}\|\psi\|_{L^2}^2 \leq& C_1|e(t)|^2 + \|\psi\|_{H^1}^2 + c_2\|\psi\|_{L^2}^2 - \|\psi_x\|_{L^2}^2 \\ \leq& C_1|e(t)|^2 + (c_2+1)\|\psi\|_{L^2}^2.\end{aligned}$$

Integrating both sides of the inequality with respect to t, by Lemma 2.1 we have

$$\begin{aligned}\|\psi\|_{L^2}^2 \leq& \|\psi(x,0)\|_{L^2}^2 + \frac{C_1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|e(\tau)|^2 d\tau + \frac{c_2+1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi\|_{L^2}^2 d\tau \\ \leq& \|\psi(x,0)\|_{L^2}^2 + \frac{C_1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{\lambda\tau}d\tau|e|_{\lambda}^2 + \frac{c_2+1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi\|_{L^2}^2 d\tau.\end{aligned}$$

Using initial condition, we can get

$$\|\psi\|_{L^2}^2 \leq \frac{C_1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{\lambda\tau}d\tau|e|_{\lambda}^2 + \frac{c_2+1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi\|_{L^2}^2 d\tau. \tag{3.7}$$

Applying Lemma 2.3, we can obtain

$$\|\psi\|_{L^2}^2 \leq C_1\frac{e^{\lambda t}}{\lambda^{\alpha}}|e|_{\lambda}^2 E_{\alpha,1}((C_F^2 + 2C_F + 1)T^{\alpha}).$$

Taking λ-norm on both sides of inequality, we can derive

$$\|\psi\|_{L^2,\lambda}^2 \leq \frac{C_1}{\lambda^{\alpha}}|e|_{\lambda}^2 E_{\alpha,1}((C_F^2 + 2C_F + 1)T^{\alpha}). \tag{3.8}$$

(ii) We then prove the formula (3.6). Multiplying both sides of the equation $D_t^{\alpha}\psi = \psi_{xx} + \delta F$ by ψ_{xx} and integrating with respect to x, it yields

$$\int_0^1 \psi_{xx}D_t^{\alpha}\psi dx = \|\psi_{xx}\|_{L^2}^2 + \int_0^1 \psi_{xx}\delta F dx.$$

By boundary condition, we get

$$\int_0^1 \psi_x D_t^\alpha \psi_x dx = -e(t) D_t^\alpha \psi(0,t) - \|\psi_{xx}\|_{L^2}^2 - \int_0^1 \psi_{xx} \delta F dx.$$

Based on Lemma 2.2, it is not hard to know

$$\frac{1}{2} D_t^\alpha \|\psi_x\|_{L^2}^2 \le -e(t) D_t^\alpha \psi(0,t) - \|\psi_{xx}\|_{L^2}^2 - \int_0^1 \psi_{xx} \delta F dx.$$

We can conclude from the formula (1.3) that

$$\frac{1}{2} D_t^\alpha \|\psi_x\|_{L^2}^2 \le -e(t) D_t^\alpha \psi(0,t) - \|\psi_{xx}\|_{L^2}^2 + C_F \int_0^1 |\psi_{xx}\psi| + |\psi_{xx}\psi_x| dx.$$

Using Young inequality (weighted form), it leads to

$$\begin{aligned} D_t^\alpha \|\psi_x\|_{L^2}^2 &\le |e(t)|^2 + |D_t^\alpha \psi(0,t)|^2 + C_F^2 (\|\psi_x\|_{L^2}^2 + \|\psi\|_{L^2}^2) \\ &\le |e(t)|^2 + M + C_F^2 \|\psi\|_{L^2}^2 + C_F^2 \|\psi_x\|_{L^2}^2, \end{aligned}$$

where $M = \max_{t \in [0,T]} |D_t^\alpha \psi(0,t)|^2$. Integrating both sides of the inequality about t and using initial condition, according to Lemma 2.1 we get

$$\begin{aligned} \|\psi_x\|_{L^2}^2 \le& \|\psi_x(x,0)\|_{L^2}^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (|e(\tau)|^2 + M + C_F^2 \|\psi\|_{L^2}^2) d\tau + \frac{C_F^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\psi_x\|_{L^2}^2 d\tau \\ \le& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{\lambda\tau} d\tau |e|_\lambda^2 + \frac{M}{\alpha\Gamma(\alpha)} t^\alpha \\ &+ \frac{C_F^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{\lambda\tau} d\tau \|\psi\|_{L^2,\lambda}^2 + \frac{C_F^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\psi_x\|_{L^2}^2 d\tau. \end{aligned}$$

Applying Lemma 2.3, we obtian

$$\|\psi_x\|_{L^2}^2 \le (|e|_\lambda^2 \frac{e^{\lambda t}}{\lambda^\alpha} + \frac{Mt^\alpha}{\alpha\Gamma(\alpha)} + C_F^2 \frac{e^{\lambda t}}{\lambda^\alpha} \|\varphi\|_{L^2,\lambda}^2) E_{\alpha,1}(C_F^2 T^\alpha).$$

Taking λ-norm on both sides of inequality, we can derive

$$\|\psi_x\|_{L^2}^2 e^{-\lambda t} \le (\frac{|e|_\lambda^2}{\lambda^\alpha} + \frac{Mt^\alpha e^{-\lambda t}}{\alpha\Gamma(\alpha)} + \frac{C_F^2}{\lambda^\alpha} \|\varphi\|_{L^2,\lambda}^2) E_{\alpha,1}(C_F^2 T^\alpha).$$

Since $t^\alpha e^{-\lambda t}$ gets the maximum value $\frac{\alpha^\alpha}{\lambda^\alpha e^\alpha}$ at $t = \frac{\alpha}{\lambda}$. Therefore, we can get

$$\|\psi_x\|_{L^2}^2 e^{-\lambda t} \le (\frac{|e|_\lambda^2}{\lambda^\alpha} + \frac{Mc_1}{\lambda^\alpha} + \frac{C_F^2}{\lambda^\alpha} \|\psi\|_{L^2,\lambda}^2) E_{\alpha,1}(C_F^2 T^\alpha) \tag{3.9}$$

where $c_1 = \frac{\alpha^\alpha}{\alpha\Gamma(\alpha) e^\alpha}$. Then, taking the maximum value on the left side of the inequality, we have

$$\|\psi_x\|_{L^2,\lambda}^2 \le (\frac{|e|_\lambda^2}{\lambda^\alpha} + \frac{Mc_1}{\lambda^\alpha} + \frac{C_F^2}{\lambda^\alpha} \|\psi\|_{L^2,\lambda}^2) E_{\alpha,1}(C_F^2 T^\alpha). \tag{3.10}$$

This completes the proof.

Lemma 3.3. *If ψ satisfies the equation*

$$\begin{cases} D_t^\alpha \psi = \psi_{xx} + \delta F, & (x,t) \in \Omega_T, \\ \psi_x(0,t) = \beta e(t) + \gamma I_t^\theta e(t), & t \in [0,T], \\ \psi_x(1,t) = 0, & t \in [0,T], \\ \psi(x,0) = 0, & x \in [0,1], \end{cases} \tag{3.11}$$

we have

$$\|\psi\|_{L^2,\lambda}^2 \leq (\frac{2C_1\beta^2}{\lambda^\alpha} + \frac{C_1c_3}{\lambda^{\alpha+2\theta-1}})|e|_\lambda^2 E_{\alpha,1}((C_F^2 + 2C_F + 1)T^\alpha),$$

$$\|\psi_x\|_{L^2,\lambda}^2 \leq (\frac{2\beta^2}{\lambda^\alpha}|e|_\lambda^2 + \frac{Mc_1}{\lambda^\alpha} + \frac{C_F^2}{\lambda^\alpha}\|\psi\|_{L^2,\lambda}^2 + \frac{c_3|e|_\lambda^2}{\lambda^{\alpha+2\theta-1}})E_{\alpha,1}(C_F^2T^\alpha),$$

where

$$\|\psi(\cdot,t)\|_{L_2,\lambda}^2 \triangleq \sup_{t\in[0,T]}\{e^{-\lambda t}\|\psi(\cdot,t)\|_{L_2}^2, \lambda > 0\},$$

$$|e(t)|_\lambda^2 \triangleq \sup_{t\in[0,T]}\{e^{-\lambda t}|e(t)|^2, \lambda > 0\},$$

$|e(t)|$ represents absolute value of $e(t)$, $M = \max_{t\in[0,T]}|D_t^\alpha\psi(0,t)|^2$, $c_1 = \frac{\alpha^\alpha}{\alpha\Gamma(\alpha)e^\alpha}$, $\delta F = F(x,t,\varphi^{k+1},\varphi_x^{k+1}) - F(x,t,\varphi^k,\varphi_x^k)$, $\psi = \varphi^{k+1} - \varphi^k$ and $c_3 = \frac{2\Gamma(2\theta-1)\gamma^2T}{\Gamma(\theta)^2}$.

Proof. (i) We firstly prove the formula (3.12). Multiplying both sides of the equation $D_t^\alpha\psi = \psi_{xx} + \delta F$ by ψ and integrating with respect to x, it yields

$$\int_0^1 \psi D_t^\alpha \psi dx = \int_0^1 \psi\psi_{xx} + \psi\delta F dx.$$

Based on Lemma 2.2 and boundary condition, it is not hard to know

$$\begin{aligned} \frac{1}{2}D_t^\alpha\|\psi\|_{L^2}^2 &\leq -\int_0^1 |\nabla\psi|^2 dx + \psi\psi_x|_0^1 + C_F\int_0^1 |\psi|(|\psi| + |\psi_x|)dx \\ &\leq -\int_0^1 |\psi_x|^2 dx + |\psi(0,t)(\beta e(t) + \gamma I_t^\theta e(t))| + C_F\|\psi\|_{L^2}^2 + C_F\int_0^1 |\psi\psi_x|dx. \end{aligned}$$

Using Young inequality (weighted form) and taking the positive constant C_1 in formula (2.1), we obtain

$$D_t^\alpha\|\psi\|_{L^2}^2 \leq 2C_1\beta^2|e(t)|^2 + 2C_1\gamma^2|I_t^\theta e(t)|^2 + \frac{1}{C_1}|\psi(0,t)|^2 + (C_F^2 + 2C_F)\|\psi\|_{L^2}^2 - \|\psi_x\|_{L^2}^2.$$

Applying Theorem2.1 and Lemma 3.1, it leads to

$$\begin{aligned} D_t^\alpha\|\psi\|_{L^2}^2 &\leq 2C_1\beta^2|e(t)|^2 + 2C_1\gamma^2|I_t^\theta e(t)|^2 + \|\psi\|_{H^1}^2 + (C_F^2 + 2C_F)\|\psi\|_{L^2}^2 - \|\psi_x\|_{L^2}^2 \\ &\leq 2C_1\beta^2|e|^2 + \frac{C_1c_3e^{\lambda t}}{\lambda^{2\theta-1}}|e|_\lambda^2 + (C_F^2 + 2C_F + 1)\|\psi\|_{L^2}^2. \end{aligned}$$

where $c_3 = \frac{2\Gamma(2\theta-1)\gamma^2 T}{\Gamma(\theta)^2}$. Integrating both sides of the inequality about t and using initial condition, by Lemma 2.1 we get

$$\begin{aligned}
\|\psi\|_{L^2}^2 &\le \|\psi(x,0)\|_{L^2}^2 + \frac{2C_1\beta^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|e|^2 d\tau + \frac{C_1 c_3 e^{\lambda t}}{\lambda^{\alpha+2\theta-1}}|e|_\lambda^2 + \frac{C_F^2+2C_F+1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi\|_{L^2}^2 d\tau \\
&\le 2C_1\beta^2\frac{e^{\lambda t}}{\lambda^\alpha}|e|_\lambda^2 + \frac{C_1 c_3 e^{\lambda t}}{\lambda^{\alpha+2\theta-1}}|e|_\lambda^2 + \frac{C_F^2+2C_F+1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi\|_{L^2}^2 d\tau.
\end{aligned}$$

It follows from Lemma 2.3 that

$$\|\psi\|_{L^2}^2 \le (2C_1\beta^2\frac{e^{\lambda t}}{\lambda^\alpha} + \frac{C_1 c_3 e^{\lambda t}}{\lambda^{\alpha+2\theta-1}})|e|_\lambda^2 E_{\alpha,1}((C_F^2+2C_F+1)T^\alpha).$$

Taking λ-norm on both sides of inequality, we can derive

$$\|\psi\|_{L^2,\lambda}^2 \le (\frac{2C_1\beta^2}{\lambda^\alpha} + \frac{C_1 c_3}{\lambda^{\alpha+2\theta-1}})|e|_\lambda^2 E_{\alpha,1}((C_F^2+2C_F+1)T^\alpha). \tag{3.12}$$

(ii) We then prove the formula (3.12). Multiplying both sides of the equation $D_t^\alpha\psi = \psi_{xx} + \delta F$ by ψ_{xx} and integrating with respect to x, it yields

$$\int_0^1 \psi_{xx} D_t^\alpha \psi dx = \|\psi_{xx}\|_{L^2}^2 + \int_0^1 \psi_{xx}\delta F dx.$$

Based on boundary condition, it is not hard to know

$$\int_0^1 \psi_x D_t^\alpha \psi_x dx = -(\beta e(t) + \gamma I_t^\theta e(t)) D_t^\alpha \psi(0,t) - \|\psi_{xx}\|_{L^2}^2 - \int_0^1 \psi_{xx}\delta F dx.$$

According to Lemma 2.2, we obtain

$$\frac{1}{2}D_t^\alpha \|\psi_x\|_{L^2}^2 \le -(\beta e(t) + \gamma I_t^\theta e(t)) D_t^\alpha \psi(0,t) - \|\psi_{xx}\|_{L^2}^2 - \int_0^1 \psi_{xx}\delta F dx.$$

Applying Lipschitz condition (1.3), we have

$$\frac{1}{2}D_t^\alpha \|\psi_x\|_{L^2}^2 \le -(\beta e(t) + \gamma I_t^\theta e(t)) D_t^\alpha \psi(0,t) - \|\psi_{xx}\|_{L^2}^2 + C_F\int_0^1 |\psi_{xx}\psi| + |\psi_{xx}\psi_x| dx.$$

Using Young inequality (weighted form), it leads to

$$\begin{aligned}
D_t^\alpha \|\psi_x\|_{L^2}^2 &\le 2\beta^2|e(t)|^2 + 2\gamma^2|I_t^\theta e(t)|^2 + |D_t^\alpha \psi(0,t)|^2 + C_F^2\|\psi_x\|_{L^2}^2 + C_F^2\|\psi\|_{L^2}^2 \\
&\le 2\beta^2|e(t)|^2 + 2\gamma^2|I_t^\theta e(t)|^2 + M + C_F^2\|\psi\|_{L^2}^2 + C_F^2\|\psi_x\|_{L^2}^2.
\end{aligned}$$

Integrating both sides of the inequality with respect to t and using initial condition, by Lemma 2.1 and Lemma 3.1, we get

$$\|\psi_x\|_{L^2}^2 \le \|\psi_x(x,0)\|_{L^2}^2 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(2\beta^2|e|^2 + M + C_F^2\|\psi\|_{L^2}^2) d\tau$$

$$
\begin{aligned}
&+\frac{C_F^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi_x\|_{L^2}^2 d\tau+\frac{c_3 e^{\lambda t}|e|_\lambda^2}{\lambda^{\alpha+2\theta-1}}\\
\leq&\frac{2\beta^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{\lambda\tau}d\tau|e|_\lambda^2+\frac{M}{\alpha\Gamma(\alpha)}t^\alpha\\
&+\frac{C_F^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{\lambda\tau}d\tau\|\psi\|_{L^2,\lambda}^2+\frac{c_3 e^{\lambda t}|e|_\lambda^2}{\lambda^{\alpha+2\theta-1}}+\frac{C_F^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi_x\|_{L^2}^2 d\tau\\
\leq&2\beta^2\frac{e^{\lambda t}}{\lambda^\alpha}|e|_\lambda^2+\frac{M}{\alpha\Gamma(\alpha)}t^\alpha+C_F^2\frac{e^{\lambda t}}{\lambda^\alpha}\|\psi\|_{L^2,\lambda}^2+\frac{C_F^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\psi_x\|_{L^2}^2 d\tau+\frac{c_3 e^{\lambda t}|e|_\lambda^2}{\lambda^{\alpha+2\theta-1}},
\end{aligned}
$$

where $c_3=\frac{2\Gamma(2\theta-1)\gamma^2 T}{\Gamma(\theta)^2}$. Using Lemma 2.3, we have

$$
\|\psi_x\|_{L^2}^2 \leq (2\beta^2|e|_\lambda^2\frac{e^{\lambda t}}{\lambda^\alpha}+\frac{M}{\alpha\Gamma(\alpha)}t^\alpha)E_{\alpha,1}(C_F^2T^\alpha)+(C_F^2\frac{e^{\lambda t}}{\lambda^\alpha}\|\psi\|_{L^2,\lambda}^2+\frac{c_3e^{\lambda t}|e|_\lambda^2}{\lambda^{\alpha+2\theta-1}})E_{\alpha,1}(C_F^2T^\alpha).
$$

Taking λ-norm on both sides of inequality, similar to Lemma (3.2), we obtain

$$
\|\psi_x\|_{L^2,\lambda}^2 \leq (\frac{2\beta^2}{\lambda^\alpha}|e|_\lambda^2+\frac{Mc_1}{\lambda^\alpha}+\frac{C_F^2}{\lambda^\alpha}\|\psi\|_{L^2,\lambda}^2+\frac{c_3|e|_\lambda^2}{\lambda^{\alpha+2\theta-1}})E_{\alpha,1}(C_F^2T^\alpha),
$$

where $c_1=\frac{\alpha^\alpha}{\alpha\Gamma(\alpha)e^\alpha}$. This completes the proof.

3.1. Open-loop P-type ILC

The open-loop P-type ILC scheme for Eq (1.2) is

$$
u^{k+1}(t)=u^k(t)+\beta e^k(t), \tag{3.13}
$$

where $e^k(t)=y^d(t)-y^k(t)$ denotes the output error and the learning gain β is an unknown parameter to be determined later.

Theorem 3.1. *For system (1.2) and the open-loop P-type law (3.13), if there exist a learning gain β and a constant $l(l>0)$ satisfying*

$$
(1+l)\overline{\rho}_1^2\leq 1, \tag{3.14}
$$

where $\overline{\rho}_1=\max_{t\in[0,T]}|1-\beta d(t)|$, then the output error e^k can converge to the ϵ-neighborhood of zero for any constant $\epsilon>0$ in the sense of λ-norm as $k\to\infty$.

Proof. From the definition of error, we get

$$
\begin{aligned}
e^{k+1}(t)&=y^d(t)-y^{k+1}(t)\\
&=y^d(t)-y^k(t)-(y^{k+1}(t)-y^k(t)).
\end{aligned} \tag{3.15}
$$

Based on the formula (3.13), it is not hard to know

$$
\begin{aligned}
e^{k+1}(t)&=e^k(t)-c(t)\delta\varphi^{k+1}(1,t)-\beta d(t)e^k(t)\\
&=(1-\beta d(t))e^k(t)-c(t)\delta\varphi^{k+1}(1,t).
\end{aligned} \tag{3.16}
$$

Squaring both sides of the equation, we get

$$|e^{k+1}(t)|^2 \le \overline{\rho}_1^2|e^k(t)|^2 + \overline{c}^2|\delta\varphi^{k+1}(1,t)|^2 + 2\overline{\rho}_1\overline{c}|e^k(t)||\delta\varphi^{k+1}(1,t)|,$$

where $\overline{\rho}_1 = \max\limits_{t\in[0,T]}|1-\beta d(t)|$ and $\overline{c} = \max\limits_{t\in[0,T]}|c(t)|$. Using Young inequality (weighted form) to ensure $(1+l)\overline{\rho}_1^2 \le 1$ and Theorem 2.1, we have

$$\begin{aligned}|e^{k+1}(t)|^2 &\le (1+l)\overline{\rho}_1^2|e^k(t)|^2 + (1+\frac{1}{l})\overline{c}^2|\delta\varphi^{k+1}(1,t)|^2\\ &\le (1+l)\overline{\rho}_1^2|e^k(t)|^2 + (1+\frac{1}{l})\overline{c}^2\max_{x\in[0,1]}|\delta\varphi^{k+1}(x,t)|^2\\ &\le (1+l)\overline{\rho}_1^2|e^k(t)|^2 + (1+\frac{1}{l})\overline{c}^2C_1\|\delta\varphi^{k+1}\|_{H^1}^2. \end{aligned} \tag{3.17}$$

Taking λ-norm on both sides of inequality, we get

$$\begin{aligned}|e^{k+1}(t)|_\lambda^2 &\le (1+l)\overline{\rho}_1^2|e^k(t)|_\lambda^2 + (1+\frac{1}{l})\overline{c}^2C_1\|\delta\varphi^{k+1}\|_{H^1,\lambda}^2\\ &\le (1+l)\overline{\rho}_1^2|e^k(t)|_\lambda^2 + (1+\frac{1}{l})\overline{c}^2C_1\big(\|\delta\varphi^{k+1}\|_{L^2,\lambda}^2 + \|\delta\varphi_x^{k+1}\|_{L^2,\lambda}^2\big).\end{aligned}$$

Using Lemma 3.2, we obtain

$$|e^{k+1}(t)|_\lambda^2 \le q_1|e^k(t)|_\lambda^2 + \mu_{1,k}, \tag{3.18}$$

where

$$q_1 = (1+l)\overline{\rho}_1^2 + (1+\frac{1}{l})\overline{c}^2C_1\beta^2(C_T + E_{\alpha,1}(C_F^2T^\alpha) + C_F^2C_TE_{\alpha,1}(C_F^2T^\alpha)\frac{1}{\lambda^\alpha})\frac{1}{\lambda^\alpha},$$

$$\mu_{1,k} = (1+\frac{1}{l})\overline{c}^2C_1\frac{E_{\alpha,1}(C_F^2T^\alpha)M_k\alpha^\alpha}{\alpha\Gamma(\alpha)e^\alpha\lambda^\alpha}, C_T = E_{\alpha,1}((C_F^2+2C_F+1)T^\alpha)$$

and $M_k = \max\limits_{t\in[0,T]}|D_t^\alpha\varphi^k(0,t)|^2$. Choosing λ large enough so that $q_1 < 1$, we get

$$\begin{aligned}|e^{k+1}(t)|_\lambda^2 \le& q_1(|e^{k-1}(t)|_\lambda^2 + \mu_{1,k-1}) + \mu_{1,k}\\ \le& q_1^{k+1}|e^0(t)|_\lambda^2 + q_1^k\mu_{1,0} + q_1^{k-1}\mu_{1,1} + \cdots + \mu_{1,k}\\ \le& q_1^{k+1}|e^0(t)|_\lambda^2 + \frac{\overline{\mu}_{1,k}}{1-q_1},\end{aligned} \tag{3.19}$$

where $\overline{\mu}_{1,k} \triangleq \max\limits_{m\in\{0,1,\cdots,k\}}\mu_{1,m}$. We select λ large enough so that $\overline{\mu}_{1,k}$ is sufficiently small. Therefore, to ensure $|e^{k+1}(t)|_\lambda^2 \le \epsilon^2$, it is sufficient to make

$$q_1^{k+1}|e^0(t)|_\lambda^2 < \epsilon^2, \tag{3.20}$$

which means that the output error converges to the ϵ-neighborhood of zero after finite step iteration ($k > \frac{2(\ln\epsilon - \ln|e^0|_\lambda)}{\ln q_1} - 1$).

Remark 3.1. *Due to* $q_1(\lambda)$ *is a monotonic decreasing function of* λ *and* $(1+l)\overline{\rho}_1^2 < 1$*, we can see that the inequality* $q_1 < 1$ *holds when* λ *is large enough. From the definition of* $\mu_{1,k}$*,* $\mu_{1,k}$ *is proportional to* $\lambda^{-\alpha}$*. The number of iterations k is finited, so* $\overline{\mu}_{1,k}$ *is also proportional to* $\lambda^{-\alpha}$ *and* $\overline{\mu}_{1,k}$ *tends to zero when* λ *is large enough.*

Remark 3.2. *In order to satisfy the convergence condition (3.14), the learning gain* β *should satisfy*

$$\frac{\sqrt{1+l}-1}{d_1\sqrt{1+l}} < \beta < \frac{\sqrt{1+l}+1}{d_2\sqrt{1+l}}.$$

To ensure that the above inequality holds, parameter l should satisfy

$$l < (\frac{d_2+d_1}{d_2-d_1})^2 - 1.$$

3.2. Closed-loop P-type ILC

The closed-loop P-type ILC control scheme for (1.2) is

$$u^{k+1}(t) = u^k(t) + \beta e^{k+1}(t), \tag{3.21}$$

where $e^{k+1}(t) = y^d(t) - y^{k+1}(t)$ is the output error and the learning gain β is an unknown parameter to be determined later.

Theorem 3.2. *For system (1.2) and the ILC law (3.21), if there exist a learning gain* β *and a constant* $l(l>0)$ *satisfying*

$$(1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 \le 1, \tag{3.22}$$

where $\overline{\rho}_2 = \max\limits_{t\in[0,T]} \frac{1}{|1+\beta d(t)|}$*, then the output error* e^k *can converge to the* ϵ*-neighborhood of zero for any constant* $\epsilon > 0$ *in the sense of* λ*-norm as* $k \to \infty$*.*

Proof. From the definition of error, we get

$$\begin{aligned} e^{k+1}(t) &= y^d(t) - y^{k+1}(t) \\ &= y^d(t) - y^k(t) - (y^{k+1}(t) - y^k(t)) \\ &= e^k(t) - c(t)\delta\varphi^{k+1}(1,t) - \beta d(t)e^{k+1}(t). \end{aligned} \tag{3.23}$$

Based on the formula (3.21), it is not hard to know

$$(1+\beta d(t))e^{k+1}(t) = e^k(t) - c(t)\delta\varphi^{k+1}(1,t). \tag{3.24}$$

Simplifying the above equation, we have

$$e^{k+1}(t) = \frac{e^k(t)}{(1+\beta d(t))} - \frac{c(t)\delta\varphi^{k+1}(1,t)}{(1+\beta d(t))}. \tag{3.25}$$

Squaring both sides of the equation, we get

$$|e^{k+1}(t)|^2 \le \overline{\rho}_2^2|e^k(t)|^2 + \overline{\rho}_2^2\overline{c}^2|\delta\varphi^{k+1}(1,t)|^2 + 2\overline{\rho}_2^2\overline{c}|e^k(t)||\delta\varphi^{k+1}(1,t)|,$$

where $\overline{\rho}_2 = \max\limits_{t\in[0,T]} \frac{1}{|1+\beta d(t)|}$ and $\overline{c} = \max\limits_{t\in[0,T]} |c(t)|$. Using Theorem 2.1 and Young inequality (weighted form) to ensure $(1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 < 1$, we have

$$\begin{aligned}|e^{k+1}(t)|^2 &\le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 |e^k(t)|^2 + (\overline{\rho}_2^2 + \frac{1}{l})\overline{c}^2 |\delta\varphi^{k+1}(1,t)|^2 \\ &\le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 |e^k(t)|^2 + (\overline{\rho}_2^2 + \frac{1}{l})\overline{c}^2 \max_{x\in[0,1]} |\delta\varphi^{k+1}(x,t)|^2 \\ &\le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 |e^k(t)|^2 + (\overline{\rho}_2^2 + \frac{1}{l})\overline{c}^2 C_1 \|\delta\varphi^{k+1}\|_{H^1}^2.\end{aligned}$$

Taking λ-norm on both sides of inequality, we get

$$|e^{k+1}(t)|_\lambda^2 \le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 |e^k(t)|_\lambda^2 + (\overline{\rho}_2^2 + \frac{1}{l})\overline{c}^2 C_1 \|\delta\varphi^{k+1}\|_{H^1,\lambda}^2.$$

According to Lemma 3.2, we obtain

$$|e^{k+1}(t)|_\lambda^2 \le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2 |e^k|_\lambda^2 + N_1 |e^{k+1}|_\lambda^2 + N_{2,k}, \tag{3.26}$$

where

$$\begin{aligned} N_1 &= (\overline{\rho}_2^2 + \frac{1}{l})\overline{c}^2 C_1 \beta^2 (C_T + E_{\alpha,1}(C_F^2 T^\alpha) + C_F^2 C_T E_{\alpha,1}(C_F^2 T^\alpha)\frac{1}{\lambda^\alpha})\frac{1}{\lambda^\alpha}, \\ N_{2,k} &= (\overline{\rho}_2^2 + \frac{1}{l})\overline{c}^2 C_1 \frac{E_{\alpha,1}(C_F^2 T^\alpha) M_k \alpha^\alpha}{\alpha\Gamma(\alpha) e^\alpha \lambda^\alpha}, \\ C_T &= E_{\alpha,1}((C_F^2 + 2C_F + 1)T^\alpha)\end{aligned}$$

and $M_k = \max\limits_{t\in[0,T]} |D_t^\alpha \varphi^k(0,t)|^2$. Selecting a sufficiently large λ such that $N_1 < 1$,we can get

$$\begin{aligned}|e^{k+1}(t)|_\lambda^2 &\le \frac{(1+\overline{\rho}_2^2 l)\overline{\rho}_2^2}{1-N_1} |e^k|_\lambda^2 + \frac{N_{2,k}}{1-N_1} \\ &\le q_2 |e^k|_\lambda^2 + \mu_{2,k},\end{aligned} \tag{3.27}$$

where $q_2 = \frac{(1+\overline{\rho}_2^2 l)\overline{\rho}_2^2}{1-N_1}$ and $\mu_{2,k} = \frac{N_{2,k}}{1-N_1}$. Using recursion,we get

$$\begin{aligned}|e^{k+1}(t)|_\lambda^2 \le& q_2(q_2|e^{k-1}(t)|_\lambda^2 + \mu_{2,k-1}) + \mu_{2,k} \\ \le& q_2^{k+1}|e^0(t)|_\lambda^2 + q_2^k \mu_{2,0} + q_2^{k-1}\mu_{2,1} + \cdots + \mu_{2,k} \\ \le& q_2^{k+1}|e^0(t)|_\lambda^2 + \frac{\overline{\mu}_{2,k}}{1-q_2},\end{aligned} \tag{3.28}$$

where $\overline{\mu}_{2,k} \triangleq \max\limits_{m\in\{0,1,\cdots,k\}} \mu_{2,m}$. We select λ large enough such that q_2 is less than 1 and $\overline{\mu}_{2,k}$ is sufficiently small. Therefore, to ensure $|e^{k+1}(t)|_\lambda^2 \le \epsilon^2$, it is sufficient to make

$$q_2^{k+1}|e^0(t)|_\lambda^2 < \epsilon^2, \tag{3.29}$$

which means that the output error converges to the ϵ-neighborhood of zero after finite step iteration ($k > \frac{2(\ln\epsilon - \ln|e^0|_\lambda)}{\ln q_2} - 1$).

Remark 3.3. *In order to satisfy the convergence condition (3.22), the learning gain β should satisfy*

$$\beta > \frac{\sqrt{1+l} - 1}{d_1}.$$

3.3. Open-loop PI^θ-type ILC

The open-loop P-type ILC scheme for (1.2) is

$$u^{k+1}(t) = u^k(t) + \beta e^k(t) + \gamma I^\theta e^k(t),\ 0.5 < \theta \leq 1\ , \tag{3.30}$$

where $e^k(t) = y^d(t) - y^k(t)$ denotes the output error and the learning gain β and γ are unknown parameters to be determined later.

Theorem 3.3. *For system (1.2) and the ILC law (3.30), if the learning gain γ is bounded, and there exist the learning gain β and the constant $l(l > 0)$ satisfying*

$$(1 + l)\overline{\rho}_1^2 \leq 1, \tag{3.31}$$

where $\overline{\rho}_1 = \max_{t\in[0,T]} |1 - \beta d(t)|$, then the output error e^k can converge to the ϵ-neighborhood of zero for any constant $\epsilon > 0$ in the sense of λ-norm as $k \to \infty$.

Proof. By the definition of error, we have

$$\begin{aligned} e^{k+1}(t) &= y^d(t) - y^{k+1}(t) \\ &= y^d(t) - y^k(t) - (y^{k+1}(t) - y^k(t)). \end{aligned} \tag{3.32}$$

Based on the formula (3.30), it is not hard to know

$$\begin{aligned} e^{k+1}(t) &= e^k(t) - c(t)\delta\varphi^{k+1}(1,t) - \beta d(t)e^k(t) - \gamma d(t)I_t^\theta e^k \\ &= (1 - \beta d(t))e^k(t) - c(t)\delta\varphi^{k+1}(1,t) - \gamma d(t)I_t^\theta e^k(t). \end{aligned}$$

Applying Young inequality (weight form), we get

$$|e^{k+1}(t)|^2 \leq (1 + l)(1 - \beta d(t))^2|e^k(t)|^2 + (2 + \frac{2}{l})(c(t)^2|\delta\varphi^{k+1}(1,t)|^2 + \gamma^2 d(t)^2|I_t^\theta e^k|^2).$$

Using Theorem 2.1, it leads to

$$\begin{aligned} |e^{k+1}(t)|^2 &\leq (1 + l)\overline{\rho}_1^2|e^k(t)|^2 + (2 + \frac{2}{l})(\overline{c}^2|\delta\varphi^{k+1}(1,t)|^2 + \gamma^2 d_2^2|I_t^\theta e^k|^2) \\ &\leq (1 + l)\overline{\rho}_1^2|e^k(t)|^2 + (2 + \frac{2}{l})(\overline{c}^2 \max_{x\in[0,1]} |\delta\varphi^{k+1}(x,t)|^2 + \gamma^2 d_2^2|I_t^\theta e^k|^2) \\ &\leq (1 + l)\overline{\rho}_1^2|e^k(t)|^2 + (2 + \frac{2}{l})(\overline{c}^2 C_1\|\delta\varphi^{k+1}(\cdot,t)\|_{H^1}^2 + \gamma^2 d_2^2|I_t^\theta e^k|^2), \end{aligned}$$

where $\overline{\rho}_1 = \max_{t\in[0,T]} |1 - \beta d(t)|$ and $\overline{c} = \max_{t\in[0,T]} |c(t)|$. Using Lemma 3.1, we obtain

$$|e^{k+1}(t)|^2 \leq (1 + l)\overline{\rho}_1^2|e^k(t)|^2 + (2 + \frac{2}{l})(\overline{c}^2 C_1\|\delta\varphi^{k+1}\|_{H^1}^2 + \frac{d_2^2 c_3 e^{\lambda t}}{\lambda^{2\theta-1}}|e^k|_\lambda^2),$$

where $c_3 = \frac{2\Gamma(2\theta-1)\gamma^2 T}{\Gamma(\theta)^2}$. Taking λ-norm on both sides of inequality, we have

$$|e^{k+1}(t)|_\lambda^2 \leq ((1 + l)\overline{\rho}_1^2 + (2 + \frac{2}{l})\frac{d_2^2 c_3}{\lambda^{2\theta-1}})|e^k(t)|_\lambda^2 + (2 + \frac{2}{l})\overline{c}^2 C_1\|\delta\varphi^{k+1}\|_{H^1,\lambda}^2.$$

According to Lemma 3.3, we get

$$|e^{k+1}(t)|_\lambda^2 \leq q_3|e^k(t)|_\lambda^2 + \mu_{3,k}, \tag{3.33}$$

where

$$q_3 = (1+l)\overline{\rho}_1^2 + (2+\frac{2}{l})\overline{c}^2 C_1(C_E C_1 C_P C_T + C_P E_{\alpha,1}(C_F^2 T^\alpha)) + (2+\frac{2}{l})\frac{d_2^2 c_3}{\lambda^{2\theta-1}},$$

$$\mu_{3,k} = (2+\frac{2}{l})\overline{c}^2 C_1 \frac{E_{\alpha,1}(C_F^2 T^\alpha) M_k \alpha^\alpha}{\alpha\Gamma(\alpha)e^\alpha \lambda^\alpha},$$

$$C_T = E_{\alpha,1}((C_F^2 + 2C_F + 1)T^\alpha),$$

$$C_P = \frac{2\beta^2}{\lambda^\alpha} + \frac{c_3}{\lambda^{\alpha+2\theta-1}},$$

$$C_E = 1 + \frac{C_F^2 E_{\alpha,1}(C_F^2 T^\alpha)}{\lambda^\alpha}$$

and

$$M_k = \max_{t\in[0,T]} |D_t^\alpha \varphi^k(0,t)|^2.$$

Choosing λ large enough such that $q_3 < 1$, it leads to

$$\begin{aligned} |e^{k+1}(t)|_\lambda^2 &\leq q_3(q_3|e^{k-1}(t)|_\lambda^2 + \mu_{3,k-1}) + \mu_{3,k} \\ &\leq q_3^{k+1}|e^0(t)|_\lambda^2 + q_3^k\mu_{3,0} + q_3^{k-1}\mu_{3,1} + \cdots + \mu_{3,k} \\ &\leq q_3^{k+1}|e^0(t)|_\lambda^2 + \frac{\overline{\mu}_{3,k}}{1-q_3}, \end{aligned} \tag{3.34}$$

where $\overline{\mu}_{3,k} \triangleq \max_{m\in\{0,1,\cdots,k\}} \mu_{3,m}$. We select λ large enough such that q_3 is less than 1 and $\overline{\mu}_{3,k}$ is sufficiently small. Therefore, to ensure $|e^{k+1}(t)|_\lambda^2 \leq \epsilon^2$, it is sufficient to make

$$q_3^{k+1}|e^0(t)|_\lambda^2 < \epsilon^2, \tag{3.35}$$

which means that the output error converges to the ϵ-neighborhood of zero after finite step iteration ($k > \frac{2(\ln\epsilon - \ln|e^0|_\lambda)}{\ln q_3} - 1$).

3.4. Closed-loop PI^θ-type ILC

The closed-loop P-type ILC scheme for (1.2) is

$$u^{k+1}(t) = u^k(t) + \beta e^{k+1}(t) + \gamma I^\theta e^{k+1}(t),\ 0.5 < \theta \leq 1, \tag{3.36}$$

where $e^k(t) = y^d(t) - y^k(t)$ denotes the output error and the learning gain β and γ are unknown parameters to be determined later.

Theorem 3.4. *For system (1.2) and the ILC law (3.36), if the learning gain γ is bounded, and there exist the learning gain β and the constant $l(l > 0)$ satisfying*

$$(1 + \overline{\rho}_2^2 l)\overline{\rho}_2^2 \leq 1, \tag{3.37}$$

where $\overline{\rho}_2 = \max_{t\in[0,T]} \frac{1}{|1+\beta d(t)|}$, then the output error e^k can converge to the ϵ-neighborhood of zero for any constant $\epsilon > 0$ in the sense of λ-norm as $k \to \infty$.

Proof. By the definition of error, we have

$$\begin{aligned} e^{k+1}(t) &= y^d(t) - y^{k+1}(t) \\ &= y^d(t) - y^k(t) - (y^{k+1}(t) - y^k(t)). \end{aligned} \tag{3.38}$$

Based on the formula (3.36), it is not hard to know

$$e^{k+1}(t) = e^k(t) - c(t)\delta\varphi^{k+1}(1,t) - \beta d(t)e^{k+1}(t) - \gamma d(t)I_t^\theta e^{k+1}.$$

Combining similar items, it leads to

$$(1+\beta d(t))e^{k+1}(t) = e^k(t) - c(t)\delta\varphi^{k+1}(1,t) - \gamma d(t)I_t^\theta e^{k+1}. \tag{3.39}$$

Simplifying the above equation, we have

$$e^{k+1}(t) = \frac{e^k(t)}{1+\beta d(t)} - \frac{c(t)\delta\varphi^{k+1}(1,t)}{1+\beta d(t)} - \frac{\gamma d(t)}{1+\beta d(t)} I_t^\theta e^{k+1}(t).$$

Applysing Young inequality (weighted form), we get

$$|e^{k+1}(t)|^2 \le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2|e^k(t)|^2 + (2\overline{\rho}_2^2 + \frac{2}{l})(\overline{c}^2|\delta\varphi^{k+1}(1,t)|^2 + \gamma^2 d_2^2 |I_t^\theta e^{k+1}(t)|^2),$$

where $\overline{\rho}_2 = \max\limits_{t\in[0,T]} \frac{1}{|1+\beta d(t)|}$ and $\overline{c} = \max\limits_{t\in[0,T]} |c(t)|$. Using Theorem 2.1, we obtain

$$\begin{aligned} |e^{k+1}(t)|^2 &\le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2|e^k(t)|^2 + (2\overline{\rho}_2^2 + \frac{2}{l})(\overline{c}^2 \max_{x\in[0,1]} |\delta\varphi^{k+1}(\cdot,t)|^2 + \gamma^2 d_2^2 |I_t^\theta e^{k+1}|^2) \\ &\le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2|e^k(t)|^2 + (2\overline{\rho}_2^2 + \frac{2}{l})(\overline{c}^2 C_1 \|\delta\varphi^{k+1}\|_{H^1}^2 + \gamma^2 d_2^2 |I_t^\theta e^{k+1}|^2). \end{aligned}$$

According to Lemma 3.1, we have

$$|e^{k+1}(t)|^2 \le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2|e^k(t)|^2 + (2\overline{\rho}_2^2 + \frac{2}{l})(\overline{c}^2 C_1 \|\delta\varphi^{k+1}\|_{H^1}^2 + \frac{d_2^2 c_3 e^{\lambda t}|e^{k+1}|_\lambda^2}{\lambda^{2\theta-1}}),$$

where $c_3 = \frac{2\Gamma(2\theta-1)\gamma^2 T}{\Gamma(\theta)^2}$. Taking λ-norm on both sides of inequality, it leads to

$$|e^{k+1}(t)|_\lambda^2 \le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2|e^k(t)|_\lambda^2 + (2\overline{\rho}_2^2 + \frac{2}{l})(\overline{c}^2 C_1 \|\delta\varphi^{k+1}\|_{H^1,\lambda}^2 + \frac{d_2^2 c_3 |e^{k+1}|_\lambda^2}{\lambda^{2\theta-1}}).$$

Using Lemma 3.3, we can get

$$|e^{k+1}(t)|_\lambda^2 \le (1+\overline{\rho}_2^2 l)\overline{\rho}_2^2|e^k(t)|_\lambda^2 + N_3|e^{k+1}(t)|_\lambda^2 + N_{4,k}, \tag{3.40}$$

where

$$N_3 = (2\overline{\rho}_2^2 + \frac{2}{l})\overline{c}^2 C_1(C_E C_1 C_P C_T + C_P E_{\alpha,1}(C_F^2 T^\alpha)) + (2\overline{\rho}_2^2 + \frac{2}{l})\frac{d_2^2 c_3}{\lambda^{2\theta-1}},$$

$$N_{4,k} = (2\overline{\rho}_2^2 + \frac{2}{l})\overline{c}^2 C_1 \frac{E_{\alpha,1}(C_F^2 T^\alpha) M_k \alpha^\alpha}{\alpha\Gamma(\alpha) e^\alpha \lambda^\alpha},$$

$$C_T = E_{\alpha,1}((C_F^2 + 2C_F + 1)T^\alpha),$$

$$C_P = \frac{2\beta^2}{\lambda^\alpha} + \frac{c_3}{\lambda^{\alpha+2\theta-1}},$$

$C_E = 1 + \frac{C_F^2 E_{\alpha,1}(C_F^2 T^\alpha)}{\lambda^\alpha}$ and $M_k = \max_{t\in[0,T]} |D_t^\alpha \varphi^k(0,t)|^2$. Selecting a sufficiently large λ such that $q_4 < 1$, we can obtain

$$\begin{aligned} |e^{k+1}(t)|_\lambda^2 &\le \frac{(1+\overline{\rho}_2^2 l)\overline{\rho}_2^2}{1-N_3}|e^k(t)|_\lambda^2 + \frac{N_{4,k}}{1-N_3} \\ &\le q_4|e^k(t)|_\lambda^2 + \mu_{4,k}, \end{aligned} \tag{3.41}$$

where $q_4 = \frac{(1+\overline{\rho}_2^2 l)\overline{\rho}_2^2}{1-N_3}$ and $\mu_{4,k} = \frac{N_{4,k}}{1-N_3}$. Using recursion, we get

$$\begin{aligned} |e^{k+1}(t)|_\lambda^2 \le& q_4(q_4|e^{k-1}(t)|_\lambda^2 + \mu_{4,k-1}) + \mu_{4,k} \\ \le& q_4^{k+1}|e^0(t)|_\lambda^2 + q_4^k\mu_{4,0} + q_4^{k-1}\mu_{4,1} + \cdots + \mu_{4,k} \\ \le& q_4^{k+1}|e^0(t)|_\lambda^2 + \frac{\overline{\mu}_{4,k}}{1-q_4}, \end{aligned} \tag{3.42}$$

where $\overline{\mu}_{4,k} \triangleq \max_{m\in\{0,1,\cdots,k\}} \mu_{4,m}$. We select λ large enough such that q_4 is less than 1 and $\overline{\mu}_{4,k}$ is sufficiently small. Therefore, to ensure $|e^{k+1}(t)|_\lambda^2 \le \epsilon^2$, it is sufficient to make

$$q_4^{k+1}|e^0(t)|_\lambda^2 < \epsilon^2, \tag{3.43}$$

which means that the output error converges to the ϵ-neighborhood of zero after finite step iteration ($k > \frac{2(\ln\epsilon - \ln|e^0|_\lambda)}{\ln q_4} - 1$).

4. Numerical examples

In this section, we use the following numerical examples to verify convergence conditions of the open-loop P-type ILC, Closed-loop P-type ILC, open-loop PI^θ-type ILC and Closed-loop PI^θ-type ILC schemes. We can also observe the convergence speed of the four iterative learning algorithms from the numerical results.

Example 4.1. *We consider a boundary tracing problem of one dimensional fractional diffusion equation*

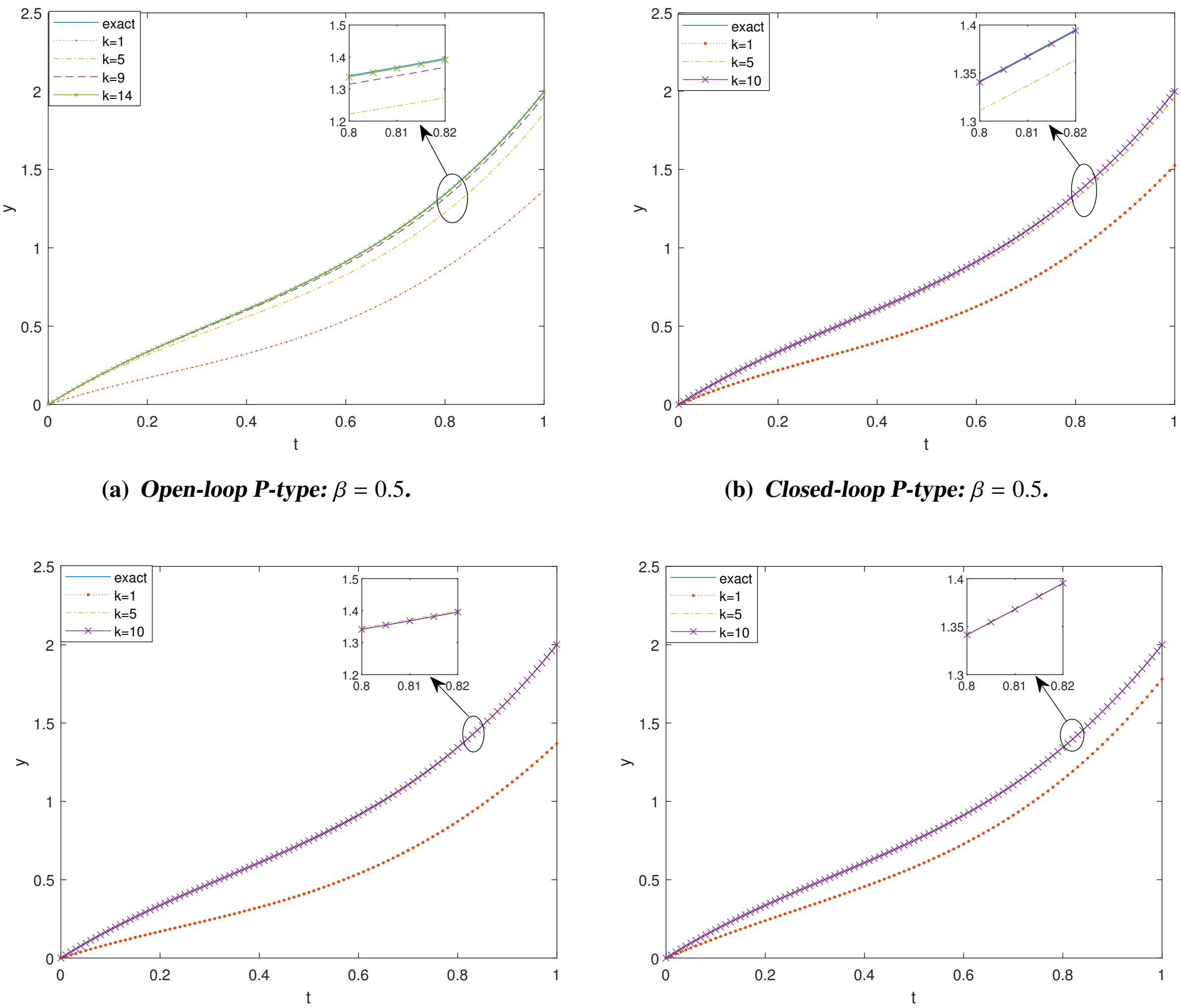
$$\begin{cases} {}_0^C D_t^\alpha \varphi^k = \varphi_{xx}^k + F(x,t,\varphi^k), & (x,t) \in (0,1)\times(0,1], \\ \varphi_x^k(0,t) = u^k(t), & t\in[0,1], \\ \varphi_x^k(1,t) = 2t^2 - 3t + 2, & t\in[0,1], \\ \varphi^k(x,0) = x^2, & x\in[0,1], \end{cases}$$

where

$$F(x,t,\varphi^k) = 2x^2(t-1)^{2-\alpha} + xt^{1-\alpha} - 2(t-1)^2 - x^2(t-1)^2 - xt - (x^2(t-1)^2 + xt)^2 + \varphi^k + |\varphi^k|^2,$$

$\alpha = 0.9$ and $T = 1$. In this simulation, the output is determined as $y^k(t) = t\varphi^k(1,t) + (t^2 - t + 1)u^k(t)$, that is $c(t) = t$, $d(t) = (t^2 - t + 1)$. The output reference trajectory is $y^d(t) = 2(t^3 - t^2 + t)$.

Figure 1a displays the tracking performance of the open-loop P-type ILC, while Figure 1b shows the tracking performance of the closed-loop P-type ILC. Additionally, Figure 1c displays the tracking performance of the open-loop PI^{θ}-type ILC, and Figure 1d shows the tracking performance of the closed-loop PI^{θ}-type ILC.



(a) ***Open-loop P-type:*** $\beta = 0.5$**.**

(b) ***Closed-loop P-type:*** $\beta = 0.5$**.**

(c) ***Open-loop*** PI^{θ}***-type:*** $\beta = 0.5$**,** $\gamma = 2$**,** $\theta = 0.95$**.**

(d) ***Closed-loop*** PI^{θ}***-type:*** $\beta = 0.5$**,** $\gamma = 2$**,** $\theta = 0.95$**.**

Figure 1. P-type and PI^{θ}-type schemes.

Figure 2 displays the maximum norm of four ILC schemes at different iteration times, including the open-loop P-type, closed-loop P-type, open-loop PI^{θ}-type, and closed-loop PI^{θ}-type ILC schemes. The results demonstrate that the closed-loop-type ILC schemes converge faster than the open-loop-type ILC schemes.

Figure 3a shows the unstable behavior of the open-loop ILC scheme. When β is set to 2, the open-loop P-type ILC scheme fails to meet the convergence conditions. Figure 3b displays that the closed-loop P-type ILC scheme satisfies the convergence conditions and achieves faster convergence speed.

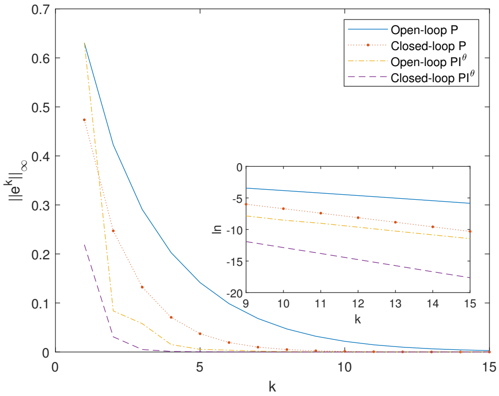
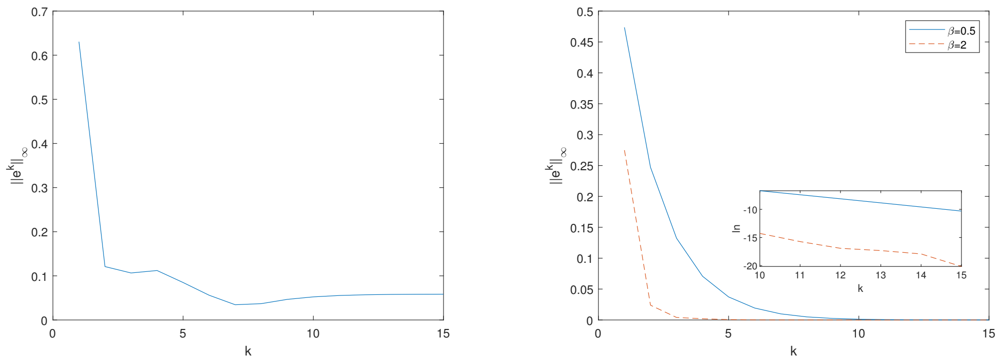


Figure 2. Maximum norm of error for $\beta = 0.5$, $\gamma = 2$, $\theta = 0.95$.



(a) ***Open-loop*** P***-type:*** $\beta = 2$**.** **(b)** ***Closed-loop*** P***-type:*** $\beta = 0.5, 2$**.**

Figure 3. Open-loop and Closed-loop P-type schemes.

Figure 4 illustrates the convergence behavior of the maximum error $||e^k||_\infty$ of the closed-loop P-type ILC scheme over 100 iterations. Although the maximum error does not decrease at iteration $k = 50$, the scheme remains stable and does not diverge.

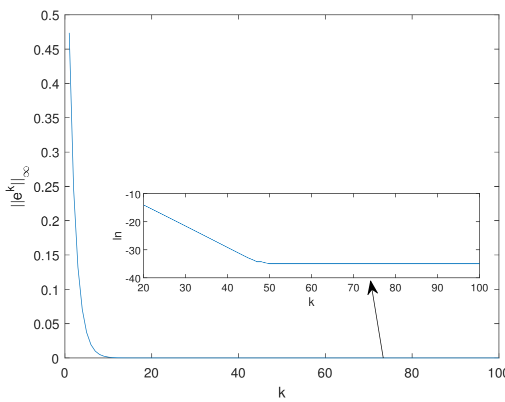


Figure 4. Closed-loop P-type scheme for $\beta = 0.5$.

Tables 1 and 2 respectively provide the maximum error of open-loop PI^{θ}-type and closed-loop PI^{θ}-type schemes. Comparing the data of PI^{θ}-type and P-type schemes in the tables, it can be observed that the PI^{θ}-type ILC scheme converges faster than the P-type ILC scheme. Comparing the data of the PI^{θ}-type ($0.5 < \theta < 1$) and the PI-type ($\theta = 1$) schemes in the tables, it can be observed that the PI^{θ}-type ILC scheme converges faster than the PI-type ILC scheme.

Table 1. Maximum norm of open-loop PI^{θ}-type scheme error: $\|e^k\|_{\infty}$.

	open-loop P-type	open-loop PI^{θ}-type				
		$\theta = 1$	$\theta = 0.95$	$\theta = 0.7$	$\theta = 0.5$	$\theta = 0.3$
$k = 1$	0.630568931	0.630568931	0.630568931	0.630568931	0.630568931	0.630568931
$k = 4$	0.202638726	0.019996924	0.015212742	0.008726639	0.005928183	0.010502699
$k = 7$	0.068332236	0.002482255	0.001857693	3.6649×10^{-4}	9.5086×10^{-5}	0.002354680
$k = 10$	0.021782898	2.8469×10^{-4}	1.9983×10^{-4}	3.1580×10^{-5}	6.5572×10^{-6}	7.2370×10^{-5}
$k = 13$	0.006572122	5.0058×10^{-5}	3.5348×10^{-5}	3.5879×10^{-6}	2.8750×10^{-7}	5.7637×10^{-6}
$k = 15$	0.002889629	1.5586×10^{-5}	1.0709×10^{-5}	8.4523×10^{-7}	3.3754×10^{-8}	1.9978×10^{-7}

Table 2. Maximum norm of closed-loop PI^{θ}-type scheme error: $\|e^k\|_{\infty}$.

	closed-loop P-type	closed-loop PI^{θ}-type				
		$\theta = 1$	$\theta = 0.95$	$\theta = 0.7$	$\theta = 0.5$	$\theta = 0.3$
$k = 1$	0.473649453	0.222303282	0.218931679	0.209854876	0.205446586	0.199953476
$k = 3$	0.132338134	0.006667802	0.005053414	0.001356560	0.002265694	0.001989692
$k = 5$	0.037264451	7.3260×10^{-4}	4.5345×10^{-4}	1.1225×10^{-4}	7.2258×10^{-5}	8.0773×10^{-5}
$k = 7$	0.009879078	7.2626×10^{-5}	4.9175×10^{-5}	4.7746×10^{-6}	6.0573×10^{-6}	7.3168×10^{-6}
$k = 9$	0.002488555	1.0026×10^{-5}	6.6767×10^{-6}	4.7218×10^{-7}	5.1262×10^{-7}	6.9890×10^{-7}
$k = 11$	6.0534×10^{-4}	1.5862×10^{-6}	1.0129×10^{-6}	4.9032×10^{-7}	4.9714×10^{-7}	4.3887×10^{-7}
$k = 13$	1.4366×10^{-4}	2.5168×10^{-7}	1.4757×10^{-7}	3.1214×10^{-8}	3.7037×10^{-8}	5.1572×10^{-8}
$k = 15$	3.3461×10^{-5}	3.8956×10^{-8}	2.1760×10^{-8}	1.2383×10^{-9}	5.7806×10^{-9}	3.3308×10^{-9}

5. Conclusions

In this paper, we investigate iterative learning algorithms for boundary tracking of nonlinear fractional diffusion equation. We provide convergence conditions for open-loop P-type, closed-loop P-type, open-loop PI^{θ}-type and closed-loop PI^{θ}-type ILC algorithms. Numerical results demonstrate the effectiveness and stability of our proposed ILC schemes. Specifically, the closed-loop ILC schemes converge faster than the open-loop ILC schemes, and the PI^{θ}-type ILC scheme outperforms the P-type and PI-type ILC schemes.

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Conflict of interest

The authors declare that there is no conflict of interest.

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