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Research article

A transformed $L1$ Legendre-Galerkin spectral method for time fractional Fokker-Planck equations

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Abstract: The numerical solutions of time α-order ($\alpha \in (0, 1)$) Caputo fractional Fokker-Planck equations is considered. The constructed method is consist of the transformed $L1$ ($TL1$) scheme in the temporal direction and the Legendre-Galerkin spectral method in the spatial direction. It has been shown that the $TL1$ Legendre-Galerkin spectral method in L^2-norm is exponential order convergent in space and $(2 - \alpha)$-th order convergent in time. Several numerical examples are given to verify the obtained theoretical results.

Keywords: time fractional Fokker-Planck equations; error estimate; Legendre-Galerkin spectral method; transformed $L1$ scheme; change of variable

1. Introduction

In the present paper, we consider numerical solution of the time fractional Fokker-Planck equations (TFFPEs):

$$\begin{cases} \partial_t^\alpha u - \Delta u + \boldsymbol{p} \cdot (\nabla u) + q(\boldsymbol{x}, t)u = f(\boldsymbol{x}, t), & (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0, & \boldsymbol{x} \in \partial\Omega, t \in (0, T], \end{cases} \quad (1.1)$$

where $\Omega \in \mathbb{R}^d$ ($d = 1, 2, 3$), $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$, $u_0(\boldsymbol{x})$ is smooth on Ω, $\boldsymbol{p} := (p_1, p_2, \ldots, p_d)$ with $p_i := p_i(\boldsymbol{x}, t)$ ($i = 1, 2, \ldots, d$) and $q := q(\boldsymbol{x}, t)$ are continuous functions. $\partial_t^\alpha u$ represents the Caputo

derivative of order $\alpha \in (0, 1)$. When $\alpha = 1$ in Eq (1.1), the corresponding equations are a class of very useful models of statistical physics to describe some practical phenomena. TFFPEs are widely used in statistical physics to describes the probability density function of position and the evolution of the velocity of a particle, see e.g., [1–4]. The TFFPEs also represent the continuous limit of a continuous time random walk with a Mittag-Leffler residence time density. For a deeper understanding of TFFPEs, we refer the readers to [5,6]. In addition, the regularity of the solutions of the TFFPE (1.1) can be found in [7].

For the past few years, many numerical methods were used to solve the TFFPEs. For example, Deng [8] proposed an efficient predictor-corrector scheme. Vong and Wang [9] constructed a compact finite difference scheme. Mahdy [10] used two different techniques to study the approximate solution of TFFPEs, namely the fractional power series method and the new iterative method. Yang et al. [11] proposed a nonlinear finite volume format to solve the two-dimensional TFFPEs. More details can refer to [12–15]. Besides, it is difficult that analysing the convergence and stability properties of the numerical schemes for TFFPEs, when convective and diffusion terms exist at the same time. In the study of TFFPEs, the conditions imposed on p and q were somewhat restrictive. For example, for solving the one-dimensional TFFPEs, Deng [16] proved the stability and convergence under the conditions that p_1 was a monotonically decreasing function and $q \leq 0$. Chen et al. [17] obtained the stability and convergence properties of the method with the conditions that p_1 was monotone or a constant and q was a constant.

To solve the time Caputo fractional equations, one of the keys is the treatment of the Caputo derivative, which raised challenges in both theoretical and numerical aspects. Under the initial singularity of the solutions of the equations, many numerical schemes are only proved to be of τ^α in temporal direction, e.g., convolution quadrature (CQ) BDF method [18], CQ Euler method [19], uniform $L1$ method et al. [20, 21]. Here τ represents the temporal stepsize. Considering the singularity of solutions, different numerical formats were established to obtain high convergence orders, e.g., the Alikhanov scheme (originally proposed in [22]) and the $L1$ scheme (see e.g., [23]) by employing the graded mesh (i.e., $t_n = T(n/K)^r, n = 1, 2, \ldots, K$, r is mesh parameter). It was proved that the optimal convergence of those methods can be 2 and $2 - \alpha$ iff $r \geq 2/\alpha$ and $r \geq (2 - \alpha)/\alpha$, respectively (see e.g., [24–29]). The $\overline{L1}$ scheme studied in [30–32] was another high-order scheme for Caputo fractional derivative. There were also some fast schemes for Caputo fractional derivative, see [33–36]. When α was small, the grids at the beginning would become very dense. It may lead to the so-called round-off errors. Recently, taking the small α and the initial singularity into account, Li et al. [37] introduced the transformation $s = t^\alpha$ for the time variable, and derived and analyzed the equivalent fractional differential equation. They constructed the $TL1$ discrete scheme, and obtained that the convergence order of the $TL1$ scheme is of $2 - \alpha$. Based on the previous research, Qin et al. [38] studied the nonlinear fractional order problem, and established the discrete fractional order Grönwall inequality. Besides, discontinuous Galerkin methods were also effective to solve the similar problems with weak singular solutions [39–41].

Much of the past study of TFFPEs (i.e., in [16, 17, 42, 43]) has been based on many restrictions on q and p_i, $i = 1, 2, \ldots, d$. This reduces the versatility of the equations. In the paper, we consider the more general TFFPE (1.1), i.e., q and p_i, $i = 1, 2, \ldots, d$, are variable coefficients, and q is independent of p_i. We draw on the treatment of the Caputo derivative in [37], introduce variable substitution, and construct the $TL1$ Legendre-Galerkin spectral scheme to solve the equivalent s-fractional equation. For

time discreteness, we take into account the initial singularity, and obtain that the optimal convergence order is $2 - \alpha$. In terms of spatial discreteness, unlike other schemes [16, 17], which impose restrictions on coefficients, the Legendre-Galerkin spectral scheme does not require p_i and q to be constants or to be monotonic. Besides, we obtain the following theoretical results. The order of convergence in L^2-norm of the method is exponential order convergent in spatial direction and $(2 - \alpha)$-th order convergent in the temporal direction. And the scheme is valid for equations with small parameter α.

The structure of the paper is as follows. In Section 2, we propose the $TL1$ Legendre-Galerkin spectral scheme for solving TFFPEs. In Section 3, the detailed proof of our main results is presented. In Section 4, two numerical examples are given to verify our obtained theoretical results. Some conclusion remarks are shown in Section 5.

2. Numerical scheme

We denote $W^{m,p}(\Omega)$ and $\| \cdot \|_{W^{m,p}(\Omega)}$ as the Sobolev space of any functions defined on Ω and the corresponding Sobolev norm, respectively, where $m \geq 0$ and $1 \leq p \leq \infty$. Especially, denote $L^2(\Omega) := W^{0,2}(\Omega)$ and $H^m(\Omega) := W^{m,2}(\Omega)$. Define $C_0^\infty(\Omega)$ as the space of infinitely differentiable functions which are nonzero only on a compact subset of Ω and $H_0^1(\Omega)$ as the completion of $C_0^\infty(\Omega)$. For convenience, denote $\| \cdot \|_0 := \| \cdot \|_{L^2(\Omega)}$, $\| \cdot \|_m := \| \cdot \|_{H^m(\Omega)}$.

For simplicity, we suppose that $\Omega = (-1, 1)^d$, and $u(\boldsymbol{x}, t) \in H_0^1(\Omega) \cap H^m(\Omega)$ for $0 \leq t \leq T$. First of all, we introduce $TL1$ scheme to discrete the Caputo fractional derivative. Introducing the change of variable as follows [21, 37, 44]:

$$t = s^{1/\alpha}, \quad w(\boldsymbol{x}, s) = u(\boldsymbol{x}, s^{1/\alpha}). \tag{2.1}$$

By this, then the Caputo derivative of $u(\boldsymbol{x}, t)$ becomes

$$
\begin{aligned}
\partial_t^\alpha u(\boldsymbol{x}, t) &= \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial u(\boldsymbol{x}, r)}{\partial r} \frac{1}{(t - r)^\alpha} dr \\
&= \frac{1}{\Gamma(1 - \alpha)} \int_0^s \frac{\partial w(\boldsymbol{x}, r)}{\partial r} \frac{1}{(s^{1/\alpha} - r^{1/\alpha})^\alpha} dr \\
&= D_s^\alpha w(\boldsymbol{x}, s). \tag{2.2}
\end{aligned}
$$

Hence, Eq (1.1) can be rewritten as

$$D_s^\alpha w(\boldsymbol{x}, s) - \Delta w + \tilde{\boldsymbol{p}} \cdot (\nabla w) + \tilde{q}(\boldsymbol{x}, s) w = \tilde{f}(\boldsymbol{x}, s), \qquad (\boldsymbol{x}, s) \in \Omega \times (0, T^\alpha], \tag{2.3}$$

$$w(\boldsymbol{x}, s) = 0, \qquad (\boldsymbol{x}, s) \in \partial\Omega \times (0, T^\alpha], \tag{2.4}$$

$$w(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{2.5}$$

where $\tilde{\boldsymbol{p}} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d)$, $\tilde{p}_d := p_d(\boldsymbol{x}, s^{1/\alpha})$, $\tilde{q} := q(\boldsymbol{x}, s^{1/\alpha})$, and $\tilde{f}(\boldsymbol{x}, s) = f(\boldsymbol{x}, s^{1/\alpha})$. Let $s_n = T^\alpha n/K$, $n = 0, 1, \ldots, K$, and the uniform mesh on $[0, T^\alpha]$ with $\tau_s = s_n - s_{n-1}$. For convenience, K_i, $i \geq 1$ represent the positive constants independent of τ_s and N, where N represents polynomial degree. In addition, we define the following notations

$$\tilde{p}_d^n := \tilde{p}_d(\boldsymbol{x}, s_n), \ \tilde{q}^n := \tilde{q}(\boldsymbol{x}, s_n), \ \tilde{f}^n := \tilde{f}(\boldsymbol{x}, s_n),$$

$$w^n := w(\boldsymbol{x}, s_n), \quad \tilde{\boldsymbol{p}}^n := (\tilde{p}_1^n, \tilde{p}_2^n, \ldots, \tilde{p}_d^n).$$

Applying the $TL1$ approximation, we have

$$
\begin{aligned}
D_s^\alpha w^n &= \frac{1}{\Gamma(1-\alpha)} \int_0^{s_n} \frac{\partial w(\boldsymbol{x}, r)}{\partial r} \frac{1}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha} dr \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{l=1}^n \frac{w^l - w^{l-1}}{\tau_s} \int_{s_{l-1}}^{s_l} \frac{dr}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha} + Q^n \\
&= \sum_{l=1}^n a_{n,n-l}(w^l - w^{l-1}) + Q^n \\
&:= D_\tau^\alpha w^n + Q^n.
\end{aligned}
\tag{2.6}
$$

Here the coefficients $a_{n,n-l} = \frac{1}{\tau_s \Gamma(1-\alpha)} \int_{s_{l-1}}^{s_l} \frac{dr}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha}$, and Q^n represents the truncation error. For more details, we refer to [37, 38]. By Eq (2.6), then Eq (2.3) arrives at

$$D_\tau^\alpha w^n - \Delta w^n + \tilde{\boldsymbol{p}}^n \cdot (\nabla w^n) + \tilde{q}^n w^n = \tilde{f}^n - Q^n.$$

For spatial discretization, we introduce the following basis functions:

$$\{\psi_{\boldsymbol{k}}(\boldsymbol{x})\} = \{\psi_{k_1}(x_1)\psi_{k_2}(x_2)\ldots\psi_{k_d}(x_d), \ k_1, k_2, \ldots, k_d \in I_N\},$$

where $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$, $I_N = \{0, 1, 2, \ldots, N-2\}$. For $\psi_{k_i}(x_i)$, $i = 1, 2, \ldots, d$, one has

$$\psi_{k_i}(x_i) = L_{k_i}(x_i) - L_{k_i+2}(x_i) \quad \text{for } \forall k_i \in I_N, \tag{2.7}$$

where $\{L_j(x)\}_{j=0}^N$ are the Legendre orthogonal polynomials, given by the following recurrence relationship [45]:

$$
\begin{cases}
(j+1)L_{j+1}(x) = (2j+1)xL_j(x) - jL_{j-1}(x) & \text{for } j \geq 1, \\
L_0(x) = 1, \quad L_1(x) = x.
\end{cases}
\tag{2.8}
$$

Define the finite-dimensional approximation space

$$X_N = span\{\psi_{\boldsymbol{k}}(\boldsymbol{x}), \ k_1, k_2, \ldots, k_d \in I_N\},$$

where $\boldsymbol{N} = (\underbrace{N, N, \ldots, N}_{d})$. For any function $w_N(x)$, write

$$w_N(x) = \sum_{k_1, k_2, \ldots, k_d \in I_N} \hat{w}_{\boldsymbol{k}} \psi_{\boldsymbol{k}}(\boldsymbol{x}).$$

By Eqs (2.7) and (2.8), we have

$$w_N(x)|_{\partial\Omega} = 0 \quad \text{for } \forall w_N(x) \in X_N.$$

Then, the $TL1$ Legendre-Galerkin spectral scheme is to seek $W^n \in X_N$, such that

$$(D_\tau^\alpha W^n, v) + (\nabla W^n, \nabla v) + (\nabla W^n, \tilde{\boldsymbol{p}}^n v) + (\tilde{q}^n W^n, v) = (\tilde{f}^n, v) \quad \text{for } \forall v \in X_N. \tag{2.9}$$

Here $W^0 = \pi_N w^0$, and π_N is the Ritz projection operator given in Lemma 2. For instance, if $d = 1$, we solve Eqs (2.3) and (2.4) by

$$A1 D_\tau^\alpha \hat{w}^n + (A2 + A3^n + A4^n)\hat{w}^n = F^n, \tag{2.10}$$

where $\hat{w}^n = (\hat{w}_0^n, \hat{w}_1^n, \hat{w}_2^n, \ldots, \hat{w}_{N-2}^n)^T$, $A1_{j,h} = (\psi_h(x), \psi_j(x))$, $j, h \in I_N$, $A2_{j,h} = (\psi_h'(x), \psi_j'(x))$, $A3_{j,h}^n = (\tilde{p}^n \psi_h'(x), \psi_j(x))$, $A4_{j,h}^n = (\tilde{q}^n \psi_h(x), \psi_j(x))$, and $F_{j,1}^n = (\tilde{f}^n, \psi_j(x))$.

The typical solution of Eq (1.1) meets [18, 46, 47]

$$\left\| \frac{\partial u}{\partial t}(\boldsymbol{x}, t) \right\|_0 \leq Ct^{\alpha-1},$$

then, with the help of the changes of variable (2.1), one has (see e.g., [38])

$$\left\| \frac{\partial^l w}{\partial s^l}(\boldsymbol{x}, s) \right\|_0 \leq C(1 + s^{1/\alpha-l}) < \infty, \quad l = 1, 2, \tag{2.11}$$

where $C > 0$ is a constant independent of s and $\boldsymbol{x}$. From [37, Lemma 2.2] and [38, Lemma 2.1], the solution becomes smoother at the beginning.

Now, the convergence results of $TL1$ Legendre-Galerkin spectral scheme (2.9) is given as follows.

Theorem 1. *Assume that $\tilde{q}$ and $\tilde{p}_i$, $i = 1, 2, \ldots, d$, in (2.3) are bounded, and that the unique solution w of Eqs (2.3) and (2.4) satisfying Eq (2.11) and $w(\boldsymbol{x}, s) \in H_0^1(\Omega) \cap H^m(\Omega)$. Then, there exist $N_0 > 0$ and $\tau_0 > 0$ such that when $N \geq N_0$ and $\tau_s \leq \tau_0$, Eq (2.9) has a unique solution W^n ($n = 0, 1, \ldots, K$), which satisfies*

$$\|w^n - W^n\|_0 \leq K^*(\tau_s^{2-\alpha} + N^{1-m}), \tag{2.12}$$

where $K^* > 0$ is a constant independent of τ_s and N.

3. Proof of the main results

3.1. Preliminaries

We will present the detailed proof of Theorem 1 in this section. For this, we first introduce the following several lemmas.

Lemma 1. *[37, 38] For $n \geq 1$, we get*

$$0 < a_{n,n-1} \leq a_{n,n-2} \leq \cdots \leq a_{n,0}. \tag{3.1}$$

Lemma 2. *If we given the Ritz projection operator $\pi_N : H_0^1(\Omega) \to X_N$ by*

$$(\nabla(\pi_N w - w), \nabla v) = 0 \quad \text{for } \forall v \in X_N,$$

then, one can get that [48]

$$\|\pi_N w - w\|_l \leq C_\Omega N^{l-m}\|w\|_m \quad \text{for } \forall w \in H_0^1(\Omega) \cap H^m(\Omega)$$

with $d \leq m \leq N + 1$, where $C_\Omega > 0$ is a constant independent of N.

Lemma 3. *[49] For any $s_K = T^\alpha > 0$ and given nonnegative sequence $\{\lambda_i\}_{i=0}^{K-1}$, assume that there exists a constant $\lambda^* > 0$ independent of τ_s such that $\lambda^* \geq \sum_{i=0}^{K-1} \lambda_i$. Assume also that the grid function $\{w^n | n \geq 0\}$ satisfies*

$$D_\tau^\alpha(w^n)^2 \leq \sum_{i=1}^n \lambda_{n-i}(w^i)^2 + w^n(Q^n + \xi) \quad for\ n \geq 1,$$

where $\{Q^n | n \geq 1\}$ is well defined in Eq (2.6). Then, there exists a constant $\tau_s^* > 0$ such that, when $\tau_s \leq \tau_s^*$,

$$w^j \leq 2E_\alpha(2\lambda^* s_j)\left[w^0 + C_1^*(\tau_s^{2-\alpha} + \xi)\right] \quad for\ 1 \leq j \leq K,$$

where C_1^* is a constant and $E_\alpha(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+k\alpha)}$.

3.2. Proof of Theorem 1

We will offer the proof of Theorem 1 in this section. The projection $\pi_N w^n$ of the exact solution w^n satisfies

$$(D_\tau^\alpha \pi_N w^n, v) = -(\nabla \pi_N w^n, \nabla v) - (\nabla \pi_N w^n, \tilde{p}^n v) - (\tilde{q}^n \pi_N w^n, v)$$
$$+ (\tilde{f}^n, v) - (Q^n, v) - (R^n, v) \quad for\ \forall v \in X_N. \tag{3.2}$$

Here $R^n = D_\tau^\alpha(w^n - \pi_N w^n) - \Delta(w^n - \pi_N w^n) + \tilde{p}^n \cdot \nabla(w^n - \pi_N w^n) + \tilde{q}^n(w^n - \pi_N w^n)$, and Q^n is the truncation error for approximating the fractional derivative defined in Eq (2.6).

The error between numerical solution W^n and exact solution w^n can be divided into

$$\|w^n - W^n\|_0 \leq \|w^n - \pi_N w^n\|_0 + \|\pi_N w^n - W^n\|_0. \tag{3.3}$$

Let

$$e^n := \pi_N w^n - W^n \quad for\ n = 0, 1, \ldots, K.$$

Subtracting Eq (2.9) from Eq (3.2), we get that

$$(D_\tau^\alpha e^n, v) = -(\nabla e^n, \nabla v) - (\nabla e^n, \tilde{p}^n v) - (\tilde{q}^n e^n, v) - (Q^n, v) - (R^n, v) \quad for\ \forall v \in X_N. \tag{3.4}$$

Setting $v = e^n$ in Eq (3.4), we obtain

$$(D_\tau^\alpha e^n, e^n) = -(\nabla e^n, \nabla e^n) - (\nabla e^n, \tilde{p}^n e^n) - (\tilde{q}^n e^n, e^n) - (Q^n, e^n) - (R^n, e^n). \tag{3.5}$$

By Lemma 1, we have

$$(D_\tau^\alpha e^n, e^n) = \left(\sum_{l=1}^n a_{n,n-l}(e^l - e^{l-1}), e^n\right)$$

$$= \left(a_{n,0}e^n - \sum_{l=1}^{n-1}(a_{n,n-l-1} - a_{n,n-l})e^l - a_{n,n-1}e^0, e^n\right)$$

$$\geq \frac{1}{2}\left(a_{n,0}\|e^n\|_0^2 - \sum_{l=1}^{n-1}(a_{n,n-l-1} - a_{n,n-l})\|e^l\|_0^2 - a_{n,n-1}\|e^0\|_0^2\right)$$

$$= \frac{1}{2} D_\tau^\alpha \|e^n\|_0^2. \tag{3.6}$$

By Cauchy-Schwartz inequality, one can obtain that

$$-(\nabla e^n, \nabla e^n) - (\nabla e^n, \tilde{\boldsymbol{p}}^n e^n) - (\tilde{q}^n e^n, e^n) \leq -\|\nabla e^n\|_0^2 + K_1 |(\nabla e^n, e^n)| + K_2 \|e^n\|_0^2$$

$$\leq -\|\nabla e^n\|_0^2 + \|\nabla e^n\|_0^2 + \frac{K_1^2}{4} \|e^n\|_0^2 + K_2 \|e^n\|_0^2$$

$$\leq \left(\frac{K_1^2}{4} + K_2 \right) \|e^n\|_0^2. \tag{3.7}$$

Here $K_1 = \max\limits_{0 \leq n \leq K} \{\|\tilde{\boldsymbol{p}}(\boldsymbol{x}, s_n)\|_0\}$, and $K_2 = \max\limits_{0 \leq n \leq K} \left\{ \max\limits_{\boldsymbol{x} \in \Omega} |\tilde{q}(\boldsymbol{x}, s_n)| \right\}$. Similarly, we see that

$$-(Q^n, e^n) \leq \|Q^n\|_0 \|e^n\|_0. \tag{3.8}$$

Noting that $e^n \in X_N$ and by Lemma 2, one has

$$(\nabla(w^n - \pi_N w^n), \nabla e^n) = 0.$$

Then

$$-(R^n, e^n) = -\left(D_\tau^\alpha(w^n - \pi_N w^n), e^n \right) - (\nabla(w^n - \pi_N w^n), \nabla e^n)$$

$$\qquad - (\nabla(w^n - \pi_N w^n), \boldsymbol{p}^n e^n) - \left(\tilde{q}^n(w^n - \pi_N w^n), e^n \right)$$

$$\leq \|D_\tau^\alpha(w^n - \pi_N w^n)\|_0 \|e^n\|_0 + K_1 \|\nabla(w^n - \pi_N w^n)\|_0 \|e^n\|_0$$

$$\qquad + K_2 \|w^n - \pi_N w^n\|_0 \|e^n\|_0$$

$$\leq C_\Omega N^{-m} \|D_\tau^\alpha w^n\|_m \|e^n\|_0 + K_1 C_\Omega N^{1-m} \|w^n\|_m \|e^n\|_0$$

$$\qquad + K_2 C_\Omega N^{-m} \|w^n\|_m \|e^n\|_0$$

$$\leq K_3 N^{1-m} \|e^n\|_0. \tag{3.9}$$

Here $K_3 = \max\limits_{0 \leq n \leq K} \{C_\Omega \|D_\tau^\alpha w^n\|_m, K_1 C_\Omega \|w^n\|_m, K_2 C_\Omega \|w^n\|_m\}$, and Lemma 2 is applied. Substituting Eqs (3.6)–(3.9) into Eq (3.5), one gets

$$D_\tau^\alpha \|e^n\|_0^2 \leq 2 \left(\frac{K_1^2}{4} + K_2 \right) \|e^n\|_0^2 + 2(\|Q^n\|_0 + K_3 N^{1-m}) \|e^n\|_0.$$

Noting that $e^0 = 0$ and by Lemma 3, it follows that

$$\|e^n\|_0 \leq 4K_3 C_1^*(\tau_s^{2-\alpha} + N^{1-m}) E_\alpha(4(K_1^2/4 + K_2)s_n).$$

By Eq (3.3), we observe

$$\|w^n - W^n\|_0 \leq \|w^n - \pi_N w^n\|_0 + \|e^n\|_0$$

$$\leq C_\Omega N^{-m} \|w^n\|_m + 4K_3 C_1^*(\tau_s^{2-\alpha} + N^{1-m}) E_\alpha(4(K_1^2/4 + K_2)s_n)$$

$$\leq K^*(\tau_s^{2-\alpha} + N^{1-m}),$$

where $K^* = \max\limits_{0 \leq n \leq K} \{C_\Omega \|w^n\|_m, 4K_3 C_1^* E_\alpha(4(K_1^2/4 + K_2)s_n)\}$. This completes the proof.

4. Numerical results

In this section, two numerical examples are given to verify our theoretical results. We define the maximal L^2 error and the convergence order in time, respectively, as

$$e(K) = \max_{0 \leq n \leq K} \|w^n - W^n\|_{L^2}, \quad \text{order} = \frac{\log\left(e(K_1)/e(K_2)\right)}{\log(K_2/K_1)}. \tag{4.1}$$

Example 1. *Consider the one-dimensional TFFPEs:*

$$\partial_t^\alpha u = u_{xx} - 2u_x + t^2 u + f(x, t), \ u \in (-1, 1) \times (0, 1], \tag{4.2}$$

where the initial-boundary conditions and the forcing term function f are choosen by the analytical solution

$$u(x, t) = (t^2 + t^\alpha)(x^3 + x^5)\sin(\pi x).$$

In this case, q is independent of p_1, furthermore, p_1 and q are not monotone functions.

We solve this problem with the $TL1$ Legendre-Galerkin spectral method. Table 1 gives the maximal L^2 errors, the convergence orders in time and the CPU times with $N = 14$. The temporal convergence orders are close to $2 - \alpha$ in Table 1. For the spatial convergence test, we set $K = 8192$. In Figure 1, we give the errors as a function of N with $\alpha = 0.3, \ 0.5, \ 0.7$ in logarithmic scale. We can observe that the errors indicate an exponential decay.
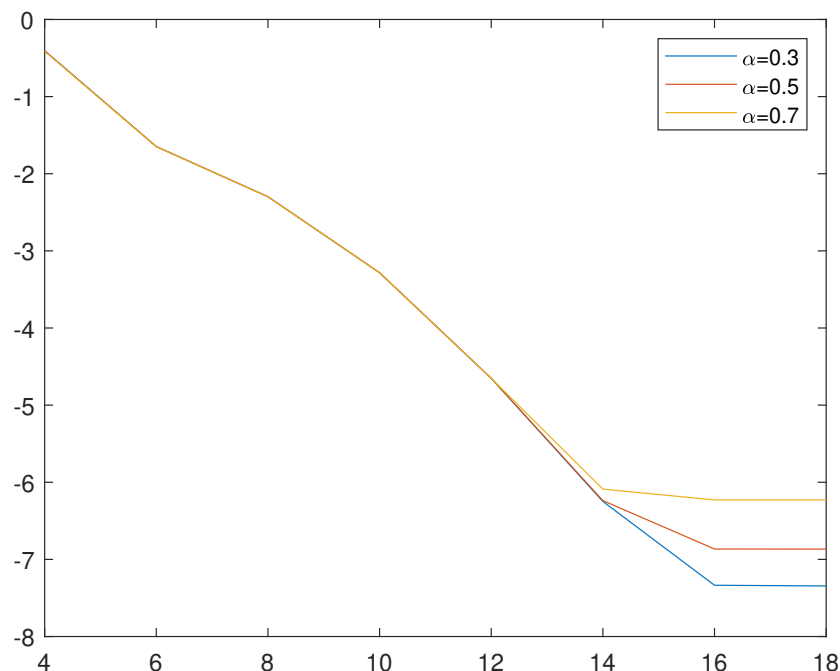
Figure 1. Errors in space with $\alpha = 0.3, 0.5, 0.7$ and different N for Example 1.

Table 1. Maximal L^2 errors, convergence orders in time and CPU times with $N = 14$ for Example 1.

|	$\alpha = 0.1$			$\alpha = 0.3$			$\alpha = 0.5$		
K	$e(K)$	order	CPU(s)	$e(K)$	order	CPU(s)	$e(K)$	order	CPU(s)
4	5.3660e-03	*	1.12e-02	7.5697e-03	*	9.76e-03	8.5571e-03	*	9.92e-03
16	1.3833e-03	0.98	2.45e-02	1.1574e-03	1.35	2.19e-02	1.3367e-03	1.34	2.25e-02
64	1.7352e-04	1.50	8.06e-02	1.3606e-04	1.54	7.53e-02	1.8311e-04	1.43	7.40e-02
256	1.6850e-05	1.68	2.90e-01	1.4476e-05	1.62	3.00e-01	2.3859e-05	1.47	3.01e-01

Example 2. *Consider the two-dimensional TFFPEs:*

$$\partial_t^\alpha u = \Delta u + t^2 x^2 y^2 (u_x + u_y) + (2t^2 xy^2 + 2t^2 x^2 y)u, \ u \in (-1, 1)^2 \times (0, 1], \tag{4.3}$$

where the initial-boundary conditions and the forcing term function f are choosen by the analytical solution

$$u(x, y, t) = E_\alpha(-t^\alpha) \sin(\pi x) \sin(\pi y).$$

Table 2 gives the maximal L^2 errors, the convergence orders in time and the CPU times with $N = 14$. The temporal convergence orders are close to $2 - \alpha$ in Table 2. For the spatial convergence test, we give the errors as a function of N for $\alpha = 0.3, \ 0.5, \ 0.7$ and $K = 8192$ in Figure 2. We use the logarithmic scale for the error-axis. Again, we observe that the errors indicate an exponential decay.
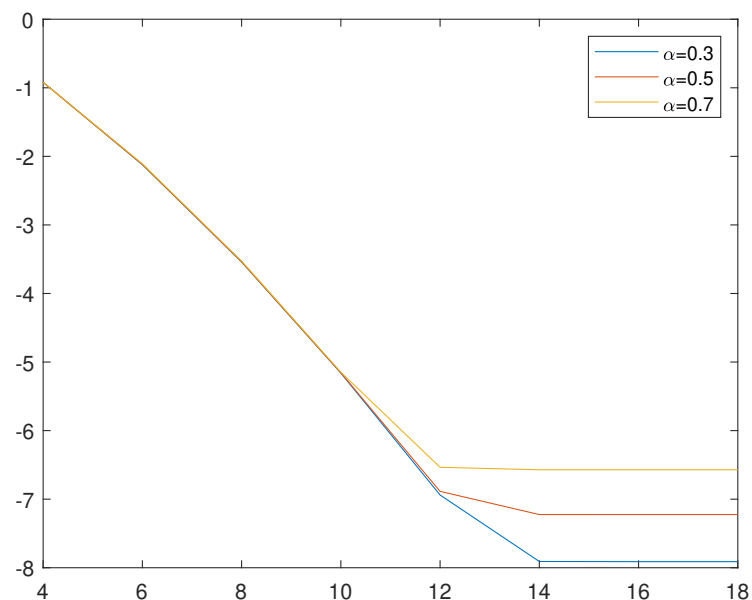
Figure 2. Errors in space with $\alpha = 0.3, 0.5, 0.7$ and different N for Example 2.

Table 2. Maximal L^2 errors, convergence orders in time and CPU times with $N = 14$ for Example 2.

K	$\alpha = 0.3$			$\alpha = 0.5$			$\alpha = 0.7$		
	$e(K)$	order	CPU(s)	$e(K)$	order	CPU(s)	$e(K)$	order	CPU(s)
32	7.0619e-05	*	2.08e-01	1.7386e-04	*	1.73e-01	3.1316e-04	*	1.69e-01
256	3.3124e-06	1.47	1.34e+00	9.7836e-06	1.38	1.28e+00	2.3617e-05	1.24	1.31e+00
2048	1.1965e-07	1.60	1.29e+01	4.6734e-07	1.46	1.28e+01	1.6199e-06	1.29	1.30e+01
8192	1.2339e-08	1.64	8.59e+01	5.9649e-08	1.48	8.54e+01	2.6824e-07	1.30	8.83e+01

5. Conclusions

We present a $TL1$ Legendre-Galerkin spectral method to solve TFFPEs in this paper. The new scheme is convergent with $O(\tau_s^{2-\alpha} + N^{1-m})$, where τ_s, N and m are the time step size, the polynomial degree and the regularity of the analytical solution, respectively. In addition, this $TL1$ Legendre-Galerkin spectral method still holds for problems with small α and gives better numerical solutions near the initial time. The new scheme can achieve a better convergence result on a relatively sparse grid point.

Acknowledgments

Conflict of interest

The authors declare that they have no conflicts of interest.

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