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Research article

Effective difference methods for solving the variable coefficient fourth-order fractional sub-diffusion equations

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Abstract: This paper is concerned with the numerical approximations for the variable coefficient fourth-order fractional sub-diffusion equations subject to the second Dirichlet boundary conditions. We construct two effective difference schemes with second order accuracy in time by applying the second order approximation to the time Caputo derivative and the sum-of-exponentials approximation. By combining the discrete energy method and the mathematical induction method, the proposed methods proved to be unconditional stable and convergent. In order to overcome the possible singularity of the solution near the initial stage, a difference scheme based on non-uniform mesh is also given. Some numerical experiments are carried out to support our theoretical results. The results indicate that the our two main schemes has the almost same accuracy and the fast scheme can reduce the storage and computational cost significantly.

Keywords: Caputo derivative; Second Dirichlet boundary condition; Variable coefficient; Unconditionally stable; Convergence

1. Introduction

In this paper we consider the numerical approximations of the following problem

$$ {}_0^C D_t^{\alpha} u(x,t) + \frac{\partial^2}{\partial x^2}\Big(\omega(x)\frac{\partial^2 u(x,t)}{\partial x^2}\Big) + \kappa u(x,t) = f(x,t),\ 0 < x < L, 0 < t \le T, \tag{1.1} $$

$$ u(x,0) = \varphi(x),\ 0 < x < L, \tag{1.2} $$

$$ u(0,t) = \alpha_1(t), u(L,t) = \alpha_2(t),\ 0 \le t \le T, \tag{1.3} $$

$$\frac{\partial^2 u(0,t)}{\partial x^2} = \beta_1(t), \frac{\partial^2 u(L,t)}{\partial x^2} = \beta_2(t), \ 0 \le t \le T, \tag{1.4}$$

where $\kappa \geq 0$ is given constant, $\varphi(x), \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t)$ and $f(x,t)$ are given sufficiently smooth functions satisfying $\varphi(0) = \alpha_1(0), \varphi(L) = \alpha_2(0), \varphi''(0) = \beta_1(0)$ and $\varphi''(L) = \beta_2(0)$, ${}_0^C D_t^\alpha u(x,t)$ denotes Caputo fractional derivative defined by

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \ \ 0 < \alpha < 1.$$

And we suppose that there exist two constants C_1 and C_2 such that $0 < C_1 \le \omega(x) \le C_2$ for $0 \le x \le L$.

More and more attention has been paid to the fractional differential equations (FDEs) due to its application foreground in chemistry, physics, finance and hydrology in the past twenty years [1–4]. As we know, the analytic solutions of FDEs are very difficult to obtain, some efficient numerical methods should be considered, especially fast algorithms with high order accuracy. Some essential definitions and properties of fractional derivatives can refer to monograph [5].

This target problem in Eq (1.1) is frequently employed to simulate some phenomena in physics, such as wave propagation in beams, brain warping, ice formation and designing special curves on surfaces and so on, e.g., [6–11] and their references.

Up to now considerable works have been done from theoretical and numerical point of view for fourth-order fractional diffusion equations. For instance, Hu and Zhang successively presented a finite difference scheme for the fourth-order fractional diffusion-wave and sub-diffusion equations, and a compact difference scheme for the former, see [12, 13]. Ji et al. [14] constructed a compact difference scheme for the fourth-order fractional sub-diffusion equation under the fist Dirichlet boundary conditions. Zhang and Pu [15] presented a compact difference scheme for such equation by $\mathcal{L}2-1_\sigma$ formula [16]. Ran and Zhang [17] presented a new compact difference schemes for the such equation of the distributed order.

However, most of the work focus on the constant coefficient case. Recently, Zhao and Xu [18] presented a compact difference scheme for the time fractional sub-diffusion equation with the variable coefficient under the Dirichlet boundary conditions. Subsequently, based on the subtle decomposition of the coefficient matrices, Vong, Lyu and Wang [19] presented a compact difference scheme to solve the equations under Neumann boundary conditions. But the above works has only accuracy of order $2-\alpha$ in time.

In this paper, our attention will be paid on the higher order difference scheme for solving the variable coefficient equations under the second Dirichlet boundary conditions For this purpose, we use the $\mathcal{L}2-1_\sigma$ formula to approximate the Caputo fractional derivative. Unlike the integer order case, the time fractional derivative requires all history information. In order to reduce the computational complexity, we also construct a fast difference scheme. The stability and convergence of both schemes are proved in detail.

The structure of this paper is as follows: In Section 2, some necessary notations and lemmas are first introduced and a second-order difference scheme for the target problem (1.1)–(1.4) is constructed. In Section 3, an important priori estimate is first proved, and the unconditional stability and convergence of scheme are obtained. In Section 4, a fast second-order difference scheme is presented, and the corresponding unconditional stability and convergence are also strictly proved. In Section 5, a difference scheme based on nonuniform time grids is first presented, and some numerical examples are provided to verify the theoretical results. A brief conclusion is given finally.

2. Derivation of the $\mathcal{L}2-1_\sigma$ scheme

Let $h = L/M$ and $\tau = T/N$, where M, N are two positive integers. Denote $x_i = ih, 0 \le i \le M, t_n = n\tau, 0 \le n \le N, \Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_n \mid 0 \le n \le N\}$. Let $\mathcal{V}_h = \{v \mid \ v = (v_0, v_1, \cdots, v_M)\}$ be grid function space on Ω_h, and $\mathring{\mathcal{V}}_h = \{v \mid \ v \in \mathcal{V}_h, v_0 = v_M = 0\}$. Also we denote $\sigma = 1 - \frac{\alpha}{2}, t_{n+\sigma} = (n+\sigma)\tau$ and $\omega(x_i) = \omega_i$.

For $u \in \mathcal{V}_h$, we define

$$\delta_x u_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i), \ \ \delta_x^2 u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}).$$

For any $u, v \in \mathring{\mathcal{V}}_h$, we define the inner products

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \ (\delta_x u, \delta_x v) = h\sum_{i=0}^{M-1}(\delta_x u_{i+\frac{1}{2}})(\delta_x v_{i+\frac{1}{2}}), \ (u, v)_\omega = h\sum_{i=1}^{M-1} u_i v_i \omega_i,$$

and norms

$$\|u\| = \sqrt{(u, u)}, \ \|u\|_\omega = \sqrt{(u, u)_\omega}, \ \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)}, \ (\delta_x^2 u, \delta_x^2 v) = h\sum_{i=1}^{M-1} \delta_x^2 u_i \delta_x^2 v_i.$$

In [16], Alikhanov developed a new second order difference formula (called $\mathcal{L}2-1_\sigma$ formula) for the Caputo fractional derivative, which can be expressed in the following lemma.

Lemma 2.1 ([16]). *Suppose $\alpha \in (0, 1), \sigma = 1 - \frac{\alpha}{2}$ and $u(t) \in C^3[0, T]$. It holds*

$$|\ {}_0^C D_t^\alpha u(t)|_{t=t_{n-1+\sigma}} - D_{\tau,\sigma}^\alpha u^n \ | = O(\tau^{3-\alpha}),$$

where

$$D_{\tau,\sigma}^\alpha u^n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Big[C_0^{(n)} u^n - \sum_{j=1}^{n-1}(C_{n-j-1}^{(n)} - C_{n-j}^{(n)})u^j - C_{n-1}^{(n)} u^0\Big],$$

in which $C_0^{(n)} = a_0 = \sigma^{1-\alpha}$ for $n = 1$, and

$$C_k^{(n)} = \begin{cases} a_0 + b_1, & k = 0, \\ a_k + b_{k+1} - b_k, & 1 \le k \le n-2, \\ a_k - b_k, & k = n-1 \end{cases}$$

for $n \ge 2$, where $a_j = (j+\sigma)^{1-\alpha} - (j-1+\sigma)^{1-\alpha}$ and $b_j = \frac{1}{2-\alpha}[(j+\sigma)^{2-\alpha} - (j-1+\sigma)^{2-\alpha}] - \frac{1}{2}[(j+\sigma)^{1-\alpha} + (j-1+\sigma)^{1-\alpha}]$ for all $j \ge 1$.

Let $v(x, t) = \frac{\partial^2 u}{\partial x^2}$. Then the problem Eqs (1.1)–(1.4) can be written in the equivalent system

$${}_0^C D_t^\alpha u(x, t) + \frac{\partial^2}{\partial x^2}(\omega(x)v(x, t)) + \kappa u(x, t) = f(x, t), \ \ 0 < x < L, \ 0 < t \le T, \tag{2.1}$$

$$v(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2},\ \ 0<x<L,\ 0<t\le T, \tag{2.2}$$
$$u(x,0)=\varphi(x),\ \ 0<x<L, \tag{2.3}$$
$$u(0,t)=\alpha_1(t),u(L,t)=\alpha_2(t),v(0,t)=\beta_1(t),v(L,t)=\beta_2(t),\ \ 0\le t\le T. \tag{2.4}$$

Suppose $u(x,t)\in C^{(6,3)}_{x,t}([0,L]\times[0,T])$. Define

$$U_i^n=u(x_i,t_n),\ V_i^n=v(x_i,t_n),\ 0\le i\le M,\ 0\le n\le N.$$

Considering the Eqs.(2.1)–(2.2) at the point $(x_i,t_{n-1+\sigma})$, we obtain

$${}_0^C D_t^\alpha u(x_i,t_{n-1+\sigma})+\frac{\partial^2}{\partial x^2}(\omega(x_i)v(x_i,t_{n-1+\sigma}))+\kappa u(x_i,t_{n-1+\sigma})=f(x_i,t_{n-1+\sigma}), \tag{2.5}$$
$$v(x_i,t_{n-1+\sigma})=\frac{\partial^2 u(x_i,t_{n-1+\sigma})}{\partial x^2}. \tag{2.6}$$

Using Taylor expansion

$$u(x_i,t_{n-1+\sigma})=\sigma U_i^n+(1-\sigma)U_i^{n-1}+O(\tau^2)=U_i^{n-1+\sigma}+O(\tau^2),$$

where $U_i^{n-1+\sigma}=\sigma U_i^n+(1-\sigma)U_i^{n-1}$. Then we obtain

$$\frac{\partial^2 u(x_i,t_{n-1+\sigma})}{\partial x^2}=\delta_x^2 U_i^{n-1+\sigma}+O(\tau^2+h^2),$$

and

$$\frac{\partial^2}{\partial x^2}(\omega(x_i)v(x_i,t_{n-1+\sigma}))=\delta_x^2(\omega_i V_i^{n-1+\sigma})+O(\tau^2+h^2).$$

Using Lemma 2.1, it follows from Eq (2.5), Eq (2.6) that

$$D_{\tau,\sigma}^\alpha U_i^n+\delta_x^2(\omega_i V_i^{n-1+\sigma})+\kappa U_i^{n-1+\sigma}=f_i^{n-1+\sigma}+(R_1)_i^n,1\le i\le M-1,1\le n\le N, \tag{2.7}$$
$$V_i^{n-1+\sigma}=\delta_x^2 U_i^{n-1+\sigma}+(R_2)_i^n,1\le i\le M-1,1\le n\le N, \tag{2.8}$$

and there exists a constant C_r such that

$$|\,(R_1)_i^n\,|+|\,(R_2)_i^n\,|\le C_r(\tau^2+h^2),1\le i\le M-1,1\le n\le N. \tag{2.9}$$

Omitting the small terms $(R_1)_i^n$ and $(R_2)_i^n$ in Eq (2.7) and Eq (2.8), we present the difference scheme (called $\mathcal{L}2-1_\sigma$ scheme) for the equivalent system (2.1)–(2.4) as follows

$$D_{\tau,\sigma}^\alpha u_i^n+\delta_x^2(\omega_i v_i^{n-1+\sigma})+\kappa u_i^{n-1+\sigma}=f_i^{n-1+\sigma},\ 1\le i\le M-1,1\le n\le N, \tag{2.10}$$
$$v_i^{n-1+\sigma}=\delta_x^2 u_i^{n-1+\sigma},\ 1\le i\le M-1,1\le n\le N, \tag{2.11}$$
$$u_i^0=\varphi(x_i),\ 0\le i\le M, \tag{2.12}$$
$$u_0^n=\alpha_1(t_n),u_M^n=\alpha_2(t_n),v_0^n=\beta_1(t_n),v_M^n=\beta_2(t_n),\ 1\le n\le N, \tag{2.13}$$

where the initial-boundary conditions Eq (2.3), Eq (2.4) have been used.

Theorem 2.2. *The above difference scheme* (2.10)–(2.13) *is equivalent to*

$$ku_1^{n-1+\sigma} + D_{\tau,\sigma}^{\alpha} u_1^n + \frac{1}{h^2}(\omega_0\beta_1(t_{n-1+\sigma}) + \omega_2\delta_x^2 u_2^{n-1+\sigma} - 2\omega_1\delta_x^2 u_1^{n-1+\sigma}) = f_1^{n-1+\sigma}, \tag{2.14}$$

$$D_{\tau,\sigma}^{\alpha} u_i^n + \delta_x^2(\omega_i\delta_x^2 u_i^{n-1+\sigma}) + ku_i^{n-1+\sigma} = f_i^{n-1+\sigma}, 2 \le i \le M-2, \tag{2.15}$$

$$ku_{M-1}^{n-1+\sigma} + D_{\tau,\sigma}^{\alpha} u_{M-1}^n + \frac{1}{h^2}(\omega_M\beta_2(t_{n-1+\sigma}) + \omega_{M-2}\delta_x^2 u_{M-2}^{n-1+\sigma} - 2\omega_{M-1}\delta_x^2 u_{M-1}^{n-1+\sigma}) = f_{M-1}^{n-1+\sigma}, \tag{2.16}$$

$$u_i^0 = \varphi(x_i), 0 \le i \le M, \tag{2.17}$$

$$u_0 = \alpha_1(t_n), u_M^n = \alpha_2(t_n). \tag{2.18}$$

Proof. Since

$$\delta_x^2\omega_1 v_1^{n-1+\sigma} = \frac{1}{h^2}(\omega_0 v_0^{n-1+\sigma} - 2\omega_1 v_1^{n-1+\sigma} + \omega_2 v_2^{n-1+\sigma}),$$
$$\delta_x^2\omega_{M-1} v_{M-1}^{n-1+\sigma} = \frac{1}{h^2}(\omega_M v_M^{n-1+\sigma} - 2\omega_{M-1} v_{M-1}^{n-1+\sigma} + \omega_{M-2} v_{M-2}^{n-1+\sigma}).$$

It follows from Eq (2.11) and Eq (2.13) that

$$\delta_x^2\omega_1 v_1^{n-1+\sigma} = \frac{1}{h^2}(\omega_0\beta_1(t_{n-1+\sigma}) - 2\omega_1\delta_x^2 u_1^{n-1+\sigma} + \omega_2\delta_x^2 u_2^{n-1+\sigma}),$$
$$\delta_x^2\omega_{M-1} v_{M-1}^{n-1+\sigma} = \frac{1}{h^2}(\omega_M\beta_2(t_{n-1+\sigma}) - 2\omega_{M-1}\delta_x^2 u_{M-1}^{n-1+\sigma} + \omega_{M-2}\delta_x^2 u_{M-2}^{n-1+\sigma}).$$

This together with Eq (2.10), we get Eq (2.14) and Eq (2.16). Eq (2.15) can be obtained by substituting Eq (2.11) into Eq (2.10). This proof is completed. □

The above equivalent form Eqs (2.14)–(2.18) will be used only in calculation.

3. Solvability, stability and convergence of the $\mathcal{L}2-1_\sigma$ scheme

We first introduce the following essential lemmas.

Lemma 3.1 ([16]). *Suppose $\alpha \in (0,1)$ and $C_k^{(n)}$ is defined in Lemma 2.1. It holds that*

$$C_0^{(n)} > C_1^{(n)} > C_2^{(n)} > \cdots > C_{n-2}^{(n)} > C_{n-1}^{(n)}, \text{ and } C_k^{(n)} > \frac{1-\alpha}{2}(k+\sigma)^{-\alpha}.$$

Lemma 3.2 ([16]). *Suppose $u = \{u^n \mid 0 \le n \le N\}$ is a grid function defined on Ω_τ. It holds that*

$$(\sigma u^n + (1-\sigma)u^{n-1})D_{\tau,\sigma}^{\alpha} u^n \ge \frac{1}{2}D_{\tau,\sigma}^{\alpha}(u^n)^2.$$

Lemma 3.3 ([20,21]). *For any $u \in \mathring{\mathcal{V}}_h$, it holds that*

$$\|u\| \le \frac{L}{\sqrt{6}}\|\delta_x u\|, \ \|\delta_x u\| \le \frac{L}{\sqrt{6}}\|\delta_x^2 u\|.$$

The following Lemma will be used in the analysis of the difference scheme.

Lemma 3.4. *For any $u \in \mathring{\mathcal{V}}_h$, it holds that*

$$C_1\|u\|^2 \le \|u\|_\omega^2 \le C_2\|u\|^2,\ C_1\|\delta_x^2 u\|^2 \le \|\delta_x^2 u\|_\omega^2 \le C_2\|\delta_x^2 u\|^2.$$

Proof. The proof is straightforward from the definition of $\|\cdot\|$ and $\|\cdot\|_\omega$. □

We next show the priori estimate of the scheme (2.10)–(2.13).

Theorem 3.5. *Suppose $\{w_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ and $\{z_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ satisfy the following difference scheme*

$$D_{\tau,\sigma}^\alpha w_i^n + \delta_x^2(\omega_i z_i^{n-1+\sigma}) + \kappa w_i^{n-1+\sigma} = p_i^{n-1+\sigma},\ 1 \le i \le M-1, 1 \le n \le N, \tag{3.1}$$

$$z_i^{n-1+\sigma} = \delta_x^2 w_i^{n-1+\sigma} + q_i^{n-1+\sigma},\ 1 \le i \le M-1, 1 \le n \le N, \tag{3.2}$$

$$w_i^n = \varphi(x_i),\ 0 \le i \le M, \tag{3.3}$$

$$w_0^n = 0, w_M^n = 0, z_0^n = 0, z_M^n = 0,\ 1 \le n \le N. \tag{3.4}$$

Then, it holds that

$$\|w^n\|^2 \le \|w^0\|^2 + 2T^\alpha\Gamma(1-\alpha)\Big(\frac{L^4}{18C_1}\max_{1\le n\le N}\|p^{n-1+\sigma}\|^2 + 2C_2\max_{1\le n\le N}\|q^{n-1+\sigma}\|^2\Big). \tag{3.5}$$

Proof. Taking the inner product of Eq (3.1) by $w^{n-1+\sigma}$, we get

$$(D_{\tau,\sigma}^\alpha w^n, w^{n-1+\sigma}) + (\delta_x^2(\omega z^{n-1+\sigma}), w^{n-1+\sigma}) + \kappa\|w^{n-1+\sigma}\|^2 = (p^{n-1+\sigma}, w^{n-1+\sigma}). \tag{3.6}$$

Taking the inner product of Eq (3.2) by $\omega z^{n-1+\sigma}$, we get

$$(z^{n-1+\sigma}, \omega z^{n-1+\sigma}) = (\delta_x^2 w^{n-1+\sigma}, \omega z^{n-1+\sigma}) + (q^{n-1+\sigma}, \omega z^{n-1+\sigma}). \tag{3.7}$$

From Eq (3.6) and Eq (3.7), it yields that

$$\begin{aligned}&(D_{\tau,\sigma}^\alpha w^n, w^{n-1+\sigma}) + (\delta_x^2(\omega z^{n-1+\sigma}), w^{n-1+\sigma}) + \kappa\|w^{n-1+\sigma}\|^2 + (z^{n-1+\sigma}, \omega z^{n-1+\sigma})\\ =\ &(p^{n-1+\sigma}, w^{n-1+\sigma}) + (\delta_x^2 w^{n-1+\sigma}, \omega z^{n-1+\sigma}) + (q^{n-1+\sigma}, \omega z^{n-1+\sigma}).\end{aligned} \tag{3.8}$$

Applying the discrete Green formula gives that

$$(\delta_x^2(\omega z^{n-1+\sigma}), w^{n-1+\sigma}) = -(\delta_x(\omega z^{n-1+\sigma}), \delta_x w^{n-1+\sigma}) = (\delta_x^2 w^{n-1+\sigma}, \omega z^{n-1+\sigma}). \tag{3.9}$$

Substituting Eq (3.9) into Eq (3.8), we obtain

$$(D_{\tau,\sigma}^\alpha w^n, w^{n-1+\sigma}) + \kappa\|w^{n-1+\sigma}\|^2 + \|z^{n-1+\sigma}\|_\omega^2 = (p^{n-1+\sigma}, w^{n-1+\sigma}) + (q^{n-1+\sigma}, z^{n-1+\sigma})_\omega. \tag{3.10}$$

From Eq (3.2), we have

$$(z_i^{n-1+\sigma})^2 = (\delta_x^2 w_i^{n-1+\sigma} + q_i^{n-1+\sigma})^2. \tag{3.11}$$

Multiplying Eq (3.11) by $h\omega_i$ and summing up for i from 1 to $M-1$, we get

$$\|z^{n-1+\sigma}\|_\omega^2 = \|\delta_x^2 w^{n-1+\sigma}\|_\omega^2 + 2(\delta_x^2 w^{n-1+\sigma}, q^{n-1+\sigma})_\omega + \|q^{n-1+\sigma}\|_\omega^2. \tag{3.12}$$

Substituting Eq (3.12) into Eq (3.10), we obtain

$$\begin{aligned}&(D^{\alpha}_{\tau,\sigma}w^n, w^{n-1+\sigma}) + \frac{1}{2}\|z^{n-1+\sigma}\|^2_{\omega} + \frac{1}{2}\|\delta_x^2 w^{n-1+\sigma}\|^2_{\omega} + \frac{1}{2}\|q^{n-1+\sigma}\|^2_{\omega} + \kappa\|w^{n-1+\sigma}\|^2\\ =\ &(p^{n-1+\sigma}, w^{n-1+\sigma}) + (q^{n-1+\sigma}, z^{n-1+\sigma})_{\omega} - (\delta_x^2 w^{n-1+\sigma}, q^{n-1+\sigma})_{\omega}.\end{aligned} \tag{3.13}$$

Using Cauchy-Schwarz inequality, we have

$$-(\delta_x^2 w^{n-1+\sigma}, q^{n-1+\sigma})_{\omega} \le \frac{1}{4}\|\delta_x^2 w^{n-1+\sigma}\|^2_{\omega} + \|q^{n-1+\sigma}\|^2_{\omega}, \tag{3.14}$$

and

$$(q^{n-1+\sigma}, z^{n-1+\sigma})_{\omega} \le \frac{1}{2}\|z^{n-1+\sigma}\|^2_{\omega} + \frac{1}{2}\|q^{n-1+\sigma}\|^2_{\omega}, \tag{3.15}$$

From Eq (3.14), Eq (3.15) and Eq (3.13), we obtain

$$(D^{\alpha}_{\tau,\sigma}w^n, w^{n-1+\sigma}) + \frac{1}{4}\|\delta_x^2 w^{n-1+\sigma}\|^2_{\omega} \le (p^{n-1+\sigma}, w^{n-1+\sigma}) + \|q^{n-1+\sigma}\|^2_{\omega}. \tag{3.16}$$

Based on Lemma 3.3 and Lemma 3.4, we have

$$\|w\|^2 \le \frac{L^4}{36C_1}\|\delta_x^2 w\|^2_{\omega},\ \|q^{n-1+\sigma}\|^2_{\omega} \le C_2\|q^{n-1+\sigma}\|^2. \tag{3.17}$$

Applying Cauchy inequality, we get

$$(p^{n-1+\sigma}, w^{n-1+\sigma}) \le \frac{9C_1}{L^4}\|w^{n-1+\sigma}\|^2 + \frac{L^4}{36C_1}\|p^{n-1+\sigma}\|^2 \le \frac{1}{4}\|\delta_x^2 w^{n-1+\sigma}\|^2_{\omega} + \frac{L^4}{36C_1}\|p^{n-1+\sigma}\|^2. \tag{3.18}$$

Substituting Eq (3.18) into Eq (3.16) yields that

$$D^{\alpha}_{\tau,\sigma}\|w^n\|^2 \le \frac{L^4}{18C_1}\|p^{n-1+\sigma}\|^2 + 2C_2\|q^{n-1+\sigma}\|^2.$$

where Lemma 3.2 has been used. That is,

$$C_0^{(n)}\|w^n\|^2 \le \sum_{k=1}^{n-1}(C^{(n)}_{n-k-1} - C^{(n)}_{n-k})\|w^k\|^2 + C^{(n)}_{n-1}\|w^0\|^2 + \mu\Big(\frac{L^4}{18C_1}\|p^{n-1+\sigma}\|^2 + 2C_2\|q^{n-1+\sigma}\|^2\Big), \tag{3.19}$$

where $\mu = \Gamma(2-\alpha)\tau^{\alpha}$. According to Lemma 3.1, we have

$$C^{(n)}_{n-1} > \frac{1-\alpha}{2}(n-1-\frac{\alpha}{2})^{-\alpha} > \frac{1-\alpha}{2}(n-\frac{\alpha}{2})^{-\alpha},\ 1 \le n \le N,$$

and

$$\mu = \tau^{\alpha}\Gamma(2-\alpha) = T^{\alpha}N^{-\alpha}\Gamma(1-\alpha)(1-\alpha) < T^{\alpha}(n-\frac{\alpha}{2})^{-\alpha}\Gamma(1-\alpha)(1-\alpha) < 2C^{(n)}_{n-1}T^{\alpha}\Gamma(1-\alpha). \tag{3.20}$$

Substituting Eq (3.20) into Eq (3.19) gives that

$$C_0^{(n)}\|w^n\|^2 \le \sum_{k=1}^{n-1}(C_{n-k-1}^{(n)} - C_{n-k}^{(n)})\|w^k\|^2 + C_{n-1}^{(n)}\Big[\|w^0\|^2 + 2T^\alpha\Gamma(1-\alpha)(\frac{L^4}{18C_1}\|p^{n-1+\sigma}\|^2 + 2C_2\|q^{n-1+\sigma}\|^2)\Big].$$

Denote

$$J = \|w^0\|^2 + 2T^\alpha\Gamma(1-\alpha)\Big(\frac{L^4}{18C_1}\max_{1\le n\le N}\|p^{n-1+\sigma}\|^2 + 2C_2\max_{1\le n\le N}\|q^{n-1+\sigma}\|^2\Big).$$

Now, we prove by the mathematical induction method that

$$\|w^n\|^2 \le J. \tag{3.21}$$

It holds obviously when $n = 0$. Assuming Eq (3.21) is valid for $n = 1, 2, \cdots, m-1$, then we have

$$C_0^{(m)}\|w^m\|^2 \le \sum_{k=1}^{m-1}(C_{m-k-1}^{(m)} - C_{m-k}^{(m)})\|w^k\|^2 + C_{m-1}^{(m)}J \le \sum_{k=1}^{m-1}(C_{m-k-1}^{(m)} - C_{m-k}^{(m)})J + C_{m-1}^{(m)}J = C_0^{(m)}J.$$

This proof is completed. □

Applying the Theorem 3.5, we can immediately obtain the stability result.

Theorem 3.6 (Stability)**.** *The difference scheme* (2.10)–(2.13) *is unconditionally stable with respect to the initial value* φ *and the source term* f.

Similarly, from Theorem 3.5, we can easily prove the solvability of the proposed scheme.

Theorem 3.7 (Solvability)**.** *The difference scheme* (2.10)–(2.13) *is uniquely solvable.*

Proof. It suffices to prove the homogeneous linear system

$$\begin{aligned}
&D_{\tau,\sigma}^{\alpha}u_i^n + \delta_x^2(\omega_i v_i^{n-1+\sigma}) + \kappa u_i^{n-1+\sigma} = 0,\ 1 \le i \le M-1, 1 \le n \le N,\\
&v_i^{n-1+\sigma} = \delta_x^2 u_i^{n-1+\sigma},\ 1 \le i \le M-1, 1 \le n \le N,\\
&u_i^0 = 0,\ 0 \le i \le M,\\
&u_0^n = u_M^n = 0, v_0^n = v_M^n = 0,\ 1 \le n \le N,
\end{aligned}$$

has only a trivial solution. Applying Theorem 3.5, we have $\|u^n\|^2 \le \|u^0\|^2 = 0$. So $u_i^n \equiv 0$ for $0 \le i \le M$, which completes the proof. □

Next, we focus on the convergence of the difference scheme (2.10)–(2.13). Denote

$$e_i^n = u(x_i, t_n) - u_i^n,\ \ \tilde{e}_i^n = v(x_i, t_n) - v_i^n,\ \ 0 \le n \le N, 0 \le i \le M.$$

Theorem 3.8 (Convergence)**.** *Assume that* $u(x,t) \in C_{x,t}^{6,3}([0,L]\times[0,T])$ *and* $\{u_i^n\}$ *are solution of the problem* (1.1)–(1.4) *and the difference scheme Eqs* (2.10)–(2.13) *respectively. Then there exists a positive constant* C *such that*

$$\|e^n\| \le C(\tau^2 + h^2),\ \ 0 \le n \le N. \tag{3.22}$$

Proof. From Eq (2.7), Eq (2.8) and Eqs (2.10)–(2.13), we have the error equations as

$$\begin{aligned}
&D^{\alpha}_{\tau,\sigma}e^n_i + \delta^2_x(\omega\tilde{e}^{n-1+\sigma})_i + \kappa e^{n-1+\sigma}_i = (R_1)^n_i,\ 1 \le i \le M-1, 1 \le n \le N,\\
&\tilde{e}^{n-1+\sigma}_i = \delta^2_x e^{n-1+\sigma}_i + (R_2)^n_i,\ 1 \le i \le M-1, 1 \le n \le N,\\
&e^0_i = 0,\ 0 \le i \le M,\\
&e^n_0 = 0, e^n_M = 0, \tilde{e}^n_0 = 0, \tilde{e}^n_M = 0,\ 1 \le n \le N.
\end{aligned}$$

Applying Theorem 3.5, we get

$$\|e^n\|^2 \le 2T^{\alpha}\Gamma(1-\alpha)\Big(\frac{L^4}{18C_1}\max_{1\le n\le N}\|R^n_1\|^2 + 2C_2\max_{1\le n\le N}\|R^n_2\|^2\Big),\ \ 1 \le n \le N.$$

Noticing Eq (2.9), we get

$$\|e^n\|^2 \le 2T^{\alpha}\Gamma(1-\alpha)(\frac{L^4}{18C_1} + 2C_2)C_r{}^2(\tau^2 + h^2)^2,\ \ 1 \le n \le N,$$

which shows that Eq (3.22) is valid with

$$C = C_r\sqrt{2T^{\alpha}\Gamma(1-\alpha)(\frac{L^4}{18C_1} + 2C_2)}.$$

This proof is completed. □

4. The $\mathcal{FL}2-1_{\sigma}$ scheme

Although the $\mathcal{L}2-1_{\sigma}$ scheme (2.10)–(2.13) has accuracy of second order in time, it is not conducive to calculation due to it needs all history data to get the solution at current time point. Also, here we present a fast scheme by applying the sum-of-exponentials approximation to the kernel function $t^{-\alpha}$.

The sum-of-exponentials approximation reads as:

Lemma 4.1 ([22]). *For the given $\alpha \in (0,1)$, tolerance error ε, cut-off time step size $\tilde{\tau}$ and final time T, there are one positive integer N_{exp}, positive points s_j and corresponding positive weights $w_j (j = 1, 2, \cdots, N_{exp})$ satisfying*

$$| t^{-\alpha} - \sum_{j=1}^{N_{exp}} w_j e^{-s_j t} | \le \varepsilon,\ \forall\ t \in [\tilde{\tau}, T],$$

and the number of exponentials needed is of the order

$$N_{exp} = O(\log(\frac{1}{\varepsilon}(\log\log\frac{1}{\varepsilon} + \log\frac{T}{\tilde{\tau}} + \log\frac{1}{\tilde{\tau}}(\log\log\frac{1}{\varepsilon} + \log\frac{T}{\tilde{\tau}})).$$

The fast evaluation of Caputo derivative, $\mathcal{FL}2-1_{\sigma}$ formula, is given as follows:

$$^{\mathcal{F}}D^{\alpha}_t u^{n+\sigma} = \sum_{j=1}^{N_{exp}} \tilde{w}_j \tilde{V}^n_j + \lambda a_0(u^{n+1} - u^n), \tag{4.1}$$

where $\lambda = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$, $\tilde{w}_j = \frac{1}{\Gamma(1-\alpha)} w_j$, and $\tilde{V}_j^n$ can be got form the following recursive relation

$$\tilde{V}_j^n = e^{-s_j\tau}\tilde{V}_j^{n-1} + A_j(u^n - u^{n-1}) + B_j(u^{n+1} - u^n),\ j = 1, 2, \cdots, N_{exp},\ n = 1, 2, \cdots, \tag{4.2}$$

with $\tilde{V}_j^0 = 0, (j = 1, 2, \cdots, N_{exp})$ and

$$A_j = \frac{(2 + \tau s_j)e^{\tau s_j} - (2 + 3\tau s_j)}{2(\tau s_j)^2 e^{(\tau s_j(\sigma+1))}},\ B_j = \frac{(\tau s_j - 2)e^{\tau s_j} + (2 + \tau s_j)}{2(\tau s_j)^2 e^{(\tau s_j(\sigma+1))}},\ \ j \geq 1.$$

The recursive relation (4.2) shows that the $\mathcal{FL}2{-}1_\sigma$ formula reduces significantly the computational complexity. Noticing that Eq (4.2) can be equivalently rewritten as the following summation form

$$\tilde{V}_j^n = e^{-(n-1)\tau s_j}A_j(u^1 - u^0) + \sum_{i=1}^{n-1}(e^{-(n-i-1)\tau s_j}A_j + e^{-(n-i)\tau s_j}B_j)(u^{i+1} - u^i) + B_j(u^{n+1} - u^n),$$

thus we have

$$^{\mathcal{F}}D_t^\alpha u^{n+\sigma} = \sum_{k=0}^{n} {}^{\mathcal{F}}g_k^{(n+1,\alpha)}(u^{k+1} - u^k), \tag{4.3}$$

in which ${}^{\mathcal{F}}g_0^{(1,\alpha)} = \lambda a_0$, and for $n \geq 1$,

$$^{\mathcal{F}}g_k^{(n+1,\alpha)} = \begin{cases} \sum_{j=1}^{N_{exp}} \tilde{w}_j e^{-(n-1)s_j\tau}A_j, & k = 0, \\ \sum_{j=1}^{N_{exp}} \tilde{w}_j(e^{-(n-k-1)s_j\tau}A_j + e^{-(n-k)s_j\tau}B_j), & 1 \leq k \leq n-1, \\ \sum_{j=1}^{N_{exp}} \tilde{w}_j B_j + \lambda a_0, & k = n. \end{cases} \tag{4.4}$$

The equivalent expression (4.3) is more applicable in stability and convergence analysis.

With respect to the $\mathcal{FL}2 - 1_\sigma$ formula, we have the following some results.

Lemma 4.2 ([22]). *For any $\alpha \in (0, 1)$, and $u(t) \in C^3[0, T]$, it holds that*

$$|\, {}_0^C D_t^\alpha u(t)|_{t=t_{n+\sigma}} - {}^{\mathcal{F}}D_t^\alpha u^{n+\sigma} \,| = O(\tau^{3-\alpha} + \varepsilon).$$

Lemma 4.3 ([22]). *Suppose $\alpha \in (0, 1)$, ${}^{\mathcal{F}}g_k^{(n+1,\alpha)}$ is defined by Eq (4.4), then it holds that*

$$^{\mathcal{F}}g_n^{(n+1,\alpha)} > {}^{\mathcal{F}}g_{n-1}^{(n+1,\alpha)} > \cdots > {}^{\mathcal{F}}g_0^{(n+1,\alpha)} \geqslant {}^{\mathcal{F}}C > 0,\ \ (2\sigma - 1){}^{\mathcal{F}}g_n^{(n+1,\alpha)} - \sigma {}^{\mathcal{F}}g_{n-1}^{(n+1,\alpha)} \geqslant 0.$$

Lemma 4.4 ([22]). *Suppose $u = \{u^n \mid 0 \leq n \leq N-1\}$ is a grid function defined on Ω_τ, then it holds that*

$$(\sigma u^{n+1} + (1-\sigma)u^n){}^{\mathcal{F}}D_t^\alpha u^{n+\sigma} \geq \frac{1}{2}{}^{\mathcal{F}}D_t^\alpha (u^{n+\sigma})^2.$$

Similar to the derivation of the $\mathcal{L}2-1_\sigma$ scheme (2.10)–(2.13), it follows from Eq (2.1), Eq (2.2) we have

$$^{\mathcal{F}}D_t^\alpha U_i^{n+\sigma} + \delta_x^2(\omega_i V_i^{n+\sigma}) + \kappa U_i^{n+\sigma} = f_i^{n+\sigma} + {}^{\mathcal{F}}(R_1)_i^n,\ 1 \le i \le M-1, 0 \le n \le N-1, \tag{4.5}$$

$$V_i^{n+\sigma} = \delta_x^2 U_i^{n+\sigma} + {}^{\mathcal{F}}(R_2)_i^n, 1 \le i \le M-1, 0 \le n \le N-1, \tag{4.6}$$

and there exists a constant $^{\mathcal{F}}C_r$ such that

$$|\, ^{\mathcal{F}}(R_1)_i^n \,| + |\, ^{\mathcal{F}}(R_2)_i^n \,| \le {}^{\mathcal{F}}C_r(\tau^2 + h^2 + \varepsilon), 1 \le i \le M-1, 0 \le n \le N-1. \tag{4.7}$$

Omitting the small terms $^{\mathcal{F}}(R_1)_i^n$ and $^{\mathcal{F}}(R_2)_i^n$ in Eq (4.5) and Eq (4.6), from the boundary and initial conditions (2.3)–(2.4), we obtain the $\mathcal{F}\mathcal{L}2-1_\sigma$ scheme for the problem (2.1)–(2.4) as follows

$$^{\mathcal{F}}D_t^\alpha u_i^{n+\sigma} + \delta_x^2(\omega_i v_i^{n+\sigma}) + \kappa u_i^{n+\sigma} = f_i^{n+\sigma},\ 1 \le i \le M-1, 0 \le n \le N-1, \tag{4.8}$$

$$v_i^{n+\sigma} = \delta_x^2 u_i^{n+\sigma},\ 1 \le i \le M-1, 0 \le n \le N-1, \tag{4.9}$$

$$u_i^0 = \varphi(x_i),\ 0 \le i \le M, \tag{4.10}$$

$$u_0^n = \alpha_1(t_n), u_M^n = \alpha_2(t_n), v_0^n = \beta_1(t_n), v_M^n = \beta_2(t_n),\ 1 \le n \le N. \tag{4.11}$$

Next, we focus on the solvability, stability and convergence of the $\mathcal{F}\mathcal{L}2-1_\sigma$ scheme.

Before the discussion, we first prove the following priori estimate.

Theorem 4.5. *Suppose* $\{w_i^n, z_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ *satisfy the difference scheme*

$$^{\mathcal{F}}D_t^\alpha w_i^{n+\sigma} + \delta_x^2(\omega_i z_i^{n+\sigma}) + \kappa w_i^{n+\sigma} = p_i^{n+\sigma},\ 1 \le i \le M-1, 0 \le n \le N-1, \tag{4.12}$$

$$z_i^{n+\sigma} = \delta_x^2 w_i^{n+\sigma} + q_i^{n+\sigma},\ 1 \le i \le M-1, 0 \le n \le N-1, \tag{4.13}$$

$$w_i^n = \varphi(x_i),\ 0 \le i \le M, \tag{4.14}$$

$$w_0^n = 0, w_M^n = 0, z_0^n = 0, z_M^n = 0,\ 1 \le n \le N. \tag{4.15}$$

Then, we have

$$\|w^n\|^2 \le \|w^0\|^2 + \frac{1}{^{\mathcal{F}}C}\left(\frac{L^4}{18C_1}\max_{1\le n\le N}\|p^{n-1+\sigma}\|^2 + 2C_2\max_{1\le n\le N}\|q^{n-1+\sigma}\|^2\right). \tag{4.16}$$

Proof. Similar to the proof of the Theorem 3.5, we can obtain from Eq (4.12) and Eq (4.13) that

$$^{\mathcal{F}}D_t^\alpha\|w^{n+\sigma}\|^2 \le \frac{L^4}{18C_1}\|p^{n+\sigma}\|^2 + 2C_2\|q^{n+\sigma}\|^2.$$

Noticing that

$$^{\mathcal{F}}D_t^\alpha\|w^{n+\sigma}\|^2 = {}^{\mathcal{F}}g_n^{(n+1,\alpha)}\|w^{n+1}\|^2 - \sum_{k=1}^{n}({}^{\mathcal{F}}g_k^{(n+1,\alpha)} - {}^{\mathcal{F}}g_{k-1}^{(n+1,\alpha)})\|w^k\|^2 - {}^{\mathcal{F}}g_0^{(n+1,\alpha)}\|w^0\|^2, \tag{4.17}$$

we get

$$^{\mathcal{F}}g_n^{(n+1,\alpha)}\|w^{n+1}\|^2 \le \sum_{k=1}^{n}({}^{\mathcal{F}}g_k^{(n+1,\alpha)} - {}^{\mathcal{F}}g_{k-1}^{(n+1,\alpha)})\|w^k\|^2 + {}^{\mathcal{F}}g_0^{(n+1,\alpha)}\|w^0\|^2 + \left(\frac{L^4}{18C_1}\|p^{n+\sigma}\|^2 + 2C_2\|q^{n+\sigma}\|^2\right). \tag{4.18}$$

From Lemma 4.4, we can further obtain

$$ {}^{\mathcal{F}}g_n^{(n+1,\alpha)}\|w^{n+1}\|^2\leq\sum_{k=1}^{n}({}^{\mathcal{F}}g_k^{(n+1,\alpha)}-{}^{\mathcal{F}}g_{k-1}^{(n+1,\alpha)})\|w^k\|^2+{}^{\mathcal{F}}g_0^{(n+1,\alpha)}[\|w^0\|^2+\frac{1}{{}^{\mathcal{F}}C}(\frac{L^4}{18C_1}\|p^{n+\sigma}\|^2+2C_2\|q^{n+\sigma}\|^2)]. $$

Denote

$$ G=\|w^0\|^2+\frac{1}{{}^{\mathcal{F}}C}(\frac{L^4}{18C_1}\max_{1\leq n\leq N}\|p^{n+\sigma}\|^2+2C_2\max_{1\leq n\leq N}\|q^{n+\sigma}\|^2). $$

Now, we prove by the mathematical induction that

$$ \|w^n\|^2\leq G. \tag{4.19} $$

It holds obviously when $n=0$. Assuming Eq (4.19) is valid for $n=1,2,\cdots,m-1$, then we have

$$ \begin{aligned} {}^{\mathcal{F}}g_m^{(m+1,\alpha)}\|w^{m+1}\|^2 &\leq \sum_{k=1}^{m}({}^{\mathcal{F}}g_k^{(m+1,\alpha)}-{}^{\mathcal{F}}g_{k-1}^{(m+1,\alpha)})\|w^k\|^2+{}^{\mathcal{F}}g_0^{(m+1,\alpha)}G \\ &\leq \sum_{k=1}^{m}({}^{\mathcal{F}}g_k^{(m+1,\alpha)}-{}^{\mathcal{F}}g_{k-1}^{(m+1,\alpha)})G+{}^{\mathcal{F}}g_0^{(m+1,\alpha)}G={}^{\mathcal{F}}g_m^{(m+1,\alpha)}G. \end{aligned} $$

This proof is completed. □

Based on Theorem 4.5, we can obtain the following stability theorems.

Theorem 4.6 (Stability)**.** *The $\mathcal{FL}2-1_\sigma$ scheme Eqs* (4.8)–(4.11) *is uniquely solvable, and unconditionally stable with respect to the initial value φ and the source term f.*

Theorem 4.7 (Convergence)**.** *Assume that $u(x,t)\in C_{x,t}^{6,3}([0,L]\times[0,T])$ and $\{u_i^n\}$ are solutions of the problem* (1.1)–(1.4) *and the $\mathcal{FL}2-1_\sigma$ scheme* (4.8)–(4.11)*, respectively. Then there exists a positive constant C such that*

$$ \|e^n\|\leq C(\tau^2+h^2+\varepsilon),\quad 0\leq n\leq N. \tag{4.20} $$

Proof. From Eq (2.7), Eq (2.8) and Eqs (4.8)–(4.11), we have the error equations as

$$ \begin{aligned} &{}^{\mathcal{F}}D_t^\alpha e_i^{n+\sigma}+\delta_x^2(\omega\tilde{e}^{n+\sigma})_i+\kappa e_i^{n+\sigma}=(R_1)_i^n,\ 1\leq i\leq M-1, 0\leq n\leq N-1,\\ &\tilde{e}_i^{n+\sigma}=\delta_x^2e_i^{n+\sigma}+(R_2)_i^n,\ 1\leq i\leq M-1, 0\leq n\leq N-1,\\ &e_i^0=0,\ 0\leq i\leq M,\\ &e_0^n=0, e_M^n=0, \tilde{e}_0^n=0, \tilde{e}_M^n=0,\ 1\leq n\leq N. \end{aligned} $$

Applying Theorem 4.5, we get

$$ \|e^n\|^2\leq\frac{1}{{}^{\mathcal{F}}C}(\frac{L^4}{18C_1}\max_{1\leq n\leq N}\|{}^{\mathcal{F}}R_1^n\|^2+2C_2\max_{1\leq n\leq N}\|{}^{\mathcal{F}}R_2^n\|^2),\quad 1\leq n\leq N. $$

Noticing Eq (4.7), we get

$$ \|e^n\|^2\leq\frac{1}{{}^{\mathcal{F}}C}(\frac{L^4}{18C_1}+2C_2){}^{\mathcal{F}}C^2(\tau^2+h^2)^2,\quad 1\leq n\leq N, $$

which shows that Eq (4.20) is valid with $C={}^{\mathcal{F}}C\sqrt{\frac{1}{{}^{\mathcal{F}}C}(\frac{L^4}{18C_1}+2C_2)}$. □

5. Numerical experiments

5.1. The nonuniform L_1 approximation

It should be pointed out that the proposed difference schemes are based on assumptions that the solution of problem is sufficiently smooth. But the singularity of the time fractional derivative may lead to weak singularity near the initial time which may influence the accuracy of numerical method. Thus, in order to overcome the possible singularity of the solution near $t = 0$, some related techniques have been developed, such as the initial correction techniques, non-uniform discretization and so on [23–26]. Because of this, a analogously scheme for the problem (1.1)–(1.4) based on the uniform mesh in space and graded mesh in time is first given as follows:

$$\Delta_N^\alpha u_i^n + \delta_x^2(\omega_i v_i^n) + \kappa u_i^n = f_i^n,\ 1 \le i \le M-1, 1 \le n \le N, \tag{5.1}$$

$$v_i^n = \delta_x^2 u_i^n,\ 1 \le i \le M-1, 1 \le n \le N, \tag{5.2}$$

$$u_i^0 = \varphi(x_i),\ 0 \le i \le M, \tag{5.3}$$

$$u_0^n = \alpha_1(t_n), u_M^n = \alpha_2(t_n), v_0^n = \beta_1(t_n), v_M^n = \beta_2(t_n),\ 1 \le n \le N, \tag{5.4}$$

where

$$\Delta_N^\alpha u_i^n = \frac{d_{n,1}}{\Gamma(2-\alpha)} u_i^n - \frac{d_{n,n}}{\Gamma(2-\alpha)} u_0^n + \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} u_i^{n-k}(d_{n,k+1} - d_{n,k}), \tag{5.5}$$

and

$$d_{n,k} = \frac{(t_n - t_{n-k})^{1-\alpha} - (t_n - t_{n-k+1})^{1-\alpha}}{\tau_{n-k+1}}, \tag{5.6}$$

with $x_i = ih, t_n = (n/N)^r T, \tau_n = t_n - t_{n-1}$, where the constant mesh grading exponent $r \ge 1$. It should be noted that the graded mesh will be simplified to a uniform grid when $r = 1$.

5.2. Numerical results

In this subsection, we rely on two numerical examples to verify the availability of the proposed methods.

Let

$$E(h,\tau) = \max_{1 \le n \le N} \|u^n - U^n\|_2,\ \ \text{Ord} = \log_2\Big(\frac{E(2h, 2\tau)}{E(h,\tau)}\Big).$$

Example 5.1. *First, we consider the following problem*

$$\begin{aligned}
&{}_0^C D_t^\alpha u(x,t) + \frac{\partial^2}{\partial x^2}\Big(\omega(x)\frac{\partial^2 u(x,t)}{\partial x^2}\Big) + u(x,t) = f(x,t),\ \ 0 < x < 1,\ 0 < t \le 1,\\
&u(x,0) = \cos(\pi x),\ \ 0 < x < 1,\\
&u(0,t) = t^{3+\alpha} + 1,\ u(1,t) = -(t^{3+\alpha} + 1),\ \ 0 \le t \le 1,\\
&\frac{\partial^2 u(0,t)}{\partial x^2} = -\pi^2(t^{3+\alpha} + 1),\ \frac{\partial^2 u(1,t)}{\partial x^2} = \pi^2(t^{3+\alpha} + 1),\ \ 0 \le t \le 1,
\end{aligned}$$

where $\omega(x) = x^2 + 1$ *and* $f(x,t) = \cos(\pi x)\frac{\Gamma(4+\alpha)}{6} t^3 + (t^{3+\alpha} + 1)\big[\cos(\pi x) - 2\pi^2\cos(\pi x) + 4x\pi^3 \sin(\pi x) + (x^2 + 1)\pi^4 \cos(\pi x)\big]$.

It is not difficult to verify that the exact solutions of the problems 5.1 is $u(x,t) = \cos(\pi x)(t^{3+\alpha} + 1)$, which satisfies the smoothness requirement in Theorems 3.8 and 4.7.

The numerical accuracy of both schemes are tested with respect to $\alpha = 0.25, 0.5, 0.75$, respectively. In calculation, we take $\varepsilon = 10^{-13}$, which is much less than τ^2. The errors and convergence orders of the suggested two schemes are showed in Table 1. We can observe that the values of Ord are always close to 2, which means that the $\mathcal{L}2-1_\sigma$ scheme and the $\mathcal{FL}2-1_\sigma$ scheme have second order accuracy both in space and time for different $\alpha \in (0,1)$. Table 2 lists the convergence orders of both schemes when $\tau = h$ and CPU time with α =0.5. Obviously, the $\mathcal{FL}2-1_\sigma$ scheme is faster than the $\mathcal{L}2-1_\sigma$ scheme, especially for small τ.

Table 1. The errors and convergence orders for Example 5.1.

		$\mathcal{FL}2-1_\sigma$ scheme			$\mathcal{L}2-1_\sigma$ scheme	
α	$h=\tau$	Nexp	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord
0.25	1/10	39	1.0510e-02	——	1.0510e-02	——
	1/20	42	2.5986e-03	2.0160	2.5986e-03	2.0160
	1/40	46	6.4786e-04	2.0040	6.4786e-04	2.0040
	1/80	49	1.6185e-04	2.0010	1.6185e-04	2.0010
	1/160	53	4.0456e-05	2.0002	4.0456e-05	2.0002
0.5	1/10	39	1.0500e-02	——	1.0500e-02	——
	1/20	42	2.5959e-03	2.0161	2.5959e-03	2.0161
	1/40	46	6.4719e-04	2.0040	6.4719e-04	2.0040
	1/80	49	1.6169e-04	2.0010	1.6169e-04	2.0010
	1/160	53	4.0416e-05	2.0002	4.0416e-05	2.0002
0.75	1/10	39	1.0472e-02	——	1.0472e-02	——
	1/20	43	2.5911e-03	2.0149	2.5911e-03	2.0149
	1/40	46	6.4598e-04	2.0040	6.4598e-04	2.0040
	1/80	50	1.6139e-04	2.0009	1.6139e-04	2.0009
	1/160	53	4.0340e-05	2.0003	4.0340e-05	2.0003

Table 2. The errors and convergence orders for Example 5.1 when $\alpha = 0.5$.

	$\mathcal{FL}2-1_\sigma$ scheme				$\mathcal{L}2-1_\sigma$ scheme		
$h=\tau$	Nexp	$E(h,\tau)$	Ord	CPU(s)	$E(h,\tau)$	Ord	CPU(s)
1/250	55	1.6549e-05	——	4.25	1.6551e-05	——	49.93
1/500	58	4.1136e-06	2.0083	17.91	4.0771e-06	2.0213	208.36
1/1000	62	1.0772e-06	1.9331	91.50	1.0253e-06	1.9915	924.27

From the Tables 1,2, we can see that these numerical results are consistent with the previous theoretical results. It shows the $\mathcal{L}2-1_\sigma$ scheme (2.10)–(2.13) and the $\mathcal{FL}2-1_\sigma$ scheme (4.8)–(4.11) are convergent with second order accuracy in space and time, and the $\mathcal{FL}2-1_\sigma$ scheme is more practical.

Example 5.2. *Now, we consider the following problem*

$$\begin{aligned}
&{}_0^C D_t^{\alpha} u(x,t) + \frac{\partial^2}{\partial x^2}\Big(\omega(x)\frac{\partial^2 u(x,t)}{\partial x^2}\Big) = f(x,t), \ \ 0 < x < \pi, \ 0 < t \le 1,\\
&u(x,0) = 0, \ \ 0 < x < \pi,\\
&u(0,t) = 0, \ u(1,t) = 0, \ \ 0 \le t \le 1,\\
&\frac{\partial^2 u(0,t)}{\partial x^2} = 0, \ \frac{\partial^2 u(1,t)}{\partial x^2} = 0, \ \ 0 \le t \le 1,
\end{aligned}$$

where $\kappa = 0, \omega(x) = e^x$ *and*

$$f(x,t) = (\Gamma(1+\alpha) + \frac{3\Gamma(3)t^{3-\alpha}}{\Gamma(4-\alpha)}) \sin x - 2e^2(t^\alpha + t^3)\cos x.$$

The exact solution of the example 5.2 is $u(x,t) = (t^\alpha + t^3)\sin x$.

The error and numerical accuracy of scheme (5.1)–(5.6) are listed in Tables 3–5 with respect to $\alpha = 0.4, 0.6, 0.8$ and some values of grading exponent r, respectively. We keep $M = 2N$ in calculation. These results show that the scheme (5.1)–(5.6) has accuracy of order α when $r = 1$, and accuracy of order $2-\alpha$ when $r \ge r_c = (2-\alpha)/\alpha$. The reason for this result is that the smoothness requirement of the solution in Theorems 3.8 and 4.7 is not satisfied.

Table 3. The errors and convergence orders for Example 5.2 when $\alpha = 0.5$.

	$r=1$		$r=r_c$		$r=2r_c$	
N	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord
32	3.3961e-02	——	4.6082e-03	——	1.3810e-02	——
64	2.8987e-02	2.2847e-01	1.8881e-03	1.2873	5.6845e-03	1.2806
128	2.4345e-02	2.5178e-01	7.1277e-04	1.4054	2.1522e-03	1.4012
256	2.0108e-02	2.7586e-01	2.5719e-04	1.4706	7.7724e-04	1.4694
512	1.6354e-02	2.9813e-01	8.8456e-05	1.5398	2.7075e-04	1.5214

Table 4. The errors and convergence orders for Example 5.2 when $\alpha = 0.6$.

	$r=1$		$r=r_c$		$r=2r_c$	
N	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord
32	2.2240e-02	——	6.2089e-03	——	1.6603e-02	——
64	1.6383e-02	4.4096e-01	2.7026e-03	1.2001	7.1400e-03	1.2174
128	1.1717e-02	4.8360e-01	1.1103e-03	1.2832	2.9149e-03	1.2925
256	8.1872e-03	5.1716e-01	4.4202e-04	1.3288	1.1560e-03	1.3343
512	5.6228e-03	5.4208e-01	1.7147e-04	1.3662	4.5076e-04	1.3587

Table 5. The errors and convergence orders for Example 5.2 when $\alpha = 0.8$.

	$r=1$		$r=r_c$		$r=2r_c$	
N	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord
32	9.0995e-03	——	1.0251e-02	——	2.3075e-02	——
64	5.6954e-03	6.7599e-01	4.8305e-03	1.0855	1.0763e-02	1.1003
128	3.5455e-03	6.8381e-01	2.1936e-03	1.1389	4.8645e-03	1.1457
256	2.1527e-03	7.1984e-01	9.7736e-04	1.1663	2.1620e-03	1.1699
512	1.2788e-03	7.5136e-01	4.3093e-04	1.1814	9.5272e-04	1.1822

Table 6. The errors and convergence orders for Example 5.3.

		$\mathcal{FL}2-1_\sigma$ scheme			$\mathcal{L}2-1_\sigma$ scheme	
α	$h=\tau$	Nexp	$E(h,\tau)$	Ord	$E(h,\tau)$	Ord
0.25	1/10	33	1.2114e-02	——	1.2114e-02	——
	1/20	36	3.0215e-03	2.0033	3.0215e-03	2.0033
	1/40	39	7.5667e-04	1.9975	7.5667e-04	1.9975
	1/80	42	1.8946e-04	1.9978	1.8946e-04	1.9978
	1/160	45	4.7412e-05	1.9986	4.7412e-05	1.9986
0.5	1/10	33	1.3545e-02	——	1.3545e-02	——
	1/20	36	3.3864e-03	1.9999	3.3864e-03	1.9999
	1/40	39	8.4892e-04	1.9961	8.4892e-04	1.9961
	1/80	42	2.1266e-04	1.9971	2.1266e-04	1.9971
	1/160	45	5.3229e-05	1.9983	5.3229e-05	1.9983
0.75	1/10	33	1.4683e-02	——	1.4683e-02	——
	1/20	36	3.6621e-03	2.0034	3.6621e-03	2.0034
	1/40	39	9.1663e-04	1.9983	9.1663e-04	1.9983
	1/80	42	2.2943e-04	1.9983	2.2943e-04	1.9983
	1/160	45	5.7401e-05	1.9989	5.7401e-05	1.9989

Example 5.3. *Finally, we consider the following space-time variable coefficient problem*

$$\begin{aligned}
&{}_0^C D_t^\alpha u(x,t) + \frac{\partial^2}{\partial x^2}\Big(((xt)^2+1)\frac{\partial^2 u(x,t)}{\partial x^2}\Big) + u(x,t) = f(x,t),\ \ 0<x<1,\ 0<t\le 1,\\
&u(x,0) = \cos(\pi x),\ \ 0<x<1,\\
&u(0,t) = t^{3+\alpha}+1,\ u(1,t) = -(t^{3+\alpha}+1),\ \ 0\le t\le 1,\\
&\frac{\partial^2 u(0,t)}{\partial x^2} = -\pi^2(t^{3+\alpha}+1),\ \frac{\partial^2 u(1,t)}{\partial x^2} = \pi^2(t^{3+\alpha}+1),\ \ 0\le t\le 1,
\end{aligned}$$

where

$$f(x,t) = \cos(\pi x)\frac{\Gamma(4+\alpha)}{6}t^3 + (t^{3+\alpha}+1)[\cos(\pi x) - 2t^2\pi^2\cos(\pi x) + 4xt^2\pi^3\sin(\pi x) + (x^2t^2+1)\pi^4\cos(\pi x)].$$

The exact solution of above problem is also $u(x, t) = \cos(\pi x)(t^{3+\alpha} + 1)$, while the variable coefficient function $\omega(x, t) = (xt)^2 + 1$ which depends on the variables x and t.

Similar to the spatially variable coefficient problem, we apply the $\mathcal{L}2-1_\sigma$ scheme and the $\mathcal{F}\mathcal{L}2-1_\sigma$ scheme to solve the problem in Example 5.3. Table 6 presents the numerical results. In calculation, we take $\varepsilon = 10^{-11}$. It is shown that the $\mathcal{L}2 - 1_\sigma$ scheme and the $\mathcal{F}\mathcal{L}2 - 1_\sigma$ scheme are convergent with second order accuracy in space and time.

6. Conclusion

In this paper, we propose two second order difference schemes in both space and time for solving the variable coefficient fourth-order fractional sub-diffusion equation subject to the second Dirichlet boundary conditions. The $\mathcal{L}2 - 1_\sigma$ formula and $\mathcal{F}\mathcal{L}2 - 1_\sigma$ formula are applied to approximation the time Caputo fractional derivative. Compared with $\mathcal{L}2 - 1_\sigma$ scheme, the $\mathcal{F}\mathcal{L}2 - 1_\sigma$ scheme shows the better computational efficiency. The unconditional stability, solvability and convergence of the two schemes are strictly proved by the discrete energy method. The nonuniform L_1 approximation for the such problem is also given. Numerical examples are given to verify the effectiveness of both schemes. It should be pointed out that the results in this paper can be directly extended to time-space variable coefficient problems if we constrain the coefficient function $\omega(w, t)$ satisfying that $0 < C_1 \leq \omega(w, t) \leq C_2$.

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Conflict of interest

The authors declare there is no conflict of interest.

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