Networks and Heterogeneous Media

http://www.aimspress.com/journal/nhm

NHM, 18(1): 29–47.
DOI: 10.3934/nhm.2023002
Received: 19 September 2022
Revised: 05 October 2022
Accepted: 11 October 2022
Published: 19 October 2022

Research article

Asymptotic flocking of the relativistic Cucker–Smale model with time delay

Hyunjin Ahn*

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul, Republic of Korea

* **Correspondence:** yagamelaito@snu.ac.kr.

Abstract: This paper presents various sufficient conditions for asymptotic flocking in the relativistic Cucker–Smale (RCS) model with time delay. This model considers a self-processing time delay. We reduce the time-delayed RCS model to its dissipative structure for relativistic velocities. Then, using this dissipative structure, we demonstrate several sufficient frameworks in terms of the initial data and system parameters for asymptotic flocking of the proposed model.

Keywords: Asymptotic flocking; dissipative structure; relativistic Cucker–Smale; time delay

1. Introduction

Collective behaviors are omnipresent in everyday life (i.e., the flocking of moving birds [11, 28], herding of sheep and schooling of fish [12, 31], colonies of bacteria [32], and synchronization of fireflies and pacemaker cells [33, 34]). Among these phenomena, *flocking* refers to the phenomenon in which all agents governed by an autonomous system move at the same velocity under simple rules with the surrounding environmental information. In this paper, we are primarily interested in the flocking dynamics. The Cucker–Smale (CS) model proposed in [11] is considered as a successful model representing the flocking, see [7] for an introduction on the CS model. Moreover, the reader may refer to the following papers for the CS model and its variants regarding flocking behavior [11], on a general digraph [15], a temperature field extension [21], collision avoidance [27], bicluster flocking [10], Riemannian manifold extension [17], mean-field limit [20, 22, 23], hydrodynamic description [16, 25], unit-speed constraint [6], rooted leadership [24, 26, 29, 30], time-delay setting [8, 9, 13], and time-delay setting in a temperature field [5, 14].

However, because the CS model is a flocking model proposed based on classical Newtonian mechanics, the authors of [19] have focused on the situation that no relativistic correction to the CS model exists. When the speeds of agents governed by a dynamical system are high, close to the speed of light, classical mechanics is limited in explaining the motions of neutrinos, spacecrafts, and satellites, for example. Therefore, to ensure a suitable relativistic correction to the CS model, the authors of [19] rigorously proposed the relativistic thermomechanical Cucker–Smale (RTCS) model from the relativistic gas mixture equations with the theory of a principal subsystem. Thereafter, they derived the relativistic Cucker–Smale (RCS) model by reducing the RTCS model using a suitable ansatz. The RCS model in terms of *position–relativistic velocity*, $(\boldsymbol{x}_i, \boldsymbol{w}_i)$, is given by Eq (1.1)

$$\begin{cases} \dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \quad t > 0, \quad i \in [N], \\ \dot{\boldsymbol{w}}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) \left(\boldsymbol{v}_j - \boldsymbol{v}_i\right), \\ \boldsymbol{w}_i := F_i \boldsymbol{v}_i, \quad F_i := \Gamma_i \Big(1 + \dfrac{\Gamma_i}{c^2}\Big), \quad \Gamma_i := \dfrac{c}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}}, \\ (\boldsymbol{x}_i(0), \boldsymbol{w}_i(0)) = (\boldsymbol{x}_i^{\text{in}}, \boldsymbol{w}_i^{\text{in}}) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $[N] := \{1, \cdots, N\}$, $\|\cdot\|$ denotes the standard Euclidean ℓ^2-norm, N is the number of particles, c is the speed of light, and Γ_i is the Lorentz factor. In addition $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$ are called the i^{th} velocity and the i^{th} relativistic velocity, respectively, and $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a nonnegative communication weight that is locally Lipschitz continuous and monotonically decreasing. For the well-definedness of Γ_i, and global well-posedness, $\max_{i \in [N]} \|\boldsymbol{v}_i\|$ cannot exceed the speed of light c. For the detailed descriptions, refer to Proposition 2.4 or previous papers [1, 3].

Subsequently, if we consider

$$g(x) := \frac{cx}{\sqrt{c^2 - x^2}} + \frac{x}{c^2 - x^2}, \quad x \in (-c, c),$$

then the function g is a strictly increasing bijective odd function from $(-c, c)$ to $\mathbb{R}$, satisfying Eq (1.2),

$$\|\boldsymbol{w}_i\| = F_i \|\boldsymbol{v}_i\| = \Gamma_i \left(1 + \frac{\Gamma_i}{c^2}\right) \|\boldsymbol{v}_i\| = \frac{c\|\boldsymbol{v}_i\|}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}} + \frac{\|\boldsymbol{v}_i\|}{c^2 - \|\boldsymbol{v}_i\|^2} = g(\|\boldsymbol{v}_i\|). \tag{1.2}$$

Using this, the authors of [3] demonstrated the existence of an odd and bijective function, $\hat{\boldsymbol{w}} : B_c(0) := \{x \in \mathbb{R}^d \mid \|x\| < c\} \to \mathbb{R}^d$, such that Eq (1.3),

$$\hat{\boldsymbol{w}}(\boldsymbol{v}_i) = \boldsymbol{w}_i = F_i \boldsymbol{v}_i = \frac{c\boldsymbol{v}_i}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}} + \frac{\boldsymbol{v}_i}{c^2 - \|\boldsymbol{v}_i\|^2}, \quad \text{and we set} \quad \hat{\boldsymbol{v}} = (\hat{\boldsymbol{w}})^{-1}. \tag{1.3}$$

Then, in terms of $\{(\boldsymbol{x}_i, \boldsymbol{w}_i)\}_{i=1}^N$, Eq (1.1) can be represented by

$$\begin{cases} \dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \quad t > 0, \quad i \in [N], \\ \dot{\boldsymbol{w}}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) \left(\hat{\boldsymbol{v}}(\boldsymbol{w}_j) - \hat{\boldsymbol{v}}(\boldsymbol{w}_i)\right), \\ \boldsymbol{w}_i := F_i \boldsymbol{v}_i, \quad F_i := \Gamma_i \Big(1 + \dfrac{\Gamma_i}{c^2}\Big), \quad \Gamma_i := \dfrac{c}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}}, \\ (\boldsymbol{x}_i(0), \boldsymbol{w}_i(0)) = (\boldsymbol{x}_i^{\text{in}}, \boldsymbol{w}_i^{\text{in}}) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases}$$

In contrast, from the perspective of $\{(\boldsymbol{x}_i, \boldsymbol{v}_i)\}_{i=1}^N$, Eq (1.1) can also be rewritten as:

$$\begin{cases} \dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \quad t > 0, \quad i \in [N], \\ \dot{\hat{\boldsymbol{w}}}(\boldsymbol{v}_i) = \dfrac{1}{N} \displaystyle\sum_{j=1}^N \phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) \left(\boldsymbol{v}_j - \boldsymbol{v}_i\right), \\ \boldsymbol{w}_i := F_i \boldsymbol{v}_i, \quad F_i := \Gamma_i \left(1 + \dfrac{\Gamma_i}{c^2}\right), \quad \Gamma_i := \dfrac{c}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}}, \\ (\boldsymbol{x}_i(0), \boldsymbol{w}_i(0)) = (\boldsymbol{x}_i^{\text{in}}, \boldsymbol{w}_i^{\text{in}}) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases}$$

Therefore, by the definition of $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$, Eq (1.1) is a second-order ODE system in terms of $\{(\boldsymbol{x}_i, \boldsymbol{v}_i)\}_{i=1}^N$ or $\{(\boldsymbol{x}_i, \boldsymbol{w}_i)\}_{i=1}^N$. However, due to the complexity of the explicit expressions of $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$, and the simple relation,

$$\boldsymbol{v}_i = \frac{\boldsymbol{w}_i}{F_i},$$

throughout the paper, we consider Eq (1.1) to be a system regarding $\{(\boldsymbol{x}_i, \boldsymbol{w}_i)\}_{i=1}^N$ rather than $\{(\boldsymbol{x}_i, \boldsymbol{v}_i)\}_{i=1}^N$ for convenience and ease of mathematical results.

As a CS model with a relativistic correction, the RCS model Eq (1.1) and its variants have received increasing attention from the mathematical community. Examples include the derivation of RCS and its flocking behavior [19], hierarchical leadership Riemannian manifold extension uniform-in-time mean-field limit on $\mathbb{R}^d$ [3], mean-field limit on the Riemannian manifold bonding feedback force on the Riemannian manifold collision avoidance [4], uniform-in-time nonrelativistic limit of particle and kinetic models [2], and kinetic and hydrodynamic descriptions [18].

However, although Eq (1.1) is the dynamical system with a relativistic correction, this model still neglects time-delayed interactions. Indeed, because the speed of light is always finite, $c > 0$, not infinite, the delayed time due to information transfer between agents cannot be ignored. Hence, we propose the following modified RCS model with time delay from Eq (1.1):

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{v}_i(t), \quad t > 0, \quad i \in [N], \\ \dot{\boldsymbol{w}}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t - \tau_{ij}(t))\|) \left(\boldsymbol{v}_j(t - \tau_{ij}(t)) - \boldsymbol{v}_i(t)\right), \\ \boldsymbol{w}_i := F_i \boldsymbol{v}_i, \quad F_i := \Gamma_i \left(1 + \dfrac{\Gamma_i}{c^2}\right), \quad \Gamma_i := \dfrac{c}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}}, \\ (\boldsymbol{x}_i(s), \boldsymbol{w}_i(s)) = (\boldsymbol{x}_i^{\text{in}}(s), \boldsymbol{w}_i^{\text{in}}(s)) \in \mathbb{R}^d \times \mathbb{R}^d, \quad s \in [-\tau, 0], \end{cases} \tag{1.4}$$

where $\boldsymbol{x}_i^{\text{in}}(s)$ and $\boldsymbol{w}_i^{\text{in}}(s)$ are continuously differentiable functions on $[-\tau, 0]$, and we define

$$C_0 := \sup_{s \in [-\tau, 0]} \max_{i \in [N]} \|\dot{\boldsymbol{w}}_i^{\text{in}}(s)\|.$$

We assume that $\tau_{ij}(t)$ is the time it takes for the i^{th} agent to receive the information delivered from the other j^{th} agent at time t. To demonstrate global well-posedness and the mathematical results, we suppose that

1. *Nonnegativity and local Lipschitz continuity*

$$\tau_{ij}(t) \in C^{0,1}_{\text{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0}), \quad i, j \in [N].$$

2. *Uniform boundedness*

$$\sup_{t\in[0,\infty)} \max_{i,j\in[N]} \tau_{ij}(t) \le \tau.$$

The second condition that τ_{ij} is uniformly bounded by τ is reasonable because, if any two agents are sufficiently far apart, the time it takes to interact with each other is sufficiently long, so they may not represent collective behavior. In addition, for $i, j \in [N],\ i \neq j$, we do not suppose that τ_{ij} is symmetric and no self-processing delay; in other words, it can be asymmetric and self-processing:

$$\text{For each time } t \ge 0, \qquad \tau_{ij}(t) = \tau_{ji}(t) \quad \text{or} \quad \tau_{ij}(t) \neq \tau_{ji}(t), \quad \text{and} \quad \tau_{ii}(t) \ge 0.$$

These are reasonable, considering the time it takes for one agent to process and respond to information on its own when receiving information from another agent. Next, concerning the well-definedness of the Lorentz factor Γ_i and global well-posedness of Eq (1.4), we refer to Proposition 2.4.

Now, we provide basic concepts for the asymptotic flocking of Eq (1.4).

Definition 1.1. *Suppose that* $Z = \{(\boldsymbol{x}_i, \boldsymbol{w}_i)\}_{i=1}^N$ *is a global solution to Eq* (1.4).

1. *Configuration Z exhibits group formation if*

$$\sup_{t\in[0,\infty)} \max_{i,j\in[N]} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| < \infty.$$

2. *Configuration Z exhibits asymptotic velocity alignment if*

$$\lim_{t\to\infty} \max_{i,j\in[N]} \|\boldsymbol{v}_j(t) - \boldsymbol{v}_i(t)\| = 0.$$

3. *Configuration Z exhibits asymptotic relativistic velocity alignment if*

$$\lim_{t\to\infty} \max_{i,j\in[N]} \|\boldsymbol{w}_j(t) - \boldsymbol{w}_i(t)\| = 0.$$

4. *Configuration Z exhibits asymptotic flocking if*

$$\sup_{t\in[0,\infty)} \max_{i,j\in[N]} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| < \infty, \qquad \lim_{t\to\infty} \max_{i,j\in[N]} \|\boldsymbol{v}_j(t) - \boldsymbol{v}_i(t)\| = 0.$$

In fact, the velocity alignment and relativistic velocity alignment are equivalent under an appropriate assumption. Therefore, it suffices to demonstrate the relativistic velocity alignment to reveal the velocity alignment in Eq (1.4). For detailed descriptions, we refer to Proposition 2.1 and 2.4.

Throughout the paper, we are primarily concerned with the following issue:

- (Main issue): Can we determine a nonempty admissible set of initial data and system parameters that cause asymptotic flocking in Eq (1.4)?

The rest of this paper is organized as follows. Section 2 introduces previous key estimates frequently used to study the RCS type models. Subsequently, we provide a uniform upper bound for the maximum speed of all agents and demonstrate basic estimates for three time-difference terms in Eq (1.4). Section 3 demonstrates a reduction from Eq (1.4) to its dissipative structure for relativistic velocities. Using this structure, we present a sufficient framework for the asymptotic flocking of Eq (1.4) under an admissible set in terms of the initial data and system parameters. Section 4 briefly summarizes the main results and future work.

Notation. Throughout this paper, we employ the following simple notation:

$$
\begin{aligned}
&[N] := \{1, \cdots, N\}, \quad \|\cdot\| := \text{the standard Euclidean norm},\\
&X := (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N), \quad V := (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_N), \quad W := (\boldsymbol{w}_1, \cdots, \boldsymbol{w}_N),\\
&D_Z := \max_{i,j\in[N]} \|z_i - z_j\|, \quad \text{for} \quad Z = (z_1, \cdots, z_N) \in \{X, V, W\},\\
&\Delta_Z^\tau(t) := \max_{i,j\in[N]} \|z_j(t - \tau_{ij}(t)) - z_j(t)\|, \quad \text{for} \quad Z = (z_1, \cdots, z_N) \in \{X, V, W\},\\
&\Delta_Z^\tau \text{ is the time-difference term for } Z, \quad \text{a.e.:=almost everywhere.}
\end{aligned}
$$

2. Preliminaries

This section introduces previous results studied in [2] and demonstrates the uniform boundedness of $\max_{i\in[N]} \|\boldsymbol{v}_i\|$ using a physical constraint in terms of the initial data in Eq (1.4). In addition, it provides several basic estimates to study the asymptotic flocking dynamics of Eq (1.4).

2.1. Previous estimates

This subsection presents two key estimates from [2]. The estimates perform an important role in reducing Eq (1.4) to a dissipative structure regarding a diameter for relativistic velocities and obtaining a velocity alignment from a relativistic velocity alignment.

Proposition 2.1. [2] *Assume that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are two vectors in $\mathbb{R}^d$ such that, for a positive constant, $V^0 > 0$,*

$$\boldsymbol{w}_i := \hat{\boldsymbol{w}}(\boldsymbol{v}_i), \quad \|\boldsymbol{v}_i\| = \|\hat{\boldsymbol{v}}(\boldsymbol{w}_i)\| \le V^0 < c, \quad i = [2],$$

where $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$ are defined in (1.3)*. Then, we have that $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|$ and $\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|$ are equivalent. Moreover,*

$$\frac{c^2+1}{c^2}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \le \|\boldsymbol{w}_1 - \boldsymbol{w}_2\| \le (g'(V^0)V^0 + g(V^0))\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|,$$

where g is defined in Eq (1.2)*.*

Next, we also give the following relationship between $\|\boldsymbol{w}_i - \boldsymbol{w}_j\|$ and $\left|\frac{1}{F_i} - \frac{1}{F_j}\right|$ in Eq (1.4):

Proposition 2.2. [2] *Suppose that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are two vectors in $\mathbb{R}^d$ such that, for a positive constant, $V^0 > 0$,*

$$\boldsymbol{w}_i := \hat{\boldsymbol{w}}(\boldsymbol{v}_i), \quad \|\boldsymbol{v}_i\| = \|\hat{\boldsymbol{v}}(\boldsymbol{w}_i)\| \le V^0 < c, \quad i = [2],$$

where $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$ are defined in (1.3)*. Then, it follows that*

$$\left|\frac{1}{F_1} - \frac{1}{F_2}\right| \le \Omega\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|,$$

where $\Omega > 0$ is a positive constant expressed by

$$\Omega := \frac{c^2(2V^0 + cV^0\sqrt{c^2 - (V^0)^2})}{(c^2+1)(c\sqrt{c^2-(V^0)^2}+1)^2}.$$

Before we finish this subsection, we provide a uniform upper bound for an operator norm of the Jacobian $\nabla_{\boldsymbol{w}}\hat{\boldsymbol{v}}$.

Proposition 2.3. [2] *Let* $\|\cdot\|_{op}$ *be an operator norm of a matrix. Assume that* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are two vectors in* $\mathbb{R}^d$ *such that*

$$\boldsymbol{v} := \hat{\boldsymbol{v}}(\boldsymbol{w}), \quad \|\boldsymbol{v}\| < c,$$

where $\hat{\boldsymbol{v}}$ *is defined in Eq* (1.3). *Then, we attain*

$$\|\nabla_{\boldsymbol{w}}\hat{\boldsymbol{v}}\|_{\text{op}} = \left(\frac{c}{\sqrt{c^2 - \|\boldsymbol{v}\|^2}} + \frac{1}{c^2 - \|\boldsymbol{v}\|^2} \right)^{-1} \le \frac{c^2}{c^2+1}.$$

Remark 2.1. *Although not used in this paper, the authors of [2] also obtained the following estimate:*

$$\begin{aligned} \|\nabla_{\boldsymbol{v}}\hat{\boldsymbol{w}}\|_{\text{op}} &= \frac{c\|\boldsymbol{v}\|^2}{(c^2 - \|\boldsymbol{v}\|^2)^{\frac{3}{2}}} + \frac{2\|\boldsymbol{v}\|^2}{(c^2 - \|\boldsymbol{v}\|^2)^2} + \frac{c}{\sqrt{c^2 - \|\boldsymbol{v}\|^2}} + \frac{1}{c^2 - \|\boldsymbol{v}\|^2} \\ &\le g'(\|\boldsymbol{v}\|)\|\boldsymbol{v}\| + g(\|\boldsymbol{v}\|), \end{aligned}$$

where g is defined in Eq (1.2).

2.2. *Preparatory estimates*

This subsection demonstrates that the maximum speed of all agents is uniformly bounded by a physical constraint regarding the initial data based on the idea used in and provides several estimates for time-difference terms Δ_X^τ, Δ_V^τ, and Δ_W^τ in Eq (1.4). These are crucially used to study the asymptotic flocking behavior of Eq (1.4) in Section 3.

Proposition 2.4. (Uniform upper bound of maximum speed) *Assume that* $\{(\boldsymbol{x}_i, \boldsymbol{w}_i)\}_{i=1}^N$ *is a solution to Eq* (1.4) *such that, for a positive constant,* $V^0 > 0$,

$$\sup_{t\in[-\tau,0]} \max_{i\in[N]} \|\boldsymbol{v}_i^{\text{in}}(t)\| \le V^0 < c.$$

Then, we obtain

$$\sup_{t\in[-\tau,\infty)} \max_{i\in[N]} \|\boldsymbol{v}_i(t)\| \le V^0 < c.$$

Proof. For a fixed positive number $\epsilon > 0$, for notational simplicity, we set:

$$V^{\text{in},\tau} := \sup_{t\in[-\tau,0]} \max_{i\in[N]} \|\boldsymbol{v}_i^{\text{in}}(t)\| \quad \text{and} \quad V^{\text{in},\tau,\epsilon} := \sup_{t\in[-\tau,0]} \max_{i\in[N]} \|\boldsymbol{v}_i^{\text{in}}(t)\| + \epsilon.$$

We also define the following set:

$$\mathcal{T}_\epsilon := \{t > 0 : L_V(s) < V^{\text{in},\tau,\epsilon}, \quad \forall s \in [0,t)\}, \quad \text{where} \quad L_V(s) := \max_{i\in[N]} \|\boldsymbol{v}_i(s)\|.$$

Further, we observe that $\mathcal{T}_\epsilon$ is nonempty because, from $L_V(0) < V^{\text{in},\tau,\epsilon}$ and the continuity of L_V, there exists a positive number ϵ' such that

$$L_V(s) < V^{\text{in},\tau,\epsilon}, \ \forall s \in [0,\epsilon').$$

Next, we set:

$$\sup \mathcal{T}_\epsilon := T_\epsilon^\infty > 0.$$

For the proof by contradiction, we suppose that $T_\epsilon^\infty < \infty$. Then, we obtain

$$\lim_{s \to T_\epsilon^\infty -} L_V(s) = V^{\text{in},\tau,\epsilon}, \qquad L_V(s) < V^{\text{in},\tau,\epsilon}, \quad \forall s \in [-\tau, T_\epsilon^\infty).$$

For a.e. $t \in (0, T_\epsilon^\infty)$, using $(1.4)_2$, we demonstrate that

$$\begin{aligned}
\frac{1}{2}\frac{d\|\boldsymbol{w}_i(t)\|^2}{dt} &= \|\boldsymbol{w}_i(t)\| \cdot \frac{d\|\boldsymbol{w}_i(t)\|}{dt} = \langle \boldsymbol{w}_i(t), \dot{\boldsymbol{w}}_i(t)\rangle \\
&= \Big\langle \boldsymbol{w}_i(t), \frac{1}{N}\sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t-\tau_{ij}(t))\|)\Big(\boldsymbol{v}_j(t-\tau_{ij}(t)) - \boldsymbol{v}_i(t)\Big)\Big\rangle \\
&= \frac{1}{N}\sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t-\tau_{ij}(t))\|)(\langle \boldsymbol{w}_i(t), \boldsymbol{v}_j(t-\tau_{ij}(t))\rangle - \langle \boldsymbol{w}_i, \boldsymbol{v}_i\rangle) \\
&= \frac{1}{N}\sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t-\tau_{ij}(t))\|)(\langle \boldsymbol{w}_i(t), \boldsymbol{v}_j(t-\tau_{ij}(t))\rangle - \|\boldsymbol{v}_i(t)\|\|\boldsymbol{w}_i(t)\|) \\
&\le \frac{1}{N}\sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t-\tau_{ij}(t))\|)(\|\boldsymbol{w}_i(t)\|\|\boldsymbol{v}_j(t-\tau_{ij}(t))\| - \|\boldsymbol{v}_i(t)\|\|\boldsymbol{w}_i(t)\|) \\
&\le \frac{1}{N}\sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t-\tau_{ij}(t))\|)(V^{\text{in},\tau,\epsilon} - \|\boldsymbol{v}_i(t)\|)\|\boldsymbol{w}_i(t)\| \\
&\le \frac{\phi(0)}{N}\sum_{j=1}^N (V^{\text{in},\tau,\epsilon} - \|\boldsymbol{v}_i(t)\|)\|\boldsymbol{w}_i(t)\| = \phi(0)(V^{\text{in},\tau,\epsilon} - \|\boldsymbol{v}_i(t)\|)\|\boldsymbol{w}_i(t)\|,
\end{aligned}$$

due to $\boldsymbol{w}_i = F_i\boldsymbol{v}_i$ and $L_V(s) \le V^{\text{in},\tau,\epsilon}$, $\forall s \in (-\tau, T_\epsilon^\infty)$. Therefore, this implies that, for a.e. $t \in (0, T_\epsilon^\infty)$,

$$\frac{d\|\boldsymbol{w}_i(t)\|}{dt} \le \phi(0)(V^{\text{in},\tau,\epsilon} - \|\boldsymbol{v}_i(t)\|).$$

From the following relation, for a.e. $t \in (0, T_\epsilon^\infty)$:

$$\begin{aligned}
\frac{d\|\boldsymbol{w}_i\|}{dt} &= \frac{d(F_i\|\boldsymbol{v}_i\|)}{dt} = \frac{d\|\boldsymbol{v}_i\|}{dt}F_i + \|\boldsymbol{v}_i\|\frac{dF_i}{dt} \\
&= \frac{d\|\boldsymbol{v}_i\|}{dt}\left(\frac{c}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}} + \frac{1}{c^2 - \|\boldsymbol{v}_i\|^2}\right) \\
&\quad + \|\boldsymbol{v}_i\|\frac{d\|\boldsymbol{v}_i\|}{dt}\left(\frac{2\|\boldsymbol{v}_i\|}{(c^2 - \|\boldsymbol{v}_i\|^2)^2} + \frac{c\|\boldsymbol{v}_i\|}{(c^2 - \|\boldsymbol{v}_i\|^2)^{\frac{3}{2}}}\right) \\
&= \frac{d\|\boldsymbol{v}_i\|}{dt}\left(\frac{c}{\sqrt{c^2 - \|\boldsymbol{v}_i\|^2}} + \frac{1}{c^2 - \|\boldsymbol{v}_i\|^2} + \frac{2\|\boldsymbol{v}_i\|^2}{(c^2 - \|\boldsymbol{v}_i\|^2)^2} + \frac{c\|\boldsymbol{v}_i\|^2}{(c^2 - \|\boldsymbol{v}_i\|^2)^{\frac{3}{2}}}\right),
\end{aligned}$$

it follows that, for a.e. $t \in (0, T_\epsilon^\infty)$,

$$\begin{aligned}\frac{d\|\boldsymbol{v}_i(t)\|}{dt} &\le \frac{\phi(0)(V^{\mathrm{in},\tau,\epsilon}-\|\boldsymbol{v}_i(t)\|)}{\left(\frac{c}{\sqrt{c^2-\|\boldsymbol{v}_i\|^2}}+\frac{1}{c^2-\|\boldsymbol{v}_i\|^2}+\frac{2\|\boldsymbol{v}_i\|^2}{(c^2-\|\boldsymbol{v}_i\|^2)^2}+\frac{c\|\boldsymbol{v}_i\|^2}{(c^2-\|\boldsymbol{v}_i\|^2)^{\frac{3}{2}}}\right)}\\ &\le \frac{c^2\phi(0)(V^{\mathrm{in},\tau,\epsilon}-\|\boldsymbol{v}_i(t)\|)}{c^2+1},\end{aligned}$$

because $x\mapsto \frac{c}{\sqrt{c^2-x^2}}+\frac{1}{c^2-x^2}+\frac{2x^2}{(c^2-x^2)^2}+\frac{cx^2}{(c^2-x^2)^{\frac{3}{2}}}$ is strictly increasing on $[0,c)$. Hence,

$$\frac{c}{\sqrt{c^2-x^2}}+\frac{1}{c^2-x^2}+\frac{2x^2}{(c^2-x^2)^2}+\frac{cx^2}{(c^2-x^2)^{\frac{3}{2}}}\ge\frac{c^2+1}{c^2}.$$

Applying the Grönwall lemma yields that, for $t\in[0,T_\epsilon^\infty)$,

$$\|\boldsymbol{v}_i(t)\|\le(\|\boldsymbol{v}_i(0)\|-V^{\mathrm{in},\tau,\epsilon})\exp\left(-\frac{c^2\phi(0)t}{c^2+1}\right)+V^{\mathrm{in},\tau,\epsilon},$$

leading to the following estimate for $t\in[0,T_\epsilon^\infty)$:

$$L_V(t)\le(L_V(0)-V^{\mathrm{in},\tau,\epsilon})\exp\left(-\frac{c^2\phi(0)t}{c^2+1}\right)+V^{\mathrm{in},\tau,\epsilon}.$$

Accordingly,

$$\lim_{t\to T_\epsilon^\infty-}L_V(t)\le(L_V(0)-V^{\mathrm{in},\tau,\epsilon})\exp\left(-\frac{c^2\phi(0)T_\epsilon^\infty}{c^2+1}\right)+V^{\mathrm{in},\tau,\epsilon}<V^{\mathrm{in},\tau,\epsilon}.$$

This result causes a contradiction to the definition of T_ϵ^∞. In conclusion, $T_\epsilon^\infty=\infty$. Finally, by taking $\epsilon\to 0$, we demonstrate the desired result. □

Remark 2.2. *From the standard Cauchy–Lipschitz theory with Proposition 2.4, the local Lipschitz continuity and uniform boundedness of ϕ, and the locally Lipschitz continuity of τ_{ij}, the global well-posedness of Eq* (1.4) *can be guaranteed.*

Subsequently, we study three time-difference estimates for the *position–velocity–relativistic velocity*, that is, Δ_X^τ, Δ_V^τ, and Δ_W^τ.

Proposition 2.5. (Crucial estimates for Δ_X^τ, Δ_V^τ, and Δ_W^τ) *Let $\{(\boldsymbol{x}_i,\boldsymbol{w}_i)\}_{i=1}^N$ be a solution to Eq* (1.4) *such that, for a positive constant, $V^0>0$,*

$$\sup_{t\in[-\tau,0]}\max_{i\in[N]}\|\boldsymbol{v}_i^{\mathrm{in}}(t)\|\le V^0<c.$$

Then, we have the following assertions:

1. *(Estimate of Δ_X^τ) For $t\in[0,\infty)$,*
$$\Delta_X^\tau(t)\le V^0\tau.$$
2. *(Estimate of Δ_V^τ) For $t\in[0,\infty)$,*
$$\Delta_V^\tau(t)\le\frac{c^2}{c^2+1}\Delta_W^\tau(t).$$

3. *(Estimate of Δ^τ_W) For $t \in [\tau, \infty)$,*

$$\Delta^\tau_W(t) \le 2V^0\phi(0)\tau, \qquad \Delta^\tau_W(t) \le \phi(0)\int_{t-\tau}^t (\Delta^\tau_V(s) + D_V(s))ds.$$

Proof. • (Proof of the first assertion) We apply Proposition 2.4 and the second property for τ_{ij} to obtain

$$\|\boldsymbol{x}_j(t-\tau_{ij}(t)) - \boldsymbol{x}_j(t)\| = \left\| \int_{t-\tau_{ij}}^t \boldsymbol{v}_j(s)ds \right\| \le \int_{t-\tau}^t \left\|\boldsymbol{v}_j(s)\right\| ds \le V^0\tau.$$

• (Proof of the second assertion) We employ Proposition 2.3 to yield the following relations:

$$\|\boldsymbol{v}_j(t-\tau_{ij}(t)) - \boldsymbol{v}_j(t)\| = \|\hat{\boldsymbol{v}}(\boldsymbol{w}_j(t-\tau_{ij}(t))) - \hat{\boldsymbol{v}}(\boldsymbol{w}_j(t))\| \le \frac{c^2}{c^2+1}\|\boldsymbol{w}_j(t-\tau_{ij}(t)) - \boldsymbol{w}_j(t)\|.$$

• (Proof of the third assertion) From Eq $(1.4)_2$ and the monotonicity of ϕ, we observe that, for $t \in [\tau, \infty)$,

$$\begin{aligned}
&\|\boldsymbol{w}_j(t-\tau_{ij}(t)) - \boldsymbol{w}_j(t)\| \\
&\le \frac{1}{N}\int_{t-\tau_{ij}}^t \left\| \sum_{k=1}^N \phi(\|\boldsymbol{x}_j(s) - \boldsymbol{x}_k(s-\tau_{jk}(s))\|)\left(\boldsymbol{v}_k(s-\tau_{jk}(s)) - \boldsymbol{v}_j(s)\right)\right\| ds \\
&\le \frac{1}{N}\int_{t-\tau}^t \left\| \sum_{k=1}^N \phi(\|\boldsymbol{x}_j(s) - \boldsymbol{x}_k(s-\tau_{jk}(s))\|)\left(\boldsymbol{v}_k(s-\tau_{jk}(s)) - \boldsymbol{v}_j(s)\right)\right\| ds \\
&\le \frac{1}{N}\int_{t-\tau}^t \sum_{k=1}^N \phi(0)\left\|\boldsymbol{v}_k(s-\tau_{jk}(s)) - \boldsymbol{v}_j(s)\right\| ds \\
&\le \frac{1}{N}\int_{t-\tau}^t \sum_{k=1}^N \phi(0)(\|\boldsymbol{v}_k(s-\tau_{jk}(s))\| + \|\boldsymbol{v}_j(s)\|)ds \\
&\le 2V^0\phi(0)\tau.
\end{aligned}$$

In contrast, we use the following relation for $i, j \in [N]$:

$$\begin{aligned}
&\|\boldsymbol{v}_j(s-\tau_{ij}(s)) - \boldsymbol{v}_i(s)\| \\
&= \|\boldsymbol{v}_j(s-\tau_{ij}(s)) - \boldsymbol{v}_j(s) + \boldsymbol{v}_j(s) - \boldsymbol{v}_i(s)\| \\
&\le \|\boldsymbol{v}_j(s-\tau_{ij}(s)) - \boldsymbol{v}_j(s)\| + \|\boldsymbol{v}_j(s) - \boldsymbol{v}_i(s)\| \\
&\le \Delta^\tau_V(s) + D_V(s),
\end{aligned}$$

to demonstrate that, for $t \in [\tau, \infty)$,

$$\begin{aligned}
&\|\boldsymbol{w}_j(t-\tau_{ij}(t)) - \boldsymbol{w}_j(t)\| \\
&\le \frac{1}{N}\int_{t-\tau_{ij}}^t \left\| \sum_{k=1}^N \phi(\|\boldsymbol{x}_j(s) - \boldsymbol{x}_k(s-\tau_{jk}(s))\|)\left(\boldsymbol{v}_k(s-\tau_{jk}(s)) - \boldsymbol{v}_j(s)\right)\right\| ds \\
&\le \frac{1}{N}\int_{t-\tau}^t \sum_{k=1}^N \phi(0)\left\|\boldsymbol{v}_k(s-\tau_{jk}(s)) - \boldsymbol{v}_j(s)\right\| ds
\end{aligned}$$

$$\leq \frac{1}{N}\int_{t-\tau}^{t}\sum_{k=1}^{N}\phi(0)(\Delta_V^\tau(s)+D_V(s))ds$$
$$\leq \phi(0)\int_{t-\tau}^{t}(\Delta_V^\tau(s)+D_V(s))ds.$$

Therefore, we obtain the desired assertions. □

Remark 2.3. *The third result of Proposition 2.5 holds for* $t \in [\tau,\infty)$ *because we did not define the velocity coupling equation,* $(1.4)_2$, *in terms of* $\dot{\boldsymbol{w}}_i$ *on* $t \in (-\tau, 0)$ *for each* $i \in [N]$.

3. Asymptotic flocking result

This section first presents a reduction from Eq (1.4) to its dissipative structure. Then, we demonstrate several sufficient frameworks for the asymptotic flocking of Eq (1.4) with this dissipative structure and the previous results studied from Section 2. We begin with the following lemma, deriving two dissipative differential inequalities for D_X, D_W, Δ_W^τ, and the system parameters.

Lemma 3.1. (Dissipative inequalities) *Let* $\{(\boldsymbol{x}_i,\boldsymbol{w}_i)\}_{i=1}^N$ *be a solution to Eq* (1.4) *such that, for a positive constant,* $V^0>0$,

$$\sup_{s\in[-\tau,0]}\max_{i\in[N]}\|\boldsymbol{v}_i^{\mathrm{in}}(s)\|\leq V^0<c.$$

We recall the function g and constant Ω *defined in Eq* (1.2) *and Proposition 2.2, respectively,*

$$g(x):=\frac{cx}{\sqrt{c^2-x^2}}+\frac{x}{c^2-x^2}\quad on\quad (-c,c),\qquad \Omega:=\frac{c^2(2V^0+cV^0\sqrt{c^2-(V^0)^2})}{(c^2+1)(c\sqrt{c^2-(V^0)^2}+1)^2}.$$

If we set the following four constants, $C_i,\ i\in[4]$:

$$C_1:=\frac{c^2}{c^2+1},\qquad C_2:=\left(\frac{g(V^0)}{V^0}\right)^{-1},\qquad C_3:=2\phi(0)g(V^0)\Omega,\qquad C_4:=2C_1\phi(0),$$

then we have that, for a.e. $t\in(0,\infty)$,

1. (Differential inequality for D_X)

$$\left|\frac{d}{dt}D_X(t)\right|\leq D_V(t)\leq C_1D_W(t).$$

2. (Differential inequality for D_W)

$$\frac{d}{dt}D_W(t)\leq\Big(-C_2\phi(D_X(t)+V^0\tau)+C_3\Big)D_W(t)+C_4\Delta_W^\tau(t).$$

Proof. To verify the first assertion, we use Propositions 2.1 and 2.4 to obtain, for $i,j\in[N]$ and a.e. $t\in(0,\infty)$,

$$\frac{1}{2}\frac{d\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^2}{dt}=\frac{d\|\boldsymbol{x}_i-\boldsymbol{x}_j\|}{dt}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|=\langle\boldsymbol{x}_i-\boldsymbol{x}_j,\boldsymbol{v}_i-\boldsymbol{v}_j\rangle$$

$$\leq \|\boldsymbol{x}_i - \boldsymbol{x}_j\|\|\boldsymbol{v}_i - \boldsymbol{v}_j\| \leq C_1\|\boldsymbol{x}_i - \boldsymbol{x}_j\|\|\boldsymbol{w}_i - \boldsymbol{w}_j\|.$$

Then, it follows that, for a.e. $t \in (0, \infty)$,

$$\frac{d\|\boldsymbol{x}_i - \boldsymbol{x}_j\|}{dt} \leq \|\boldsymbol{v}_i - \boldsymbol{v}_j\| \leq C_1\|\boldsymbol{w}_i - \boldsymbol{w}_j\|.$$

By selecting two maximal indices, $i_t, j_t \in [N]$, dependent on time t, such that

$$D_X(t) = \|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_{j_t}(t)\|,$$

we obtain the following first assertion for a.e. $t \in (0, \infty)$:

$$\left|\frac{d}{dt}D_X(t)\right| \leq D_V(t) \leq C_1 D_W(t).$$

To verify the second assertion, we choose two maximal indices, $i_t, j_t \in [N]$, depending on time t, satisfying

$$D_W(t) = \|\boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t)\|.$$

Then, from Eq $(1.4)_2$, we demonstrate, for a.e. $t \in (0, \infty)$, that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}D_W^2(t)\\
&= \langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \dot{\boldsymbol{w}}_{i_t}(t) - \dot{\boldsymbol{w}}_{j_t}(t)\rangle\\
&= \langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \dot{\boldsymbol{w}}_{i_t}(t)\rangle - \langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \dot{\boldsymbol{w}}_{j_t}(t)\rangle\\
&= \frac{1}{N}\left\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N}\phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\left(\boldsymbol{v}_k(t - \tau_{i_t k}(t)) - \boldsymbol{v}_{i_t}(t)\right)\right\rangle\\
&\quad - \frac{1}{N}\left\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N}\phi(\|\boldsymbol{x}_{j_t}(t) - \boldsymbol{x}_k(t - \tau_{j_t k}(t))\|)\left(\boldsymbol{v}_k(t - \tau_{j_t k}(t)) - \boldsymbol{v}_{j_t}(t)\right)\right\rangle\\
&:= \mathcal{I} + \mathcal{J}.
\end{aligned}$$

• (Estimate of $\mathcal{I}$) To estimate $\mathcal{I}$,

$$\begin{aligned}
\mathcal{I} &= \frac{1}{N}\left\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N}\phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\left(\boldsymbol{v}_k(t - \tau_{i_t k}(t)) - \boldsymbol{v}_{i_t}(t)\right)\right\rangle\\
&= \frac{1}{N}\left\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N}\phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\left(\boldsymbol{v}_k(t) - \boldsymbol{v}_{i_t}(t)\right)\right\rangle\\
&\quad + \frac{1}{N}\left\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N}\phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\left(\boldsymbol{v}_k(t - \tau_{i_t k}(t)) - \boldsymbol{v}_k(t)\right)\right\rangle\\
&\leq \frac{1}{N}\left\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N}\phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\left(\boldsymbol{v}_k(t) - \boldsymbol{v}_{i_t}(t)\right)\right\rangle\\
&\quad + \frac{\phi(0)}{N}D_W(t)\sum_{k=1}^{N}\|\boldsymbol{v}_k(t - \tau_{i_t k}(t)) - \boldsymbol{v}_k(t)\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{N}\Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N} \phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\,(\boldsymbol{v}_k(t) - \boldsymbol{v}_{i_t}(t))\Big\rangle \\
&\quad + \phi(0)\Delta_V^\tau(t) D_W(t) \\
&\le \frac{1}{N}\Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N} \phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|)\,(\boldsymbol{v}_k(t) - \boldsymbol{v}_{i_t}(t))\Big\rangle \\
&\quad + C_1\phi(0)\Delta_W^\tau(t) D_W(t),
\end{aligned}
$$

where we applied the second assertion of Proposition 2.5 to the last estimate, equivalently,

$$
\Delta_V^\tau(t) \le \frac{c^2}{c^2+1}\Delta_W^\tau(t) = C_1 \Delta_W^\tau(t).
$$

From the relation,

$$
\boldsymbol{v}_k - \boldsymbol{v}_{i_t} = \frac{\boldsymbol{w}_k}{F_k} - \frac{\boldsymbol{w}_{i_t}}{F_{i_t}},
$$

we observe that

$$
\begin{aligned}
&\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{v}_k(t) - \boldsymbol{v}_{i_t}(t)\rangle \\
&= \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \frac{\boldsymbol{w}_k(t)}{F_k(t)} - \frac{\boldsymbol{w}_{i_t}(t)}{F_{i_t}(t)}\Big\rangle \\
&= \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \frac{1}{F_{i_t}(t)}(\boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t)) + \boldsymbol{w}_k(t)\Big(\frac{1}{F_k(t)} - \frac{1}{F_{i_t}(t)}\Big)\Big\rangle \\
&= \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \frac{1}{F_{i_t}(t)}(\boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t))\Big\rangle + \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t)\Big(\frac{1}{F_k(t)} - \frac{1}{F_{i_t}(t)}\Big)\Big\rangle \\
&:= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

• (Estimate of $\mathcal{I}_1$) Employing the following relation:

$$
\begin{aligned}
&\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t)\rangle \le 0 \\
&\qquad \Longleftrightarrow \langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t) - \boldsymbol{w}_{j_t}(t)\rangle \le D_W^2(t),
\end{aligned}
$$

we attain

$$
\begin{aligned}
\mathcal{I}_1 &\le \left(\frac{c}{\sqrt{c^2 - (V^0)^2}} + \frac{1}{c^2 - (V^0)^2}\right)^{-1} \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t)\Big\rangle \\
&= \left(\frac{g(V^0)}{V^0}\right)^{-1} \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t)\Big\rangle \\
&= C_2\Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t)\Big\rangle \le 0,
\end{aligned}
$$

because the definition of F_{i_t} and Proposition 2.4 yield

$$
F_{i_t} = \Gamma_{i_t}\left(1 + \frac{\Gamma_{i_t}}{c^2}\right) = \frac{c}{\sqrt{c^2 - \|\boldsymbol{v}_{i_t}\|^2}} + \frac{1}{c^2 - \|\boldsymbol{v}_{i_t}\|^2} \le \frac{c}{\sqrt{c^2 - (V^0)^2}} + \frac{1}{c^2 - (V^0)^2}.
$$

• (Estimate of $\mathcal{I}_2$) Propositions 2.2 and 2.4, with the strict monotonicity of g in Eq (1.2), lead to

$$\mathcal{I}_2 \le D_W \|\boldsymbol{w}_k\| \left| \frac{1}{F_k(t)} - \frac{1}{F_{i_t}(t)} \right| = D_W g(\|\boldsymbol{v}_k\|) \left| \frac{1}{F_k(t)} - \frac{1}{F_{i_t}(t)} \right| \le \Omega g(V^0) D_W^2.$$

Therefore, combining $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\mathcal{I}$ induces

$$\begin{aligned}
\mathcal{I} &\le \frac{1}{N} \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \sum_{k=1}^{N} \phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|) \left(\boldsymbol{v}_k(t) - \boldsymbol{v}_{i_t}(t)\right) \Big\rangle \\
&\quad + C_1 \phi(0) \Delta_W^\tau(t) D_W(t) \\
&= \frac{1}{N} \sum_{k=1}^{N} \phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|) \mathcal{I}_1 + \frac{1}{N} \sum_{k=1}^{N} \phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|) \mathcal{I}_2 \\
&\quad + C_1 \phi(0) \Delta_W^\tau(t) D_W(t) \\
&\le \frac{C_2 \phi(D_X(t) + V^0 \tau)}{N} \sum_{k=1}^{N} \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_k(t) - \boldsymbol{w}_{i_t}(t) \Big\rangle \\
&\quad + \phi(0) g(V^0) \Omega D_W^2 + C_1 \phi(0) \Delta_W^\tau(t) D_W(t),
\end{aligned}$$

because $\mathcal{I}_1 \le 0$ and the monotonicity of ϕ and the first assertion of Proposition 2.5 derive

$$\begin{aligned}
\phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|) &\ge \phi(\|\boldsymbol{x}_{i_t}(t) - \boldsymbol{x}_k(t)\| + \|\boldsymbol{x}_k(t) - \boldsymbol{x}_k(t - \tau_{i_t k}(t))\|) \\
&\ge \phi(D_X(t) + \Delta_X^\tau(t)) \ge \phi(D_X(t) + V^0 \tau).
\end{aligned}$$

Similar to the method above, we can also demonstrate that

$$\begin{aligned}
\mathcal{J} \le \frac{C_2 \phi(D_X(t) + V^0 \tau)}{N} \sum_{k=1}^{N} \Big\langle \boldsymbol{w}_{i_t}(t) - \boldsymbol{w}_{j_t}(t), \boldsymbol{w}_{j_t}(t) - \boldsymbol{w}_k(t) \Big\rangle \\
+ \phi(0) g(V^0) \Omega D_W^2 + C_1 \phi(0) \Delta_W^\tau(t) D_W(t).
\end{aligned}$$

Hence, we sum $\mathcal{I}$ and $\mathcal{J}$ to obtain, for a.e. $t \in (0, \infty)$,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} D_W^2(t) &\le -C_2 \phi(D_X(t) + V^0 \tau) D_W^2(t) \\
&\quad + 2\phi(0) g(V^0) \Omega D_W^2 + 2C_1 \phi(0) \Delta_W^\tau(t) D_W(t) \\
&= \left(-C_2 \phi(D_X(t) + V^0 \tau) + C_3 \right) D_W^2(t) + C_4 \Delta_W^\tau(t) D_W(t).
\end{aligned}$$

This outcome implies, for a.e. $t \in (0, \infty)$, that

$$\frac{dD_W(t)}{dt} \le \left(-C_2 \phi(D_X(t) + V^0 \tau) + C_3 \right) D_W(t) + C_4 \Delta_W^\tau(t).$$

Consequently, we prove the desired assertions. □

Accordingly, with two differential dissipative inequalities of Lemma 3.1, we can construct an admissible set in terms of the initial data and system parameters for the asymptotic flocking estimate of Eq (1.4). To do this, we apply continuous arguments to derive the desired results.

Theorem 3.1. (Flocking dynamics) *Let* $\{(\boldsymbol{x}_i, \boldsymbol{w}_i)\}_{i=1}^N$ *be a solution to Eq* (1.4) *satisfying, for a positive constant,* $V^0 > 0$,

$$\sup_{s\in[-\tau,0]} \max_{i\in[N]} \|\boldsymbol{v}_i^{\text{in}}(s)\| \le V^0 < c.$$

We recall the definition of C_0 *as follows.*

$$C_0 := \sup_{s\in[-\tau,0]} \max_{i\in[N]} \|\dot{\boldsymbol{w}}_i^{\text{in}}(s)\|.$$

Suppose that

$$C_1 := \frac{c^2}{c^2+1}, \qquad C_2 := \left(\frac{g(V^0)}{V^0}\right)^{-1}, \qquad C_3 := 2\phi(0)g(V^0)\Omega, \qquad C_4 := 2C_1\phi(0).$$

Assume that there exist positive constants $D_X^\infty > 0$, $\alpha > 0$, $\beta > 0$, *and* $\gamma \in (0,1)$ *such that*

$$\begin{aligned}
\alpha := C_2\phi(D_X^\infty + V^0\tau) - C_3 > 0, \quad D_X(0) + \frac{C_1 D_W(0)}{\alpha} + \frac{C_1 C_4 \beta}{\alpha^2\gamma(1-\gamma)} \le D_X^\infty, \\
C_1\phi(0)\left(\beta + D_W(0) + \frac{C_4\beta}{\alpha(1-\gamma)}\right)(\exp(\alpha\gamma\tau) - 1) \le \beta, \\
\exp(\alpha\gamma\tau)\tau < \min\left(\frac{C_1\beta}{C_4 V^0}, \frac{\beta}{\max(C_0, 2V^0\phi(0))}\right), \quad \tau < \frac{1}{2V^0}\left(\frac{C_1 D_W(0)}{\alpha} + \frac{C_1 C_4\beta}{\alpha^2\gamma(1-\gamma)}\right).
\end{aligned} \tag{3.1}$$

Then, we demonstrate the following asymptotic flocking result for $t \in [0,\infty)$:

1. (Group formation)
$$D_X(t) \le D_X^\infty.$$
2. (Exponential decay of the time-difference for relativistic velocity)
$$\Delta_W^\tau(t) \le \beta\exp(-\alpha\gamma t).$$
3. (Relativistic velocity alignment)
$$D_W(t) \le D_W(0)\exp(-\alpha t) + \frac{C_4\beta}{\alpha(1-\gamma)}\exp(-\alpha\gamma t).$$

Proof. If $D_W(0) = 0$, then there is nothing to prove using the standard Cauchy–Lipschitz theory. Thus, we assume that

$$D_W(0) > 0.$$

Now, we define the set $\mathcal{S}_1$ and number $\mathcal{S}_1^*$ by

$$\mathcal{S}_1 := \{t > 0 \mid (1) \text{ is true}, \quad \forall s \in [\tau, t)\}, \qquad S_1^* := \sup \mathcal{S}_1.$$

Then, $\mathcal{S}_1 \neq \emptyset$ because D_X is continuous and for the following inequality holds using the second and fifth conditions of Eq (3.1) and Proposition 2.4:

$$D_X(\tau) \le D_X(0) + \int_0^\tau D_V(s)ds \le D_X(0) + 2V^0\tau$$

$$< D_X(0) + \frac{C_1 D_W(0)}{\alpha} + \frac{C_1 C_4 \beta}{\alpha^2 \gamma (1-\gamma)} \le D_X^\infty.$$

Hence, $S_1^* > \tau$. Next, we define the set $\mathcal{S}_2$ and number $\mathcal{S}_2^*$ by

$$\mathcal{S}_2 := \{t > 0 \mid (2) \text{ is true}, \quad \forall s \in [\tau, t), \quad \text{where} \quad t \in (\tau, S_1^*]\}, \qquad S_2^* := \sup \mathcal{S}_2.$$

Here, $\mathcal{S}_2 \neq \emptyset$ because Δ_W^τ is continuous and the following inequality holds due to the third assertion of Proposition 2.5 and the fourth condition of Eq (3.1):

$$\Delta_W^\tau(\tau) \le \frac{C_4}{C_1} V^0 \tau < \beta \exp(-\alpha\gamma\tau).$$

Then, we obtain $S_2^* > \tau$. Subsequently, we assume that $S_2^* < S_1^* \le \infty$. From the definition of S_2^*, it follows that

$$\Delta_W^\tau(t) \le \beta \exp(-\alpha\gamma t), \quad \forall t \in [\tau, S_2^*), \qquad \Delta_W^\tau(S_2^*) = \beta \exp(-\alpha\gamma S_2^*).$$

In addition, using the fourth condition of (3.1), definitions of C_0 and τ, and the following relation for $t \in (0, \infty)$:

$$\begin{aligned}
\|\dot{\boldsymbol{w}}_i(t)\| &= \left\| \frac{1}{N} \sum_{j=1}^N \phi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t-\tau_{ij}(t))\|) \left(\boldsymbol{v}_j(t-\tau_{ij}(t)) - \boldsymbol{v}_i(t) \right) \right\| \\
&\le \frac{1}{N} \sum_{j=1}^N \phi(0) \left(\|\boldsymbol{v}_j(t-\tau_{ij}(t))\| + \|\boldsymbol{v}_i(t)\| \right) \\
&\le \frac{1}{N} \sum_{j=1}^N 2V^0 \phi(0) = 2V^0 \phi(0),
\end{aligned}$$

we obtain, for $t \in [0, \tau]$,

$$\begin{aligned}
\Delta_W^\tau(t) &= \max_{i,j \in [N]} \|\boldsymbol{w}_j(t-\tau_{ij}(t)) - \boldsymbol{w}_j(t)\| \\
&\le \max\left(C_0, 2V^0\phi(0) \right) \tau < \beta \exp(-\alpha\gamma\tau) \le \beta \exp(-\alpha\gamma t),
\end{aligned}$$

we can demonstrate that

$$\Delta_W^\tau(t) \le \beta \exp(-\alpha\gamma t), \quad \forall t \in [0, S_2^*), \qquad \Delta_W^\tau(S_2^*) = \beta \exp(-\alpha\gamma S_2^*).$$

This outcome and the second assertion of Lemma 3.1 yield, for a.e. $t \in (0, S_2^*)$,

$$\begin{aligned}
\frac{d}{dt} D_W(t) &\le (-C_2 \phi(D_X(t)) + C_3) D_W(t) + C_4 \Delta_W^\tau(t) \\
&\le (-C_2 \phi(D_X^\infty) + C_3) D_W(t) + C_4 \beta \exp(-\alpha\gamma t) \\
&= -\alpha D_W(t) + C_4 \beta \exp(-\alpha\gamma t).
\end{aligned}$$

Therefore, the Grönwall lemma leads to the following estimate for $t \in [0, S_2^*]$:

$$D_W(t) \le D_W(0) \exp(-\alpha t) + \frac{C_4 \beta}{\alpha(1-\gamma)} \exp(-\alpha\gamma t).$$

We combine this estimate, the third condition of Eq (3.1), the definition of S_2^*, the first assertion of Lemma 3.1, and the second assertion of Proposition 2.5 to get the following inequalities for $t \in [\tau, S_2^*]$:

$$\begin{aligned}
\Delta_W^\tau(t) &\le \phi(0) \int_{t-\tau}^{t} (\Delta_V^\tau(s) + D_V(s)) ds \\
&\le C_1 \phi(0) \int_{t-\tau}^{t} (\Delta_W^\tau(s) + D_W(s)) ds \\
&< C_1 \phi(0) \int_{t-\tau}^{t} \left(\beta + D_W(0) + \frac{C_4 \beta}{\alpha(1-\gamma)}\right) \exp(-\alpha\gamma s) ds \\
&= C_1 \phi(0) \left(\beta + D_W(0) + \frac{C_4 \beta}{\alpha(1-\gamma)}\right) (\exp(\alpha\gamma\tau) - 1) \exp(-\alpha\gamma t) \\
&\le \beta \exp(-\alpha\gamma t).
\end{aligned}$$

This result yields a contradiction to the definition of S_2^*. Therefore, $S_1^* = S_2^* \le \infty$. To prove that $S_1^* = S_2^* = \infty$, we suppose that $S_1^* = S_2^* < \infty$ for the proof by contradiction. Then, the definition of S_1^* deduces that

$$D_X(t) \le D_X^\infty, \quad \forall t \in [\tau, S_1^*), \qquad D_X(S_1^*) = D_X^\infty.$$

Indeed, using the following estimate for $t \in [0, \tau]$:

$$\begin{aligned}
D_X(t) &\le D_X(0) + \int_0^\tau D_V(s) ds \\
&\le D_X(0) + 2V^0 \tau < D_X(0) + \frac{C_1 D_W(0)}{\alpha} + \frac{C_1 C_4 \beta}{\alpha^2 \gamma (1-\gamma)} \le D_X^\infty,
\end{aligned}$$

we obtain

$$D_X(t) \le D_X^\infty, \quad \forall t \in [0, S_1^*), \qquad D_X(S_1^*) = D_X^\infty.$$

We apply the first assertion of Lemma 3.1 and the estimate for D_W to attain

$$\begin{aligned}
D_X(S_1^*) &\le D_X(0) + \int_0^{S_1^*} D_V(s) ds \\
&\le D_X(0) + C_1 \int_0^{S_1^*} D_W(s) ds \\
&\le D_X(0) + C_1 \int_0^{S_1^*} \left(D_W(0) \exp(-\alpha s) + \frac{C_4 \beta}{\alpha(1-\gamma)} \exp(-\alpha\gamma s)\right) ds \\
&< D_X(0) + C_1 \int_0^{\infty} \left(D_W(0) \exp(-\alpha s) + \frac{C_4 \beta}{\alpha(1-\gamma)} \exp(-\alpha\gamma s)\right) ds
\end{aligned}$$

$$= D_X(0) + \frac{C_1 D_W(0)}{\alpha} + \frac{C_1 C_4 \beta}{\alpha^2 \gamma(1-\gamma)} \leq D_X^\infty.$$

This outcome contradicts the definition of S_1^*. Finally, we demonstrate that

$$S_1^* = S_2^* = \infty,$$

and conclude the desired results. □

Remark 3.1. *The admissible data, Eq* (3.1)*, is reasonable by taking* τ *and* V^0 *to be smaller and smaller, and* β *and* $\phi(D_X^\infty + V^0\tau)$ *to be larger and larger with a suitable* ϕ*.*

4. Conclusion

This paper demonstrates several sufficient frameworks for the asymptotic flocking of the relativistic Cucker–Smale (RCS) model with time delay that allows for self-processing time delays. We first derived dissipative inequalities for *position–relativistic velocity* diameters to do this. Subsequently, we employed the double continuous argument with these inequalities to prove the asymptotic flocking of the proposed model under an admissible set in terms of the initial data and system parameters. Some topics remain to study in the future, which include the mean-field limit of Eq (1.4), extension Eq (1.4) to a Riemannian manifold setting, and generalization of Eq (1.4) to a general digraph.

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (2022R1C12007321).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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