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RELAXATION APPROXIMATION OF FRIEDRICHS' SYSTEMS UNDER CONVEX CONSTRAINTS

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Abstract. This paper is devoted to present an approximation of a Cauchy problem for Friedrichs' systems under convex constraints. It is proved the strong convergence in L^2_{loc} of a parabolic-relaxed approximation towards the unique constrained solution.

1. **Introduction.** The aim of this paper is to prove the convergence of a relaxation approximation of weak solutions to Friedrichs' systems under convex constraints. The well-posedness has been established in [3] by means of a numerical scheme. We present here another way to construct such weak solutions thanks to a model that relaxes the constraints. We consider the following Cauchy problem: find $W : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\begin{cases} \partial_t W + \sum_{j=1}^n B_j \partial_j W = 0 & \text{in }]0,T] \times \mathbb{R}^n, \\ W(t,x) \in K & \text{if } (t,x) \in [0,T] \times \mathbb{R}^n, \\ W(0,x) = W^0(x) & \text{if } x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where K is a fixed (*i.e.* independent of the time and space variables) non empty closed and convex subset of $\mathbb{R}^m$ containing 0 in its interior, the matrices B_j are $m \times m$ symmetric matrices independent of time and space, and $T > 0$. The main difficulty is due to the constraints which introduce nonlinear effects to the linear Friedrichs' system [5]. This type of hyperbolic problems has been introduced in [3] where a notion of weak solutions to problem (1) has been defined.

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Definition 1.1. Let $W^0 \in L^2(\mathbb{R}^n, K)$, and $T > 0$. A function W is a weak constrained solution of (1) if $W \in L^2([0,T] \times \mathbb{R}^n, K)$ satisfies

$$\int_0^T \int_{\mathbb{R}^n} \Big(|W-\kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle W-\kappa; B_j(W-\kappa)\rangle \, \partial_j \phi \Big) \, dt \, dx + \int_{\mathbb{R}^n} |W^0(x)-\kappa|^2 \phi(0,x) \, dx \geq 0, \quad (2)$$

for all $\kappa \in K$ and $\phi \in \mathcal{C}_c^\infty([0,T[\times\mathbb{R}^n)$ with $\phi(t,x) \geq 0$ for all $(t,x) \in [0,T[\times\mathbb{R}^n$.

We recall here the main result of [3].

Theorem 1.2. *Assume that $W^0 \in L^2(\mathbb{R}^n, K)$. There exists a unique weak constrained solution $W \in L^2([0,T]\times\mathbb{R}^n, K)$ to (1) in the sense of Definition 1.1. In addition, this solution belongs to $\mathcal{C}([0,T], L^2(\mathbb{R}^n, K))$, and if further $W^0 \in H^1(\mathbb{R}^n, K)$, then $W \in L^\infty([0,T], H^1(\mathbb{R}^n, K))$.*

As already mentioned, the well-posedness of problem (1) has been established in [3] thanks to a numerical method. We relax here the constraints $W(t,x) \in K$ for a.e. $(t,x) \in]0,T[\times\mathbb{R}^n$ as

$$\partial_t W_\epsilon + \sum_{j=1}^n B_j \partial_{x_j} W_\epsilon = \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}, \quad (3)$$

where P_K denotes the projection onto the closed convex set K, and $\epsilon > 0$ is a small parameter. Formally, if we multiply equation (3) by ϵ, and let ϵ tend to 0, we get that the "limit" of W_ϵ, denoted by W, satisfies $P_K(W) = W$, which ensures that $W \in K$. In addition, Definition 1.1 has been motivated in [3] by a formal derivation from the relaxation system (3). To see it, it suffices to take the scalar product of equation of (3) with $W_\epsilon - \kappa$, where $\kappa \in K$ is arbitrary. We then use the first order characterization of the projection which ensures that the right hand side is non-positive, to get the inequality of Definition 1.1. The purpose of this work is to rigorously justify these formal steps.

The relaxation model presented here is very similar to viscous approximation of constrained models found in mechanics, and especially in plasticity. We start from the system of dynamical linear elasticity in three space dimensions which can be written as

$$\begin{cases} \partial_t F + \nabla_x v &= 0, \\ \partial_t v + \operatorname{div} \sigma &= 0, \end{cases} \quad (4)$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^3$. In the previous system, $F(t,x)$ is a 3×3 matrix which stands for the displacement gradient, $v(t,x) \in \mathbb{R}^3$ is the velocity, *i.e.* the displacement time derivative, and $\sigma = \mu\left(F + F^T\right) + \lambda\,(\mathrm{tr}\ F)\, I_3$ is the symmetric Cauchy stress tensor (here λ and μ are *the Lamé coefficients*, and I_3 is the identity matrix in $\mathbb{R}^3$). This system can be rewritten (thanks to a change of variables - see [7]) in the Friedrichs' framework as

$$\partial_t U + A_1 \partial_{x_1} U + A_2 \partial_{x_2} U + A_3 \partial_{x_3} U = 0,$$

where U is a vector in $\mathbb{R}^9$ (containing the three components of the velocity v, and the six components of the stress σ) and A_1, A_2 and A_3 are symmetric matrices.

We now introduce the convex constraint coming from plasticity, see [11]. Indeed, the theory of perfect plasticity is characterized by the fact that the stress tensor σ is constrained to stay inside a fixed closed convex set K of symmetric 3×3 matrices.

The total strain is then additively decomposed as the sum of (i) the elastic strain, denoted by e, which is still related to the stress by the linear relation $\sigma = \lambda \mathrm{tr} e + 2\mu e$; (ii) and the plastic strain, denoted by p, whose rate is oriented in a normal direction to K at σ. Summarizing, one has

$$\frac{F+F^T}{2} = e + p,\ \sigma = \lambda \mathrm{tr} e + 2\mu e \in K,\ \partial_t p \in \partial I_K(\sigma), \tag{5}$$

where $\partial I_K(\sigma)$ denotes the subdifferential of I_K, the indicator function of K, at the point σ. Using Fenchel-Moreau regularization of I_K (see [8]), the last condition in (5) can be relaxed as

$$\partial_t p = \frac{1}{\epsilon}\left(P_K(\sigma) - \sigma\right),$$

where $\epsilon > 0$ is a viscosity parameter. We can now reformulate, at least formally, the dynamical problem of visco-plasticity (see [9, 11]) as

$$\begin{cases} \partial_t U + A_1 \partial_{x_1} U + A_2 \partial_{x_2} U + A_3 \partial_{x_3} U = \dfrac{P_{\tilde{K}}(U) - U}{\epsilon} & \text{on }]0,T] \times \mathbb{R}^3, \\ U(t,x) \in \tilde{K} & \text{if } (t,x) \in [0,T] \times \mathbb{R}^3, \\ U(0,x) = U^0(x) & \text{if } x \in \mathbb{R}^3, \end{cases} \tag{6}$$

where, again, $U \in \mathbb{R}^9$, and $\tilde{K} = \left\{ u \in \mathbb{R}^9 : \sigma \in K \right\}$, and A_1, A_2 and A_3 are the same matrices than in the elasto-dynamic case. As ϵ tends to zero, one expects the solution to (6) to converge to that of the model of perfect plasticity (see [12] in the quasistatic case).

Notation. In the sequel, we denote by $\langle \, | \, \rangle$ the scalar product of $L^2(\mathbb{R}^n, \mathbb{R}^m)$ and by $\langle ; \rangle$ the canonical scalar product of $\mathbb{R}^m$ (and $|.|$ the associated norm). Also, to shorten notation, we write $L^2_{t,x}$ (resp. $H^1_{t,x}$) instead of $L^2(0,T; L^2(\mathbb{R}^n, \mathbb{R}^m))$ (resp. $H^1((0,T) \times \mathbb{R}^n, \mathbb{R}^m)$), L^2_x instead of $L^2(\mathbb{R}^n, \mathbb{R}^m)$.

This paper is organized as follows. In the first section, we establish the existence and uniqueness of the relaxation model thanks to a parabolic approximation. In the second section, we prove that the relaxed solution W_ϵ to (3) satisfies the inequalities of Definition 1.1. Finally, to get the existence of a solution as a limit when ϵ tends to zero of relaxed solutions W_ϵ, we prove the strong convergence of the sequence $(W_\epsilon)_{\epsilon>0}$ in the space $L^2((0,T) \times \omega)$, where ω is a open bounded subset of $\mathbb{R}^n$, to a weak solution to the constrained Friedrichs' systems.

2. Parabolic approximation.

In order to find a solution to the relaxation problem (3), we use a parabolic type regularization. To this aim, we consider a classical sequence of mollifiers in $\mathbb{R}^n$, denoted by $(\rho_\eta)_{\eta>0}$.

Theorem 2.1. *Let $W^0 \in H^1(\mathbb{R}^n, K)$. For every $\epsilon > 0$ and $\eta > 0$, the system*

$$\begin{cases} \partial_t W_{\epsilon,\eta} - \eta \Delta W_{\epsilon,\eta} + \displaystyle\sum_{j=1}^n B_j \partial_j W_{\epsilon,\eta} = \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}, & \textit{on }]0,T] \times \mathbb{R}^n, \\ W_{\epsilon,\eta}(0,x) = W^0(x) * \rho_\eta, & \textit{if } x \in \mathbb{R}^n, \end{cases} \tag{7}$$

admits a unique solution $W_{\epsilon,\eta}$ with the following properties:

$$W_{\epsilon,\eta} \in L^2(0,T; H^2(\mathbb{R}^n, \mathbb{R}^m)), \quad \partial_t W_{\epsilon,\eta} \in L^2(0,T; H^1(\mathbb{R}^n, \mathbb{R}^m)),$$

and

$$\partial_{tt} W_{\epsilon,\eta} \in L^2(0,T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m)).$$

Furthermore, we have the following estimates

$$\sup_{0\le t\le T} \|W_{\epsilon,\eta}(t)\|^2_{L^2_x} \le \left\|W^0\right\|^2_{L^2_x}, \tag{8}$$

$$\sup_{0\le t\le T} \|W_{\epsilon,\eta}(t)\|_{H^1_x} \le C_\epsilon \left\|W^0\right\|_{H^1_x}, \tag{9}$$

$$\sup_{0\le t\le T} \|\partial_t W_{\epsilon,\eta}(t)\|_{L^2_x} \le C_\epsilon \left\|W^0\right\|_{H^1_x}, \tag{10}$$

for some constant $C_\epsilon > 0$ independent of η.

Proof. The proof essentially follows that of Theorem 1, Part II, Section 7.3.2 in [4]. Let $X = L^\infty(0,T;H^1(\mathbb{R}^n,\mathbb{R}^m))$ and $V \in X$. We consider the problem

$$\begin{cases} \partial_t U - \eta\Delta U = \dfrac{P_K(V)-V}{\epsilon} - \displaystyle\sum_{j=1}^n B_j\partial_j V, & \text{on }]0,T]\times\mathbb{R}^n, \\ U(0,x) = W^0(x) * \rho_\eta, & \text{if } x\in\mathbb{R}^n. \end{cases} \tag{11}$$

Since $0\in K$, we have the following inequality

$$\forall k\in\mathbb{R}^m, \qquad |(P_K(k)-k)| \le |k|$$

which shows that $\frac{P_K(V)-V}{\epsilon} - \sum_{j=1}^n B_j\partial_j V \in L^2(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$. Using the theory of parabolic equations, we get that equation (11) admits a unique solution U with $U\in L^2(0,T;H^2(\mathbb{R}^n,\mathbb{R}^m))$ and $\partial_t U \in L^2(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$. Let $\tilde V \in X$ and $\tilde U$ be the solution to (11) associated with $\tilde V$. The function $\hat U = U - \tilde U$ is a solution to

$$\begin{cases} \partial_t \hat U - \eta\Delta \hat U = \dfrac{P_K(V)-P_K(\tilde V)-(V-\tilde V)}{\epsilon} - \displaystyle\sum_{j=1}^n B_j\partial_j (V-\tilde V), & \text{on }]0,T]\times\mathbb{R}^n, \\ \hat U(0,x) = 0, & \text{if } x\in\mathbb{R}^n. \end{cases} \tag{12}$$

Thanks to the theory of parabolic equations, we have the following estimate,

$$\begin{aligned} &\operatorname*{ess\text{-}sup}_{0\le t\le T} \left\|\hat U(t)\right\|_{H^1_x} \\ &\quad \le C(\eta)\left\| \frac{P_K(V)-P_K(\tilde V)-(V-\tilde V)}{\epsilon} - \sum_{j=1}^n B_j\partial_j(V-\tilde V)\right\|_{L^2_{t,x}} \\ &\quad \le C(\eta)\max\left(\frac{2}{\epsilon}, \|B_j\|\right)\left\|V-\tilde V\right\|_{L^2_t H^1_x} \\ &\quad \le C(\eta)\max\left(\frac{2}{\epsilon}, \|B_j\|\right)\sqrt{T}\operatorname*{ess\text{-}sup}_{0\le t\le T}\left\|V(t)-\tilde V(t)\right\|_{H^1_x}. \end{aligned}$$

Therefore, the mapping

$$\psi : \begin{cases} X \to \left\{U\in L^2_t H^2_x \text{ and } \partial_t U \in L^2_{t,x}\right\} \subset X \\ \hat V \mapsto \hat U \end{cases}$$

is Lipschitz-continuous with Lipschitz constant bounded by $C(\eta)\max\left(\frac{2}{\epsilon},\|B_j\|\right)\sqrt{T}$.

We now divide $[0,T]$ into sub-intervals $[0,T_1],[T_1,2T_1],[2T_1,3T_1],\dots,[NT_1,T]$ such that

$$C(\eta)\max\left(\frac{2}{\epsilon},\|B_j\|\right)\sqrt{\max(T_1, T-NT_1)} < 1.$$

For every $i \in \{0, \dots N-1\}$, the Banach fixed-point Theorem ensures the existence and uniqueness of a solution to (7) on the interval $[iT_1, (i+1)T_1]$ (with initial condition $W^0 * \rho_\eta$ if $i = 0$, and $W_{\epsilon,\eta}(iT_1)$ if $i \geq 1$ obtained at the previous step). We then obtain a solution $W_{\epsilon,\eta}$ on the entire interval $[0,T]$ by gluing the solutions on each sub-intervals, so that $W_{\epsilon,\eta} \in L^2(0,T; H^2(\mathbb{R}^n, \mathbb{R}^m))$. According to the initial condition on each sub-intervals, the function $t \mapsto W_{\epsilon,\eta}(t)$ is continuous in L^2_x at every $t = iT_1$, so that $\partial_t W_{\epsilon,\eta} \in L^2(0,T; L^2(\mathbb{R}^n, \mathbb{R}^m))$, and, in particular $W_{\epsilon,\eta} \in H^1(]0,T[\times\mathbb{R}^n, \mathbb{R}^m)$.

To obtain the announced regularity, we use the following result (whose proof relies on the chain rule in Sobolev spaces).

Lemma 2.2. *Let $U \in H^1(]0,T[\times\mathbb{R}^n, \mathbb{R}^m)$. Then the function $P_K \circ U - U$ also belongs to $H^1(]0,T[\times\mathbb{R}^n, \mathbb{R}^m)$, and there exists a constant $M > 0$, independent of U, such that*

$$\|P_K \circ U - U\|_{H^1_{t,x}} \leq M \|U\|_{H^1_{t,x}}.$$

This result ensures that $\frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon} \in H^1(]0,T[\times\mathbb{R}^n, \mathbb{R}^m)$ and then, using again to the regularity theory of parabolic equations, we obtain that

$$W_{\epsilon,\eta} \in L^2(0,T; H^2(\mathbb{R}^n, \mathbb{R}^m)), \qquad \partial_t W_{\epsilon,\eta} \in L^2(0,T; H^1(\mathbb{R}^n, \mathbb{R}^m)),$$

and

$$\partial_{tt} W_{\epsilon,\eta} \in L^2(0,T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m)).$$

Now we derive the estimates. We are going to use the following result (see [4]).

Lemma 2.3. *Let $U \in L^2(0,T, H^1(\mathbb{R}^n, \mathbb{R}^m))$ with $\partial_t U \in L^2(0,T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m))$. Then, the function*

$$t \mapsto \|U(t)\|^2_{L^2_x},$$

is absolutely continuous, and for a.e. $t \in [0,T]$,

$$\frac{d}{dt}\left(\frac{1}{2}\|U(t)\|^2_{L^2_x}\right) = \langle U(t), \partial_t U(t)\rangle_{H^1_x, H^{-1}_x}.$$

Applying this result to $W_{\epsilon,\eta}$, we get that for a.e. $t \in [0,T]$,

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|W_{\epsilon,\eta}(t)\|^2_{L^2_x}\right) &= \langle W_{\epsilon,\eta}(t) \,|\, \partial_t W_{\epsilon,\eta}(t)\rangle_{L^2_x} \\ &= \left\langle W_{\epsilon,\eta}(t) \,\middle|\, \frac{P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)}{\epsilon} - \sum_{j=1}^n B_j \partial_j W_{\epsilon,\eta}(t) + \eta \Delta W_{\epsilon,\eta}(t)\right\rangle, \quad (13)\end{aligned}$$

where we used the fact that $W_{\epsilon,\eta}$ is a solution to the partial differential equation (7). Since $0 \in K$, we have

$$\begin{aligned}&\langle W_{\epsilon,\eta}(t) \,|\, P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)\rangle_{L^2_x} \\ &= \langle P_K(W_{\epsilon,\eta})(t) \,|\, P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)\rangle - \|P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)\|^2_{L^2_x} \leq 0. \quad (14)\end{aligned}$$

On the other hand, if $v \in \mathcal{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^m)$, an integration by parts shows that

$$\langle v \,|\, \eta \Delta v\rangle_{L^2} = -\eta \|Dv\|^2_{L^2_x} \leq 0,$$

and

$$\left\langle v \,\middle|\, \sum_{j=1}^n B_j \partial_j v\right\rangle = 0,$$

since the matrices B_j are symmetric and independent of the space variables. By approximation, these formulas are true for $v \in H^1(\mathbb{R}^n, \mathbb{R}^m)$ as well, and in particular,

$$\left\langle W_{\epsilon,\eta}(t) \,\middle|\, \sum_{j=1}^{n} B_j \partial_j W_{\epsilon,\eta}(t) + \eta \Delta W_{\epsilon,\eta}(t) \right\rangle \leq 0. \tag{15}$$

Gathering (13), (14) and (15), we obtain that

$$\frac{d}{dt}\left(\frac{1}{2}\left\|W_{\epsilon,\eta}\right\|_{L_x^2}^2\right) \leq 0,$$

and using Gronwall's Lemma, we derive the first estimate (8).

$$\sup_{0\leq t\leq T} \left\|W_{\epsilon,\eta}(t)\right\|_{L_x^2}^2 = \left\|W^0 * \rho_\eta\right\|_{L_x^2}^2 \leq \left\|W^0\right\|_{L_x^2}^2. \tag{16}$$

We apply the same argument to the spatial weak derivatives $\partial_k W_{\epsilon,\eta}$ of $W_{\epsilon,\eta}$. Deriving the partial differential equation (7) in the sense of distribution, we infer that

$$\partial_t \partial_k W_{\epsilon,\eta} - \eta \Delta \partial_k W_{\epsilon,\eta} + \sum_{j=1}^{n} B_j \partial_j \partial_k W_{\epsilon,\eta} = \partial_k \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}.$$

The previous equality actually holds in $L^2(0,T,L^2(\mathbb{R}^n,\mathbb{R}^m))$ thanks to the regularity of $W_{\epsilon,\eta}$, and we can apply Lemma 2.3 to obtain that Êfor a.e. $t \in [0,T]$,

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\partial_k W_{\epsilon,\eta}(t)\right\|_{L_x^2}^2\right) = \langle \partial_k W_{\epsilon,\eta}(t), \partial_t \partial_k W_{\epsilon,\eta}(t)\rangle_{L_x^2}$$

$$= \left\langle \partial_k W_{\epsilon,\eta}(t) \,\middle|\, \partial_k \frac{P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)}{\epsilon} - \sum_{j=1}^{n} B_j \partial_j \partial_k W_{\epsilon,\eta}(t) + \eta \Delta \partial_k W_{\epsilon,\eta}(t) \right\rangle.$$

Arguing as in (15), we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\left\|\partial_k W_{\epsilon,\eta}(t)\right\|_{L_x^2}^2\right) &\leq \left\langle \partial_k W_{\epsilon,\eta}(t) \,\middle|\, \partial_k \frac{P_K(W_{\epsilon,\eta}(t)) - W_{\epsilon,\eta}(t)}{\epsilon} \right\rangle \\ &\leq \frac{M}{\epsilon}\left\|\partial_k W_{\epsilon,\eta}(t)\right\|_{L_x^2}^2,\end{aligned}$$

where we used Lemma 2.2. Using again Gronwall's Lemma, it yields

$$\begin{aligned}\sup_{0\leq t\leq T}\left\|\partial_k W_{\epsilon,\eta}(t)\right\|_{L_x^2}^2 &\leq \exp\left(\frac{2TM}{\epsilon}\right)\left\|\partial_k W^0 * \rho_\eta\right\|_{L_x^2}^2 \\ &\leq \exp\left(\frac{2TM}{\epsilon}\right)\left\|\partial_k W^0\right\|_{L_x^2}^2,\end{aligned} \tag{17}$$

which completes the proof of estimate (9).

We finally derive the last estimate for $\partial_t W_{\epsilon,\eta}$. Again, we derive the partial differential equation (7) with respect to t in the distributional sense to get

$$\partial_{tt} W_{\epsilon,\eta} - \eta \Delta \partial_t W_{\epsilon,\eta} + \sum_{j=1}^{n} B_j \partial_j \partial_t W_{\epsilon,\eta} = \partial_t \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}$$

in $L^2(0,T,H^{-1}(\mathbb{R}^n,\mathbb{R}^m))$. Using again Lemma 2.3, we get that for a.e. $t\in[0,T]$,

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\partial_t W_{\epsilon,\eta}(t)\right\|^2_{L^2_x}\right)=\langle\partial_t W_{\epsilon,\eta}(t)\,|\,\partial_{tt}W_{\epsilon,\eta}(t)\rangle_{H^1_x,H^{-1}_x}$$
$$=\langle\partial_t W_{\epsilon,\eta}(t)|\eta\Delta\partial_t W_{\epsilon,\eta}(t)-\sum_{j=1}^n B_j\partial_j\partial_t W_{\epsilon,\eta}(t)+\partial_t\frac{P_K(W_{\epsilon,\eta})(t)-W_{\epsilon,\eta}(t)}{\epsilon}\rangle_{H^1_x,H^{-1}_x}.$$

As before, we infer that

$$\left\langle\partial_t W_{\epsilon,\eta}(t)\,\middle|\,\sum_{j=1}^n B_j\partial_j\partial_t W_{\epsilon,\eta}(t)\right\rangle_{L^2_x}=0,\quad\langle\partial_t W_{\epsilon,\eta}(t)\,|\,\eta\Delta\partial_t W_{\epsilon,\eta}(t)\rangle_{H^1_x,H^{-1}_x}\le 0,$$

which shows, thanks to Lemma 2.2, that

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\partial_t W_{\epsilon,\eta}(t)\right\|^2_{L^2_x}\right)\le\frac{1}{\epsilon}\left\|\partial_t W_{\epsilon,\eta}(t)\right\|_{L^2_x}\left\|\partial_t(P_K(W_{\epsilon,\eta})(t)-W_{\epsilon,\eta}(t))\right\|_{L^2_x}$$
$$\le\frac{M}{\epsilon}\left\|\partial_t W_{\epsilon,\eta}(t)\right\|^2_{L^2_x}.$$

At this point, we would like to use Gronwall's Lemma. To do that, we need to know the value of $\partial_t W_{\epsilon,\eta}$ at $t=0$. To this aim, let us take a test function $\phi\in\mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$ with $\phi(T,\cdot)=0$. On the one hand, according to Fubini's Theorem and Green's formula on $[0,T]$, we have

$$\int_0^T\int_{\mathbb{R}^n}\langle\partial_t W_{\epsilon,\eta};\partial_t\phi\rangle=\int_{\mathbb{R}^n}\left[-\langle\partial_t W_{\epsilon,\eta}(0,\cdot);\phi(0,\cdot)\rangle-\int_0^T\langle\partial_{tt}W_{\epsilon,\eta};\phi\rangle\right],$$

since $\partial_t W_{\epsilon,\eta}\in H^1(0,T;H^{-1}(\mathbb{R}^n,\mathbb{R}^m))$ and ϕ is smooth. On the other hand, according to equation (7)

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}\langle\partial_t W_{\epsilon,\eta};\partial_t\phi\rangle\\&\qquad=\int_0^T\int_{\mathbb{R}^n}\left\langle\eta\Delta W_{\epsilon,\eta}-\sum_{j=1}^n B_j\partial_j W_{\epsilon,\eta}+\frac{P_K(W_{\epsilon,\eta})-W_{\epsilon,\eta}}{\epsilon};\partial_t\phi\right\rangle\\&\quad=\int_{\mathbb{R}^n}\left\{-\int_0^T\left\langle\partial_t\left[\eta\Delta W_{\epsilon,\eta}-\sum_{j=1}^n B_j\partial_j W_{\epsilon,\eta}+\frac{P_K(W_{\epsilon,\eta})-W_{\epsilon,\eta}}{\epsilon}\right];\phi\right\rangle\right.\\&\quad\left.-\left\langle\eta\Delta W^0*\rho_\eta-\sum_{j=1}^n B_j\partial_j W^0*\rho_\eta+\frac{P_K(W^0*\rho_\eta)-W^0*\rho_\eta}{\epsilon};\phi(0,\cdot)\right\rangle\right\}.\end{aligned}$$

Since $\phi(0,\cdot)$ is arbitrary, we obtain that

$$\partial_t W_{\epsilon,\eta}(0,\cdot)=\eta\Delta W^0*\rho_\eta-\sum_{j=1}^n B_j\partial_j W^0*\rho_\eta+\frac{P_K(W^0*\rho_\eta)-W^0*\rho_\eta}{\epsilon}.$$

We are now in position to apply Gronwall's Lemma which implies that

$$\sup_{0\le t\le T}\left\|\partial_t W_{\epsilon,\eta}(t)\right\|^2_{L^2_x}\le\exp\left(\frac{2TM}{\epsilon}\right)\left\|\partial_t W_{\epsilon,\eta}(0,\cdot)\right\|^2_{L^2_x}.$$

Since,

$$\left\|\Delta W^0 * \rho_\eta\right\|_{L^2_x} \leq \frac{C}{\eta} \left\|\nabla W^0\right\|_{L^2_x},$$

for some constant $C > 0$ independent of η, we deduce that

$$\sup_{0\leq t\leq T} \|\partial_t W_{\epsilon,\eta}(t)\|_{L^2_x} \leq C_\epsilon \left\|W^0\right\|_{H^1_x},$$

where $C_\epsilon > 0$ is another constant independent of η, which completes the proof of the last estimate (10). $\square$

3. **Approximation of the convex constraints.** We now consider the relaxation problem

$$\begin{cases} \partial_t W_\epsilon + \displaystyle\sum_{j=1}^n B_j \partial_j W_\epsilon = \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}, & \text{on }]0,T] \times \mathbb{R}^n, \\ W_\epsilon(0,x) = W^0(x), & \text{if } x \in \mathbb{R}^n. \end{cases} \tag{18}$$

Thanks to Theorem 2.1, we will construct the solution to the previous problem as the limit of the solution of the parabolic problem (7) when η tends to zero.

Theorem 3.1. *There exists a unique solution $W_\epsilon \in H^1(]0,T[\times\mathbb{R}^n,\mathbb{R}^m))$ to (18) satisfying, for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$,*

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t W_\epsilon + \sum_{j=1}^n B_j \partial_j W_\epsilon ; \phi \right\rangle dx\, dt = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon} ; \phi \right\rangle dx\, dt, \tag{19}$$

and $W_\epsilon(0,\cdot) = W_0$ in $L^2(\mathbb{R}^n,\mathbb{R}^m)$. In addition,

$$\sup_{0\leq t\leq T} \|W_\epsilon(t)\|^2_{L^2_x} \leq \left\|W^0\right\|^2_{L^2_x}. \tag{20}$$

Proof. Thanks to Theorem 2.1, the sequence $(W_{\epsilon,\eta})_{\eta>0}$ is bounded in the space $H^1(]0,T[\times\mathbb{R}^n,\mathbb{R}^m)$. We can thus extract a subsequence (not relabeled) such that

$$W_{\epsilon,\eta} \rightharpoonup W_\epsilon \text{ weakly in } H^1(]0,T[\times\mathbb{R}^n,\mathbb{R}^m).$$

In particular, (20) is a consequence of (8) by the lower semicontinuity of the norm with respect to weak convergence. Since the embedding of $H^1(]0,T[\times\mathbb{R}^n,\mathbb{R}^m)$ into $L^2_{\text{loc}}([0,T]\times\mathbb{R}^n,\mathbb{R}^m)$ is compact (cf [1]), we deduce that

$$W_{\epsilon,\eta} \to W_\epsilon \text{ strongly in } L^2(]0,T[\times]-R,R[^n,\mathbb{R}^m),$$

for each $R > 0$. Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$ and R such that the support of ϕ is contained in $[-R,R]^{n+1}$. Since $W_{\epsilon,\eta}$ is a weak solution of (11), we have

$$\int_0^T \int_{\mathbb{R}^n} \Big(\langle \partial_t W_{\epsilon,\eta};\phi\rangle + \eta \sum_{j=1}^n \langle \partial_j W_{\epsilon,\eta};\partial_j\phi\rangle + \sum_{j=1}^n \langle B_j\partial_j W_{\epsilon,\eta};\phi\rangle \Big)\, dx\, dt$$
$$= \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon};\phi \right\rangle dx\, dt.$$

Using the weak convergence, we infer that

$$\int_0^T \int_{\mathbb{R}^n} \Big(\langle \partial_t W_{\epsilon,\eta};\phi\rangle + \eta \sum_{j=1}^n \langle \partial_j W_{\epsilon,\eta};\partial_j\phi\rangle + \sum_{j=1}^n \langle B_j\partial_j W_{\epsilon,\eta};\phi\rangle \Big)\, dx\, dt$$
$$\to \int_0^T \int_{\mathbb{R}^n} \Big(\langle \partial_t W_\epsilon;\phi\rangle + \sum_{j=1}^n \langle B_j\partial_j W_\epsilon;\phi\rangle \Big)\, dx\, dt.$$

On the other hand, the strong convergence yields

$$\left\langle \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}; \phi \right\rangle \to \left\langle \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}; \phi \right\rangle \text{ strongly in } L^1(]0,T[\times\mathbb{R}^n),$$

and consequently, we obtain that

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t W_\epsilon + \sum_{j=1}^n B_j \partial_j W_\epsilon; \phi \right\rangle dx\, dt = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}; \phi \right\rangle dx\, dt. \quad (21)$$

We next focus on the initial condition. We take $\phi \in \mathcal{C}_c^\infty(]-\infty,T[\times\mathbb{R}^n,\mathbb{R}^m)$ (in particular $\phi(T)=0$). An integration by parts shows that

$$-\int_0^T \int_{\mathbb{R}^n} \langle W_{\epsilon,\eta}; \partial_t \phi \rangle \, dx\, dt + \int_{\mathbb{R}^n} \left\langle W^0 * \rho_\eta(x); \phi(0,x) \right\rangle dx = \int_0^T \int_{\mathbb{R}^n} \langle \partial_t W_{\epsilon,\eta}; \phi \rangle \, dx\, dt.$$

Letting η tend to zero, and using (21) leads to

$$\int_0^T \int_{\mathbb{R}^n} \Big(-\langle W_\epsilon; \partial_t \phi\rangle + \sum_{j=1}^n \langle B_j \partial_j W_\epsilon; \phi\rangle \Big) dx\, dt + \int_{\mathbb{R}^n} \left\langle W^0(x); \phi(0,x) \right\rangle dx = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}; \phi \right\rangle dx\, dt, \quad (22)$$

since $W^0 * \rho_\eta \to W^0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^n,\mathbb{R}^m)$. We now integrate by parts in (21), using the fact that $W_\epsilon \in H^1(0,T,L^2(\mathbb{R}^n,\mathbb{R}^m))$,

$$\int_0^T \int_{\mathbb{R}^n} \Big(\langle W_\epsilon; \partial_t \phi\rangle + \sum_{j=1}^n \langle B_j \partial_j W_\epsilon; \phi\rangle \Big) dx\, dt + \int_{\mathbb{R}^n} \langle W_\epsilon(0,x); \phi(0,x) \rangle \, dx = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}; \phi \right\rangle dx\, dt. \quad (23)$$

Using the equations (22) and (23), it gives that

$$\int_{\mathbb{R}^n} \langle W_\epsilon(0,x); \phi(0,x) \rangle \, dx = \int_{\mathbb{R}^n} \left\langle W^0(x); \phi(0,x) \right\rangle dx,$$

and consequently the initial condition is satisfied in $L^2(\mathbb{R}^n,\mathbb{R}^m)$.

It remains to show the uniqueness. Let us consider two solutions W_ϵ and $\tilde{W}_\epsilon$ associated with the same initial condition W^0. Using the partial differential equation (18), we obtain that

$$\partial_t \left(W_\epsilon - \tilde{W}_\epsilon \right) + \sum_{j=1}^n B_j \partial_j \left(W_\epsilon - \tilde{W}_\epsilon \right) = \frac{P_K(W_\epsilon) - W_\epsilon - P_K(\tilde{W}_\epsilon) + \tilde{W}^\epsilon}{\epsilon}.$$

As already observed, we know that for a.e. $t \in [0,T]$,

$$\int_{\mathbb{R}^n} \sum_{j=1}^n \left\langle B_j \partial_j W_\epsilon(t) - \tilde{W}_\epsilon(t); W_\epsilon(t) - \tilde{W}_\epsilon(t) \right\rangle dx = 0,$$

and also

$$\frac{1}{2} \frac{d}{dt} \left\| W_\epsilon(t) - \tilde{W}_\epsilon(t) \right\|_{L^2_x}^2 = \left\langle \partial_t W_\epsilon(t) - \partial_t \tilde{W}_\epsilon(t) \,\middle|\, W_\epsilon(t) - \tilde{W}_\epsilon(t) \right\rangle.$$

Consequently, we have that

$$\frac{1}{2}\frac{d}{dt}\left\|W_\epsilon(t)-\tilde{W}_\epsilon(t)\right\|^2_{L^2_x} = \left\langle \frac{P_K(W_\epsilon)(t)-W_\epsilon(t)-P_K(\tilde{W}_\epsilon)(t)+\tilde{W}_\epsilon(t)}{\epsilon}\,\middle|\, W_\epsilon(t)-\tilde{W}_\epsilon(t)\right\rangle \leq \frac{2}{\epsilon}\left\|W_\epsilon(t)-\tilde{W}_\epsilon(t)\right\|^2_{L^2_x},$$

since the projection is 1-Lipschitz. Gronwall's Lemma thus implies that $W_\epsilon = \tilde{W}_\epsilon$ since they satisfy the same initial condition. As a consequence of the uniqueness, we deduce that there is no need to extract a subsequence from $(W_{\epsilon,\eta})_{\eta>0}$ to get the convergences as $\eta \to 0$. □

4. **Convergence of the relaxed formulation.** In this section, we first show that the solution W_ϵ to the relaxation problem (18) satisfies the inequality of Definition 1.1, and then we prove that we can pass to the limit in this inequality to get a solution to the initial problem (1).

Lemma 4.1. *Let W_ϵ be the unique solution to (18). For all $\kappa \in K$ and for all $\phi \in W^{1,\infty}((-\infty,T)\times\mathbb{R}^n,\mathbb{R}^+)$ with compact support in $]-\infty,T[\times\mathbb{R}^n$, one has*

$$\int_0^T\int_{\mathbb{R}^n}\Big(|W_\epsilon-\kappa|^2\partial_t\phi+\sum_{j=1}^n\langle W_\epsilon-\kappa;B_j(W_\epsilon-\kappa)\rangle\,\partial_j\phi\Big)dt\,dx + \int_{\mathbb{R}^n}|W^0(x)-\kappa|^2\phi(0,x)\,dx \geq 0. \quad (24)$$

Proof. Since W_ϵ is a solution to (18), we know that

$$\int_0^T\int_{\mathbb{R}^n}\left\langle \partial_t W_\epsilon+\sum_{j=1}^n B_j\partial_j W_\epsilon-\frac{P_K(W_\epsilon)-W_\epsilon}{\epsilon};W_\epsilon-\kappa\right\rangle\phi dx\,dt=0.$$

By the first order characterization of the projection, one has

$$\left\langle\frac{P_K(W_\epsilon)-W_\epsilon}{\epsilon};W_\epsilon-\kappa\right\rangle(t,x)\leq 0$$

for a.e. $(t,x)\in[0,T]\times\mathbb{R}^n$. On the other hand, since $W_\epsilon\in H^1(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$ and $W_\epsilon\in L^2(0,T;H^1(\mathbb{R}^n,\mathbb{R}^m)))$ we can integrate by parts to obtain the desired result. □

Remark 1. Let us stress that, although the function W_ϵ satisfies the same inequality than the weak constrained solution, it is not a weak constrained solution in the sense of Definition 1.1 because it does not *a priori* belong to K.

To get a weak constrained solution from the sequence of solutions $(W_\epsilon)_{\epsilon>0}$ to the relaxation problem (18), we need to pass to the limit as $\epsilon\to 0$ in the previous inequality. This is the purpose of the following result.

Theorem 4.2. *For every bounded open set $\omega\subset\mathbb{R}^n$, the sequence $(W_\epsilon)_{\epsilon>0}$ converges strongly in $L^2((0,T)\times\omega,\mathbb{R}^m)$ to some function W which is a weak constrained solution to problem (1).*

Proof. Let ω be an open bounded subset of $\mathbb{R}^n$. We are going to prove the existence of a subsequence of $(W_\epsilon)_{\epsilon>0}$ (associated with the same initial data W^0) which converges in $L^2(0,T,L^2(\omega,\mathbb{R}^m))$. We will use the following compactness criterion (see [10]).

Theorem 4.3. *Let B be a Banach space. A subset F of $L^2(0,T;B)$ is relatively compact if and only if both conditions are fulfilled:*

- *the set $\left\{\int_{t_1}^{t_2} f(t)\,dt : f \in F\right\}$ is relatively compact in B for all $0<t_1<t_2<T$;*
- *we have*
$$\sup_{f\in F} \|\tau_h f - f\|_{L^2(0,T-h;B)} \underset{h\to 0}{\to} 0,$$
where $\tau_h f : (t,x) \mapsto \tau_h f(t,x) := f(t+h,x)$.

We are going to apply this result to $F = (W_{\epsilon|\omega})_\epsilon$ where $W_{\epsilon|\omega}$ is the restriction to $[0,T]\times\omega$ of W_ϵ. We first show that the set $\mathcal{F} = \left\{\int_{t_1}^{t_2} W_{\epsilon|\omega}(t,\cdot)\,dt : \epsilon>0\right\}$ is relatively compact in $L^2(\omega,\mathbb{R}^m)$ for all $0<t_1<t_2<T$. We first observe that $\mathcal{F}$ is bounded in $L^2(\omega,\mathbb{R}^m)$ by $(t_2-t_1)\left\|W^0\right\|_{L^2_x}$ thanks to estimate (20). To show that $\mathcal{F}$ is relatively compact in $L^2(\omega,\mathbb{R}^m)$, it is enough to check the validity of the Riesz-Fréchet-Kolmogorov compactness criterion (see [2] remark 13 page 74), *i.e.*,

$$\lim_{h\to 0}\sup_{\epsilon>0}\left\|\int_{t_1}^{t_2} (W_\epsilon(t,x+h) - W_\epsilon(t,x))\,dt\right\|_{L^2_x} = 0. \tag{25}$$

Note that $W_{\epsilon,h} := W_\epsilon(\cdot,\cdot+h)$ is a solution to the problem (18) associated with the initial condition $W^0(\cdot+h)$. Consequently, since the projection is 1-Lipschitz, we have for all $\phi \in W^{1,\infty}((-\infty,T)\times\mathbb{R}^n,\mathbb{R}^+)$ with compact support in $]-\infty,T[\times\mathbb{R}^n$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}&\left\langle \partial_t(W_{\epsilon,h}-W_\epsilon) + \sum_{j=1}^n B_j\partial_j(W_{\epsilon,h}-W_\epsilon), W_{\epsilon,h}-W_\epsilon\right\rangle\varphi\,dt\,dx\\ &= \frac{1}{\epsilon}\int_0^T\int_{\mathbb{R}^n}\langle P_K(W_{\epsilon,h}) - P_K(W_\epsilon) - (W_{\epsilon,h}-W_\epsilon), W_{\epsilon,h}-W_\epsilon\rangle\,\varphi\,dt\,dx \le 0,\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\Big(|W_{\epsilon,h}-W_\epsilon|^2\partial_t\varphi + \sum_{i=1}^n\langle W_{\epsilon,h}-W_\epsilon, B_i(W_{\epsilon,h}-W_\epsilon)\rangle\,\partial_{x_i}\varphi\Big)\,dt\,dx\\ + \int_{\mathbb{R}^n}|W^0-W^0_h|^2(x)\varphi(0,x)\,dx \ge 0,\end{aligned} \tag{26}$$

where $W^0_h = W^0(\cdot+h)$. Let $r>0$, we define the function φ as

$$\varphi(t,x) = \begin{cases} \frac{T-t}{T} + \frac{r-|x|}{nLT}, & \text{if } t\in[0,T] \text{ and } r \le |x| \le r + nL(T-t),\\ \frac{T-t}{T}, & \text{if } t\in[0,T] \text{ and } x \in B(0,r),\\ 0, & \text{otherwise,}\end{cases}$$

where L is the maximum of the spectral radii of the matrices B_i. We claim that for a.e. $(t,x)\in[0,T]\times\mathbb{R}^n$

$$\left(|W_{\epsilon,h}-W_\epsilon|^2\partial_t\varphi + \sum_{i=1}^n\langle W_{\epsilon,h}-W_\epsilon, B_i(W_{\epsilon,h}-W_\epsilon)\rangle\,\partial_{x_i}\varphi\right)(t,x)\le 0.$$

This inequality is obviously satisfied as soon as $x \in B(0,r)$ and $t \in [0,T]$. In the case where $r \leq |x| \leq r + nL(T-t)$, we get for all $1 \leq i \leq n$

$$\langle W_{\epsilon,h} - W_\epsilon, B_i(W_{\epsilon,h} - W_\epsilon)\rangle (t,x) \geq -L|W_{\epsilon,h} - W_\epsilon|^2(t,x).$$

Consequently multiplying by $\partial_{x_i}\varphi$, it yields

$$\left(\sum_{i=1}^n \langle W_{\epsilon,h} - W_\epsilon, B_i(W_{\epsilon,h} - W_\epsilon)\rangle \partial_{x_i}\varphi\right)(t,x) \leq -|W_{\epsilon,h} - W_\epsilon|^2(t,x)\partial_t\varphi(t,x),$$

and the inequality is true also in that case. According to (26), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^n} |W^0 - W^0_h|^2(x)\varphi(0,x)\,dx\\ &\geq -\int_0^T\int_{B(0,r)} \Big(|W_{\epsilon,h} - W_\epsilon|^2\partial_t\varphi + \sum_{i=1}^n \langle W_{\epsilon,h} - W_\epsilon, B_i(W_{\epsilon,h} - W_\epsilon)\rangle \partial_{x_i}\varphi\Big)\,dt\,dx.\end{aligned}$$

Thanks to the definition of φ, we get

$$\int_0^T\int_{B(0,r)} |W_{\epsilon,h} - W_\epsilon|^2\,dx\,dt \leq T\int_{B(0,r+nLT)} |W^0 - W^0_h|^2\,dx,$$

and the regularity of W_0 together with [2, Proposition 9.3] yields

$$\int_0^T\int_{B(0,r)} |W_{\epsilon,h} - W_\epsilon|^2\,dx\,dt \leq T|h|^2\int_{\mathbb{R}^n} |\nabla W^0|^2\,dx.$$

Therefore, (25) holds, and consequently the set $\mathcal{F}$ is relatively compact in $L^2(\omega,\mathbb{R}^m)$ for all $0 < t_1 < t_2 < T$.

It remains to show that

$$\lim_{h\to 0}\sup_{\epsilon>0} \|\tau_h W_\epsilon - W_\epsilon\|_{L^2(0,T-h;L^2(\omega,\mathbb{R}^m))}\, 0.$$

For all $\phi \in W^{1,\infty}(]-\infty, T-h[\times\mathbb{R}^n, \mathbb{R}^+)$ with compact support in $]-\infty, T-h[\times\mathbb{R}^n$, one has

$$\begin{aligned}&\int_0^{T-h}\int_{\mathbb{R}^n} \langle \partial_t(\tau_h W_\epsilon - W_\epsilon), (\tau_h W_\epsilon - W_\epsilon)\rangle\,\varphi\,dt\,dx\\ &\qquad + \int_0^{T-h}\int_{\mathbb{R}^n} \left\langle \sum_{j=1}^n B_j\partial_j(\tau_h W_\epsilon - W_\epsilon), (\tau_h W_\epsilon - W_\epsilon)\right\rangle \varphi\,dt\,dx\\ &= \frac{1}{\epsilon}\int_0^{T-h}\int_{\mathbb{R}^n} \langle P_K(\tau_h W_\epsilon) - P_K(W_\epsilon) - (\tau_h W_\epsilon - W_\epsilon), \tau_h W_\epsilon - W_\epsilon\rangle\,\varphi\,dt\,dx \leq 0,\end{aligned}$$

since the projection is 1-Lipschitz. Arguing as before, we obtain

$$\begin{aligned}&\int_0^{T-h}\int_{\mathbb{R}^n} |\tau_h W_\epsilon - W_\epsilon|^2\partial_t\varphi\,dt\,dx\\ &\qquad + \int_0^{T-h}\int_{\mathbb{R}^n}\sum_{i=1}^n \langle \tau_h W_\epsilon - W_\epsilon, B_i(\tau_h W_\epsilon - W_\epsilon)\rangle\,\partial_{x_i}\varphi\,dt\,dx\\ &\qquad\qquad + \int_{\mathbb{R}^n} |W_\epsilon(h,x) - W^0(x)|^2\varphi(0,x)\,dx \geq 0.\end{aligned}$$

Using a similar test function

$$\varphi(t,x) = \begin{cases} \frac{T-h-t}{T-h} + \frac{r-|x|}{nL(T-h)}, & \text{if } t \in [0, T-h] \text{ and } r \leq |x| \leq r + nL(T-t), \\ \frac{T-h-t}{T-h}, & \text{if } t \in [0,T-h] \text{ and } x \in B(0,r), \\ 0, & \text{otherwise,} \end{cases}$$

we get that for a.e. $(t,x) \in [0,T-h] \times \mathbb{R}^n$,

$$\left(|\tau_h W_\epsilon - W_\epsilon|^2 \partial_t \varphi + \sum_{i=1}^{n} \langle \tau_h W_\epsilon - W_\epsilon, B_i(\tau_h W_\epsilon - W_\epsilon) \rangle \, \partial_{x_i} \varphi \right)(t,x) \leq 0,$$

and then

$$\begin{aligned} \int_0^{T-h} \int_{B(0,r)} |\tau_h W_\epsilon - W_\epsilon|^2 \, dx \, dt & \\ \leq (T-h) \int_{B(0,r+nL(T-h))} |W_\epsilon(h,x) - W^0(x)|^2 \varphi(0,x) \, dx & \\ \leq T \int_{B(0,r+nLT)} |W_\epsilon(h,x) - W^0(x)|^2 \, dx. & \quad (27) \end{aligned}$$

The conclusion then follows from the following result whose proof is very close to that of [3, Proposition 7].

Lemma 4.4. *For all $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^+)$, one has*

$$\lim_{h \to 0} \sup_{\epsilon > 0} \int_{\mathbb{R}^n} |W_\epsilon(h,x) - W^0(x)|^2 \xi(x) \, dx = 0.$$

According to Theorem 4.3 and estimate (20), the sequence $(W_\epsilon)_{\epsilon>0}$ admits a subsequence (not relabeled) which converges strongly in $L^2([0,T]; L^2(\omega, \mathbb{R}^m))$ to some $\tilde{W}$ and weakly in $L^2([0,T]; L^2(\mathbb{R}^n, \mathbb{R}^m))$ to some $W \in L^2([0,T]; L^2(\mathbb{R}^n, \mathbb{R}^m))$. By uniqueness of the limit, we infer that $\tilde{W} = W \in L^2([0,T]; L^2(\mathbb{R}^n, \mathbb{R}^m)$. Let us take a test function $\phi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^n)$ in (19). Multiplying this inequality by ϵ and passing to the limit as $\epsilon \to 0$ yields $W = P_K(W)$ a.e. in $]0,T[\times\mathbb{R}^n$ which shows that $W \in L^2([0,T]; L^2(\mathbb{R}^n, K))$. Finally, passing to the limit as $\epsilon \to 0$ in (24) shows that W is a solution in the sense of Definition 1.1 to the problem (1). Note finally that, by uniqueness of the solution to (1) (see [3, Lemma 9]), there is no need to extract a subsequence to get the above convergences as $\epsilon \to 0$. $\square$

The construction of the solution W to (1) rests on the assumption that the initial data $W_0 \in H^1(\mathbb{R}^n, K)$. Let us now explain how to construct a solution W when W_0 only belongs to $L^2(\mathbb{R}^n, K)$. We use here the following result whose proof can be found in [3]

Theorem 4.5. *Let W^0 and $\tilde{W^0} \in H^1(\mathbb{R}^n, K)$. We denote by W (resp. $\tilde{W}$) the solution in $L^2(0,T; L^2(\mathbb{R}^n, K))$ to problem (1) in the sense of Definition 1.1 associated with W^0 (resp. $\tilde{W^0}$). Then, W and $\tilde{W}$ belong to $\mathcal{C}([0,T]; L^2(\mathbb{R}^n, K))$, and, in addition, we have the following estimate*

$$\forall t \in [0,T], \forall r > 0, \quad \left\| W(t,\cdot) - \tilde{W}(t,\cdot) \right\|_{L^2(B(0,r))} \leq \left\| W^0 - \tilde{W^0} \right\|_{L^2(B(0,r+nLT))},$$

where L is the maximum of the spectral radii of the matrices B_i.

By mollification, let us construct a sequence $(W_k^0)_{k\in\mathbb{N}}$ such that $W_k^0 \in H^1(\mathbb{R}^n, K)$ for all $k \in \mathbb{N}$, which converges to W^0 in $L^2(\mathbb{R}^n, K)$. The estimates of Theorem 4.5 imply that

$$\sup_{t\in[0,T]} \|W_k(t,\cdot) - W_l(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \left\|W_k^0 - W_l^0\right\|_{L^2(\mathbb{R}^n)}, \tag{28}$$

where W_k (resp. W_l) is the solution to (1) associated with the initial condition W_k^0 (resp. W_l^0). It follows that the sequence $(W_k)_{k\in\mathbb{N}}$ is of Cauchy type in $L^\infty(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$, and therefore it converges strongly in $L^\infty(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$ to some function $W \in L^\infty(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$. Thanks to the strong convergence, we find that W satisfies the inequality (2). In addition, since $W_k = P_K(W_k)$ for all $k \in \mathbb{N}$, we deduce that $W = P_K(W)$ which ensures that $W \in L^\infty(0,T;L^2(\mathbb{R}^n,K))$. The following result has thus been established.

Theorem 4.6. *Let $W^0 \in L^2(\mathbb{R}^n, K)$, then there exists a unique solution $W \in L^\infty(0,T,L^2(\mathbb{R}^n,K))$ to (1) in the sense of Definition 1.1.*

Remark 2. In this paper, we construct an approximation of the Friedrichs' constrained solution by sending firstly the viscosity parameter η to 0 and then the parameter ϵ (related to the constraint) to 0. If one proves that the sequence $(W_{\epsilon,\eta})$ is bounded in $H^1_{t,x}$ uniformly with respect to the two parameters ϵ and η, one will also obtain by sending first the parameter ϵ to 0 and then the parameter η to 0 an approximation of the Friedrichs' constrained solution.

5. Conclusion.

Lastly, the relaxed problem (18), that was used in [3] to derive formally a definition of weak solutions of hyperbolic constrained problems, is in fact a rigorous way to construct weak solutions of hyperbolic constrained problems. It is worth noting that this relaxation procedure is deeply related to viscoplastic models. In order to fully apply this theory to mechanical problems, one should consider problems that are posed in bounded spatial domain. To do so, a new formulation of weak solutions to Friedrichs' systems posed in bounded domains is proposed in [6], without constraints. It remains now to investigate the interactions between the boundary conditions which are considered in [6] and the convex constraints.

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