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ASYMPTOTIC ANALYSIS OF A SIMPLE MODEL OF FLUID-STRUCTURE INTERACTION

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Abstract. This paper is devoted to the asymptotic analysis of simple models of fluid-structure interaction, namely a system between the heat and wave equations coupled via some transmission conditions at the interface. The heat part induces the dissipation of the full system. Here we are interested in the behavior of the model when the thickness of the heat part and/or the heat diffusion coefficient go to zero or to infinity. The limit problem is a wave equation with a boundary condition at the interface, this boundary condition being different according to the limit of the above mentioned parameters. It turns out that some limit problems are dissipative but some of them are non dissipative or their behavior is unknown.

1. **Introduction.** This paper is concerned with the asymptotic analysis of a simple model of fluid-structure interaction. More precisely we consider a coupled system between the heat equation and the wave equation, the coupling being made through some transmission conditions along the interface. It is well known that the heat component induces the dissipation of the full system, see [25, 30, 31, 32], where it is shown that the energy of the system decays polynomially under some geometrical conditions between the heat and wave parts. Such a system is a simplified and linearized version of a fluid-structure interaction. More realistic models should consist in the coupling between the Navier-Stokes (or Stokes) and the elasticity systems, but for such systems some basic mathematical questions remain open [5, 8, 9, 10, 13, 14, 15, 29]. Furthermore in a first attempt we have preferred to analyze the simplest model.

A complete analysis of a system which couples at the interface the linear version of the Navier-Stokes equations with the equations of linear elasticity (wave-like) has been recently done by Avalos and Triggiani [2, 3]. Probably our approach could be extended to such a model. This will be investigated in the future.

Here we are interested in the behavior of the system when the thickness of the heat component and/or the heat diffusion coefficient go to zero or to infinity. To our

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knowledge, such an analysis is not yet done (for stationary heat problem, we refer to [12]). The main question is to see if the limit problem inherits the decay property of the family of coupled systems. We actually show that the limit problem is a wave equation with a boundary condition at the interface. Four types of boundary conditions are obtained according to the limit of the above mentioned parameters. Namely we find either a Dirichlet boundary condition, or a Neumann boundary condition, or a standard dissipative boundary condition or a non standard boundary condition of memory type. Hence, for the first two boundary conditions the limit problem is not dissipative, while for the third one, it is dissipative. Finally for the non standard boundary condition of memory type, we do not know if the system is dissipative or not.

Our main idea is to use the Neumann to Dirichlet and a variant of the Dirichlet to Neumann operators in the heat part in order to transform the transmission condition into a boundary condition with memory for the wave unknown. Hence, in a second step, we are able to analyze the limit procedure in this boundary condition with memory.

Our paper is clearly connected with the problem of *perfectly matching layers.* The absorbing boundary conditions, introduced by Engquist and Majda [11] and Bayliss and Turkel [4] to truncate infinite domains in order to carry out computations of wave propagation phenomena in acoustic and fluid dynamics, are almost always nonlocal. Thus, they are difficult to deal with and require pseudodifferential analysis. It is then interesting to substitute absorbing boundary conditions with a partial differential equation on a close domain, easier to analyse and especially to use numerically (see [7]).

The paper is mainly divided in two parts. The first part treats the one-dimensional situation, while the second one is devoted to the multidimensional case. Even if some similarities exist between these two parts, we have kept this subdivision because the one-dimensional case is more simple to treat and then allows to understand the underlying ideas.

The paper is organized as follows. In section 2, we first recall the model problem, transform it by using a standard scaling argument and then show that it is well posed using semi-group theory. In section 3, in the one-dimensional case we give explicit expressions for the Neumann to Dirichlet and the variant of the Dirichlet to Neumann operators. Hence the limit process is made in section 4 in the one-dimensional case. We go on with the multidimensional case, with a similar scheme. Namely the Dirichlet to Neumann and Neumann to Dirichlet operators are given in section 5 and we end up with the limit process.

In the whole paper, we will use the following notations. As usual, we denote by $L^2(.)$ the Lebesgue space and by $H^s(.)$, $s \geq 0$, the standard Sobolev space. The usual norm and seminorm of $H^s(D)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$. Hence the $L^2(D)$-norm will be denoted by $\|\cdot\|_{0,D}$.

2. **The problem.** We consider the following hyperbolic-parabolic problem. We suppose that the heat equation is set in

$$\Omega_h(\epsilon) = (0, \epsilon) \times O,$$

where O is a bounded domain of $\mathbb{R}^{n-1}$, $n \geq 1$ with a Lipschitz boundary (in the case $n = 1$, this means that $\Omega_h(\epsilon) = (0, \epsilon)$). We further assume that the wave equation holds in a domain Ω_w of $\mathbb{R}^n$ with a Lipschitz boundary such that its boundary contains

$$I = \{0\} \times O,$$

called the interface (between $\Omega_h(\epsilon)$ and Ω_w). If $n = 1$, we simply take $\Omega_w = (-1, 0)$. Finally the two equations are coupled through the interface I leading to the following system

$$y_{tt} - \Delta y = 0 \quad \text{in} \quad \Omega_w \times (0, +\infty) \tag{1}$$
$$z_t - c^2 \Delta z = 0 \quad \text{in} \quad \Omega_h(\epsilon) \times (0, +\infty) \tag{2}$$
$$y_t(0, x', t) = z(0, x', t) \quad x' \in O, t \in (0, +\infty) \tag{3}$$
$$y_{x_1}(0, x', t) = c^2 z_{x_1}(0, x', t) \quad x' \in O, t \in (0, +\infty) \tag{4}$$
$$y(x, t) = 0 \quad x \in \partial\Omega_w \setminus I, t \in (0, +\infty) \tag{5}$$
$$z(x, t) = 0 \qquad x \in \partial\Omega_h(\epsilon) \setminus I, t \in (0, +\infty) \tag{6}$$
$$y(x, 0) = y_0(x) \quad \text{and} \quad y_t(x, 0) = y_1(x) \quad \text{in} \quad \Omega_w \tag{7}$$
$$z(x, 0) = z_0(x) \quad \text{in} \quad \Omega_h(\epsilon), \tag{8}$$

where $\epsilon > 0$ is the thickness of the heat component and $c > 0$ is the heat diffusion coefficient, which are two variable parameters.

Our main goal is to analyze the limit problem obtained as ϵ and/or c go to zero or to infinity.

As mentioned in the introduction, this system is a simplified and linearized model for a fluid-structure interaction. The unknown z corresponds to the velocity of the fluid, while y and y_t represent the displacement and velocity of the structure respectively. We refer to [31, 32] for the long time behavior of this system for fixed parameters ϵ and c, see also [25, 30] for a variant of this system.

For the sake of simplicity $x \in \Omega_h(\epsilon)$ will be often written $x = (x_1, x')$, with $x_1 \in (0, \epsilon)$ and $x' \in O$ (if $n = 1$, the variable x' has to be ignored). We further write

$$\Omega_h = \Omega_h(1).$$

By the change of variables $x_1 = \epsilon \hat{x}_1$, $\hat{x}_1 \in (0, 1)$ and the change of unknown

$$\hat{z}(\hat{x}_1, x', t) = z(x_1, x', t),$$

the above problem is equivalent to

$$y_{tt} - \Delta y = 0 \quad \text{in} \quad \Omega_w \times (0, +\infty) \tag{9}$$
$$\hat{z}_t - (k^2 \partial_{\hat{x}_1 \hat{x}_1} + \frac{1}{\alpha^2} \Delta_{x'}) \hat{z} = 0 \quad \text{in} \quad \Omega_h \times (0, +\infty) \tag{10}$$
$$y_t(0, x' t) = \hat{z}(0, x' t) \quad x' \in O, t \in (0, +\infty) \tag{11}$$
$$y_{x_1}(0, t) = \frac{k}{\alpha} \hat{z}_{\hat{x}_1}(0, x', t) \quad x' \in O, t \in (0, +\infty) \tag{12}$$
$$y(x, t) = 0 \quad x \in \partial\Omega_w \setminus I, t \in (0, +\infty) \tag{13}$$
$$\hat{z}(x, t) = 0 \qquad x \in \partial\Omega_h \setminus I, t \in (0, +\infty) \tag{14}$$
$$y(x, 0) = y_0(x) \quad \text{and} \quad y_t(x, 0) = y_1(x) \quad \text{in} \quad \Omega_w \tag{15}$$
$$\hat{z}(\hat{x}, 0) = z_0(\epsilon \hat{x}) \quad \text{in} \quad \Omega_h, \tag{16}$$

where $k = c/\epsilon > 0$ will be one of our new parameters that may tend to zero or to infinity. The second parameter will be $\alpha = \frac{1}{c} = \frac{1}{k\epsilon}$ that may also tend to zero or to infinity. Note further that in the case $n = 1$, the variable x' and the operator $\frac{1}{\alpha^2} \Delta_{x'}$ disappear.

For shortness from now on, we write z instead of $\hat{z}$ and assume that $z_0 \equiv 0$, therefore the above problem depends only on the parameter k and α. We are interested in the behavior of the problem as k and α go to 0 or to infinity.

Now we give existence and regularity results for the system (9) to (16). We further give a priori bounds that will be useful for our limit processes. Denoting by

$$U = (y, y_t, z)\,,$$

we see that (9) and (10) imply that

$$U_t = (y_t, y_{tt}, z_t) = \left(y_t, \Delta y, k^2 z_{x_1 x_1} + \frac{1}{\alpha^2}\Delta_{x'} z\right) = \mathcal{A}^{(k,\alpha)}U, \tag{17}$$

where we set (formally)

$$\mathcal{A}^{(k,\alpha)}\,(u, v, z) = \left(v, \Delta u, k^2 z_{x_1 x_1} + \frac{1}{\alpha^2}\Delta_{x'} z\right). \tag{18}$$

From these considerations, we introduce

$$\mathcal{H} = V \times L^2(\Omega_w) \times L^2(\Omega_h),$$

where

$$V = \{u \in H^1(\Omega_w) : u = 0 \text{ on } \partial\Omega_w \setminus I\}.$$

The space $\mathcal{H}$ is a Hilbert space for the inner product

$$((u, v, z)\,, (\tilde{u}, \tilde{v}, \tilde{z}))_{\mathcal{H}} = \int_{\Omega_w} (\nabla u(x) \cdot \nabla \tilde{u}(x) + v(x)\tilde{v}(x))\, dx + \frac{1}{k\alpha}\int_{\Omega_h} z(x)\tilde{z}(x)\, dx.$$

We further introduce the domain of the operator $\mathcal{A}^{(k,\alpha)}$ as

$$\begin{aligned} D(\mathcal{A}^{(k,\alpha)}) = \{(u, v, z) \in \mathcal{H} : \quad & v \in V; \Delta u \in L^2(\Omega_w); k^2 z_{x_1 x_1} + \frac{1}{\alpha^2}\Delta_{x'} z \in L^2(\Omega_h); \\ & z = 0 \text{ on } \partial\Omega_h \setminus I; \\ & v = z, u_{x_1} = \frac{k}{\alpha} z_{x_1} \text{ on } I\}. \end{aligned}$$

Finally for $U \in D(\mathcal{A}^{(k,\alpha)})$, $\mathcal{A}^{(k,\alpha)}U$ is defined by (18).

Note that if $n = 1$, then

$$\begin{aligned} D(\mathcal{A}^{(k,\alpha)}) = \{(u, v, z) \in H^2(-1, 0) \times H^1(-1, 0) \times H^2(0, 1) : \\ u(-1) = v(-1) = z(1) = 0,\ v(0) = z(0), u_x(0) = \frac{k}{\alpha} z_x(0)\}. \end{aligned}$$

On the contrary if $n \geq 2$, if (u, v, z) belongs to $D(\mathcal{A}^{(k,\alpha)})$, then we only have $u \in E(\Delta, L^2(\Omega_w))$ where

$$E(\Delta, L^2(\Omega_w)) := \left\{u \in H^1(\Omega_w)\ :\ \Delta u \in L^2(\Omega_w)\right\}, \tag{19}$$

which does not guarantee that $u \in H^2(\Omega_w)$. Nevertheless by Theorem 1.5.3.10 of [16], we deduce that $u_{x_1}(0, x') = \frac{\partial u}{\partial x_1}(0, x')$ belongs to $\tilde{H}^{1/2}(I)'$ (see below). Similarly for z, one get $z_{x_1}(0, x') \in \tilde{H}^{1/2}(I)'$.

Above $\tilde{H}^{1/2}(I)'$ is the dual space of

$$\tilde{H}^{1/2}(I) := \{u \in H^{1/2}(I)\ :\ \tilde{u} \in H^{1/2}(\{0\} \times \mathbb{R}^{n-1})\} \tag{20}$$

where $\tilde{u}$ is the extension of u defined by

$$\tilde{u} := \begin{cases} u & \text{on } I \\ 0 & \text{on } (\{0\} \times \mathbb{R}^{n-1}) \setminus I. \end{cases} \tag{21}$$

The operator $\mathcal{A}^{(k,\alpha)}$ is dissipative since $\forall\ U=(u,v,z)\in D(\mathcal{A}^{(k,\alpha)})$,

$$(\mathcal{A}^{(k,\alpha)}U,U)=-\frac{k}{\alpha}\int_{\Omega_h} z_{x_1}^2\,dx-\frac{1}{k\alpha^3}\int_{\Omega_h}|\nabla_{x'}z|^2\,dx\le 0.$$

Moreover the operator $\mathcal{A}^{(k,\alpha)}$ has domain dense in $\mathcal{H}$ and is surjective (see Theorem 1 of [31]). Then it generates a C_0-semigroup of contraction in $\mathcal{H}$ and therefore one deduces the following results (using the fact that, for smooth initial data, $U=(y,y_2,z)$ is solution of (17) if and only if $y_2=y_t$ and (y,z) is solution of (9) to (16)):

Theorem 2.1. *For all $(y_0,y_1,z_0)\in\mathcal{H}$, problem (9) to (16) has a unique weak solution (y,z) with the regularity $y\in C([0,\infty);H^1(\Omega_w))\cap C^1([0,\infty);L^2(\Omega_w))$ and $z\in C([0,\infty);L^2(\Omega_h))$. If moreover $(y_0,y_1,z_0)\in D(\mathcal{A}^{(k,\alpha)})$, then problem (9) to (16) has a unique strong solution (y,z) that satisfies*

$$y\in C([0,\infty);E(\Delta,L^2(\Omega_w)))\cap C^1([0,\infty);H^1(\Omega_w))\cap C^2([0,\infty);L^2(\Omega_w))$$

and

$$z\in C^1([0,\infty);L^2(\Omega_h))\cap C([0,\infty);E(k^2\partial_{x_1x_1}+\frac{1}{\alpha^2}\Delta_{x'},L^2(\Omega_h))),$$

where

$$E(k^2\partial_{x_1x_1}+\frac{1}{\alpha^2}\Delta_{x'},L^2(\Omega_h)):=\{\ z\in L^2(\Omega_h)\ :\ k^2z_{x_1x_1}+\frac{1}{\alpha^2}\Delta_{x'}z\in L^2(\Omega_h)\ \}. \tag{22}$$

If $(y_0,y_1,z_0)\in D((\mathcal{A}^{(k,\alpha)})^2)$, then problem (9) to (16) has a unique strong solution (y,z) that satisfies

$$y\in C([0,\infty);E(\Delta,H^1(\Omega_w)))\cap C^1([0,\infty);E(\Delta,L^2(\Omega_w)))\cap C^2([0,\infty);H^1(\Omega_w))$$

and

$$z\in C^1([0,\infty);L^2(\Omega_h))\cap C([0,\infty);E^2(k^2\partial_{x_1x_1}+\frac{1}{\alpha^2}\Delta_{x'},L^2(\Omega_h))),$$

where

$$E(\Delta,H^1(\Omega_w)):=\{\ u\in H^1(\Omega_w)\ :\ \Delta u\in H^1(\Omega_w)\ \} \tag{23}$$

and

$$E^2(k^2\partial_{x_1x_1}+\frac{1}{\alpha^2}\Delta_{x'},L^2(\Omega_h)):=\Big\{\ z\in L^2(\Omega_h)\ :\ k^2z_{x_1x_1}+\frac{1}{\alpha^2}\Delta_{x'}z\in L^2(\Omega_h),\ k^2\frac{\partial^4 z}{\partial x_1^4}+\frac{1}{\alpha^2}\Delta_{x'}(\Delta_{x'}z)+2\frac{k^2}{\alpha^2}(\Delta_{x'}z)_{x_1x_1}\in L^2(\Omega_h)\ \Big\}. \tag{24}$$

Note that the solution of the system (9) to (16) has a poor regularity if $n\ge 2$, this lack of regularity was already pointed out in [32] and seems to be responsible of a weaker decay of the energy than in dimension 1 (see Remarks 7.2 and 7.3 of [32]). This lack of regularity also renders our analysis below more delicate if $n\ge 2$ than in the one-dimensional case.

Let us now define the standard energy of our system (9)–(16):

$$E(t)=\frac{1}{2}\|(y,y_t,z)\|_{\mathcal{H}}^2=\frac{1}{2}\left(\int_{\Omega_w}(|\nabla y|^2+y_t^2)dx+\frac{1}{k\alpha}\int_{\Omega_h}z^2(x,t)dx\right). \tag{25}$$

Next, for $U_0=(y_0,y_1,z_0)\in D((\mathcal{A}^{(k,\alpha)})^l), l=1,2$, from the previous theorem we know that $U=(y,y_t,z)\in C([0,\infty);D((\mathcal{A}^{(k,\alpha)})^l))$ and therefore we can define the modified energy

$$\tilde{E}^{(l)}(t) = \frac{1}{2}\|(\mathcal{A}^{(k,\alpha)})^l(y, y_t, z)\|^2_{\mathcal{H}}, \tag{26}$$

explicitly given by

$$\tilde{E}^{(1)}(t) = \frac{1}{2}\left(\int_{\Omega_w}(|\nabla y_t|^2 + (\Delta y)^2)dx + \frac{1}{k\alpha}\int_{\Omega_h}(k^2 z_{x_1x_1} + \frac{1}{\alpha^2}\Delta_{x'}z)^2dx\right),$$

and

$$\tilde{E}^{(2)}(t) = \frac{1}{2}\left(\int_{\Omega_w}(|\nabla(\Delta y)|^2 + (\Delta y_t)^2)dx \right.$$
$$\left. + \frac{1}{k\alpha}\int_{\Omega_h}\left(k^4 z_{x_1x_1x_1x_1} + \frac{1}{\alpha^2}\nabla^2_{x'}z + 2\frac{k^2}{\alpha^2}(\Delta^2_{x'}z)\right)^2 dx\right).$$

Lemma 1. *For all* $(y_0, y_1, z) \in \mathcal{H}$*, the energy* $E(t)$ *of the weak solution* (y, z) *of problem* (9) − (16) *is decreasing, i.e.,*

$$E(t) \le E(s), \quad \forall\, t \ge s \ge 0. \tag{27}$$

If moreover $(y_0, y_1, z) \in D((\mathcal{A}^{(k,\alpha)})^l)$*,* $l = 1, 2$*, then the modified energy* $\tilde{E}^{(l)}(t)$ *of the strong solution* (y, z) *of problem* (9) − (16) *is decreasing, i.e.,*

$$\tilde{E}^{(l)}(t) \le \tilde{E}^{(l)}(s), \quad \forall\, t \ge s \ge 0. \tag{28}$$

Proof. The first assertion follows from the dissipativeness of $\mathcal{A}^{(k,\alpha)}$ since for strong solution U, we have

$$E'(t) = (U_t, U)_{\mathcal{H}} = (\mathcal{A}^{(k,\alpha)}U, U)_{\mathcal{H}} \le 0. \tag{29}$$

This last estimate implies (27) for strong solutions and then for weak solutions by the density of $D(\mathcal{A}^{(k,\alpha)})$ into $\mathcal{H}$.

For the second assertion, we first take $U_0 \in D((\mathcal{A}^{(k,\alpha)})^{l+1})$, then the solution $U = (y, y_t, z)$ of (17) has the regularity

$$U \in C^1([0,\infty); D((\mathcal{A}^{(k,\alpha)})^l)) \cap C([0,\infty); D((\mathcal{A}^{(k,\alpha)})^{l+1})).$$

Therefore

$$\frac{d}{dt}\tilde{E}^{(l)}(t) = ((\mathcal{A}^{(k,\alpha)})^l U_t, (\mathcal{A}^{(k,\alpha)})^l U)_{\mathcal{H}} = (\mathcal{A}^{(k,\alpha)}((\mathcal{A}^{(k,\alpha)})^l U), (\mathcal{A}^{(k,\alpha)})^l U)_{\mathcal{H}},$$

and the conclusion follows from (29). As before this last estimate implies (28) by the density of $D((\mathcal{A}^{(k,\alpha)})^{l+1})$ into $D((\mathcal{A}^{(k,\alpha)})^l)$. □

If $n = 1$, problem (9) and (10) reduces to (with the previous notations)

$$y_{tt} - y_{xx} = 0 \quad \text{in} \quad (-1, 0) \times (0, +\infty) \tag{30}$$
$$z_t - k^2 z_{xx} = 0 \quad \text{in} \quad (0, 1) \times (0, +\infty) \tag{31}$$
$$y_t(0, t) = z(0, t) \quad t \in (0, +\infty) \tag{32}$$
$$y_x(0, t) = \frac{k}{\alpha} z_x(0, t) \quad t \in (0, +\infty) \tag{33}$$
$$y(-1, t) = z(1, t) = 0 \quad t \in (0, +\infty) \tag{34}$$
$$y(x, 0) = y_0(x) \quad \text{and} \quad y_t(x, 0) = y_1(x) \quad \text{in} \quad (-1, 0) \tag{35}$$
$$z(x, 0) = z_0(\epsilon x) \quad \text{in} \quad (0, 1). \tag{36}$$

Since the domain of $D(\mathcal{A}^{(k,\alpha)})$ is more regular than in the case $n \ge 2$, its limit process is simpler to consider. We then start by considering this problem first and postponed to the end of the paper the analysis of the case $n \ge 2$.

3. **The Neumann to Dirichlet and Dirichlet to Neumann operators in the case** $n = 1$**.** Let us denote by G_k the following Neumann to Dirichlet operator associated with the heat equation in $(0, 1)$. Namely for $h \in H_0^1(0, \infty)$, let w be the unique solution of (see below)

$$w_t - k^2 w_{xx} = 0 \quad \text{in} \quad (0,1) \times (0, +\infty) \tag{37}$$
$$w_x(0,t) = h(t) \quad t \in (0, +\infty) \tag{38}$$
$$w(1,t) = 0 \quad t \in (0, +\infty) \tag{39}$$
$$w(x,0) = 0 \quad \text{in} \quad (0,1), \tag{40}$$

then $G_k h$ is the function defined by

$$G_k h(t) = w(0,t) \quad t \in (0, +\infty). \tag{41}$$

The introduction of this operator is motivated by the fact that if $(y^{(k,\alpha)}, z^{(k,\alpha)})$ is solution of (30) to (36) with $z_0 \equiv 0$, then

$$y_t^{(k,\alpha)}(0,t) = z^{(k,\alpha)}(0,t) = G_k(z_x^{(k,\alpha)}(0,t)) = \frac{\alpha}{k} G_k(y_x^{(k,\alpha)}(0,t)) \quad t \in (0, +\infty). \tag{42}$$

Let us first give explicitly the solution of problem (37)–(40):

Lemma 2. *The solution w of problem (37) − (40) is given by*

$$w(\cdot, t) = \sqrt{2} k^2 \sum_{l=0}^{\infty} \int_0^t e^{-k^2 \lambda_{M,l}^2 (t-s)} h(s)\, ds (\sin \lambda_{M,l}) \varphi_l, \tag{43}$$

where for all $l \in \mathbb{N}$, we have set $\lambda_{M,l} = \frac{\pi}{2} + l\pi$ and $\varphi_l(x) = \sqrt{2} \sin(\lambda_{M,l}(x - 1))$ ($-\lambda_{M,l}^2$ are the eigenvalues with eigenfunction φ_l of the Laplace operator with Dirichlet boundary condition at 1 and Neumann boundary condition at 0). Therefore we get

$$G_k(h)(t) = -k \int_0^t F_k(s) h(t-s)\, ds, \tag{44}$$

where the kernel F_k is defined by

$$F_k(s) = 2k \sum_{l=0}^{\infty} e^{-k^2 \lambda_{M,l}^2 s}, \forall s > 0, \tag{45}$$

and satisfies for all $s > 0$

$$0 \le F_k(s) \le \frac{C}{\sqrt{s}}, \tag{46}$$
$$F_k(s) \to 0 \text{ as } k \to +\infty, \tag{47}$$
$$F_k(s) \to \frac{1}{\sqrt{\pi s}} \text{ as } k \to 0, \tag{48}$$

for some $C > 0$ independent of k and s.

Proof. Setting

$$\tilde{w}(x,t) = w(x,t) - h(t)(x-1),$$

we see that $\tilde{w}$ is solution of (reminding that $h(0) = 0$)

$$\tilde{w}_t - k^2 \tilde{w}_{xx} = -h_t(t)(x-1) \quad \text{in} \quad (0,1) \times (0, +\infty) \tag{49}$$
$$\tilde{w}_x(0,t) = 0 \quad t \in (0, +\infty) \tag{50}$$
$$\tilde{w}(1,t) = 0 \quad t \in (0, +\infty) \tag{51}$$
$$\tilde{w}(x,0) = 0 \quad \text{in} \quad (0,1). \tag{52}$$

As $\{\varphi_l\}_{l\in\mathbb{N}}$ forms an orthonormal basis of $L^2(0,1)$ and reminding that φ_l is an eigenvector of the operator $k^2\tilde{w}_{xx}$ with eigenvalue $-k^2\lambda_{M,l}^2$, we can write

$$\tilde{w}(\cdot,t) = -\sum_{l=0}^{\infty}\int_0^t e^{-k^2\lambda_{M,l}^2(t-s)}h_t(s)\,ds\alpha_l\varphi_l, \tag{53}$$

where

$$\alpha_l = \int_0^1 (x-1)\varphi_l(x)\,dx = \sqrt{2}\frac{\sin\lambda_{M,l}}{\lambda_{M,l}^2}.$$

An integration by parts in the time yields (43).

From the expansion (43) we can say that

$$\begin{aligned} w(0,t) &= \sqrt{2}k^2\sum_{l=0}^{\infty}\int_0^t e^{-k^2\lambda_{M,l}^2(t-s)}h(s)\,ds\sin\lambda_{M,l}\varphi_l(0)\\ &= -2k^2\sum_{l=0}^{\infty}\int_0^t e^{-k^2\lambda_{M,l}^2(t-s)}h(s)\,ds. \end{aligned}$$

By the dominate convergence theorem of Lebesgue, the above identity implies that (44) holds.

It then remains to consider the asymptotic behavior of F_k. For that purpose introduce the function $f_k(x) = e^{-k^2sx^2}$, for $x \geq 0$. This function is decreasing on $[0,\infty)$ and therefore

$$f_k(\lambda_{M,l}) \geq f_k(x) \geq f_k(\lambda_{M,l+1}) \text{ for } \lambda_{M,l} \leq x \leq \lambda_{M,l+1}, \quad l \in \mathbb{N}$$

These estimates allow to say that

$$\begin{aligned} 0 \leq F_k(s) &= 2k\sum_{l=0}^{\infty} f_k(\lambda_{M,l})\\ &= 2kf_k(\lambda_{M,0}) + 2k\sum_{l=1}^{\infty} f_k(\lambda_{M,l})\\ &\leq 2kf_k(\lambda_{M,0}) + \frac{2k}{\pi}\sum_{l=0}^{\infty}\int_{\lambda_{M,l}}^{\lambda_{M,l+1}} f_k(x)\,dx\\ &= 2kf_k(\frac{\pi}{2}) + \frac{2k}{\pi}\int_{\lambda_{M,0}}^{\infty} f_k(x)\,dx\\ &= 2ke^{-k^2s\pi^2/4} + \frac{2}{\pi\sqrt{s}}\int_{\frac{\pi k\sqrt{s}}{2}}^{\infty} e^{-y^2}\,dy. \end{aligned}$$

This last estimate directly implies (47) because each term of this right-hand side tends to zero as k goes to infinity. It also proves (46) because there exists $C > 0$ such that

$$\sqrt{x}e^{-x} \leq C, \forall x > 0,$$

and

$$\int_{\frac{\pi k\sqrt{s}}{2}}^{\infty} e^{-y^2}\,dy \leq \int_0^{\infty} e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}.$$

For the case $k \to 0$, we need a lower bound for F_k, namely we write

$$\begin{aligned} F_k(s) &= 2k\sum_{l=0}^{\infty} f_k(\lambda_{M,l}) \\ &\geq \frac{2k}{\pi}\sum_{l=0}^{\infty}\int_{\lambda_{M,l}}^{\lambda_{M,l+1}} f_k(x)\,dx \\ &= \frac{2k}{\pi}\int_{\lambda_{M,0}}^{\infty} f_k(x)\,dx \\ &= \frac{2}{\pi\sqrt{s}}\int_{\frac{\pi k\sqrt{s}}{2}}^{\infty} e^{-y^2}\,dy. \end{aligned}$$

These two estimates lead to (48). □

Similarly we denote by H_k the Dirichlet to Neumann operator associated with the heat equation in $(0,1)$. Namely for $g \in H_0^1(0,\infty)$, let w be the unique solution of (see below)

$$w_t - k^2 w_{xx} = 0 \quad \text{in} \quad (0,1)\times(0,+\infty) \tag{54}$$

$$w(0,t) = g(t) \quad t \in (0,+\infty) \tag{55}$$

$$w(1,t) = 0 \quad t \in (0,+\infty) \tag{56}$$

$$w(x,0) = 0 \quad \text{in} \quad (0,1), \tag{57}$$

then $H_k g$ is the function defined by

$$H_k g(t) = w_x(0,t) \quad t \in (0,+\infty). \tag{58}$$

As before if $(y^{(k,\alpha)}, z^{(k,\alpha)})$ is solution of (30) to (36) with $z_0 \equiv 0$, then

$$y_x^{(k,\alpha)}(0,t) = \frac{k}{\alpha} H_k(y_t^{(k,\alpha)}(0,t)) \quad t \in (0,+\infty). \tag{59}$$

Unfortunately this operator seems to have a bad behaviour as k goes to infinity, therefore we use the following approach.

Lemma 3. *Assume that $g \in H_0^1(0,\infty) \cap H^2(0,\infty)$. The solution w of problem (54) − (57) is given by*

$$w(\cdot,t) = -\sqrt{2}\sum_{l=1}^{\infty}\int_0^t e^{-k^2\lambda_{D,l}^2(t-s)} g_t(s)\,ds \frac{1}{\lambda_{D,l}}\varphi_l - g(t)(\cdot - 1), \tag{60}$$

where for all $l \geq 1$, we have set $\lambda_{D,l} = l\pi$ and $\varphi_l(x) = \sqrt{2}\sin(\lambda_{D,l}x)$ ($-\lambda_{D,l}^2$ are the eigenvalues with eigenfunction φ_l of the Laplace operator with Dirichlet boundary condition at 0 and 1). Moreover we have

$$w_x(0,t) = \int_0^t K_k(t-s) g_t(s)\,ds - g(t), \tag{61}$$

where the kernel K_k satisfies

$$k|K_k(s)| \leq \frac{C}{\sqrt{s}}, \forall\ s > 0, \tag{62}$$

for some $C > 0$.

Proof. As before we set

$$\tilde{w}(x,t) = w(x,t) + g(t)(x-1),$$

and see that $\tilde{w}$ is solution of

$$\tilde{w}_t - k^2\tilde{w}_{xx} = g_t(t)(x-1) \quad \text{in} \quad (0,1)\times(0,+\infty) \tag{63}$$

$$\tilde{w}(0,t) = 0 \quad t\in(0,+\infty) \tag{64}$$

$$\tilde{w}(1,t) = 0 \quad t\in(0,+\infty) \tag{65}$$

$$\tilde{w}(x,0) = 0 \quad \text{in} \quad (0,1). \tag{66}$$

As $\{\varphi_l\}_{l\in\mathbb{N}}$ is the sequence of eigenfunctions of the operator $k^2\tilde{w}_{xx}$ with eigenvalues $-k^2\lambda_{D,l}^2$, we can write

$$\tilde{w}(\cdot,t) = -\sqrt{2}\sum_{l=1}^{\infty}\int_0^t e^{-k^2\lambda_{D,l}^2(t-s)}g_t(s)\,ds\frac{1}{\lambda_{D,l}}\varphi_l. \tag{67}$$

This expansion and the expression of $\tilde{w}$ directly give (60).

Differentiating in x the expansion (60) we can say that

$$\begin{aligned} w_x(0,t) &= -2\sum_{l=1}^{\infty}\int_0^t e^{-k^2\lambda_{D,l}^2(t-s)}g_t(s)\,ds - g(t) \\ &= \int_0^t K_k(t-s)g_t(s)\,ds - g(t) \end{aligned}$$

where

$$K_k(s) = -2\sum_{l=1}^{\infty}e^{-k^2\lambda_{D,l}^2 s},$$

the above identity being meaningful by the dominate convergence theorem of Lebesgue.

The estimate (62) follows from the fact that

$$k|K_k(s)| \leq \tilde{F}_k(s),$$

where $\tilde{F}_k(s) = 2k\sum_{l=1}^{\infty}e^{-k^2l^2\pi^2 s}$ and is bounded by $\frac{C}{\sqrt{s}}$ as in the proof of the previous lemma. □

4. **The limit problems in the case $n = 1$.** In this section, we consider the limit problems of (30) to (36) as k,α go to zero or to ∞.

Coming back to the solution $(y^{(k,\alpha)}, z^{(k,\alpha)})$ of (30) to (36) with $z_0 \equiv 0$, and making use of (42), we can say that $y^{(k,\alpha)}$ satisfies the boundary condition with memory:

$$y_t^{(k,\alpha)}(0,t) = \frac{\alpha}{k}G_k(y_x^{(k,\alpha)}(0,\cdot))(t) \quad t\in(0,+\infty). \tag{68}$$

The problem consists in justifying the passage to the limit in (68). We first treat the case α tending to 0 or α_0:

Theorem 4.1. *Assume that $y_0 \in H^2(-1,0)$ and $y_1 \in H^1(-1,0)$ are such that*

$$y_0(-1) = y_{0x}(0) = y_1(-1) = y_1(0) = 0.$$

Let $(y^{(k,\alpha)}, z^{(k,\alpha)})$ be the strong solution of (30) to (36) with initial data y_0, y_1 and $z_0 \equiv 0$. For all $T > 0$, let us set $Q_T = (-1,0)\times(0,T)$. Then for all $T > 0$, there exist $y \in H^2(Q_T)$ and a subsequence of $y^{(k,\alpha)}$, still denoted by $y^{(k,\alpha)}$ for the sake of shorthness, such that $y^{(k,\alpha)}$ tends to y weakly in $H^2(Q_T)$ as $k \to k_0$, $\alpha \to \alpha_0$, with

$k_0 \in [0, \infty]$, $\alpha_0 \in [0, \infty)$. *Moreover* y *is the weak solution of the wave equation with Dirichlet boundary condition at* -1*:*

$$y_{tt} - y_{xx} = 0 \quad \text{in} \quad (-1, 0) \times (0, T), \tag{69}$$

$$y(-1, t) = 0 \quad t \in (0, T), \tag{70}$$

$$y(x, 0) = y_0(x) \quad \text{and} \quad y_t(x, 0) = y_1(x) \quad \text{in} \quad (-1, 0). \tag{71}$$

For the boundary condition at 0*, we distinguish the following cases:*
1. If $\alpha \to 0$*, then* y *satisfies the Dirichlet boundary condition at* 0 *:*

$$y(0, t) = y_0(0) \quad t \in (0, +\infty). \tag{72}$$

2. If $\alpha \to \alpha_0 \in (0, \infty)$*, then the boundary condition at* 0 *depends on the limit on* k*:*
a. If $k \to \infty$*, then* y *satisfies the Dirichlet boundary condition* (72).
b. If $k \to k_0 \in (0, \infty)$*, then* y *satisfies the boundary condition with memory*

$$y_t(0, t) = -\alpha_0 \int_0^t F_{k_0}(s) y_x(0, t - s)\, ds, \quad t \in (0, +\infty), \tag{73}$$

c. If $k \to 0$*, then* y *satisfies the boundary condition with memory*

$$y_t(0, t) = K(y_x(0, t)) \quad t \in (0, +\infty), \tag{74}$$

where K *is the integral operator defined by*

$$Kh(t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{s}} h(t - s)\, ds. \tag{75}$$

Proof. The assumptions on the data guarantee that the triple $(y_0, y_1, 0)$ belongs to $D(\mathcal{A}^{(k,\alpha)})$. Therefore by Theorem 2.1 and Lemma 1 the strong solution $(y^{(k,\alpha)}, z^{(k,\alpha)})$ of (30) to (36) satisfies for all $t \geq 0$

$$\int_{-1}^0 (|y_x^{(k,\alpha)}(x, t)|^2 + |y_t^{(k,\alpha)}(x, t)|^2)\, dx + \frac{1}{k\alpha} \int_0^1 |z^{(k,\alpha)}(x, t)|^2\, dx \\ \leq \int_{-1}^0 (|y_{0x}(x)|^2 + |y_1(x)|^2\, dx, \tag{76}$$

$$\int_{-1}^0 (|y_{tx}^{(k,\alpha)}(x, t)|^2 + |y_{xx}^{(k,\alpha)}(x, t)|^2)\, dx + \frac{k^3}{\alpha} \int_0^1 |z_{xx}^{(k,\alpha)}(x, t)|^2\, dx \\ \leq \int_{-1}^0 (|y_{1x}(x)|^2 + |y_{0xx}(x)|^2\, dx. \tag{77}$$

As the right-hand sides of these two estimates are independent of t, we deduce that there exists $C > 0$ such that

$$|y^{(k,\alpha)}|_{2,Q_T} \leq CT, \tag{78}$$

$$|y^{(k,\alpha)}|_{1,Q_T} \leq CT, \tag{79}$$

reminding that $y_{tt}^{(k,\alpha)} = y_{xx}^{(k,\alpha)}$. Since the seminorm $H^1(Q_T)$ is a norm on $H^1(Q_T)$ due the Dirichlet boundary condition on $\{-1\} \times (0, T)$, we deduce that the sequence $(y^{(k,\alpha)})_{k,\alpha}$ is bounded in $H^2(Q_T)$. Consequently there exists $y \in H^2(Q_T)$ such that

$$y^{(k,\alpha)} \to y \text{ weakly in } H^2(Q_T) \text{ as } k \to k_0 \text{ and } \alpha \to \alpha_0. \tag{80}$$

Moreover from the compact embedding of $H^2(Q_T)$ into $H^{2-\eta}(Q_T)$ for any $\eta > 0$, we further have

$$y^{(k,\alpha)} \to y \text{ strongly in } H^{2-\eta}(Q_T) \text{ as } k \to k_0 \text{ and } \alpha \to \alpha_0. \tag{81}$$

for any $\eta > 0$.

From the first property (80), we see that y satisfies (69) in the distributional sense. On the other hand from the second property we see that y fulfils the boundary condition (70) and the initial condition (71). It then remains to prove the boundary condition at 0. For that purpose, we recall that the identity (42) showed that

$$y_t^{(k,\alpha)}(0,t) = G_k(\frac{\alpha}{k} y_x^{(k,\alpha)}(0,t)) = -\alpha \int_0^t F_k(s) y_x^{(k,\alpha)}(0,t-s)\, ds, \quad t \in (0,+\infty), \tag{82}$$

owing to (44). We therefore need to justify the passage to the limit in the above identity.

First if α tends to 0, since (81) and a trace theorem lead to

$$y_x^{(k,\alpha)}(0,\cdot) \to y_x(0,\cdot) \text{ in } H^{1/2-\eta}(0,T),$$

we deduce that

$$y_t^{(k,\alpha)}(0,t) \to 0,$$

which yields

$$y_t(0,\cdot) = 0. \tag{83}$$

and then (72).

Secondly assume that α tends to α_0, then we distinguish the cases $k \to 0$, $k \to k_0$ and $k \to \infty$.

If $k \to +\infty$, then by the dominate convergence theorem of Lebesgue, we see that

$$\int_0^t F_k(s) y_x^{(k,\alpha)}(0,t-s)\, ds \to 0,$$

and by (82) we conclude the boundary condition (83) and then (72).

If $k \to k_0$, then by (82) applying the dominate convergence theorem we directly get (73).

Finally if $k \to 0$, writing $F(s) = \frac{1}{\sqrt{\pi s}}$, we readily see that

$$-3k \le F(s) - F_k(s) \le k. \tag{84}$$

Consequently we write

$$\int_0^t F_k(s) y_x^{(k,\alpha)}(0,t-s)\, ds = \int_0^t (F_k(s) - F(s)) y_x^{(k,\alpha)}(0,t-s)\, ds + \int_0^t F(s) y_x^{(k,\alpha)}(0,t-s)\, ds.$$

As $y_x^{(k,\alpha)}(0,\cdot)$ is uniformly bounded in $L^2(0,T)$, we directly deduce that

$$\int_0^t (F_k(s) - F(s)) y_x^{(k,\alpha)}(0,t-s)\, ds \to 0 \text{ as } k \to 0.$$

On the other hand, since the sequence $(y^{(k,\alpha)})_{k,\alpha}$ is bounded in $H^2(Q_T)$, by a trace theorem we deduce that $(y_x^{(k,\alpha)}(0,\cdot))_{k,\alpha}$ is bounded in $H^{1/2}(0,T)$, and therefore

$$y_x^{(k,\alpha)}(0,\cdot) \to y_x(0,\cdot) \text{ weakly in } H^{1/2}(0,T),$$

and by the compact embedding of $H^{1/2}(0,T)$ into $L^r(0,T)$, for all $r > 1$, we deduce that

$$y_x^{(k,\alpha)}(0,\cdot) \to y_x(0,\cdot) \text{ strongly in } L^r(0,T), \forall r > 1.$$

	$\alpha \to \infty$
$k \to 0$	$\frac{k}{\alpha} \to 0$
$k \to k_0$	$\frac{k}{\alpha} \to 0$
$k \to \infty$	$\frac{k}{\alpha} \to ?$

TABLE 1. Limit cases for the ratio $\frac{k}{\alpha}$

As F belongs to $L^p(0,T), \forall p \in [1,2)$, by Hölder's inequality we deduce that

$$\int_0^t F(s) y_x^{(k,\alpha)}(0,t-s)\, ds \to \int_0^t F(s) y_x(0,t-s)\, ds \text{ as } k \to 0.$$

All together we have shown that

$$\int_0^t F_k(s) y_x^{(k,\alpha)}(0,t-s)\, ds \to \int_0^t F(s) y_x(0,t-s)\, ds \text{ as } k \to 0.$$

This property and the fact that $y_t^{(k,\alpha)}(0,\cdot)$ tends to $y_t(0,\cdot)$ weakly in $H^{1/2}(0,T)$ allow to pass to the limit in the identity (82) and to obtain (74). □

It remains to consider the case when α tends to ∞. In that case the ratio $\frac{k}{\alpha}$ has different behaviors according to the limit on k, see table 1. From this table, we see that in the case $k \to \infty$, the limit of $\frac{k}{\alpha}$ is undetermined. Therefore in that case, we distinguish two cases. Either the ratio admits a limit in $[0,\infty]$, or not. But in that last case, we can always assume that a subsequence admits a limit in $[0,\infty]$. Indeed, if the ratio is uniformly bounded, then we can subtract a subsequence that converges to $\kappa_0 \in [0,\infty)$; on the other hand, if the sequence is not bounded, then we can subtract a subsequence that converges to ∞. So from now on, we work either with the convergent sequence or with such a convergent subsequence.

Theorem 4.2. *Assume that $y_0 \in H^3(-1,0)$ and $y_1 \in H^2(-1,0)$ are such that*

$$y_0(-1) = y_{0xx}(-1) = y_{0x}(0) = y_{0xx}(0) = y_1(-1) = y_1(0) = y_{1x}(0) = 0.$$

Let $(y^{(k,\alpha)}, z^{(k,\alpha)})$ be the strong solution of (30) to (36) with initial data y_0, y_1 and $z_0 \equiv 0$. For all $T > 0$, let us set $Q_T = (-1,0) \times (0,T)$. Then for all $T > 0$, there exist $y \in H^2(Q_T)$ and a subsequence of $y^{(k,\alpha)}$, still denoted by $y^{(k,\alpha)}$ for the sake of shorthness, such that $y^{(k,\alpha)}$ tends to y weakly in $H^2(Q_T)$ as $k \to k_0$ and $\alpha \to \infty$, with $k_0 \in [0,\infty]$. Moreover y is the weak solution of the wave equation with Dirichlet boundary condition at -1, namely satisfies (69), (70) and (71). For the boundary condition at 0, we distinguish the following cases:

1. If $\frac{k}{\alpha} \to 0$, then y satisfies the Neumann boundary condition at 0 :

$$y_x(0,\cdot) = 0. \tag{85}$$

2. If $\frac{k}{\alpha} \to \infty$, then y satisfies the Dirichlet boundary condition (72).

3. If $\frac{k}{\alpha} \to \kappa_0 \in (0,\infty)$, then y satisfies the dissipative boundary condition

$$y_x(0,\cdot) + \kappa_0 y_t(0,\cdot) = 0. \tag{86}$$

Proof. The proof starts as the one of Theorem 4.1. Namely, the estimates (78) and (79) being valid there exists $y \in H^2(Q_T)$ weak limit in $H^2(Q_T)$ and strong limit in

$H^{2-\eta}(Q_T)$ of $y^{(k,\alpha)}$ as $k \to k_0$ and $\alpha \to \infty$. As before these properties imply that y satisfies (69) to (71). It remains to analyze the boundary condition at 0.

First we use the fact that the initial datum $(y_0, y_1, 0)$ belongs to $D((\mathcal{A}^{(k,\alpha)})^2)$, that yields by Lemma 1

$$\tilde{E}^{(2)}(t) \leq \tilde{E}^{(2)}(0), \ \forall t > 0,$$

or equivalently

$$\int_{-1}^{0} (|y_{txx}^{(k,\alpha)}(x,t)|^2 + |y_{xxx}^{(k,\alpha)}(x,t)|^2)\, dx + \frac{k^7}{\alpha} \int_0^1 |z_{xxxx}^{(k,\alpha)}(x,t)|^2\, dx \\ \leq \int_{-1}^{0} (|y_{1xx}(x)|^2 + |y_{0xxx}(x)|^2)\, dx$$

From this estimate and the estimate (77) we see that the sequence $(y_{tt}^{(k,\alpha)})_{k,\alpha}$ is bounded in $H^1(Q_T)$ (recalling that $y_{tt}^{(k,\alpha)} = y_{xx}^{(k,\alpha)}$) and therefore by the Sobolev embedding theorem it admits a subsequence, still denoted by $y_{tt}^{(k,\alpha)}$, that converges to y_{tt} in $H^{1-\eta}(Q_T)$. By a standard trace theorem, we conclude that

$$y_{tt}^{(k,\alpha)}(0,\cdot) \to y_{tt}(0,\cdot) \text{ in } H^{1/2-\eta}(0,T). \tag{87}$$

Now we make use of Lemma 3. Indeed from this lemma we may write

$$z_x^{(k,\alpha)}(0,t) = \int_0^t K_k(t-s) y_{tt}^{(k,\alpha)}(0,s)\, ds - y_t^{(k,\alpha)}(t). \tag{88}$$

Now since

$$y_x^{(k,\alpha)}(0,t) = \frac{k}{\alpha} z_x^{(k,\alpha)}(0,t),$$

we deduce that

$$y_x^{(k,\alpha)}(0,t) = \frac{k}{\alpha} \int_0^t K_k(t-s) y_{tt}^{(k,\alpha)}(0,s)\, ds - \frac{k}{\alpha} y_t^{(k,\alpha)}(0,t). \tag{89}$$

Now, using the estimate (62) and the fact that

$$\|y_{tt}^{(k,\alpha)}(0,s)\|_{L^r(0,T)} \leq CT, \quad \forall\, r > 1,$$

consequence of the property (87), we deduce that

$$\frac{k}{\alpha} \int_0^t K_k(t-s) y_{tt}^{(k,\alpha)}(0,s)\, ds = \frac{1}{\alpha} \int_0^t k K_k(t-s) y_{tt}^{(k,\alpha)}(0,s)\, ds \to 0 \text{ as } \alpha \to \infty. \tag{90}$$

If $\frac{k}{\alpha} \to 0$, then the property (90) and the identity (89) lead to

$$y_x^{(k,\alpha)}(0,\cdot) \to 0 \text{ as } \alpha \to \infty,$$

and then to the boundary condition (85).

On the contrary in the case $\frac{k}{\alpha} \to \infty$, if we note that identity (89) may be equivalently written as

$$y_t^{(k,\alpha)}(0,t) = -\frac{\alpha}{k} \left(y_x^{(k,\alpha)}(0,t) - \frac{k}{\alpha} \int_0^t K_k(t-s) y_{tt}^{(k,\alpha)}(0,s)\, ds \right), \tag{91}$$

then property (90) leads to

$$y_t^{(k,\alpha)}(t) \to 0,$$

and then to the boundary condition

$$y_t(0,\cdot) = 0,$$

	$\alpha \to 0$	$\alpha \to \alpha_0$	$\alpha \to \infty$
$k \to 0$	Dirichlet bc	memory bc	Neumann bc
$k \to k_0$	Dirichlet bc	memory bc	Neumann bc
$k \to \infty$	Dirichlet bc	Dirichlet bc	$\frac{k}{\alpha} \to 0$:Neumann bc
$k \to \infty$			$\frac{k}{\alpha} \to \kappa_0$: dissipative bc
$k \to \infty$			$\frac{k}{\alpha} \to \infty$: Dirichlet bc

TABLE 2. Summary of the limit problems

which leads to the Dirichlet boundary condition

$$y(0,\cdot) = y_0(0).$$

It remains to consider the case $\frac{k}{\alpha} \to \kappa_0$. In that case, from (89) and (90) and passage to the limit we get (86). Note that this boundary condition leads to an exponential decay of the energy. □

Remark 1. The different limit problems are summarized in Table 2. When Dirichlet or Neumann boundary condition appear at 0, then the limit problem is not dissipative. The only cases where we can guarantee that the limit problem is dissipative is the two following cases:
1. when α tends to infinity and $\frac{k}{\alpha}$ tends to a positive real number, then the limit problem is the wave equation with the standard dissipative law (86) at 0, and the exponential decay of this system was proved for instance in [19, 20, 21].
2. when α tends to a real number α_0 and k tends to a positive real number k_0, then the limit problem is still the starting problem with the parameters α_0 and k_0, and its polynomial dissipativeness follows from the considerations from [30].

Finally in the case when α tends to a real number α_0 and k tends to zero, then the limit problem is the following one

$$y_{tt} - y_{xx} = 0 \quad \text{in} \quad (-1,0) \times (0,+\infty) \tag{92}$$

$$y(-1,t) = 0 \quad t \in (0,+\infty) \tag{93}$$

$$y_t(0,t) = K(y_x(0,t)) \quad t \in (0,+\infty), \tag{94}$$

$$y(x,0) = y_0(x) \quad \text{and} \quad y_t(x,0) = y_1(x) \quad \text{in} \quad (-1,0). \tag{95}$$

Actually the dissipation law (94) is a boundary condition with memory but is not a standard one and seems not studied in the literature. For different stability results of the wave equation with memory boundary conditions we refer to the papers [1, 26, 6, 17, 18, 24, 27, 28, 23].

The stability of the system (92) to (95) is not clear at all. Nevertheless if we denote by

$$Y(\cdot,s) = (\mathcal{L}y)(s) = \int_0^\infty e^{-st} y(\cdot,t)\,dt,$$

the Laplace transform of y. Then Y is solution of

$$s^2 Y - Y_{xx} = sy_0 + y_1 \quad \text{in} \quad (-1,0) \tag{96}$$

$$y(-1,s) = 0 \tag{97}$$

$$sY(0,s) + y_0(0) = -\frac{\sqrt{\pi}}{\sqrt{s}} Y_x(0,s). \tag{98}$$

The eigenvalues associated with this problem (corresponding to $y_0 = y_1 = 0$) are the complex numbers s, roots of

$$\sqrt{s}\sinh s + \sqrt{\pi}\cosh s = 0.$$

This equation has complex roots approaching the imaginary axis ($s \sim il\pi$), so we can expect a polynomial decay for the system (92) to (95).

5. **The Neumann to Dirichlet and Dirichlet to Neumann operators for the case $n \geq 2$.** Now we pass to the general case $n \geq 2$ (the case $n = 1$ is not relevant since it was treated in the previous sections). Let us denote by G_k the following Neumann to Dirichlet operator associated with the heat equation in Ω_h. Namely for $h \in H_0^1(O \times (0, \infty))$, let w be the unique solution of

$$w_t - (k^2\partial_{\hat{x}_1\hat{x}_1} + \frac{1}{\alpha^2}\Delta_{x'})w = 0 \quad \text{in} \quad \Omega_h \times (0, +\infty) \tag{99}$$

$$w_{x_1}(0, x', t) = h(x', t) \quad x' \in O, t \in (0, +\infty) \tag{100}$$

$$w(x, t) = 0 \quad x \in \partial\Omega_h \setminus I, t \in (0, +\infty) \tag{101}$$

$$w(x, 0) = 0 \quad \text{in} \quad \Omega_h, \tag{102}$$

then $G_k h$ is the function defined by

$$G_k h(x', t) = w(0, x', t) \quad x' \in O, t \in (0, +\infty). \tag{103}$$

In order to give explicitly the solution of problem (99)–(102) we recall that the operator $-\Delta_{x'}$ with Dirichlet boundary condition on ∂O is a positive selfadjoint operator with a discrete spectrum $\{\mu_j^2\}_{j=1}^{\infty}$ (repeated according to their multiplicity). For all $j \geq 1$ we denote by ψ_j the eigenfunction of $-\Delta_{x'}$ associated with the eigenvalue μ_j^2.

For shortness for all $s > 0$ we denote by $\Psi_h(\cdot, s, x')$, the function

$$\Psi_h(t, s, \cdot) = e^{(t-s)\frac{1}{\alpha^2}\Delta_{x'}} h(s, \cdot), \forall\ t \geq s.$$

This function exists (and is unique) since $\Delta_{x'}$ is the infinitesimal generator of a contraction semigroup $e^{t\Delta_{x'}}$. Note that this function $\Psi_h(\cdot, s)$ is the unique solution of

$$\partial_t \Psi_h - \frac{1}{\alpha^2}\Delta_{x'}\Psi_h = 0 \quad \text{in} \quad O \times (s, +\infty),$$
$$\Psi_h = 0 \quad \text{on} \quad \partial O \times (s, +\infty),$$
$$\Psi_h(s, s, x') = h(s, x'), \quad x' \in O.$$

Lemma 4. *The solution w of problem (99) – (102) is given by*

$$w(x_1, x', t) = \sqrt{2}k^2 \sum_{l=0}^{\infty} \int_0^t e^{-k^2\lambda_{M,l}^2(t-s)} \Psi_h(t, s, x')\, ds (\sin \lambda_{M,l})\varphi_l(x_1), \tag{104}$$

where for all $l \in \mathbb{N}$, we have set $\lambda_{M,l} = \frac{\pi}{2} + l\pi$ and $\varphi_l(x_1) = \sqrt{2}\sin(\lambda_{M,l}(x_1 - 1))$. Therefore we get

$$G_k(h)(x', t) = -k \int_0^t F_k(s)\Psi_h(t, s, x')\, ds, \tag{105}$$

where the kernel F_k is the same as the one from Lemma 2.

Proof. Setting

$$\tilde{w}(x_1, x', t) = w(x_1, x', t) - h(t, x')(x_1 - 1),$$

we see that $\tilde{w}$ is solution of

$$\tilde{w}_t - (k^2\partial_{\hat{x}_1\hat{x}_1} + \frac{1}{\alpha^2}\Delta_{x'})\tilde{w} = -(x_1 - 1)g \quad \text{in} \quad \Omega_h \times (0, +\infty) \tag{106}$$

$$\tilde{w}_{x_1}(0, x', t) = 0 \quad x' \in O, \; t \in (0, +\infty) \tag{107}$$

$$\tilde{w}(x_1, x', t) = 0 \quad x_1 \in (0, 1), \; x' \in \partial O, \; t \in (0, +\infty) \tag{108}$$

$$\tilde{w}(x, 0) = 0 \quad \text{in} \quad \Omega_h, \tag{109}$$

where $g = h_t - \frac{1}{\alpha^2}\Delta_{x'}h$. As $\{\varphi_l\psi_j\}_{l\in\mathbb{N}, j\geq 1}$ forms an orthonormal basis of $L^2(\Omega_h)$ and recalling $\varphi_l\psi_j$ is an eigenvector of the operator $k^2\partial_{\hat{x}_1\hat{x}_1} + \frac{1}{\alpha^2}\Delta_{x'}$ with eigenvalue $-k^2\lambda_{M,l}^2 - \frac{1}{\alpha^2}\mu_j^2$, we can write

$$\tilde{w}(\cdot, t) = -\sum_{l=0}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{M,l}^2 + \frac{1}{\alpha^2}\mu_j^2)(t-s)}(g(s, \cdot), \psi_j)\, ds\alpha_l\varphi_l\psi_j, \tag{110}$$

where $(g(s, \cdot), \psi_j)$ means the L^2-inner product in O of $g(s, \cdot)$ with ψ_j (or the duality bracket), i.e.

$$(g(s, \cdot), \psi_j) = \int_O g(s, x')\psi_j(x')dx',$$

and $\alpha_l = \int_0^1 (x - 1)\varphi_l(x)\, dx = \sqrt{2}\frac{\sin\lambda_{M,l}}{\lambda_{M,l}^2}$. Integrations by parts in time and space yield (104).

From the expansion (104) we can say that

$$w(0, x', t) \quad = \quad -2k^2\sum_{l=0}^{\infty}\int_0^t e^{-k^2\lambda_{M,l}^2(t-s)}\Psi_h(t, s, x')\, ds.$$

The end of the proof is similar to the one of Lemma 2. ☐

Similarly for $g \in H_0^1((0, \infty); L^2(O))$, let w be the unique solution of

$$w_t - (k^2\partial_{\hat{x}_1\hat{x}_1} + \frac{1}{\alpha^2}\Delta_{x'})w = 0 \quad \text{in} \quad \Omega_h \times (0, +\infty) \tag{111}$$

$$w(0, x', t) = g(x', t) \quad x' \in O, t \in (0, +\infty) \tag{112}$$

$$w(x, t) = 0 \quad x \in \partial\Omega_h \setminus I, t \in (0, +\infty) \tag{113}$$

$$w(x, 0) = 0 \quad \text{in} \quad \Omega_h. \tag{114}$$

Due to the lack of regularity, we slightly modify the arguments from Lemma 3.

Lemma 5. *Assume that $g \in H_0^1((0, \infty); L^2(O))$. The solution w of problem (111)–(114) satisfies*

$$\Delta_{x'}^{-1}w_{x_1}(0, x', t) = -\int_0^t K_k(t - s)\Psi_h(t, s, x')\, ds - \Delta_{x'}^{-1}g(t), \tag{115}$$

where the kernel K_k is the one from Lemma 3 and $h = \Delta_{x'}^{-1}\partial_t g - \frac{1}{\alpha^2}g$.

Proof. First assume that $g \in H_0^1(O \times (0, \infty)) \cap H^2(O \times (0, \infty))$. Then setting

$$\tilde{w} = w + (x_1 - 1)g,$$

and using a Fourier expansion, we get

$$\tilde{w}(x,t) = -\sqrt{2}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{D,l}^2+\frac{1}{\alpha^2}\mu_j^2)(t-s)}(h_1(s,\cdot),\psi_j)\,ds\frac{1}{\lambda_{D,l}}\varphi_l\psi_j,$$

where $h_1 = \partial_t g - \frac{1}{\alpha^2}\Delta_{x'}g$, $\lambda_{D,l} = l\pi$ and $\varphi_l(x_1) = \sqrt{2}\sin(\lambda_{D,l}x)$ (see Lemma 3). Differentiating with respect to x_1, we obtain

$$\tilde{w}_{x_1}(0,x',t) = -2\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{D,l}^2+\frac{1}{\alpha^2}\mu_j^2)(t-s)}(h_1(s,\cdot),\psi_j)\,ds\psi_j.$$

As a consequence, we obtain

$$\Delta_{x'}^{-1}\tilde{w}_{x_1}(0,x',t) = -2\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{D,l}^2+\frac{1}{\alpha^2}\mu_j^2)(t-s)}\mu_j^{-2}(h_1(s,\cdot),\psi_j)\,ds\psi_j.$$

Since $-\Delta_{x'}^{-1}\psi_j = \mu_j^{-2}\psi_j$, we obtain

$$\begin{aligned}\Delta_{x'}^{-1}\tilde{w}_{x_1}(0,x',t) &= 2\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{D,l}^2+\frac{1}{\alpha^2}\mu_j^2)(t-s)}(h_1(s,\cdot),\Delta_{x'}^{-1}\psi_j)\,ds\psi_j\\ &= 2\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{D,l}^2+\frac{1}{\alpha^2}\mu_j^2)(t-s)}(\Delta_{x'}^{-1}h_1(s,\cdot),\psi_j)\,ds\psi_j\\ &= 2\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{-(k^2\lambda_{D,l}^2+\frac{1}{\alpha^2}\mu_j^2)(t-s)}(h(s,\cdot),\psi_j)\,ds\psi_j.\end{aligned}$$

Since $H_0^1(O\times(0,\infty))\cap H^2(O\times(0,\infty))$ is dense in $H_0^1((0,\infty);L^2(O))$, the last identity remains valid for $g\in H_0^1((0,\infty);L^2(O))$.

The remainder of proof is similar to the one of Lemma 3. □

6. **The limit problems for the case $n\geq 2$.** In this section, we consider the limit problems of (9) to (16) as k,α go to zero or to ∞.

Coming back to the solution $(y^{(k,\alpha)}, z^{(k,\alpha)})$ of (9) to (16) with $z_0\equiv 0$, and making use of (12) and (103), we can say that $y^{(k,\alpha)}$ satisfies the boundary condition with memory:

$$y_t^{(k,\alpha)}(0,x',t) = \frac{\alpha}{k}G_k(y_{x_1}^{(k,\alpha)})(x',t)\quad x'\in O, t\in(0,+\infty). \tag{116}$$

As before we need to justify the passage to the limit in (116). We first treat the case α tending to 0 or α_0:

Theorem 6.1. *Assume that the triple $(y_0,y_1,0)\in D(\mathcal{A}^{(k,\alpha)})$ or equivalently that $y_0\in E(\Delta,L^2(\Omega_w))$ and $y_1\in H^1(\Omega_w)$ with*

$$y_0\equiv y_1\equiv 0\quad\text{on }\partial\Omega_w\setminus I,\quad y_{0x_1}\equiv y_1\equiv 0\quad\text{on }I.$$

Let $(y^{(k,\alpha)}, z^{(k,\alpha)})$ be the strong solution of (9) to (16) with initial data y_0,y_1 and $z_0\equiv 0$. For all $T>0$, let us set $Q_T=\Omega_w\times(0,T)$. Then for all $T>0$, there exist $y\in H^1(Q_T)$ and a subsequence of $y^{(k,\alpha)}$, still denoted by $y^{(k,\alpha)}$ for the sake of shorthness, such that $y^{(k,\alpha)}$ tends to y weakly in $H^1(Q_T)$ as $k\to k_0$, $\alpha\to\alpha_0$, with

$k_0 \in [0,\infty]$, $\alpha_0 \in [0,\infty)$. *Moreover* y *is the weak solution of the wave equation with Dirichlet boundary condition on* $\partial\Omega_w \setminus I$

$$y_{tt} - \Delta y = 0 \quad \text{in} \quad \Omega_w \times (0,T), \tag{117}$$

$$y(x,t) = 0 \quad x \in \partial\Omega_w \setminus I, t \in (0,T), \tag{118}$$

$$y(x,0) = y_0(x) \quad \text{and} \quad y_t(x,0) = y_1(x) \quad \text{in} \quad \Omega_w. \tag{119}$$

For the boundary condition on I*, we distinguish the following cases:*
1. If $\alpha \to 0$*, then* y *satisfies the Dirichlet boundary condition on* I *:*

$$y(0,x',t) = y_0(0,x') \quad x' \in O, t \in (0,T). \tag{120}$$

2. If $\alpha \to \alpha_0 \in (0,\infty)$*, then the boundary condition on* I *depends on the limit on* k*:*
a. If $k \to \infty$*, then* y *satisfies the Dirichlet boundary condition (120).*
b. If $k \to k_0 \in (0,\infty)$*, then* y *satisfies the boundary condition with memory*

$$y_t(0,x',t) = -\alpha_0 \int_0^t F_{k_0}(s)\Psi_{y_{x_1}}(t,s,x')\,ds, \quad x' \in O, t \in (0,T), \tag{121}$$

c. If $k \to 0$*, then* y *satisfies the boundary condition with memory*

$$y_t(0,t) = K(\Psi_{y_{x_1}}(t,s,x')\,ds) \quad t \in (0,+\infty), \tag{122}$$

where K *is the integral operator defined by (75).*

Proof. The assumptions on the data guarantee that the triple $(y_0, y_1, 0)$ belongs to $D(\mathcal{A}^{(k,\alpha)})$. Therefore by Theorem 2.1 and Lemma 1 the strong solution $(y^{(k,\alpha)}, z^{(k,\alpha)})$ of (9) to (16) satisfies for all $t \geq 0$

$$\int_{\Omega_w} (|\nabla y^{(k,\alpha)}(x,t)|^2 + |y_t^{(k,\alpha)}(x,t)|^2)\,dx + \frac{1}{k\alpha}\int_{\Omega_h} |z^{(k,\alpha)}(x,t)|^2\,dx \\ \leq \int_{\Omega_w} (|\nabla y_0(x)|^2 + |y_1(x)|^2\,dx, \tag{123}$$

$$\int_{\Omega_w} (|\nabla y_t^{(k,\alpha)}(x,t)|^2 + |\Delta y^{(k,\alpha)}(x,t)|^2)\,dx \\ + \frac{1}{k\alpha}\int_{\Omega_h} |(k^2\partial_{\hat{x}_1\hat{x}_1} + \frac{1}{\alpha^2}\Delta_{x'})z^{(k,\alpha)}(x,t)|^2\,dx \\ \leq \int_{\Omega_w} (|\nabla y_1(x)|^2 + |\Delta y_0(x)|^2\,dx. \tag{124}$$

As the right-hand side of these two estimates are independent of t, we deduce that there exists $C > 0$ such that

$$|y^{(k,\alpha)}|_{1,Q_T} + |y_t^{(k,\alpha)}|_{1,Q_T} \leq CT, \tag{125}$$

reminding that $y_{tt}^{(k,\alpha)} = \Delta y^{(k,\alpha)}$. Unfortunately (contrary to the 1d-case), we cannot deduce that the sequence is bounded in $H^2(Q_T)$. But we can say that

$$\|\Delta y^{(k,\alpha)}\|_{0,Q_T} \leq CT. \tag{126}$$

Now we recall that Theorem 1.5.3.10 of [16] shows that if $u \in E(\Delta, L^2(\Omega_w))$, then $\frac{\partial u}{\partial x_1}(0,\cdot)$ belongs to $\tilde{H}^{1/2}(O)'$ with the estimate

$$\|\frac{\partial u}{\partial x_1}(0,\cdot)\|_{\tilde{H}^{1/2}(O)'} \leq C(\|u\|_{1,\Omega_w} + \|\Delta u\|_{0,\Omega_w}). \tag{127}$$

By the estimates (125), (126) and (127), we deduce that

$$\|\frac{\partial y^{(k,\alpha)}}{\partial x_1}\|_{L^\infty(0,T;\tilde{H}^{1/2}(I)')} \leq C. \tag{128}$$

Now as $\tilde{H}^{1/2}(I) = D(\Delta_{x'}^{1/4})$ (because Corollary 2.7.2 of [22] yields $\tilde{H}^{1/2}(I) = [H_0^1(I), L^2(I)]_{1/2}$ and hence $\tilde{H}^{1/2}(I) = [D(\Delta_{x'}^{1/2}), D(\Delta_{x'}^0)]_{1/2} = D(\Delta_{x'}^{1/4})$), we can say that

$$\begin{aligned}
\|\Psi_{y_{x_1}^{(k,\alpha)}}(t,s)\|^2_{\tilde{H}^{1/2}(I)'} &\sim \sum_{j=1}^\infty e^{-\frac{1}{\alpha^2}\mu_j^2(t-s)}\mu_j^{-1/2}| < y_{x_1}^{(k,\alpha)}(s);\psi_j > |^2 \\
&\leq C\sum_{j=1}^\infty \mu_j^{-1/2}| < y_{x_1}^{(k,\alpha)}(s);\psi_j > |^2 = \|y_{x_1}^{(k,\alpha)}(s)\|^2_{\tilde{H}^{1/2}(I)'}.
\end{aligned}$$

By the estimate (128) we conclude that

$$\|\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\|_{\tilde{H}^{1/2}(I)'} \leq C, \quad \forall\, t \geq s. \tag{129}$$

Since the seminorm $H^1(Q_T)$ is a norm on $H^1(Q_T)$ due the Dirichlet boundary condition on $(\{-1\}\times O)\times(0,T)$, from (125) we deduce that the sequences $(y^{(k,\alpha)})_{k,\alpha}$ and $(y_t^{(k,\alpha)})_{k,\alpha}$ are bounded in $H^1(Q_T)$. Consequently there exists $y \in H^1(Q_T)$ such that

$$y^{(k,\alpha)} \to y \text{ weakly in } H^1(Q_T) \text{ as } k \to k_0 \text{ and } \alpha \to \alpha_0, \tag{130}$$

$$y_t^{(k,\alpha)} \to y_t \text{ weakly in } H^1(Q_T) \text{ as } k \to k_0 \text{ and } \alpha \to \alpha_0. \tag{131}$$

Moreover from the compact embedding of $H^1(Q_T)$ into $H^{1-\eta}(Q_T)$ for any $\eta > 0$, we further have

$$y^{(k,\alpha)} \to y \text{ strongly in } H^{1-\eta}(Q_T) \text{ as } k \to k_0 \text{ and } \alpha \to \alpha_0, \tag{132}$$

$$y_t^{(k,\alpha)} \to y_t \text{ strongly in } H^{1-\eta}(Q_T) \text{ as } k \to k_0 \text{ and } \alpha \to \alpha_0, \tag{133}$$

for any $\eta > 0$.

From the first property (130), we see that y satisfies (117) in the distributional sense, while from the second properties we see that y fulfils the boundary condition (118) and the initial condition (119). In order to obtain the boundary condition on I, we recall that the identity (116) showed that

$$y_t^{(k,\alpha)}(0,x',t) = -\alpha\int_0^t F_k(s)\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\,ds, \quad x' \in O, t \in (0,+\infty), \tag{134}$$

owing to (105). We therefore need to justify the passage to the limit in the above identity.

First if α tends to 0, then as

$$F_k(s) \leq \frac{C}{\sqrt{s}}, \tag{135}$$

and making use of the estimate (129), we deduce that

$$\begin{aligned}
\|\int_0^t F_k(s)\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\,ds\|_{\tilde{H}^{1/2}(I)'} &\leq \int_0^t F_k(s)\|\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\|_{\tilde{H}^{1/2}(I)'}\,ds \\
&\leq C\int_0^t \frac{1}{\sqrt{s}}\,ds.
\end{aligned}$$

This shows that, for $\alpha \to 0$,

$$y_t^{(k,\alpha)} \to 0 \text{ in } L^\infty((0,T); \tilde{H}^{1/2}(I)').$$

As (133) implies that $y_t^{(k,\alpha)} \to y_t$ in $L^2(I \times (0,T))$, we deduce that

$$y_t(0,\cdot) = 0. \tag{136}$$

and then (120).

As in $1-d$ when α tends to α_0, we distinguish the cases $k \to 0$, $k \to k_0$ and $k \to \infty$.

If $k \to +\infty$, then we first have

$$F_k(s) \to 0.$$

The estimate (135) and the dominate convergence theorem of Lebesgue yields

$$\int_0^t F_k(s)\, ds \to 0.$$

This property combined with the estimate (129) gives

$$\| \int_0^t F_k(s) \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\, ds \|_{\tilde{H}^{1/2}(I)'} \le C \int_0^t F_k(s)\, ds \to 0,$$

and therefore

$$\int_0^t F_k(s) \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\, ds \to 0 \text{ in } \tilde{H}^{1/2}(I)' \text{ a. e. in } (0,T),$$

and by (134) we conclude the boundary condition (136) and then (120).

If $k \to k_0$, then by (134) we directly get (121).

Finally if $k \to 0$, as before setting $F(s) = \frac{1}{\sqrt{\pi s}}$, we may write

$$\int_0^t F_k(s) \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\, ds = \\ \int_0^t (F_k(s) - F(s)) \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\, ds + \int_0^t F(s) \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\, ds.$$

For the first term using the estimates (84) and (129), we directly deduce that

$$\int_0^t (F_k(s) - F(s)) \Psi_{y_x^{(k,\alpha)}}(t,s,x')\, ds \to 0 \text{ in } L^\infty((0,T); \tilde{H}^{1/2}(I)').$$

On the other hand, by the estimate (128) and the Banach-Alaoglu theorem, we deduce that

$$y_{x_1}^{(k,\alpha)} \to y_{x_1} \text{ weakly in } L^q(0,T; \tilde{H}^{1/2}(I)'), \quad \forall\, q > 1.$$

By the definition of Ψ_h, we deduce that

$$\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x') \to \Psi_{y_{x_1}}(t,s,x') \text{ weakly in } L^q(0,T; \tilde{H}^{1/2}(I)').$$

Indeed for $w \in L^p(0,T; \tilde{H}^{1/2}(I))$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} &< \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x') - \Psi_{y_{x_1}}(t,s,x'); w > \\ &\qquad = \int_0^t < e^{\frac{1}{\alpha^2}(t-s)\Delta_{x'}} (y_{x_1}^{(k,\alpha)} - y_{x_1})(s), w(s) > \, ds \\ &\qquad = \int_0^t < (y_{x_1}^{(k,\alpha)} - y_{x_1})(s), e^{\frac{1}{\alpha^2}(t-s)\Delta_{x'}} w(s) > \, ds \to 0. \end{aligned}$$

As F belongs to $L^p(0,T), \forall p \in [1,2)$, we deduce that

$$\int_0^t F(s)\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\,ds \to \int_0^t F(s)\Psi_{y_{x_1}}(t,s,x')\,ds \text{ weakly in } \tilde{H}^{1/2}(I)' \text{ as } k \to 0.$$

Indeed for any $\varphi \in \tilde{H}^{1/2}(I)$, we may write

$$\begin{aligned} &< \int_0^t F(s)(\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x') - \Psi_{y_{x_1}}(t,s,x'))\,ds; \varphi > \\ &\qquad = \int_0^t < \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x') - \Psi_{y_{x_1}}(t,s,x')); F(s)\varphi > \,ds \\ &\qquad =< \Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x') - \Psi_{y_{x_1}}(t,s,x')); F(s)\varphi > \to 0, \end{aligned}$$

since $F(\cdot)\varphi$ belongs to $L^p(0,T;\tilde{H}^{1/2}(I))$.

Alltogether we have shown that

$$\int_0^t F_k(s)\Psi_{y_{x_1}^{(k,\alpha)}}(t,s,x')\,ds \to \int_0^t F(s)\Psi_{y_{x_1}}(t,s,x')\,ds \text{ weakly in } \tilde{H}^{1/2}(I)' \text{ as } k \to 0.$$

This property and the fact that $y_t^{(k,\alpha)}(0,x',t)$ tends to $y_t(0,x',t)$ weakly in $H^{1/2}(I \times (0,T))$ allow to pass to the limit in the identity (134) and to obtain (122). ☐

It remains to consider the case when α tends to ∞.

Theorem 6.2. *Assume that $(y_0, y_1, 0)$ belongs to $D((\mathcal{A}^{(k,\alpha)})^2)$, or equivalently that $y_0 \in E(\Delta, H^1(\Omega_w))$ and $y_1 \in E(\Delta, L^2(\Omega_w))$ with*

$$y_0 \equiv \Delta y_0 \equiv y_1 = 0 \quad \text{on } \partial\Omega_w \setminus I, \quad y_{0x_1} \equiv \Delta y_0 \equiv y_1 \equiv y_{1x_1} \equiv 0 \quad \text{on } I.$$

Let $(y^{(k,\alpha)}, z^{(k,\alpha)})$ be the strong solution of (9) to (16) with initial data y_0, y_1 and $z_0 \equiv 0$. Then for all $T > 0$, there exist $y \in H^1(Q_T)$ and a subsequence of $y^{(k,\alpha)}$, still denoted by $y^{(k,\alpha)}$ for the sake of shorthness, such that $y^{(k,\alpha)}$ tends to y weakly in $H^1(Q_T)$ as $k \to k_0$ and $\alpha \to \infty$, with $k_0 \in [0,\infty]$. Moreover y is the weak solution of the wave equation with Dirichlet boundary condition at the exterior boundary of Ω_w, namely satisfies (117), (118) and (119). For the boundary condition on I, we distinguish the following cases:

1. If $\frac{k}{\alpha} \to 0$, then y satisfies the Neumann boundary condition on I:

$$y_{x_1}(0,\cdot) = 0 \text{ on } I. \tag{137}$$

2. If $\frac{k}{\alpha} \to \infty$, then y satisfies the Dirichlet boundary condition (120).

3. If $\frac{k}{\alpha} \to \kappa_0 \in (0,\infty)$, then y satisfies the dissipative boundary condition

$$y_{x_1} + \kappa_0 y_t = 0 \text{ on } I. \tag{138}$$

Proof. As in Theorem 6.1, since the estimate (125) is still valid, there exists $y \in H^1(Q_T)$ weak limit in $H^1(Q_T)$ and strong limit in $H^{1-\eta}(Q_T)$ of $y^{(k,\alpha)}$ as $k \to k_0$ and $\alpha \to \infty$. As before these properties imply that y satisfies (117) to (119). It remains to analyze the boundary condition on I.

Here we use the fact that the initial datum $(y_0, y_1, 0)$ belongs to $D((\mathcal{A}^{(k,\alpha)})^2)$, that yields by Lemma 1

$$\tilde{E}^{(2)}(t) \le \tilde{E}^{(2)}(0), \ \forall t > 0,$$

which implies

$$\int_{\Omega_w} (|\nabla \Delta y^{(k,\alpha)}(x,t)|^2 + |\Delta y_t^{(k,\alpha)}(x,t)|^2)\,dx \le \int_{\Omega_w} (|\nabla \Delta y_0(x)|^2 + |\Delta y_1(x)|^2)\,dx.$$

From this estimate and the estimate (124) we see that the sequence $(y_{tt}^{(k,\alpha)})_{k,\alpha}$ is bounded in $H^1(Q_T)$ (reminding that $y_{tt}^{(k,\alpha)} = \Delta y^{(k,\alpha)}$) and therefore by the Sobolev embedding theorem it admits a subsequence, still denoted by $y_{tt}^{(k,\alpha)}$, that converges to y_{tt} in $H^{1-\eta}(Q_T)$. By a standard trace theorem, we conclude that

$$y_{tt}^{(k,\alpha)}(0,\cdot) \to y_{tt}(0,\cdot) \text{ in } H^{1/2-\eta}(I \times (0,T)). \tag{139}$$

By (133) and again a trace theorem we deduce that

$$h^{k,\alpha} := \Delta_{x'}^{-1} y_{tt}^{(k,\alpha)} - \frac{1}{\alpha^2} y_t^{(k,\alpha)} \to \Delta_{x'}^{-1} y_{tt} \text{ in } L^q((0,T);L^2(I)), \tag{140}$$

for some $q > 2$, recalling that $1/\alpha \to 0$, since $\alpha \to \infty$ and that $H^{1/2-\eta}(I \times (0,T)) \hookrightarrow L^q(I \times (0,T)) \hookrightarrow L^q((0,T);L^2(I))$, for some $q > 2$ (close enough to 2). Note finally that the definition of Ψ_h leads to

$$\|\Psi_{h^{(k,\alpha)}}(t,s,x')\|_{0,I} \le \|h^{(k,\alpha)}(s,x')\|_{0,I}, \text{ for a.e. } s \in (0,t),$$

and then, by (140)

$$\int_0^t \|\Psi_{h^{(k,\alpha)}}(t,s,x')\|_{0,I}^q \, ds \le \int_0^t \|h^{(k,\alpha)}(s,x')\|_{0,I}^q \, ds \le C. \tag{141}$$

Now using Lemma 5, we may write

$$\Delta_{x'}^{-1} z_{x_1}^{(k,\alpha)}(0,x',t) = -\int_0^t K_k(t-s)\Psi_{h^{(k,\alpha)}}(t,s,x')\,ds - \Delta_{x'}^{-1} y_t^{(k,\alpha)}(x',t). \tag{142}$$

Since

$$y_{x_1}^{(k,\alpha)}(0,x',t) = \frac{k}{\alpha} z_{x_1}^{(k,\alpha)}(0,x',t),$$

we deduce that

$$\Delta_{x'}^{-1}(y_{x_1}^{(k,\alpha)}(0,x',t) + \frac{k}{\alpha} y_t^{(k,\alpha)}(x',t)) = -\frac{k}{\alpha}\int_0^t K_k(t-s)\Psi_{h^{(k,\alpha)}}(t,s,x')\,ds. \tag{143}$$

Now using the estimates (62) and (141) we deduce that

$$\frac{k}{\alpha}\|\int_0^t K_k(t-s)\Psi_{h^{(k,\alpha)}}(t,s,x')\,ds\|_{0,I} \le \frac{C}{\alpha} t^{1-p/2} \to 0 \text{ as } \alpha \to \infty, \tag{144}$$

where $1/p + 1/q = 1$ (and then $p < 2$). Consequently we have obtained

$$\|\Delta_{x'}^{-1}(y_{x_1}^{(k,\alpha)}(0,x',t) + \frac{k}{\alpha} y_t^{(k,\alpha)}(x',t))\|_{0,I} \to 0 \text{ as } \alpha \to \infty. \tag{145}$$

If $\frac{k}{\alpha} \to 0$, then the property (145) and the property (133) lead to

$$\Delta_{x'}^{-1} y_{x_1}^{(k,\alpha)}(0,x',t) \to 0 \text{ in } L^\infty(0,T;L^2(I)) \text{ as } \alpha \to \infty,$$

and then to

$$y_{x_1}^{(k,\alpha)}(0,x',t) \to 0 \text{ in } L^\infty(0,T;D(\Delta_{x'}^{-1})) \text{ as } \alpha \to \infty.$$

As

$$y_{x_1}^{(k,\alpha)}(0,x',t) \to y_{x_1}(0,x',t) \text{ weakly in } L^q(0,T;D(\Delta_{x'}^{-1/4})) \text{ as } \alpha \to \infty,$$

we deduce that the boundary condition (137) holds.

On the contrary in the case $\frac{k}{\alpha} \to \infty$, then by the triangular inequality, we have

$$\begin{aligned}\|\Delta_{x'}^{-1} y_t^{(k,\alpha)}(x',t))\|_{0,I} \le \frac{\alpha}{k}\|\Delta_{x'}^{-1}(y_{x_1}^{(k,\alpha)}(0,x',t) + \frac{k}{\alpha} y_t^{(k,\alpha)}(x',t))\|_{0,I} \\ + \frac{\alpha}{k}\|\Delta_{x'}^{-1} y_{x_1}^{(k,\alpha)}(0,x',t)\|_{0,I}.\end{aligned}$$

Then the property (145) and the estimate (128) lead to

$$\Delta_{x'}^{-1} y_t^{(k,\alpha)}(0, x', t) \to 0,$$

and then to the boundary condition

$$y_t(0, x', t) = 0,$$

which leads to the Dirichlet boundary condition

$$y(0, x', t) = y_0(0, x').$$

It remains to consider the case $\frac{k}{\alpha} \to \kappa_0$. In that case, from (145) and passage to the limit we get (138). Note that this boundary condition leads to an exponential decay of the energy. □

Note that the different limit problems are analogous to the one-dimensional case and are listed in Table 2. Furthermore the same comments than the ones from Remark 1 can be made.

7. **Coming back to the original problem.** In this section we will state the limit problems of (1) – (8) as c, ϵ go to zero, to infinity or to constant values. The proofs of the theorems below follow from Theorems 4.1, 4.2, 6.1, 6.2, recalling that $k = c/\epsilon$ and $\alpha = \frac{1}{c}$. For shorthness, we do not distinguish between the case $n = 1$ and the case $n \geq 2$.

Theorem 7.1. *Assume that $(y_0, y_1, 0) \in D(\mathcal{A}^{(k,\alpha)})$ or equivalently that*

$$y_0 \in E(\Delta, L^2(\Omega_w)) \quad \text{and} \quad y_1 \in H^1(\Omega_w)$$

with

$$y_0 \equiv y_1 \equiv 0 \quad \text{on } \partial\Omega_w \setminus I, \quad y_{0x_1} \equiv y_1 \equiv 0 \quad \text{on } I.$$

Let $(y^{(\epsilon,c)}, z^{(\epsilon,c)})$ be the strong solution of (9) to (16) with initial data y_0, y_1 and $z_0 \equiv 0$. For all $T > 0$, let us set $Q_T = \Omega_w \times (0,T)$. Then for all $T > 0$, there exist $y \in H^1(Q_T)$ and a subsequence of $y^{(\epsilon,c)}$, still denoted by $y^{(\epsilon,c)}$ for the sake of shorthness, such that $y^{(\epsilon,c)}$ tends to y weakly in $H^1(Q_T)$ as $c \to c_0$ and $\epsilon \to \epsilon_0$, with $c_0 \in (0, \infty]$, $\epsilon_0 \in [0, \infty]$. Moreover y is the weak solution of the wave equation with Dirichlet boundary condition on $\partial\Omega_w \setminus I$

$$y_{tt} - \Delta y = 0 \quad \text{in} \quad \Omega_w \times (0, T), \tag{146}$$

$$y(x, t) = 0 \quad x \in \partial\Omega_w \setminus I, t \in (0, T), \tag{147}$$

$$y(x, 0) = y_0(x) \quad \text{and} \quad y_t(x, 0) = y_1(x) \quad \text{in} \quad \Omega_w. \tag{148}$$

For the boundary condition on I, we distinguish the following cases:
1. If $c \to \infty$, then y satisfies the Dirichlet boundary condition (120) on I.
2. If $c \to c_0 \in (0, \infty)$, then the boundary condition on I depends on the limit on ϵ:
a. If $\epsilon \to 0$, then y satisfies the Dirichlet boundary condition (120).
b. If $\epsilon \to \epsilon_0 \in (0, \infty)$, then y satisfies the boundary condition with memory (121).
c. If $\epsilon \to \infty$, then y satisfies the boundary condition with memory (122).

Theorem 7.2. *Assume that $(y_0, y_1, 0)$ belongs to $D((\mathcal{A}^{(k,\alpha)})^2)$, or equivalently that $y_0 \in E(\Delta, H^1(\Omega_w))$ and $y_1 \in E(\Delta, L^2(\Omega_w))$ with*

$$y_0 \equiv \Delta y_0 \equiv y_1 = 0 \quad \text{on } \partial\Omega_w \setminus I, \quad y_{0x_1} \equiv \Delta y_0 \equiv y_1 \equiv y_{1x_1} \equiv 0 \quad \text{on } I.$$

Let $(y^{(\epsilon,c)}, z^{(\epsilon,c)})$ be the strong solution of (9) to (16) with initial data y_0, y_1 and $z_0 \equiv 0$. Then for all $T > 0$, there exist $y \in H^1(Q_T)$ and a subsequence of $y^{(\epsilon,c)}$, still denoted by $y^{(\epsilon,c)}$ for the sake of shorthness, such that $y^{(\epsilon,c)}$ tends to y weakly

	$\epsilon \to \epsilon_0$	$\epsilon \to +\infty$	$\epsilon \to 0$
$c \to c_0$	memory bc	memory bc	Dirichlet bc
$c \to +\infty$	Dirichlet bc	Dirichlet bc	Dirichlet bc
$c \to 0$	Neumann bc	Neumann bc	$\frac{c^2}{\epsilon} \to 0$:Neumann bc
$c \to 0$			$\frac{c^2}{\epsilon} \to \kappa_0$: dissipative bc
$c \to 0$			$\frac{c^2}{\epsilon} \to \infty$: Dirichlet bc

TABLE 3. Summary of the limit problems

in $H^1(Q_T)$ as $c \to 0$ and $\epsilon \to \epsilon_0$, with $\epsilon_0 \in [0, \infty]$. Moreover y is the weak solution of the wave equation with Dirichlet boundary condition at the exterior boundary of Ω_w, namely satisfies (117), (118) and (119). For the boundary condition on I, we distinguish the following cases:

1. If $\frac{c^2}{\epsilon} \to 0$, then y satisfies the Neumann boundary condition (137) on I.

2. If $\frac{c^2}{\epsilon} \to \infty$, then y satisfies the Dirichlet boundary condition (120).

3. If $\frac{c^2}{\epsilon} \to \kappa_0 \in (0, \infty)$, then y satisfies the dissipative boundary condition (138).

The results of these theorems are summarized in Table 3. The interpretations of these results are the following ones:

- If the diffusion coefficient becomes very large (i.e., $c \to \infty$), then the solution of the heat equation becomes too small in order to influence the wave part. Hence the limit problem is no more dissipative.
- If the diffusion coefficient tends to some $c_0 \in (0, \infty)$, then the limit process depends on the limit of the thickness. If the thickness tends to zero, again the solution of the heat equation becomes too small in order to influence the wave part. On the contrary if the thickness tends to infinity, we may expect a decay of the limit problem, but as said in Remark 1, the decay in this case is an open problem.
- If the diffusion coefficient becomes very small (i.e., $c \to 0$), then in order to have a decay in the wave part, the thickness of the heat part has to be of the order of c^2; in the other cases, the limit problem is no more dissipative.

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