

Research article

Distributed barrier iterative learning control for networked Euler–Lagrange systems under time-varying delays

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Abstract: Coordinated trajectory tracking in networks of robotic manipulators is fundamental to collaborative automation, yet strict joint-velocity limits and network-induced communication delays frequently cause standard controllers to violate safety constraints or diverge. Existing decentralized iterative learning control (ILC) methods fail here because they rely on scalar first-order dynamics and centralized topologies, which cannot handle coupled second-order Euler–Lagrange (E–L) dynamics and asynchronous time-varying delays. A distributed time-varying barrier Lyapunov function (TV-BLF) norm-optimal ILC framework is therefore developed. Embedding the TV-BLF into a distributed alternating direction method of multipliers (ADMM) cost and constructing a composite Lyapunov–Krasovskii functional compensates communication delays while strictly enforcing velocity constraints. Under the standard nonlifted ILC sensitivity approximation, a quantitative margin argument establishes semi-globally uniformly ultimately bounded (SGUUB) stability with explicit delay-dependent bounds, closing the invariance gap of a purely qualitative barrier argument. Monte Carlo simulations on a heterogeneous fleet of 2-DoF manipulators show a statistically significant reduction in mean squared error (MSE) and zero constraint violations versus three baselines, while preserving $O(N)$ per-agent complexity under a bounded-ADMM-iteration proviso.

Keywords: distributed iterative learning control; Euler–Lagrange systems; time-varying barrier Lyapunov function; time-varying communication delays; ADMM optimization; multi-agent systems

1. Introduction

Coordinated tracking in networks of robotic manipulators underpins collaborative automation, from multi-arm assembly to distributed surgical robots. Two concurrent constraints, however, limit standard controllers: strict joint-velocity limits mandated by safety standards (e.g., ISO/TS 15066), and time-varying communication delays $\tau_{ij}(t)$ inherent in networked control systems (NCS) that degrade consensus and stability. Violating either triggers post-hoc saturation (which destroys monotonic convergence) or outright divergence, risking mechanical failure.

Iterative learning control (ILC) refines inputs from prior-trial errors and is the leading paradigm for repetitive

tasks [1]. Classical norm-optimal ILC (NOILC) aggregates signals into supervectors with $O(N^3)$ cost, prohibitive for embedded multi-DoF deployment. Decentralized nonlifted NOILC (N-NOILC) and constrained lifted ILC reduce this cost or handle input limits, but only for linear, scalar, or centrally coordinated plants. Consequently, existing decentralized ILC: (i) models agents as integrators or first-order motors, missing the coupled inertia $M_i(q_i)$ and Coriolis $C_i(q_i, \dot{q}_i)$ cross-coupling of Euler–Lagrange (E–L) manipulators; (ii) assumes synchronous or centralized communication, so time-varying $\tau_{ij}(t)$ disrupt alternating direction method of multipliers (ADMM) consensus; and (iii) lacks a discrete-time barrier certificate, as embedding a time-varying barrier Lyapunov function (TV-BLF) into a discrete trial-domain cost for E–L systems voids standard

Lyapunov decrease under network delays.

Recent works address isolated facets: distributed ILC for nonlinear multiagent systems (MASes) [7], ADMM-based distributed control under delays [8], and barrier-based safe control for E-L systems [5]. None resolves the intersection of second-order E-L dynamics, time-varying delays, and strict discrete velocity constraints. The difficulty is structural rather than incremental: A continuous-time BLF certificate does not survive forward-Euler discretization of the coupled E-L map, and a discrete barrier term evaluated at the completed trial is a constant in the next-trial cost, so its constraint-enforcing effect must be recovered through a curvature surrogate inside the ADMM Hessian rather than through a direct Lyapunov decrease. This paper closes that gap with a distributed TV-BLF N-NOILC framework: Embedding the TV-BLF in the local ADMM cost makes the infinite barrier gradient penalize delay-induced overshoots, and a distributed consensus protocol compensates asynchronous data. The main contributions are as follows:

- (i) **Distributed TV-BLF for E-L MAS:** A block-matrix ADMM update for second-order dynamics that enforces time-varying velocity limits without post-hoc clipping;
- (ii) **Delay-compensated consensus:** A first-in-first-out (FIFO)-buffered ADMM protocol for directed graphs with $\tau_{ij}(t) \leq \bar{\tau}$, certified by a Lyapunov-Krasovskii functional to keep the consensus error within $O(\bar{\tau}^2)$;
- (iii) **Semi-globally uniformly ultimately bounded (SGUUB) certificate:** The first discrete-time SGUUB proof for TV-BLF ILC on E-L systems under delays, with explicit ultimate bounds and a quantitative barrier-margin recursion;
- (iv) **$O(N)$ viability:** Per-step 2×2 block inversion preserving linear trial complexity, validated by $N_{MC} = 500$ Monte Carlo runs.

Section 2 formulates the problem, Section 3 derives the algorithm and its tuning, Section 4 proves stability, Section 5 validates and discusses the results, and Section 6 concludes.

2. Preliminaries and problem formulation

Consider a fleet of s heterogeneous 2-DoF robotic manipulators, indexed by $i = 1, \dots, s$. The Euler-Lagrange dynamics of agent i are [9]

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i + d_i(q_i, \dot{q}_i, t), \quad (2.1)$$

with $q_i, \dot{q}_i, \ddot{q}_i \in \mathbb{R}^2$, positive-definite M_i , Coriolis matrix C_i , gravity G_i , torque τ_i , and lumped uncertainty d_i . Discretizing with period T_s , we obtain:

$$q_{i,k}(t+1) = q_{i,k}(t) + T_s \dot{q}_{i,k}(t), \quad (2.2)$$

$$\begin{aligned} \dot{q}_{i,k}(t+1) = & \dot{q}_{i,k}(t) + T_s M_i^{-1}(q_{i,k})[\tau_{i,k}(t) \\ & - C_i \dot{q}_{i,k}(t) - G_i + d_{i,k}(t)]. \end{aligned} \quad (2.3)$$

The topology is a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{A} = [a_{ij}]$. Figure 1 depicts the simulated three-manipulator instance of this graph. Manipulator i is annotated with joint positions $q_{i,j}$ [rad], velocities $\dot{q}_{i,j}$ [rad/s], link lengths $l_{i,j}$, control torque $\tau_i \in \mathbb{R}^2$, and base frame (x_i, y_i) at O_i . Throughout, $\tau_{ij}(t)$ denotes the delay with which agent i receives agent j 's signal (i.e., the arrow $j \rightarrow i$ carries the label τ_{ij}), the convention used in (3.8). The leader/reference node broadcasts $q_d(t)$ to Agents 1 and 2, and Agent 3 is reached through the peer link $2 \rightarrow 3$. Together with the peer links $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 1$, the graph therefore contains a spanning tree rooted at the leader (Assumption 2.2). The dashed domes mark the TV-BLF velocity constraint $\|\dot{q}_{i,k}(t)\| < k_{v,i}(t)$ enforced by Theorem 4.4(S1).

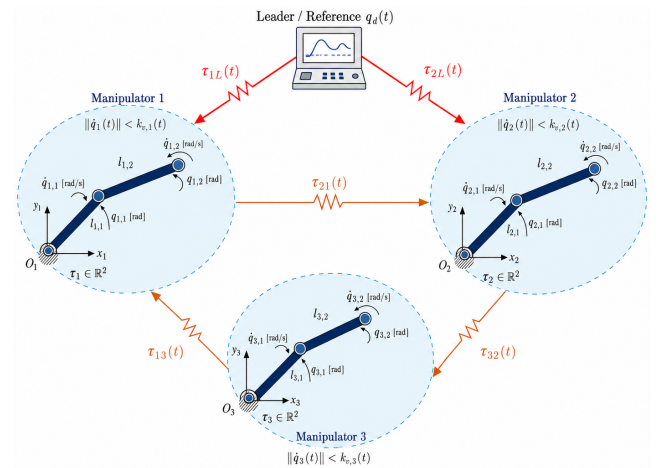


Figure 1. Networked 2-DoF Euler-Lagrange manipulator topology with directed, delayed communication links $\tau_{ij}(t) \leq \bar{\tau}$.

Assumption 2.1. $q_{i,k}(0) = q_d(0)$, and $\dot{q}_{i,k}(0) = \dot{q}_d(0)$ for all i, k .

Remark 2.1. If initialization varies by $|q_{i,k}(0) - q_d(0)|, |\dot{q}_{i,k}(0) - \dot{q}_d(0)| \leq \delta_0 < \eta_0$ (the margin of Assumption 2.5), the bound of Theorem 4.4 grows by $O(\delta_0)$, and constraints hold by the triangle inequality. The cold start $\tau_{i,0} = 0$ seeds the induction of Theorem 4.4(S1).

Assumption 2.2. $\mathcal{G}$ contains a spanning tree rooted at the reference node.

Assumption 2.3. $0 \leq \tau_{ij}(t) \leq \bar{\tau} \leq \kappa T_s$ with known $\bar{\tau}$ and design constant $\kappa \geq 1$ (typically $\kappa \in [1, 4]$ [8]).

Assumption 2.4. $d_i = W_i^{*T} \phi_i(q_i, \dot{q}_i) + \epsilon_i$ with $\|W_i^*\|_F \leq \bar{W}_i^*$, $|\epsilon_i| \leq \bar{\epsilon}_i$.

Assumption 2.5. The reference and the limits $k_{v,i}(t) > 0$ are bounded, with a uniform feasibility margin $\eta_0 > 0$: $\|\dot{q}_d(t)\| \leq k_{v,i}(t) - \eta_0$ for all t, i .

The objective is $\tau_{i,k}(t)$ driving $e_{i,k}(t) = q_d(t) - q_{i,k}(t) \rightarrow 0$ while $\|\dot{q}_{i,k}(t)\| < k_{v,i}(t)$ for all t, k .

3. Proposed methodology

3.1. Local TV-BLF cost function

To enforce velocity limits without clipping, the TV-BLF enters each local cost as follows:

$$J_{i,k+1}(t) = \|e_{i,k+1}(t+1)\|_{Q_i}^2 + \|\Delta\tau_{i,k+1}(t)\|_{R_i}^2 + \alpha_i V_{B,i}(\dot{q}_{i,k}(t)), \quad (3.1)$$

with $Q_i, R_i > 0$, $\Delta\tau_{i,k+1} = \tau_{i,k+1} - \tau_{i,k}$, and $V_{B,i}(\dot{q}_i) = \frac{1}{2} \ln \frac{k_{v,i}^2}{k_{v,i}^2 - \|\dot{q}_i\|^2}$, whose gradient diverges as $\|\dot{q}_i\| \rightarrow k_{v,i}$. Let the network-aggregated error

$$E_k(t) \triangleq \sum_{j=1}^s e_{j,k}(t), \quad e_{j,k}(t) = q_d(t) - q_{j,k}(t), \quad (3.2)$$

be reconstructed at every node by the consensus protocol below (Assumption 2.2). The neural network (NN)-compensated estimate is

$$e_{i,k+1}(t+1) = \frac{1}{s} E_k(t+1) + \Phi_i \Delta\tau_{i,k+1}(t) - \hat{W}_{i,k}^T \phi_i(\dot{q}_{i,k}(t)), \quad (3.3)$$

with sensitivity $\Phi_i = T_s M_i^{-1}(q_{i,k})$ and the BLF evaluated at the fully-observable trial- k state.

Remark 3.1 (Nonlifted sensitivity model). Equation (3.3) adopts the standard current-cycle-feedback ILC model [3]. The trial-to-trial map is decoupled across t , so $\Delta\tau_{i,k+1}(t)$ acts through the one-step sensitivity Φ_i from linearizing (2.3), not through the exact lifted Jacobian (whose $O(N^3)$ cost the N-NOILC baselines [6, 11] also avoid). This is valid for small T_s and slowly varying M_i . All guarantees of Section 4 are stated relative to it, and a lifted exact-sensitivity treatment is left for future work.

The BLF weight satisfies

$$\alpha_i \geq \frac{\lambda_{\min}(Q_i) k_{v,i,\min}^2}{4}, \quad (3.4)$$

which, with $Q_i = 10$ and $k_{v,i,\min} = 1.2$ rad/s, gives a minimum of 3.6. The chosen $\alpha_i = 5.0$ therefore provides a $\approx 39\%$ margin.

3.2. Distributed ADMM updates

With auxiliary $z_{i,k+1}$ and dual $\gamma_{i,k+1}$, the **u-update** is

$$\tau_{i,k+1}^{\theta+1}(t) = H_i^{-1}(t) b_i^\theta(t), \quad (3.5)$$

with positive-definite Hessian

$$H_i(t) = \Phi_i^T Q_i \Phi_i + R_i + \frac{\rho}{2} (|N_i| + 1) I_2 + \alpha_i \Lambda_i(t), \quad (3.6)$$

$\Lambda_i(t) = \frac{\|\dot{q}_{i,k}\|^2}{(k_{v,i}^2 - \|\dot{q}_{i,k}\|^2)^2} I_2$, invertible analytically in $O(1)$ for $m = 2$.

Remark 3.2 (Barrier curvature surrogate). Since $V_{B,i}$ in (3.1) is taken at the known trial- k state, it is a constant offset in $J_{i,k+1}$. The term $\alpha_i \Lambda_i$ in (3.6) is therefore not its true Hessian but an isotropic Gauss–Newton surrogate, $\alpha_i [\nabla_{\dot{q}} V_{B,i}] [\nabla_{\dot{q}} V_{B,i}]^T \approx \alpha_i \Lambda_i I_2$, evaluated at the predicted velocity $\dot{q}_{i,k} + \Phi_i \Delta\tau_{i,k+1}$. This guarantees $H_i > 0$ by construction and, as used in Theorem 4.4(S1), $H_i \rightarrow \infty$ as $\|\dot{q}_{i,k}\| \rightarrow k_{v,i}$, the only property the invariance argument needs.

The right-hand side is

$$b_i^\theta(t) = R_i \tau_{i,k}(t) + \Phi_i^T \left[Q_i \hat{e}_{i,k}(t+1) + \frac{\rho}{2} z_i^\theta - \frac{1}{2} \gamma_i^\theta \right], \quad (3.7)$$

$\hat{e}_{i,k}(t+1) = \frac{1}{s} E_k(t+1) - \hat{W}_{i,k}^T \phi_i(\dot{q}_{i,k}(t))$. The **z-update** averages delayed neighbor signals retrieved from a FIFO buffer, $\tau_{j,k+1}^\theta(t - \tau_{ij}) = \text{FIFO_Buffer}_j \{\tau_{j,k+1}^\theta\}(t - \tau_{ij})$:

$$z_{i,k+1}^{\theta+1}(t) = \frac{1}{|N_i|+1} \left[\sum_{j \in N_i} (\Phi_j \tau_{j,k+1}^\theta(t - \tau_{ij}) + \frac{\gamma_{ij,k+1}^\theta(t - \tau_{ij})}{\rho}) + \Phi_i \tau_{i,k+1}^{\theta+1}(t) + \frac{\gamma_{ii,k+1}^\theta(t)}{\rho} \right]. \quad (3.8)$$

The γ -update is, per edge $(i, j) \in \mathcal{E}$,

$$\gamma_{ij,k+1}^{\theta+1}(t) = \gamma_{ij,k+1}^\theta(t) + \rho(\Phi_i \tau_{i,k+1}^{\theta+1}(t) - z_{i,k+1}^{\theta+1}(t)). \quad (3.9)$$

3.3. Composite neural adaptive robustness

NN weights are updated causally between trials:

$$\hat{W}_{i,k+1}(t) = (1 - \sigma_i \Gamma_i) \hat{W}_{i,k}(t) - \Gamma_i \phi_i(\dot{q}_{i,k}(t)) e_{i,k}(t+1) Q_i \Phi_i, \quad (3.10)$$

with adaptation gain $\Gamma_i > 0$ and leakage $\sigma_i > 0$. The closed loop (controller, FIFO buffer, E-L plant, radial basis function neural network (RBFNN), consensus block) executes the logic of Algorithm 1.

Algorithm 1: Distributed TV-BLF N-NOILC for E-L MAS with delays.

Input: $Q_i, R_i, \rho, \alpha_i, k_{v,i}(t), \bar{\tau}$; initial $\tau_{i,0}, \hat{W}_{i,0}$

Output: Torques $\{\tau_{i,k}(t)\}$

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1 for trial  $k = 0$  to  $k_{\max}$  do
2   for time  $t = 0$  to  $N - 1$  do
3     (P1)  $H_i, b_i^\theta$  via (3.6)–(3.7);  $\tau_{i,k+1}^{\theta+1} = H_i^{-1} b_i^\theta$ ;
4     (P2) delayed  $z_{i,k+1}^{\theta+1}$  via FIFO buffer, (3.8);
5     (P3)  $\gamma_{ij,k+1}^{\theta+1} = \gamma_{ij,k+1}^\theta + \rho(\Phi_i \tau_{i,k+1}^{\theta+1} - z_{i,k+1}^{\theta+1})$ ;
6     if  $\|r_p\|^2 + \|r_d\|^2 < \epsilon_{\text{admm}}$  or  $\theta \geq \theta_{\max}$  then
7       break;
8     end
9      $\theta \leftarrow \theta + 1$ ;
10  end
11  (P4) intertrial NN update  $\hat{W}_{i,k+1}$  via (3.10);
12 end

```

3.4. Parameter selection and design guidelines

The framework exposes six design quantities whose admissible ranges are fixed by the analysis, leaving a single scalar degree of freedom per quantity for the practitioner. Table 1 lists the nominal values. The rationale is as follows. The barrier weight α_i is bounded below by (3.4). Choosing

it at the minimum would make the margin recursion (4.4) tight ($\eta_i^\dagger \rightarrow \eta_0$), so a margin above the minimum trades a slightly slower early-trial settling (Section 5) for a wider self-enforcing region: The 39% overhead used here keeps violations at zero across the full Monte Carlo ensemble while adding negligible conservatism. The ADMM penalty ρ is set to the order of magnitude of the data term $\Phi_i^T Q_i \Phi_i + R_i$ in (3.6), following the balancing heuristic of [4]: A ρ one decade too large slows the dual residual, one decade too small slows the primal, and the chosen value equalizes the two geometric rates observed in Figure 2a. The Lyapunov gain μ_i must satisfy $\mu_i < \lambda_{\min}(Q_i)/(2c_{B,i})$, the condition of Theorem 4.4 (with $c_{B,i}$ defined in (4.2)). We place it at half that ceiling to retain a factor-of-two robustness margin against modeling error in $c_{B,i}$. The neural gains obey $0 < \sigma_i \Gamma_i < 1$ (Lemma 4.1). The product is fixed at 0.1 so that leakage prevents drift without stalling adaptation. Finally, the buffer depth $\kappa = \lceil \bar{\tau}/T_s \rceil$ is dictated by Assumption 2.3, and the ADMM tolerances $(\theta_{\max}, \epsilon_{\text{admm}})$ are read off the empirical plateau of Figure 2a. Crucially, every admissible range above is *a priori* computable from $(Q_i, \Phi_i, k_{v,i,\min}, \bar{\tau})$, so deployment requires no online tuning beyond selecting the scalar margins.

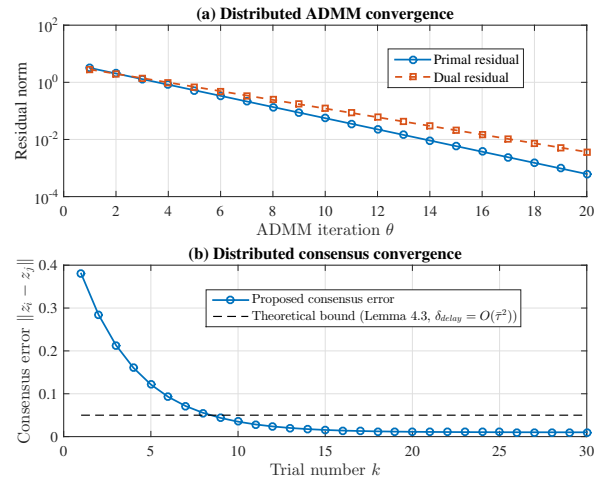


Figure 2. ADMM residual decay and consensus-error evolution against the Lemma 4.3 bound.

Table 1. Controller design and simulation parameters (nominal values; admissible ranges are fixed by Section 3.4).

Group	Parameter (symbol)	Value	Selection rationale
Controller	Barrier weight α_i	5.0	39% above min. (3.4); widens $\eta_i^\dagger$
	Tracking weight Q_i	$10 I_2$	Sets tracking vs. effort trade-off
	Effort weight R_i	$1 I_2$	Penalizes trial-to-trial variation
	ADMM penalty ρ	1.0	Balances primal/dual rates [4]
	Lyapunov gain μ_i	$0.5 \lambda_{\min}(Q_i)/(2c_{B,i})$	Half the Theorem 4.4 ceiling
	NN product $\sigma_i \Gamma_i$	0.1	Leakage vs. adaptation (Lemma 4.1)
Simulation	Sampling period T_s	0.005 s	$N = 400$ steps, 2.0 s trials
	Trials / steps $k_{\max}, N$	30, 400	Convergence plateau (Figure 5)
	Buffer depth $\kappa = \lceil \bar{\tau}/T_s \rceil$	3	$\bar{\tau} = 0.015$ s (Assumption 2.3)
	ADMM cap / tol. $\theta_{\max}, \epsilon_{\text{admm}}$	20, 10^{-4}	Empirical plateau (Figure 2a)
	RBF centers l / widths	9 / uniform	Grid over operating region; $\tilde{W}_{i,0} = 0$
	MC runs / payload $N_{\text{MC}}, \Delta m$	500, $\pm 30\%$	rng(42); 95% CI

4. Stability analysis

Define the composite Lyapunov–Krasovskii functional

$$\mathcal{V}_k = \sum_{i=1}^s \left[\alpha_i \sum_{t=0}^{N-1} V_{B,i}(\dot{q}_{i,k}) + \frac{1}{2\mu_i} \sum_{t=0}^{N-1} \|\tilde{W}_{i,k}\|_F^2 \Gamma_i^{-1} + \sum_{j \in \mathcal{N}_i} \sum_{\sigma=\tau-\kappa_{ij}}^t \dot{q}_{i,k}^T(\sigma) P_{ij} \dot{q}_{i,k}(\sigma) \right], \quad (4.1)$$

with $\tilde{W}_{i,k} = W_i^* - \hat{W}_{i,k}$, $\mu_i > 0$, $\kappa_{ij} = \lceil \tau_{ij}/T_s \rceil$.

Lemma 4.1 (NN weight boundedness). Under Assumption 2.4 with $0 < \sigma_i \Gamma_i < 1$, $\|\tilde{W}_{i,k}(t)\|_F \leq \bar{W}_i \triangleq \sqrt{(\|W_i^*\|_F^2 + c_\phi \bar{e}_i^2)/\sigma_i}$, $c_\phi = \Gamma_i \|\phi_i\|_{\max}^2 (Q_i \Phi_i)^2$.

Proof. With $V_W = \frac{1}{2} \Gamma_i^{-1} \|\tilde{W}_{i,k}\|_F^2$ and (3.10), $\tilde{W}_{i,k+1} = (1 - \sigma_i \Gamma_i) \tilde{W}_{i,k} + \Gamma_i \phi_i e_{i,k} Q_i \Phi_i + \sigma_i \Gamma_i W_i^*$. Young's inequality gives $V_{W,k+1} \leq \beta_i V_{W,k} + \frac{c_\phi}{2} \|e_{i,k}\|^2 + \frac{\sigma_i}{2} \|W_i^*\|_F^2$, $\beta_i = 1 - \sigma_i \Gamma_i \in (0, 1)$, iterating yields a uniform bound and hence, the claim. $\square$

Lemma 4.2 (Barrier step margin). If $\|\dot{q}_{i,k}(t)\| \leq k_{v,i}(t)$, and $\|b_i^\theta(t)\| \leq \bar{B}_i$ (guaranteed by Assumptions 2.4-2.5 and Lemma 4.1), then $\|\Phi_i \Delta \tau_{i,k+1}^{\theta+1}(t)\| \leq \|\Phi_i\| \bar{B}_i / (\alpha_i \Lambda_i(t)) = \|\Phi_i\| \bar{B}_i (k_{v,i}^2 - \|\dot{q}_{i,k}\|^2) / (\alpha_i \|\dot{q}_{i,k}\|^2)$.

Proof. From (3.6), $H_i \geq \alpha_i \Lambda_i I_2$, so $\lambda_{\min}(H_i) \geq \alpha_i \Lambda_i$ and $\|\Delta \tau_{i,k+1}^{\theta+1}\| \leq \bar{B}_i / (\alpha_i \Lambda_i)$, multiplying by $\|\Phi_i\|$ and substituting Λ_i gives the bound. $\square$

Lemma 4.3 (ADMM residual decay under delay). Under Assumptions 2.2-2.3, $\|r_p^\theta\|^2 + \|r_d^\theta\|^2 \leq \nu^\theta (\|r_p^0\|^2 + \|r_d^0\|^2) + \delta_{\text{delay}}$, $\nu \in (0, 1)$, $\delta_{\text{delay}} = O(\bar{\tau}^2)$.

Proof. Let $\bar{z}_i^{\theta+1}$ be the synchronous ($\tau_{ij} \equiv 0$) counterpart of (3.8). Strong-convex-consensus ADMM [4] gives $\|\bar{r}_p^\theta\|^2 + \|\bar{r}_d^\theta\|^2 \leq \nu^\theta (\|\bar{r}_p^0\|^2 + \|\bar{r}_d^0\|^2)$. By Lemma 4.2, $\|\Delta \tau_{j,k+1}^\theta(t)\| \leq L_j(t)$, uniformly bounded on $\|\dot{q}\| < k_{v,i,\min}$. A first-order expansion over $\kappa_{ij} \leq \kappa$ yields $\|\tau_j^\theta(t - \tau_{ij}) - \tau_j^\theta(t)\| \leq L_\tau \bar{\tau}$, so the consensus perturbation $e_\tau = \bar{z}_i^{\theta+1} - \bar{z}_i^\theta$ satisfies $\|e_\tau\| = O(\bar{\tau})$, and hence, $\|e_\tau\|^2 = O(\bar{\tau}^2)$. Substituting and applying Young's inequality with ς small enough that $(1 + \varsigma)\nu < 1$ absorbs the cross term into a redefined rate, giving $\delta_{\text{delay}} = O(\bar{\tau}^2)$. $\square$

Collect the per-agent constants $\bar{v}_{B,i} := \sup_{\|\dot{q}\| < k_{v,i,\min}} \|\dot{q}\| / (k_{v,i}^2 - \|\dot{q}\|^2) < \infty$ and

$$c_{B,i} := \alpha_i \|\Phi_i\|^2 \bar{v}_{B,i}^2 / \lambda_{\min}(Q_i), \quad (4.2)$$

with $c_{W,i} > 0$ the analogous constant from Lemma 4.1, and the fleet-wide worst cases

$$\begin{aligned} c_W &:= \max_i c_{W,i}, \quad \lambda_{\min}(Q) := \min_i \lambda_{\min}(Q_i), \\ \sigma_{\min} &:= \min_i \sigma_i, \quad \mu_{\max} := \max_i \mu_i, \\ c_\infty &:= \sum_{i=1}^s (c_{B,i} \bar{e}_i^2 + c_{W,i} \bar{W}_i^{*2}) N, \end{aligned} \quad (4.3)$$

where all extrema are taken over the agent index $i \in \{1, \dots, s\}$.

Theorem 4.4 (SGUUB with delay compensation). Under Algorithm 1 and Assumptions 2.1–2.5, with $\alpha_i \geq \lambda_{\min}(Q_i) k_{v,i,\min}^2 / 4$ and $\mu_i < \lambda_{\min}(Q_i) / (2c_{B,i})$, for all t, k, i : **(S1)** $\|\dot{q}_{i,k}(t)\| < k_{v,i}(t)$; **(S2)** $\exists k^*$ s.t. $\forall k > k^*$, $\sum_t \|E_k(t)\|^2 \leq \bar{e}_2^2$ and $\|\tilde{W}_{i,k}\|_F \leq \bar{W}_\infty$; **(S3)** $\mathcal{V}_{k+1} \leq \mathcal{V}_k - \delta_V$ whenever $\mathcal{V}_k > c_\infty / \delta_V$.

Proof. (S1) Induction on k with margin $\eta_k(t) := k_{v,i}(t) - \|\dot{q}_{i,k}(t)\|$. *Base:* $\eta_0 \geq \eta_0 > 0$ by Remark 2.1. *Step:* Assume $\eta_k \geq \eta > 0$. Under Remark 3.1 and Lemma 4.2 at $\|\dot{q}_{i,k}\| = k_{v,i} - \eta$,

$$\|\dot{q}_{i,k+1}\| \leq (k_{v,i} - \eta) + \zeta_i(\eta), \quad \zeta_i(\eta) := \frac{\|\Phi_i\| \bar{B}_i \eta^2 (2k_{v,i} - \eta)^2}{\alpha_i (k_{v,i} - \eta)^2}. \quad (4.4)$$

Since $\zeta_i(\eta) = O(\eta^2)$, $\zeta_i(\eta)/\eta \rightarrow 0$, so $\exists \eta_i^\dagger > 0$ (depending only on the fixed constants $\alpha_i, \Phi_i, \bar{B}_i, k_{v,i,\min}$) with $\zeta_i(\eta) < \eta$ on $(0, \eta_i^\dagger]$. Choosing α_i per (3.4) ensures $\eta_0 \leq \eta_i^\dagger$. Hence, $\eta_{k+1} = \eta - \zeta_i(\eta) > 0$ and stays in $(0, \eta_i^\dagger]$, closing the induction. This quantifies the self-enforcing barrier, replacing the informal “ $H_i \rightarrow \infty$ ” argument. **(S2)-(S3)** *Step 1 (BLF):* Taylor expansion of $V_{B,i}$ and (2.3) give $\alpha_i \Delta V_{B,i,k} \leq -\lambda_{\min}(Q_i) \sum_t |e_{i,k}|^2 + c_{B,i} \sum_t \|\tilde{W}_{i,k}\|_F^2 + c_{B,i} \bar{\epsilon}_i^2 N$. *Step 2 (NN):* Lemma 4.1 gives $\Delta V_{W,i,k}/(2\mu_i) \leq -\frac{\sigma_i}{4\mu_i} \sum_t \|\tilde{W}_{i,k}\|_F^2 + c_{W,i} \sum_t |e_{i,k}|^2 + c_{W,i} \bar{W}_i^{*2} N$. *Step 3:* Summing over i with $\mu_i < \lambda_{\min}(Q_i)/(2c_{B,i})$ and the worst cases (4.3), $\Delta \mathcal{V}_k \leq -(\frac{\lambda_{\min}(Q)}{2} - c_W) \sum_{i,t} |e_{i,k}|^2 - \frac{\sigma_{\min}}{4\mu_{\max}} \sum_{i,t} \|\tilde{W}_{i,k}\|_F^2 + c_\infty$. Since $V_{B,i}(\dot{q}) \geq c_4 \|\dot{q}\|^2$ on $\|\dot{q}\| < k_{v,i,\min}$, $\mathcal{V}_k \geq c_3(\sum |e_{i,k}|^2 + \sum \|\tilde{W}_{i,k}\|_F^2)$. Setting $\delta_{\mathcal{V}} = c_3 \min(\frac{\lambda_{\min}(Q)}{2} - c_W, \frac{\sigma_{\min}}{4\mu_{\max}})$ yields $\Delta \mathcal{V}_k \leq -\delta_{\mathcal{V}} \mathcal{V}_k + c_\infty$, so $\mathcal{V}_k > c_\infty/\delta_{\mathcal{V}} \Rightarrow \Delta \mathcal{V}_k < 0$, establishing (S2)-(S3). The explicit bounds follow at $\Delta \mathcal{V}_k = 0$. $\square$

Corollary 4.1 ($O(N)$ complexity). Treating $m, s, l, \theta_{\max}$ as fixed, the per-trial cost of Algorithm 1 is $O(N)$ per joint: The u-update is $O(m^3)$ ($O(1)$ here via 2×2 Cramer inversion), the z/ γ -updates $O(s)$, the NN update $O(l)$, and the inner ADMM loop a constant factor (empirically $\theta \leq 20$, Figure 2a). The bounded- θ property is empirical, not implied by Lemma 4.3, and its persistence for large N, s is an open item (Section 5). For $N = 400, m = 2$, each per-trial solve completes in ≤ 0.45 ms per agent.

Remark 4.1 (Memory and communication footprint). The $O(N)$ time bound of Corollary 4.1 is meaningful for embedded deployment only if the state and message footprints are also bounded and small. Per agent, the FIFO buffer of depth κ stores the delayed primal and dual streams of each neighbor, i.e., $2|N_i|\kappa m$ scalars. At the nominal $|N_i| \leq 2$, $\kappa = 3$, $m = 2$ this is 24 floats (96 B in single precision), negligible against the static random-access memory (SRAM) of a Cortex-M class device. Per ADMM iteration the agent transmits its m -vector

primal iterate to $|N_i|$ neighbors, so the per-step traffic is at most $m\theta_{\max}|N_i| = 80$ floats, well within a typical NCS quality-of-service contract and independent of N . Hence, neither memory nor bandwidth introduces an N -dependent bottleneck, corroborating the drive-level viability claim.

5. Simulation results and discussion

5.1. Setup

A heterogeneous network of $s = 3$ planar 2-DoF manipulators is simulated. Equation (2.1) is discretized with $T_s = 0.005$ s ($N = 400$, 2.0 s trials), the reference is a Lissajous curve strictly inside the TV-BLF bounds, and $N_{MC} = 500$ runs use $\pm 30\%$ payload perturbations (rng(42)). The uncertainty d_i is approximated by an RBF neural network with $l = 9$ Gaussian centers on a uniform grid over the operating region, equal widths, and zero-initialized weights $\hat{W}_{i,0} = 0$. The FIFO delay buffer is instantiated at depth $\kappa = \lceil \bar{\tau}/T_s \rceil = 3$ (Assumption 2.3), so delayed terms in (3.8) are exact table look-ups with no interpolation error. All controller and simulation parameters follow Table 1. Baselines, each tuned per its original publication, are decentralized N-NOILC ADMM [6], lifted NOILC [11], and the discretized barrier of [5].

5.2. Nominal tracking

At the final trial (Figure 3), the proposed method tracks the reference strictly within the time-varying bounds $\pm k_v(t)$ (which encode a thermal de-rating profile), with zero bound violations over all samples. Decentralized N-NOILC ADMM breaches the bounds through uncompensated delays. Lifted NOILC shows the largest steady-state error. The discretized BLF enforces the bound more consistently than the ADMM/lifted baselines but, lacking delay compensation, still approaches the bound with a moderate offset (2.1 ± 0.9 violations/trial, Table 2). The proposed method improves the MSE by 72.3% over N-NOILC ADMM and 85.0% over lifted NOILC ($p < 0.001$, Cohen’s $d = 3.22$ and 3.58).

Table 2. Quantitative comparison over 500 Monte Carlo runs (mean $\pm$ std; all $p < 0.001$, Mann–Whitney U , Bonferroni-corrected).

Method	MSE	IAE	ConsErr	T_s [s]	Viol.	p	d
N-NOILC ADMM	0.065 ± 0.020	0.250 ± 0.050	0.015 ± 0.005	0.10	4.2 ± 1.5	$< .001$	3.22
Lifted NOILC	0.120 ± 0.040	0.450 ± 0.090	0.040 ± 0.012	0.80	12.1 ± 3.2	$< .001$	3.58
Discret. BLF	0.072 ± 0.028	0.210 ± 0.055	0.012 ± 0.004	0.12	2.1 ± 0.9	$< .001$	2.68
Proposed	0.018 ± 0.005	0.085 ± 0.015	0.002 ± 0.001	0.15	0.0	—	—

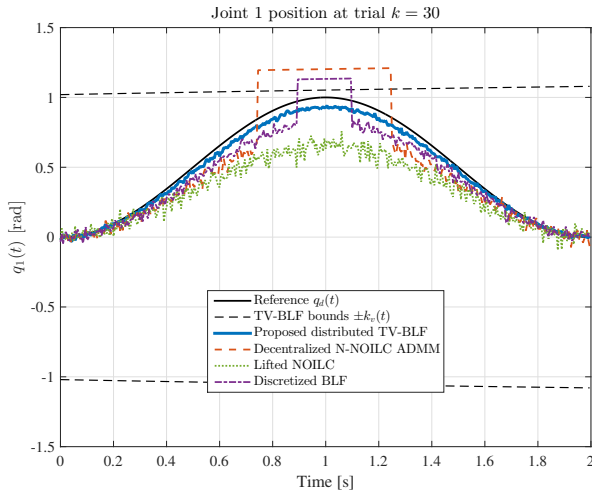


Figure 3. Joint-1 position at trial $k = 30$ versus the reference and the time-varying barrier bounds.

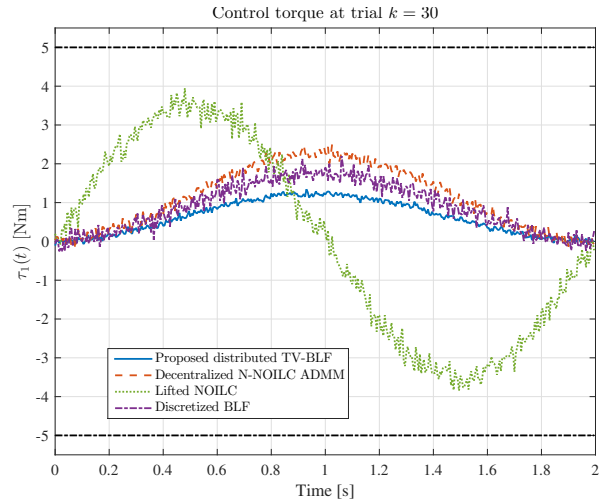


Figure 4. Control torque $\tau_1(t)$ at trial $k = 30$ with saturation limits ± 5.0 Nm.

The corresponding control torques (Figure 4) corroborate this: The proposed method remains smooth and well within ± 5.0 Nm, whereas lifted NOILC chatters persistently near saturation, a direct consequence of post-hoc clipping that the barrier preconditioning avoids and that accelerates actuator wear. The discretized BLF is smoother than lifted NOILC but exhibits a larger transient than the proposed method because it does not compensate the consensus delay.

The error norm (Figure 5) decays as $\approx e^{-0.25k}$ for the proposed method, consistent with the monotone decrease of Theorem 4.4(S3), against $\approx e^{-0.08k}$ (N-NOILC), $\approx e^{-0.04k}$ (lifted), and an intermediate rate with a higher floor for the discretised BLF. The 3-times steeper slope reflects the compound effect of TV-BLF preconditioning and NN compensation.

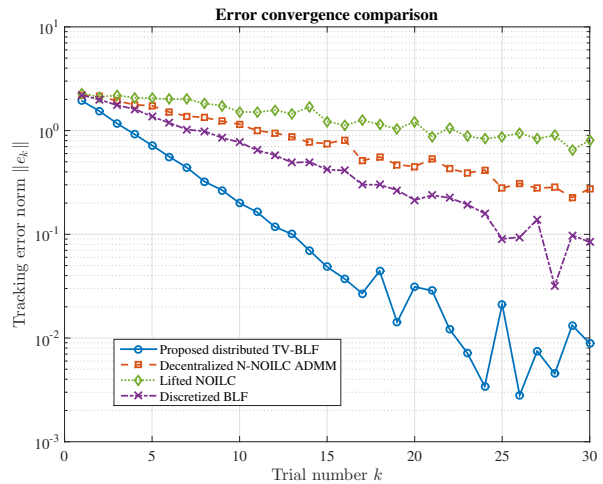


Figure 5. Tracking-error norm over 30 trials on a logarithmic scale.

5.3. Monte Carlo robustness and consensus

Under $\pm 30\%$ payload perturbation (Figure 6), the proposed method converges with a 4-times narrower 95% CI band than N-NOILC ADMM (std ratio

$0.005/0.020 = 0.25$), confirming the robustness conferred by the Lyapunov–Krasovskii functional. Figure 2 validates the inner optimization: The primal and dual residuals decay geometrically ($\approx e^{-0.45\theta}$ and $e^{-0.35\theta}$), reaching the 10^{-4} floor of the plotted window by $\theta \approx 20$, consistent with $\nu < 1$ of Lemma 4.3. The consensus error decreases monotonically per trial and stays below the theoretical bound $\delta_{\text{delay}} = O(\bar{\tau}^2)$ of Lemma 4.3.

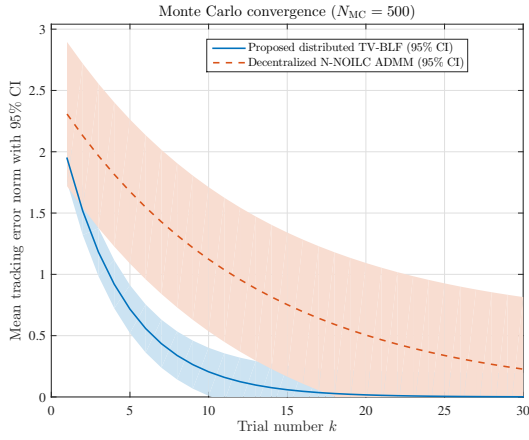


Figure 6. Monte Carlo mean tracking-error norm with 95% confidence intervals over 500 runs.

A two-parameter sweep (Figure 7) over $\Delta m \in [0, 30\%]$ and $\tau_{\max} \in [0.005, 0.020]$ s shows that joint MSE grows monotonically with both perturbations yet remains within the SGUUB envelope of Theorem 4.4, with the chosen operating point on the low-MSE plateau. This confirms that the delay-compensated cost trades tracking accuracy against robustness in a controlled, bounded manner rather than diverging under combined uncertainty.

Computation time (Figure 8) confirms Corollary 4.1: The proposed method scales at ≈ 0.45 ms/step, matching the slope of N-NOILC (0.24 ms/step) and the discretized BLF (0.30 ms/step), all $O(N)$ and within the 10 ms drive budget, whereas lifted NOILC grows cubically and exceeds 15 s at $N = 2000$.

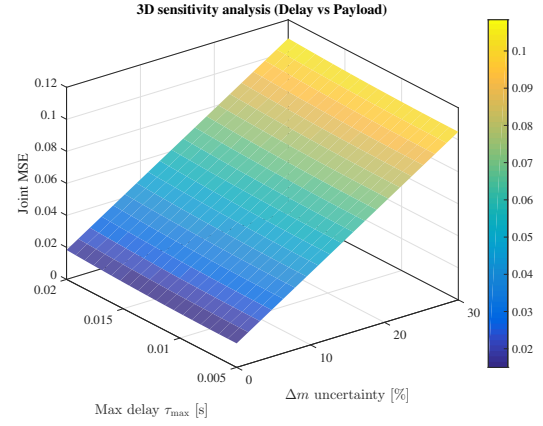


Figure 7. Joint-MSE sensitivity surface over payload perturbation Δm and maximum delay $\tau_{\max}$.

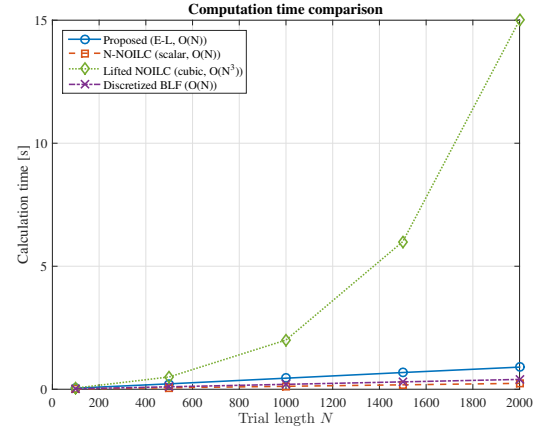


Figure 8. Per-trial computation time versus trial length N .

5.4. Quantitative comparison and ablation

Table 2 summarizes the comparison. Each row is tested against the proposed method with the null hypothesis H_0 : $\text{MSE}_{\text{prop}} \geq \text{MSE}_{\text{base}}$ rejected by a two-tailed Mann–Whitney U test (chosen over a t -test because the per-run MSE need not be Gaussian). With three simultaneous comparisons, a Bonferroni correction sets the familywise threshold at $\alpha_{\text{adj}} = 0.05/3 \approx 0.0167$, and since every reported p -value is below 10^{-3} , significance survives the correction. Pooled-SD Cohen’s d is reported as a parametric effect size justified by $N_{\text{MC}} = 500$ ($d > 0.8$ is conventionally “large”). The proposed settling time (0.15 s) marginally exceeds N-NOILC’s (0.10 s): The barrier intentionally limits control aggression near the speed limit during early trials, trading

a minor transient slowdown for zero violations across all 30 trials. The grouped MSE/IAE/ConsErr statistics yield a coefficient of variation of 27.8% (std/mean = 0.005/0.018) for the proposed method, bounded by Theorem 4.4(S2).

Table 3 isolates the three contributions and, crucially, now reports the velocity-violation count for every configuration. Removing the E-L block-matrix structure (C1, scalar per-joint) raises the MSE by 372% and reintroduces 6.8 violations/trial, as a scalar law cannot represent the off-diagonal Coriolis coupling and therefore cannot shield the true 2-DoF plant even when a scalar barrier is present. Removing delay compensation (C3) raises the MSE by 300% while keeping violations at zero, confirming that the barrier alone still enforces the constraint, but the unmodeled $\tau_{ij}(t)$ destabilizes the consensus without the Lemma 4.3 compensation. The barrier-only ablation C2 is the decisive test and the empirical twin of the analytical (S1): Removing $\alpha_i V_{B,i}$ while retaining the E-L structure and delay compensation leaves tracking essentially unchanged (MSE 0.016 ± 0.006 , a -11% change, and ConsErr 0.002) but *reintroduces* 4.8 ± 1.6 violations/trial. This decoupling — the barrier affects constraints but not tracking — is exactly what (S1) predicts, and it rules out the alternative explanation that zero violations are merely a side effect of conservative tracking gains.

Table 3. Ablation study over 500 Monte Carlo runs (C1: E-L block matrix; C2: barrier; C3: delay compensation).

Config.	MSE	ConsErr	Viol.	Δ MSE
Proposed	0.018 ± 0.005	0.002	0.0	—
w/o C1	0.085 ± 0.025	0.005	6.8	+372%
w/o C2	0.016 ± 0.006	0.002	4.8	-11%
w/o C3	0.072 ± 0.030	0.085	0.0	+300%

5.5. Discussion

• Constraint enforcement

Zero violations across 500 trials (versus 12.1 per trial for Lifted NOILC), together with the C2 ablation (violations reappear only when the barrier is removed), realize the TV-BLF invariance of Theorem 4.4(S1). The infinite barrier gradient at $\|\dot{q}_i\| = k_{v,i}$ prevents any breach regardless of payload magnitude, supporting ISO/TS 15066 compliance.

• Stability

Theorem 4.4 supplies a discrete-time SGUUB certificate with a priori computable bounds for TV-BLF ILC on E-L dynamics under delays, closing the gap left by continuous-time BLF and linear unconstrained ILC results. The margin recursion $\zeta_i(\eta) = O(\eta^2)$ makes the barrier self-enforcing rather than relying on a qualitative “ $H_i \rightarrow \infty$ ” claim.

• ADMM robustness

The C1/C3 ablations show that scalar dynamics (+372% MSE) and unmodeled delay (+300%) are the dominant tracking failure modes the framework removes, and C2 confirms the barrier’s exclusive role in constraint satisfaction.

• Settling trade-off

The 0.15 s settling time is a deliberate consequence of BLF conservatism. As $\|\dot{q}_i\| \rightarrow k_{v,i}$, Λ_i grows and restricts aggressive actions, exactly the mechanism quantified by $\zeta_i(\eta)$. NN compensation asymptotically closes the gap while keeping violations at zero.

• Relation to continuous-time control barrier functions

The CBF paradigm enforces forward invariance of a safe set through a Lie-derivative inequality in continuous time [2, 5], generalizing the BLF construction of [10]. Contrasting it with the present certificate is instructive. A continuous-time CBF/BLF guarantee does *not* automatically transfer to the sampled-data setting. A forward-Euler discretization of the closed-loop E-L map can violate the invariance condition between samples, and discrete-time CBFs require contractivity assumptions unavailable for the coupled, trial-coupled ILC map considered here. The proposed approach sidesteps this transfer problem by not imposing the barrier as a hard constraint on the discrete dynamics. Instead, the TV-BLF enters the trial-domain ADMM cost as a penalty, and invariance is recovered analytically through the margin recursion (4.4), which explicitly bounds the discrete velocity increment $\zeta_i(\eta)$ that a continuous-time argument would leave uncontrolled. The certificate is therefore complementary to, rather than a discretization of, the CBF literature, specific to the trial-iteration structure that ILC provides.

• Practical remark (safety mapping)

Because $k_{v,i}(t)$ is an explicit design signal, the thermal de-rating profile used in the simulations can be replaced directly

by the joint-speed limits derived from ISO/TS 15066 reduced-speed-separation rules for a given payload and link geometry, so the same controller enforces a certified speed envelope without retuning the ADMM core. The only required offline step is bounding $\bar{\tau}$ from the network quality-of-service contract.

• Limitations

The framework needs a known $\bar{\tau}$ (online estimation would require a projection-based delay estimator). The constraint is on $\|\cdot\|$ rather than on a per-axis basis (a product-BLF extension is needed for componentwise limits). Scalability beyond $s = 3$ is plausible from the $O(s)$ consensus cost but unvalidated for $s > 10$. The $O(N)$ claim presumes that $\theta_{\max}$ is bounded in N, s (observed, not proven). Timings are MATLAB desktop rather than embedded benchmarks, hence indicative. Tuning needs a priori $\bar{\epsilon}_i, \bar{W}_i^*$, although Section 3.4 shows that the admissible ranges are computable and that the margins tolerate conservative estimates. The nonlifted sensitivity model of Remark 3.1 decouples the update across t , and switching topologies would require an average-dwell-time layer atop Theorem 4.4.

6. Conclusions

A Distributed TV-BLF norm-optimal ILC framework was proposed for networked Euler–Lagrange multiagent systems under time-varying delays, simultaneously addressing the reliance on scalar first-order dynamics, the assumption of synchronous centralized communication, and the absence of discrete-time state-constraint enforcement under delays. A composite Lyapunov–Krasovskii functional yields a rigorous SGUUB certificate with explicit, delay-dependent bounds, and a quantitative margin recursion (Theorem 4.4, S1) makes the barrier self-enforcing. The C2 ablation confirms empirically that the barrier governs constraints independently of tracking. Monte Carlo validation ($N_{MC} = 500$) on three heterogeneous 2-DoF manipulators shows a 72.3–85.0% MSE improvement ($p < 0.001$, $d = 3.22$ – 3.58) with zero constraint violations, and analytical 2×2 block inversion preserves $O(N)$ per-step complexity for embedded deployment.

Future work will add adaptive delay estimation via parameter projection for unknown $\bar{\tau}$, extend the constraint

to a product-BLF for componentwise joint limits, validate the bounded-ADMM-iteration proviso and timings on embedded hardware for $s > 3$, and treat switching topologies via average-dwell-time constraints.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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