

Theory article

Threshold dynamics and global stability of a SIRS epidemic model with information intervention and vaccination

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Abstract: This paper investigates the dynamic behavior of a susceptible–infected–recovered–susceptible (SIRS) epidemic model, taking information intervention and vaccination measures into account. We find that, both in the deterministic model and the stochastic model, the basic reproduction number $\mathcal{R}$ acts as a critical threshold for the disease's persistence and extinction. If $\mathcal{R} < 1$, the deterministic system has a disease-free equilibrium which is globally asymptotically stable. Moreover, the disease will eventually disappear with probability one in stochastic system, while if $\mathcal{R} > 1$, in deterministic model, the endemic equilibrium is globally asymptotically stable; similarly, within the stochastic system, a stationary distribution implies the disease will be prevailing.

Keywords: SIRS epidemic model; information intervention; extinction; persistence; threshold

1. Introduction

Infectious disease has plagued our society from its beginning and also influences human health, social welfare, and other related developments [1–3]. Therefore, the study of epidemics is particularly important. Many studies in the fields of theoretical and mathematical research have explored methods of disease control [4–6]. In particular, vaccination has been considered as an important approach in eliminating diseases [7–9]. Lahrouz et al. [10] proposed the susceptible–infected–recovered–susceptible (SIRS) epidemic model to describe the effect of vaccination.

During disease explosion, awareness created by the media coverage or education plays an important role in changing human's behavior, which can reduce the transmission of infectious disease [11–13]. Thus, a large number of tools,

including the mass media, have been applied to control and eliminate epidemic diseases [14]. The height of the epidemic peak and the severity of the disease were reduced as a result of media coverage in Xiao et al. [15]. Individuals can better curb disease spread through improved preventive knowledge, and the effect of media coverage can be significant [16, 17]. Motivated by the discussion of Kumar [18], we derived a deterministic model with an information intervention that we will discuss in this paper as follows:

$$\begin{cases} dS = \left((1-p)\Lambda - \delta S - \frac{\beta SI}{\varphi(I)} + \eta R - \mu m Z S \right) dt, \\ dI = \left(-(\delta + c + \theta)I + \frac{\beta SI}{\varphi(I)} \right) dt, \\ dR = \left(p\Lambda - (\delta + \eta)R + \theta I + \mu m Z S \right) dt, \\ dZ = \left(\frac{gI}{1+g_1 I} - g_0 Z \right) dt, \end{cases} \quad (1.1)$$

where S, I, R , and Z denote the number of susceptible,

infected, recovered, as well as the information density. Every parameter in Model (1.1) is taken to be non-negative, with the rest presented in the list below:

- Λ : Rate of recruitment in the general population;
- p : Proportion of the population receiving vaccination;
- δ : Natural death rate;
- c : Mortality caused by disease;
- β : Coefficient of disease transmission;
- θ : Recovery rate;
- η : The ratio of susceptible individuals to recovered individuals;
- $\varphi(I)$: A function with positivity;
- m : Information interaction rate;
- μ : Information intensity;
- g : Growth rate of information;
- g_1 : Saturation constant;
- g_0 : Rate at which information naturally degrades.

In addition, considering the randomness and stochasticity in reality, applying stochastic models is more useful [19–22]. May [23] revealed that, affected by environmental noise, parameters like birth and death rates within the deterministic system present random oscillations to a certain extent. As a result, many authors have discussed stochastic systems, including biological models and epidemic models. For instance, Dalal et al. [24] studied the asymptotic behavior in a stochastic human immunodeficiency (HIV) epidemic model by disturbing δ, a , and u . Tornatore et al. [25] constructed an stochastic SIRS epidemic model by disturbing β and studied the stability of disease-free equilibrium by using stochastic analysis theory. The dynamics of a stochastic SIRS epidemic model involving vaccination were analyzed in [26] and the author obtained the threshold of the disease's extinction and persistence.

We assume that the coefficient m is stochastically perturbed as follows:

$$mdt \rightarrow mdt + \sigma dB(t).$$

Namely, the rate m is perturbed by Gaussian white noise. Thus, the deterministic system (1.1) can be given by the

stochastic differential equation (SDE) as follows:

$$\begin{cases} dS = \left((1-p)\Lambda - \delta S - \frac{\beta SI}{\varphi(I)} + \eta R - \mu m Z S \right) dt \\ \quad - \sigma \mu Z S dB(t), \\ dI = \left(-(\delta + c + \theta)I + \frac{\beta SI}{\varphi(I)} \right) dt, \\ dR = (p\Lambda - (\delta + \eta)R + \theta I + \mu m Z S) dt + \sigma \mu Z S dB(t), \\ dZ = \left(\frac{gI}{1+g_1 I} - g_0 Z \right) dt, \end{cases} \quad (1.2)$$

where $B(t)$ is standard Brownian motion and σ is its intensity.

Kumar considered the effect of information intervention, but the random disturbance of the environment was not involved in the discussion of [18]. Similarly, Lahrouz et al. [10] took a stochastic factor into account but the influence of information was not discussed. The effect of information and stochastic factors will be both taken account.

In this paper, the following questions will be explored:

- (Q1) How does the rate of vaccination affect the system?
- (Q2) What role does information intensity play?

In Section 2, we investigate the positivity, boundedness, and local and global stability of E_0 and E^* of the deterministic system (1.1). In Section 3, the extinction and persistence of disease in Model (1.2), and we show the existence of a stationary distribution and ergodicity. A brief summary and the answer to be questions above are given in the last section.

2. Equilibrium analysis

The basic reproduction number of (1.1) as follows [27]:

$$\mathcal{R} = \frac{\beta \delta (1-p) + \beta \eta}{(\delta + \eta)(\delta + c + \theta)}. \quad (2.1)$$

In addition, the Model (1.1) has two equilibria: The disease-free equilibrium $E_0 = (\frac{\delta(1-p)\Lambda + \eta\Lambda}{\delta(\delta + \eta)}, 0, \frac{p\Lambda}{\delta + \eta}, 0)$ and the endemic equilibrium $E^* = (S^*, I^*, R^*, Z^*)$ whenever $\mathcal{R} > 1$, where

$$\begin{aligned} S^* &= \frac{(\delta + c + \theta)\varphi(I^*)}{\beta}, \\ R^* &= \frac{1}{\delta + \eta} \left[p\Lambda + \theta I^* + \frac{\mu m g (\delta + c + \theta)\varphi(I^*) I^*}{g_0 \beta (1 + g_1 I^*)} \right], \\ Z^* &= \frac{g I^*}{g_0 (1 + g_1 I^*)}, \end{aligned} \quad (2.2)$$

and I^* is the unique positive root of the following equation: where

$$H(I) = (1-p)\Lambda - \frac{(\delta+c+\theta)\varphi(I)}{\beta} - (\delta+c+\theta)I + \frac{\gamma p\Lambda}{\delta+\eta} + \frac{\theta\eta I}{\delta+\eta} + \frac{\mu mg\eta(\delta+c+\theta)\varphi(I)I}{g_0\beta(\delta+\eta)(1+g_1I)} = 0. \quad (2.3)$$

2.1. Positivity and boundedness

From the System (1.1), we have

$$\begin{aligned} \frac{dS}{dt} \big|_{S=0} &= (1-p)\Lambda + \eta R \geq 0, & \frac{dI}{dt} \big|_{I=0} &\geq 0, \\ \frac{dR}{dt} \big|_{R=0} &= p\Lambda + \theta I + \mu mZS \geq 0, & \frac{dZ}{dt} \big|_{Z=0} &= \frac{gI}{1+g_1I} \geq 0. \end{aligned}$$

If we consider the inner of this cone, from the Model (1.1), we can know that the total population $N = S + I + R$ obeys the equation:

$$\frac{dN}{dt} = \Lambda - \delta N - cI. \quad (2.4)$$

Thus, we have $\frac{dN}{dt} \leq \Lambda - \delta N$, which means $\sup_{t \rightarrow \infty} N(t) \leq \frac{\Lambda}{\delta}$. Hence, $\frac{\Lambda}{\delta}$ is the boundary of all the solutions S, I , and R . According to the fourth equation of Model (1.1) and using the bound of I , we know that $\limsup_{t \rightarrow \infty} Z(t) \leq \frac{g\Lambda}{g_0(\delta+g_1\Lambda)}$. Therefore, we derive the following invariant bounded set:

$$\Gamma = \{(S, I, R, Z) \in \mathbb{R}_+^4 : 0 < S + I + R \leq \frac{\Lambda}{\delta}, 0 \leq Z \leq \frac{g\Lambda}{g_0(\delta+g_1\Lambda)}\}.$$

Thus, the solution will always remain in Γ in the non-negative range of $\mathbb{R}_+^4$.

2.2. Local stability of E_0 and E^*

For convenience, we denote the variational matrix of the Model (1.1) as follows:

$$\mathbf{D} = \begin{bmatrix} -\delta - \frac{\beta I}{\varphi(I)} - c & -\frac{\beta S}{\varphi(I)} & \eta & -\mu mS \\ \frac{\beta I}{\varphi(I)} & -\delta - c - \theta + \frac{\beta S}{\varphi(I)} & 0 & 0 \\ \mu mZ & \theta & -\delta - \eta & \mu mS \\ 0 & \frac{g}{(1+g_1I)^2} & 0 & -g_0 \end{bmatrix}.$$

For infected equilibrium E^* , the characteristic of $\mathbf{D}$ is given as:

$$\lambda^4 + A_1\lambda^3 + B_1\lambda^2 + C_1\lambda + D_1 = 0,$$

$$A_1 = g_0 + 2\delta + \eta + \frac{\beta I^*}{\varphi(I^*)} + \mu mZ^*,$$

$$B_1 = g_0(\delta + \eta) + (\delta + g_0)\left(\delta + \frac{\beta I^*}{\varphi(I^*)} + \mu mZ^*\right) + \eta\left(\frac{\beta I^*}{\varphi(I^*)} + \delta\right) + \frac{\beta^2 S^* I^*}{\varphi(I^*)^2},$$

$$C_1 = \frac{\beta^2 S^* I^*}{\varphi(I^*)^2}(g_0 + \delta + \eta) + \frac{g\beta I^* \mu mS^*}{(1+g_1I^*)^2} - \frac{\beta I^*}{\varphi(I^*)}\theta\eta + g_0\delta\left(\delta + \frac{\beta I^*}{\varphi(I^*)} + \mu mZ^*\right) + g_0\eta\left(\delta + \frac{\beta I^*}{\varphi(I^*)}\right),$$

$$D_1 = \frac{g_0\beta I^*}{\varphi(I^*)}(\delta(\delta + c + \theta) + \eta(\delta + c)) + \frac{g\beta I^*}{\varphi(I^*)(1+g_1I^*)^2}\delta\mu mS^*.$$

Theorem 2.1. (i) Local asymptotically stability is exhibited by the disease-free state E_0 under $\mathcal{R} < 1$; otherwise, it is unstable.

(ii) The System (1.1) has a unique infected equilibrium E^* and if $\mathcal{R} > 1$ and it is locally asymptotically stable if the following conditions are satisfied: $A_1B_1 > C_1$ and $A_1(B_1C_1 - A_1D_1) > C_1^2$.

Proof. (i) Since the proof follows analogously to Theorem 2 of [28], the details are omitted.

(ii) Clearly $A_1 > 0, D_1 > 0$ and by Routh–Hurwitz criterion if $A_1B_1 > C_1$ and $A_1(B_1C_1 - A_1D_1) > C_1^2$, the characteristic of J_{E^*} has negative or with negative real parts of the roots. Therefore, if $\mathcal{R} > 1, A_1B_1 > C_1, A_1(B_1C_1 - A_1D_1) > C_1^2$ and using Hartman–Grobman theorem [29], we can see that E^* is locally asymptotically stable. $\square$

So far, the cases of $\mathcal{R} > 1$ and $\mathcal{R} < 1$ have been studied.

Next, we will investigate the situation of $\mathcal{R} = 1$ [30].

Theorem 2.2. When $\mathcal{R} = 1$, the system will have bifurcation.

Proof. Set $x_1 = S, x_2 = I, x_3 = R$, and $x_4 = Z$. For $\mathcal{R} = 1, \beta$ is the bifurcation parameter. Thus, because $\mathcal{R} = 1, \beta^* = \frac{\Lambda((1-p)\delta+\eta)}{\delta(\delta+\eta)(\delta+c+\theta)}$. Applying new variables, the System (1.1) becomes

$$\begin{cases} \frac{dx_1}{dt} = (1-p)\Lambda - \delta x_1 - \frac{\beta x_1 x_2}{\varphi(x_2)} + \eta x_3 - \mu m x_1 x_4 := f_1, \\ \frac{dx_2}{dt} = -(\delta + c + \theta)x_2 + \frac{\beta x_1 x_2}{\varphi(x_2)} := f_2, \\ \frac{dx_3}{dt} = p\Lambda - (\delta + \eta)x_3 + \theta x_2 + \mu m x_1 x_4 := f_3, \\ \frac{dx_4}{dt} = \frac{g x_2}{1+g_1 x_2} - g_0 x_4 := f_4. \end{cases}$$

So the Jacobian matrix associated with the disease-free equilibrium x^* can be written as follows:

$$\mathbf{D}_{x^*}(\beta^*) = \begin{bmatrix} -\delta & -\delta - c - \theta & \eta & \frac{-\mu m[\delta + \eta - \delta p]}{\varphi(x_2^*)(\delta + \eta)} \\ 0 & 0 & 0 & 0 \\ 0 & \theta & -\delta - \eta & \frac{\mu m[\delta + \eta - \delta p]}{\varphi(x_2^*)(\delta + \eta)} \\ 0 & g & 0 & -g_0 \end{bmatrix}.$$

We know that $\mathbf{D}_{x^*}(\beta^*)$ has an zero eigenvalue at $\mathcal{R} = 1$. All the other eigenvalues are negative. In association with the zero eigenvalue of $\mathbf{D}_{x^*}(\beta^*)$, the right eigenvector is denoted $y = (y_1, y_2, y_3, y_4)'$, where

$$\begin{cases} y_1 = \frac{1}{\delta} \left\{ \frac{\eta\theta}{(\delta + \eta)g} + \frac{-\delta - c - \theta}{g} - \frac{\delta\mu m[\delta + \eta - \delta p]}{\varphi(x_2^*)g_0(\delta + \eta)^2} \right\}, \\ y_2 = \frac{1}{g}, \\ y_3 = \frac{1}{(\delta + \eta)} \left\{ \frac{\theta}{g} + \frac{\mu m[(1-p)(\delta + \eta) + \eta p]}{g_0\varphi(x_2^*)(\delta + \eta)} \right\}, \\ y_4 = \frac{1}{g_0}. \end{cases}$$

Similarly, we can see that $\mathbf{D}_{x^*}(\beta^*)$ associated with the zero eigenvalue is $z = (z_1, z_2, z_3, z_4)$, where $z_1 = 0, z_2 = 1, z_3 = 0, z_4 = 0$. Further, we can get a_2 and b_2 . For the calculation of a_2 and b_2 , the non-zero second-order partial derivatives of $f = (f_1, f_2, f_3, f_4)$ at (x^*, β^*) are given as

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \beta^*, \frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \beta^*, \frac{\partial^2 f_2}{\partial x_2 \partial \beta} = x_1^*.$$

Therefore, we obtain

$$\begin{aligned} a_2 &= \frac{-2((1-p)(\delta + \eta) + \eta p)}{g_0 g^2 \delta (\delta + \eta)^3 (\delta + c + \theta) \varphi(x_2^*)} [g_0 \varphi(x_2^*)(\delta + c + \theta)(\delta + \eta) \\ &\quad + g_0 \varphi(x_2^*)\eta(\delta + c)(\delta + \eta) + g\delta\mu m[\delta + \eta - \delta p]], \\ b_2 &= \frac{\delta + \eta - \delta p}{g\varphi(x_2^*)(\delta + \eta)}. \end{aligned}$$

Clearly, $a_2 < 0, b_2 > 0$. $\square$

2.3. Global stability of E_0 and E^*

For considering the global stability of E_0 and E^* , we vary $\frac{dX}{dt} = F(X, Y)$, $\frac{dY}{dt} = G(X, Y)$, where $X = (S, R, Z) \in \mathbb{R}^3, Y(I) \in \mathbb{R}^1$ [18].

Theorem 2.3. For $\mathcal{R} < 1$, the disease-free equilibrium $(X_0, 0)$ is globally asymptotically stable of the Model (1.1).

Proof. Similar to the verification of Theorem A4 in [31], we find that

$$\begin{aligned} F(X, Y) &= ((1-p)\Lambda - \delta S - \frac{\beta S I}{\varphi(I)} + \eta R - \mu m Z S, \\ &\quad p\Lambda - (\delta + \eta)R + \theta I + \mu m Z S, \frac{gI}{1 + g_1 I} - g_0 Z), \\ G(X, Y) &= -(\delta + c + \theta)I + \frac{\beta S I}{\varphi(I)}, \end{aligned}$$

where $X = (S, R, Z)'$ and $Y = I$. The disease-free equilibrium $U_0 = E_1 = (X_0, 0)$ with $X_0 = (\frac{\delta(1-p)\Lambda + \eta\Lambda}{\delta(\delta + \eta)}, \frac{p\Lambda}{\delta + \eta}, 0)$. When $t \rightarrow \infty$, we have $X \rightarrow (\frac{\delta(1-p)\Lambda + \eta\Lambda}{\delta(\delta + \eta)}, \frac{p\Lambda}{\delta + \eta}, 0)$. Clearly X_0 is globally asymptotically stable. Further, $G(X, Y) = -(\delta + c + \theta)(1 - \mathcal{R})I - \frac{\beta I}{\varphi(I)} \left[\frac{\varphi(I)(\delta(1-p)\Lambda + \eta\Lambda)}{\delta(\delta + \eta)} - S \right]$. Here, $G(X, Y) = \frac{\beta I}{\varphi(I)} \left[\frac{\varphi(I)(\delta(1-p)\Lambda + \eta\Lambda)}{\delta(\delta + \eta)} - S \right] \geq 0$ as $S \leq \frac{\delta(1-p)\Lambda + \eta\Lambda}{\delta(\delta + \eta)}$. Therefore, the conditions are satisfied. Hence, we see that when $\mathcal{R} < 1$, the disease-free equilibrium of the System (1.1) exhibits global asymptotic stability. $\square$

Theorem 2.4. The unique infected equilibrium E^* is globally asymptotically stable, if $\mathcal{R} > 1$ and the following conditions are satisfied:

$$\begin{aligned} a^2 &< \frac{g_0}{3} \left(c + \frac{2\delta\beta S^*}{\eta\varphi(I^*)} \right), \\ \mu^2 m^2 &< \min \left\{ \frac{g_0 \eta^2}{12\delta^2 S^{*2}} \left(c + \frac{2\delta\beta S^*}{\eta\varphi(I^*)} \right), \right. \\ &\quad \frac{g_0 \theta \delta^2 (\delta + \eta)}{c\Lambda^2}, \frac{8\delta^2 (\delta + \eta) \theta}{3c\eta Z^{*2}}, \frac{g_0 \eta}{6S^{*2}}, \\ &\quad \left. \frac{2\eta g_0^2 (\delta + g_1 \Lambda)^2}{9g^2 \Lambda^2} \left(c + \frac{2\delta\beta S^*}{\eta\varphi(I^*)} \right) \right\}. \end{aligned}$$

Proof. Consider a function V in Γ :

$$\begin{aligned} V &= \frac{1}{2} [(S + I + R) - (S^* + I^* + R^*)]^2 + m_1 \left(I - I^* - I^* \log \frac{I}{I^*} \right) \\ &\quad + \frac{m_2}{2} ((S + I) - (S^* + I^*))^2 + \frac{m_3}{2} (R - R^*)^2 + \frac{1}{2} (Z - Z^*)^2. \end{aligned}$$

Now, if we choose

$$m_1 = \frac{\varphi(I^*)}{\beta} \left((2\delta + c) + \frac{2\delta}{\eta} (\delta + \frac{\beta S^*}{\varphi(I^*)}) \right), m_2 = \frac{2\delta}{\eta}, m_3 = \frac{c}{\theta},$$

we can obtain

$$\begin{aligned}\dot{V} \leq & -\delta(S - S^*)^2 - (\delta + c)(I - I^*)^2 - \delta(R - R^*)^2 \\ & - \frac{2\delta}{\eta} \left(\delta(S - S^*)^2 + \frac{\beta S^*}{\varphi(I^*)}(I - I^*)^2 + \mu m Z(S - S^*)^2 \right) \\ & - \frac{2\delta}{\eta} (\mu m S^*(S - S^*)(Z - Z^*) + \mu m Z(I - I^*)(S - S^*) \\ & + \mu m S^*(I - I^*)(Z - Z^*)) - \frac{c}{\theta} (\delta + \eta)(R - R^*)^2 \\ & + \frac{c}{\theta} (\mu m Z^*(R - R^*)(S - S^*) + \mu m S(R - R^*)(Z - Z^*)) \\ & + \frac{g(I - I^*)(Z - Z^*)}{(1 + g_1 I)(1 + g_1 I^*)} - g_0(Z - Z^*)^2.\end{aligned}$$

Applying the bound of S and Z in Γ , it can be easily seen that $LV \leq 0$ if the following conditions are satisfied:

$$\begin{aligned}a^2 & < \frac{g_0}{3} \left(c + \frac{2\delta\beta S^*}{\eta\varphi(I^*)} \right), \\ \mu^2 m^2 & < \min \left\{ \frac{g_0\eta^2}{12\delta^2 S^{*2}} \left(c + \frac{2\delta\beta S^*}{\eta\varphi(I^*)} \right), \right. \\ & \quad \frac{g_0\theta\delta^2(\delta + \eta)}{c\Lambda^2}, \frac{8\delta^2(\delta + \eta)\theta}{3c\eta Z^{*2}}, \frac{g_0\eta}{6S^{*2}}, \\ & \quad \left. \frac{2\eta g_0^2(\delta + g_1\Lambda)^2}{9g^2\Lambda^2} \left(c + \frac{2\delta\beta S^*}{\eta\varphi(I^*)} \right) \right\}.\end{aligned}$$

Thus, we obtain $LV < 0$ and $LV = 0$. Then, applying LaSalle's theorem [32], we see that E^* is globally asymptotically stable in Γ . $\square$

Example 2.5. The parameters associated with System (1.1) are given as follows:

$$\begin{aligned}p = 0.5, \quad \Lambda = 4, \quad \beta = 0.04, \quad \delta = 0.02, \quad \mu = 0.009, \\ m = 0.01, \quad c = 0.005, \quad \eta = 0.01, \quad \theta = 0.7, \quad g = 0.02, \\ g_0 = 0.045, \quad g_1 = 1.\end{aligned}\quad (2.5)$$

Set $\varphi(I) = 1 + I^2$, we can derive $\mathcal{R} = 7.3563 > 1$, $g^2 = 4.0000 \times 10^{-4} < 0.0436$, $\mu^2 m^2 = 8.10000 \times 10^{-9} < \min \{3.7023 \times 10^{-7}, 4.7250 \times 10^{-6}, 5.2238, \text{ and } 1.0196 \times 10^{-8}, 0.0330\}$. Clearly, Figure 1 support our conclusion.

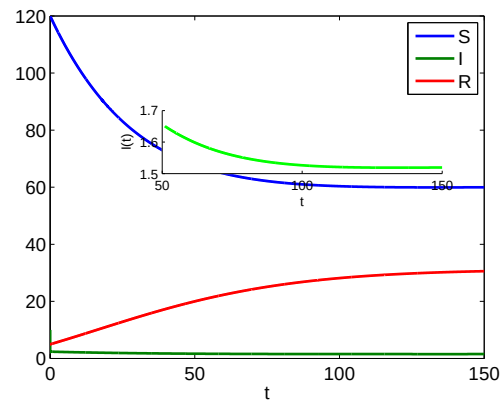


Figure 1. The path of $S(t)$, $I(t)$, and $R(t)$ for Model (1.1) and other parameter values given in (2.5).

3. The analysis of the stochastic system

3.1. Existence of a unique positive solution

The primary results are established via the following theorems.

Theorem 3.1. The System (1.2) has a unique positive solution on $t \geq 0$ that remains in the region with a probability one in $\mathbb{R}_+^4$ for all initial condition $(S_0, I_0, R_0, Z_0) \in \mathbb{R}_+^4$.

Proof. Define a C^4 -function $V : \mathbb{R}_+^4 \rightarrow \mathbb{R}$ as

$$\begin{aligned}V(S, I, R, Z) = & (S - 1 - \log S) + (I - 1 - \log I) \\ & + \frac{1}{2}R^2 + (Z - 1 - \log Z).\end{aligned}\quad (3.1)$$

This function is non-negative. According to Itô's formula [33], we can obtain

$$dV = LV dt - \left(1 - \frac{1}{S}\right)\mu\sigma ZS dB(t) + \mu\sigma ZS dB(t), \quad (3.2)$$

where

$$\begin{aligned}LV \leq & (1 - p)\Lambda + \eta R + \delta + \frac{\beta I}{N} + \mu m Z + \frac{1}{2}\mu^2\sigma^2 Z^2 + (\delta + c + \theta) \\ & + p\Lambda R + \theta I R + \mu m Z S R + \frac{1}{2}\mu^2\sigma^2 Z^2 S^2 + \frac{gI}{1 + g_1 I} + g_0 \\ \leq & (1 - p)\Lambda + 2\delta + \beta + c + \theta + g_0 + \frac{g}{g_1} + (\eta + \mu m + p\Lambda)M \\ & + \theta M^2 + \frac{\mu g m M^2}{g_0 g_1} + \frac{\mu^2\sigma^2 g^2(1 + M^2)}{2g_0^2 g_1^2} \\ := & K.\end{aligned}$$

Because $S + I + R \leq M$ and $M = \max\left\{\frac{\Lambda}{\delta}, S_0 + I_0 + R_0\right\}$. Similar to [34], we can complete the proof. $\square$

3.2. Extinction of disease

We will study the stochastic extinction of the disease of Model (1.2). In order to prove the main results of this subsection expediently, we introduce the following linear transformation:

$$X = S - S_0, Y = R - R_0,$$

and let

$$\begin{aligned} \mathbb{X}' &= \{(X, I, Y, Z) : 0 \leq X < -S_0, I \geq 0, 0 \leq Y < -R_0, Z \geq 0\}, \\ \Gamma' &= \{(X, I, Y, Z) \in \mathbb{X}' : -\frac{\Lambda c}{\delta(\delta + c)} \leq X + I + Y \leq 0, X, Y \leq 0, \\ &\text{and } I \leq \frac{\Lambda}{\delta}, 0 \leq Z \leq \frac{g\Lambda}{g_0(\delta + g_1\Lambda)}\}. \end{aligned} \quad (3.3)$$

The stochastic extinction of the disease for the Model (1.2) will be seen in the following theorem.

Theorem 3.2. (S_t, I_t, R_t, Z_t) will be the solution of the SDE model (1.2) for any given initial value $(S_0, I_0, R_0, Z_0) \in \mathbb{R}_+^4$. If

$$\mathcal{R} < 1, \quad (3.4)$$

and the solution (S_t, I_t, R_t, Z_t) has the following property:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log I_t}{t} &\leq -a' < 0, \text{ a.s.} \\ \limsup_{t \rightarrow \infty} \frac{\log Z_t}{t} &\leq -b' < 0, \text{ a.s.} \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t R_s ds &= 0, \text{ a.s.} \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_s - S_0) ds &= 0, \text{ a.s.} \end{aligned} \quad (3.5)$$

where $a' = (\delta + c + \theta)(1 - \mathcal{R})$.

Proof. From the linear transformation $S = X + S_0, R = Y + R_0$, we have

$$d \log I = \phi(X)dt$$

where $\phi(X) = \frac{\beta(X+S_0)}{\varphi(I)} - (\delta + c + \theta)$. Taking the integral of both sides of the equation above from 0 to t , we get

$$\log I_t = \log I_0 + \int_0^t \phi(X_s)ds. \quad (3.6)$$

Under Condition (3.4), we get

$$\phi(X) \leq (\delta + c + \theta)(\mathcal{R} - 1). \quad (3.7)$$

Thus, we can obtain

$$\limsup_{t \rightarrow \infty} \frac{\log I_t}{t} \leq 0 \quad \text{a.s.} \quad (3.8)$$

Thus, based on (3.8), we know that for a infinitesimal $\varsigma > 0$, for $t \geq T_1$,

$$I_t(\omega) \leq e^{(-a' + \varsigma)t}. \quad (3.9)$$

If $t \geq T_1$, the fourth equation of Model (1.2) implies that

$$\begin{aligned} Z(t) &= e^{-g_0 t} \left(\int_0^t \frac{gI}{1 + g_1 I} e^{g_0 s} ds + Z_0 \right) \\ &\leq e^{-g_0 t} \left(\int_0^t gI e^{g_0 s} ds + Z_0 \right) \\ &= e^{-g_0 t} \left(\int_0^{T_1} gI_s e^{g_0 s} ds + \int_{T_1}^t g e^{g_0 s} e^{(-a' + \varsigma + g_0)s} ds \right) \\ &\quad + e^{-g_0 t} Z_0. \end{aligned}$$

Thus, we obtain

$$\limsup_{t \rightarrow \infty} \frac{\log Z_t(\omega)}{t} \leq \min\{-g_0, -a' + \varsigma\}.$$

Fixing a positive constant ε that can be taken as small as desired, we obtain the following for $t \geq T_2$:

$$Z_t(\omega) \leq e^{(-b' + \varepsilon)t}. \quad (3.10)$$

Integrating both sides from 0 to t of the third equation of the Model (1.2), we get

$$\begin{aligned} &\frac{(\delta + \eta)}{t} \int_0^t Y_s ds \\ &\leq \frac{\theta}{t} \int_0^t I_s ds + \frac{\mu m \Lambda}{\delta t} \int_0^t Z_s ds - \frac{R_t - R_0}{t} \\ &\quad + \frac{1}{t} \int_0^t \sigma \mu Z_s (X_s + S_0) dB_s \\ &\leq \frac{\theta \Lambda T_1}{\delta t} + \frac{g \mu m \Lambda^2 T_2}{g_0 \delta (\delta + g_1 \Lambda) t} + \frac{\theta e^{(-a' + \varsigma)t}}{t} + \frac{\mu m \Lambda e^{(-b' + \varepsilon)t}}{\delta t} \\ &\quad - \frac{\theta e^{(-a' + \varsigma)t} T_1}{t} - \frac{\mu m \Lambda e^{(-b' + \varepsilon)t} T_2}{\delta t} - \frac{R_t - R_0}{t} \\ &\quad + \frac{1}{t} \int_0^t \sigma \mu Z_s (X_s + S_0) dB_s. \end{aligned}$$

For all $t > 0$, we have $X(t) + S_0 \leq \frac{\Lambda}{\delta}$ and $Z(t) \leq \frac{g\Lambda}{g_0(\delta + g_1\Lambda)}$. Applying the large numbers for local martingales, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma \mu Z_s (X_s + S_0) dB_s = 0, \quad \text{a.s.}$$

Then, according an arbitrary value of ς and ε , we obtain $\frac{1}{t} \int_0^t Y_s ds \leq 0$, and the fact $R(t) \geq 0$, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y_s ds = 0. \quad (3.11)$$

In addition

$$\begin{aligned} d(X_t + I_t + Y_t) &= (-\delta(X_t + I_t + Y_t) - a'I_t) dt \\ &\geq [-a'e^{(-a'+\varsigma)t} - \delta(X_t + I_t + Y_t)] dt. \end{aligned}$$

Integrating from 0 to t , we obtain

$$\frac{1}{t} \int_0^t (X_s + I_s + Y_s) ds \geq -\frac{a'}{\delta t} \int_0^t e^{(-a'+\varsigma)s} ds - \varphi(t),$$

where $\varphi(t) = \frac{1}{\delta} (\frac{X_t + I_t + Y_t}{t} - \frac{X_0 + I_0 + Y_0}{t})$. Because $-\frac{\Lambda}{\delta} < (X_t + I_t + Y_t) \leq 0$, then $\lim_{t \rightarrow \infty} \varphi(t) = 0$ a.s. Thus, we obtain that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t (X_s + I_s + Y_s) ds \geq 0 \quad a.s.$$

Thus

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t (X_s + I_s + Y_s) ds = 0 \quad a.s.$$

If we combine the Eqs (3.9) and (3.11), this leads to $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_s ds = 0$ a.s. $\square$

Example 3.3. For numerical simulations, the parameters of System (1.2) are chosen:

$$\begin{aligned} p &= 0.5, \quad \Lambda = 1, \quad \beta = 0.01, \quad \delta = 0.01, \quad \mu = 0.1, \\ m &= 0.017, \quad c = 0.005, \quad \eta = 0.001, \quad \theta = 0.8, \quad g = 0.01, \\ g_0 &= 0.045, \quad g_1 = 1, \quad \sigma = 0.08. \end{aligned} \quad (3.12)$$

Note that $\mathcal{R} = 0.6693 < 1$, then by Theorem 3.2, we know that the disease will extinct for initial $(S_0, I_0, R_0, Z_0) = (85, 1, 4, 10) \in \mathbb{R}_+^4$. We provide the following figure (Figure 2) to support our results.

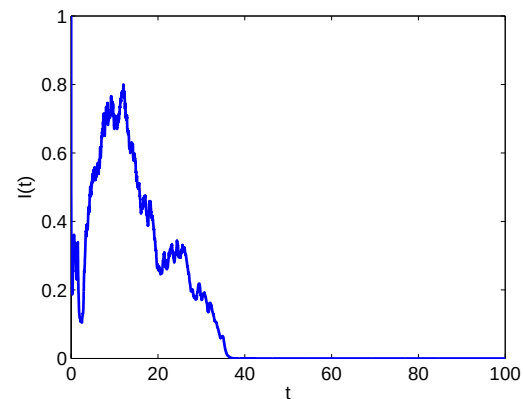


Figure 2. The path of $I(t)$ for Model (1.2) and other parameter values given in (3.12).

3.3. Persistence of disease

This section aims to derive conditions ensuring the persistence of the disease in the Model (1.2).

Theorem 3.4. For the initial value $(S_0, I_0, R_0, Z_0) \in \mathbb{R}_+^4$, let (S_t, I_t, R_t, Z_t) be the solution of the SDE model (1.2). If $\mathcal{R} > 1$, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S_s ds \geq \frac{(1-p)\Lambda}{\frac{\Lambda\beta}{\delta} + \delta + \frac{\mu mg\Lambda}{g_0(\delta + g_1\Lambda)}} > 0, \quad a.s.$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I_s ds \geq \frac{(\delta + c + \eta)(\mathcal{R} - 1)}{\frac{\beta\Lambda(g_0\beta + g\delta m)}{g_0\delta^2}} \equiv K > 0, \quad a.s.$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t R_s ds \geq \frac{\theta K}{\delta + \eta} > 0, \quad a.s.$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Z_s ds \geq \frac{g\delta K}{g_0(\delta + g_1\Lambda)} > 0, \quad a.s.$$

where K is a positive constant.

Proof. Because

$$\begin{aligned} dS_t &\geq \left((1-p)\Lambda - \left[\frac{\Lambda\beta}{\delta\varphi(0)} + \delta + \frac{\mu mg\Lambda}{g_0(\delta + g_1\Lambda)} \right] S_t \right) dt \\ &\quad - \sigma\mu Z_t S_t dB_t. \end{aligned}$$

If we integrate from 0 to t , then

$$\begin{aligned} &\left[\frac{\Lambda\beta}{\delta} + \delta + \frac{\mu mg\Lambda}{g_0(\delta + g_1\Lambda)} \right] \frac{1}{t} \int_0^t S_s ds \\ &\geq (1-p)\Lambda - \frac{S_t - S_0}{t} - \sigma\mu Z_s S_s dB_s. \end{aligned} \quad (3.13)$$

According to the law of large numbers pertaining to local martingales, then $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t Z_s S_s dB_s = 0$ a.s.

Therefore

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S_s ds \geq \frac{(1-p)\Lambda}{\frac{\Lambda\beta}{\delta} + \delta + \frac{\mu mg\Lambda}{g_0(\delta+g_1\Lambda)}} > 0, \quad a.s.$$

Computing

$$\begin{aligned} \phi(S) - \phi(S_0) &= \frac{\beta S}{\varphi(I)} - \frac{\beta \Lambda}{\mu \varphi(I)} \\ &\leq -\frac{\beta}{\varphi(I)} \left(\frac{\Lambda}{\delta} - S \right), \end{aligned}$$

where $\phi(\cdot)$ is defined by Eq (3.7). This implies that

$$\phi(S) \geq (\delta + c + \theta)(\mathcal{R} - 1) - \beta \left(\frac{\Lambda}{\delta} - S \right).$$

Thus, we can obtain

$$\log I_t \geq \log I_0 + (\delta + c + \theta)(\mathcal{R} - 1)t - \beta \int_0^t \left(\frac{\Lambda}{\delta} - S_s \right) ds. \quad (3.14)$$

Given the first and third equations of (1.2), we obtain

$$\begin{aligned} &d \left(\frac{\delta + \eta}{\delta} S_t + \frac{\eta}{\delta} R_t \right) \\ &= \left[(\delta + \eta) \left(\frac{\Lambda}{\delta} - S_t \right) - \frac{(\delta + \eta)\beta}{\delta \varphi(I)} S_t I_t - \frac{\mu m(\delta + \eta)}{\delta} Z_t S_t \right. \\ &\quad \left. + \frac{\eta \theta}{\delta} I_t + \frac{\eta \mu m}{\delta} Z_t S_t \right] dt - \sigma \mu Z_t S_t dB_t \\ &\geq \left[(\delta + \eta) \left(\frac{\Lambda}{\delta} - S_t \right) - \frac{\Lambda \beta(\delta + \eta)}{\delta^2 \varphi(0)} I_t - \frac{\Lambda \mu m(\delta + \eta)}{\delta^2} Z_t \right] dt \\ &\quad - \sigma \mu Z_t S_t dB_t. \end{aligned} \quad (3.15)$$

On the other hand, from the fourth equation of Eq (1.2)

$$Z_t dt = \frac{g I_t dt}{g_0(1 + g_1 I_t)} - \frac{1}{g_0} dZ_t \leq \frac{g}{g_0} I_t dt - \frac{1}{g_0} dZ_t.$$

Substituting the inequality above into (3.15), we can obtain

$$\begin{aligned} &d \left(\frac{\delta + \eta}{\delta} S_t + \frac{\eta}{\delta} R_t \right) \\ &\geq \left[(\delta + \eta) \left(\frac{\Lambda}{\delta} - S_t \right) - \frac{\Lambda(\delta + \eta)(g_0 \beta + g \mu m)}{g_0 \delta^2} I_t \right] dt \\ &\quad + \frac{\Lambda \mu m(\delta + \eta)}{g_0 \delta^2} dZ_t - \sigma \mu Z_t S_t dB_t. \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} &\int_0^t \left(\frac{\Lambda}{\delta} - S_s \right) ds \\ &\leq \frac{1}{\delta} (S_t - S_0) + \frac{\eta}{\delta(\delta + \eta)} (R_t - R_0) - \frac{\Lambda \mu m}{g_0 \delta^2} (Z_t - Z_0) \\ &\quad + \frac{\Lambda(g_0 \beta + g \mu m)}{g_0 \delta^2} \int_0^t I_s ds + \frac{\mu \sigma}{\delta + \eta} \int_0^t Z_s S_s dB_s. \end{aligned} \quad (3.16)$$

Substituting (3.16) into (3.14), we can get

$$\log I_t \geq (\delta + c + \theta)(\mathcal{R} - 1)t - \beta \int_0^t I_s ds + g_1(t) + g_2(t),$$

where

$$\begin{aligned} g_1(t) &= \log I_0 - \beta \left[\frac{1}{\delta} (S_t - S_0) + \frac{\eta}{\delta(\delta + \eta)} (R_t - R_0) \right. \\ &\quad \left. - \frac{\Lambda \mu m}{g_0 \delta^2} (Z_t - Z_0) \right], \\ g_2(t) &= \frac{-\beta \mu \sigma}{\delta + \eta} \int_0^t S_s I_s dB_s. \end{aligned}$$

Since $(S_t, I_t, R_t, Z_t) \in \Gamma$, through application of the large number theorem

$$\lim_{t \rightarrow \infty} \frac{g_1(t)}{t} = \lim_{t \rightarrow \infty} \frac{g_2(t)}{t} = 0, \quad a.s.$$

Therefore

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I_s ds \geq \frac{(\delta + c + \eta)(\mathcal{R} - 1)}{\frac{\beta \Lambda(g_0 \beta + g \mu m)}{g_0 \delta^2}} \equiv K > 0, \quad a.s. \quad (3.17)$$

it can be deduced immediately from $\mathcal{R} > 1$.

The third equation of (1.2) means that

$$dR_t \geq [\theta I_t - (\delta + \eta) R_t] dt + \sigma \mu Z_t S_t dB_t. \quad (3.18)$$

Applying the large number theorem for local martingales, we have

$$\lim_{t \rightarrow \infty} \sigma \mu Z_t S_t dB_t = 0.$$

In addition, by (3.17), for any $\xi > 0$ ($\xi < K$), there is a constant $T > 0$ such that

$$\frac{1}{t} \int_0^t I_s ds \geq K - \xi, \quad \text{for } t \geq T. \quad (3.19)$$

Integrating both sides of (3.18) and applying the inequality above, we find that for $t > T$,

$$\begin{aligned} \frac{1}{t} \int_0^t R_s ds &\geq \frac{\theta}{(\delta + \eta)t} \int_0^t I_s ds - \frac{R_t - R_0}{(\delta + \eta)t} \\ &\geq \frac{\theta}{(\delta + \eta)} (K - \mu) - \frac{R_t - R_0}{(\delta + \eta)t}. \end{aligned}$$

If we let $t \rightarrow \infty$, by the arbitrary nature of ξ , we get

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t R_s ds \geq \frac{\theta K}{\delta + \eta} > 0, \quad a.s.$$

Taking the integral of both sides of the final equation of (1.2), we obtain

$$\frac{Z_t - Z_0}{t} \geq \frac{g\delta}{(\delta + g_1\Lambda)t} \int_0^t I_s ds - \frac{g_0}{t} \int_0^t Z_s ds.$$

Thus, Eq (3.19) implies that for $t > T$, we have

$$\frac{1}{t} \int_0^t Z_s ds \geq \frac{g\delta}{g_0(\delta + g_1\Lambda)} (K - \xi) - \frac{Z_t - Z_0}{g_0 t}.$$

Similarly, if we let $t \rightarrow \infty$, by the arbitrary nature of ξ , we can obtain

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Z_s ds \geq \frac{g\delta K}{g_0(\delta + g_1\Lambda)} > 0, \quad a.s.$$

□

3.4. Stationary distribution and ergodicity

This section addresses the long-time behavior of the SDE model (1.2), focusing on the stationary distribution and its ergodic property.

Theorem 3.5. *For the initial values $(S_0, I_0, R_0, Z_0) \in \mathbb{R}_+^4$, we consider Model (1.2). If $\mathcal{R} > 1$, the system has a stationary distribution $\pi(\cdot)$, which is an ergodic property.*

Proof. Define k_1 and k_2 as sufficiently large numbers. Let

$$U = \{(x_1, x_2, x_3, x_4) \in \Gamma : \frac{1}{k_1} < x_1, x_2, x_3 < \frac{\Lambda}{\delta} - \frac{1}{k_1}, \\ \frac{1}{k_2} < x_4 < \frac{g\Lambda}{g_0(\delta + g_1\Lambda)} - \frac{1}{k_2}\}.$$

The positive functions V_1 , V_2 , and V_3 are defined for $(S, I, R, Z) \in \mathbb{R}_+^4$ as follows:

$$V_1(S, R, Z) = \frac{1}{S} + R + \frac{1}{Z}, \\ V_2(I) = \frac{1}{q} I^{-q}, \\ V_3(S, I) = \frac{1}{q} I^{-q} \left(\frac{\Lambda((1-p)\delta + \eta)}{\delta(\delta + \eta)} - S \right),$$

where q represents a positive constant, the value of which will be fixed later. Making use of Itô's formula, we get

$$LV_1 = -\frac{(1-p)\Lambda}{S^2} + \frac{\beta I}{S\varphi(I)} + \frac{\delta}{S} + \frac{\mu m Z}{S} - \frac{\eta R}{S^2} + \frac{\sigma^2 \mu^2 Z^2}{S} \\ + p\Lambda - (\delta + \eta)R + \theta I + \mu m ZS - \frac{gI}{(1 + g_1 I)Z^2} + \frac{g_0}{Z} \\ \leq -\frac{(1-p)\Lambda}{S^2} + p\Lambda \\ + \frac{\theta\Lambda}{\delta} + \frac{\mu m g \Lambda^2}{g_0 \delta (\delta + g_1 \Lambda)} - \frac{gI}{Z^2} + \frac{g_0}{Z} \\ + \left(\frac{\beta\Lambda}{\delta} + \delta + \frac{\mu m g \Lambda}{g_0 (\delta + g_1 \Lambda)} + \frac{\sigma^2 \mu^2 g^2 \Lambda^2}{g_0^2 (\delta + g_1 \Lambda)^2} \right) \frac{1}{S}, \quad (3.20)$$

$$LV_2 = -I^{-q} \left(\frac{\beta S}{\varphi(I)} - (\delta + c + \theta) \right) \\ \leq I^{-q} (-(\delta + c + \theta)(\mathcal{R} - 1)) \\ + I^{-q} \beta \left(\frac{\Lambda((1-p)\delta + \eta)}{\delta(\delta + \eta)} - S \right), \quad (3.21)$$

$$LV_3 = I^{-q} \left(\frac{\Lambda((1-p)\delta + \eta)}{\delta(\delta + \eta)} - S \right) \left(\delta + c + \theta - \frac{\beta S}{\varphi(I)} \right) \\ - \frac{1}{q} I^{-q} \left((1-p)\Lambda - \frac{\beta S I}{\varphi(I)} - \delta S - \mu m ZS + \eta R \right) \\ \leq I^{-q} \left(\frac{\Lambda((1-p)\delta + \eta)}{\delta(\delta + \eta)} - S \right) \left(\delta + c + \theta - \frac{\delta}{q} \right) \\ + \frac{1}{q} \mu m ZS I^{-q} + \frac{\beta \Lambda}{q \delta} I^{1-q}. \quad (3.22)$$

Since $\mathcal{R} > 1$, we can choose $q < 1$ such that

$$\delta + c + \theta + \beta - \frac{\delta}{q} < 0.$$

According to Theorem 3.4, for all $t \geq T$, there is a constant $K > 0$ such that $I(t) > \frac{K}{2}$ a.s. Thus, combining (3.20)–(3.22), we obtain

$$LV \leq -\frac{(1-p)\Lambda}{2S^2} + I^{-q} (-(\delta + c + \theta)(\mathcal{R} - 1)) \\ + I^{-q} \left(\frac{\Lambda((1-p)\delta + \eta)}{\delta(\delta + \eta)} - S \right) \left(\delta + c + \theta + \beta - \frac{\delta}{q} \right) \\ - \frac{gK}{4Z^2} + \lambda_0, \quad (3.23)$$

where

$$\lambda_0 = \sup_{(S, I, R, Z) \in \Gamma} \left\{ -\frac{(1-p)\Lambda}{2S^2} + \left(\frac{\beta\Lambda}{\delta} + \delta + \frac{\mu m g \Lambda}{g_0 (\delta + g_1 \Lambda)} \right. \right. \\ \left. \left. + \frac{\sigma^2 \mu^2 g^2 \Lambda^2}{a_0^2 (\delta + g_1 \Lambda)^2} \right) \frac{1}{S} + \frac{2^q \Lambda^2 \mu m g}{g_0 \delta (\delta + \Lambda d) q K^q} + \frac{\beta}{q} \left(\frac{\Lambda}{\delta} \right)^{2-q} \right. \\ \left. + p\Lambda + \frac{\theta\Lambda}{\delta} + \frac{\mu m g \Lambda^2}{g_0 \delta (\delta + g_1 \Lambda)} - \frac{gK}{4Z^2} + \frac{g_0}{Z} \right\}.$$

Since $\frac{\Lambda}{\delta+c} \leq S + I + R \leq \frac{\Lambda}{\delta}$, $0 \leq Z \leq \frac{g\Lambda}{g_0(\delta+g_1\Lambda)}$, then for (3.23), we have

$$LV \leq -1.$$

□

Example 3.6. *Throughout the persistence of the disease and stationary distribution and ergodicity, we assign the following parameters for System (1.2):*

$$p = 0.5, \quad \Lambda = 4, \quad \beta = 0.04, \quad \delta = 0.02, \quad \mu = 0.009, \\ m = 0.01, \quad c = 0.005, \quad \eta = 0.01, \quad \theta = 0.7, \quad g = 0.02, \\ g_0 = 0.045, \quad g_1 = 1. \quad (3.24)$$

Note that $\mathcal{R} = 7.3563 > 1$, and thus by Theorem 3.4, we know that the disease will persist for any initial $(S_0, I_0, R_0, Z_0) = (120, 10, 5, 10) \in \mathbb{R}_+^4$. From Figure 3(c)–(f), we can see that the fluctuation range of $S(t)$, $I(t)$, and $R(t)$ become stronger with the increase in stochastic noise σ . Comparing the left column pictures in Figure 3, we can derive the distribution of $I(t)$ becomes skewed with the increase in stochastic noise σ .

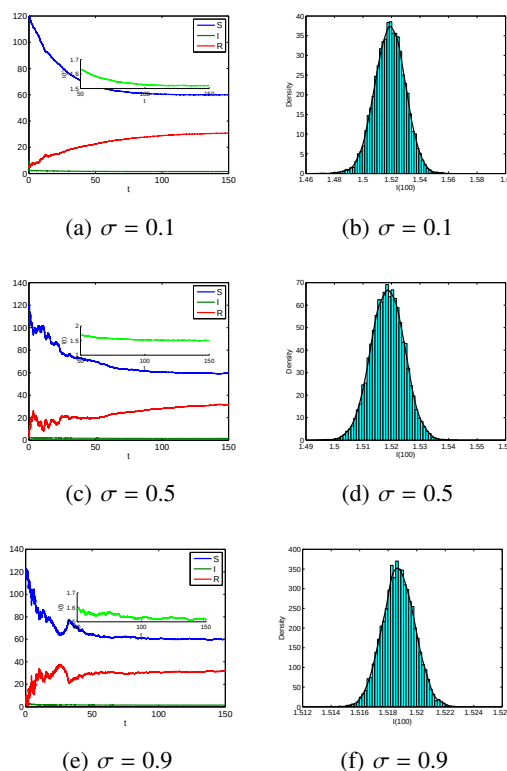


Figure 3. The left column displays the time path of $S(t)$, $I(t)$, and $R(t)$ for Model (1.2) and other parameter values given in (3.24) under a range of noise intensities $\sigma = 0.01, \sigma = 0.04$, and $\sigma = 0.07$, respectively. The histogram for the density function is illustrated in the right column with the above mentioned values of σ respectively.

4. Conclusions

In deterministic model, we analyse a equilibria, namely the disease-free equilibrium and the unique endemic equilibrium, and their local and global stability. Considering the randomness in most real-world problems, we also study

the stochastic system. When $\mathcal{R} < 1$, the disease will extinct, while $\mathcal{R} > 1$, we testify the stationary distribution and ergodicity. In addition, through the basic reproduction number $\mathcal{R}$, we can further know how vaccination influences the disease transmission. When we increase the rate of vaccination p , the basic reproduction number $\mathcal{R}$ will reduce, which means an increase in vaccination can suppress the disease spreading.

For future studies, the logarithmic Ornstein–Uhlenbeck process, an effective technique for incorporating environmental noise into the stochastic epidemic system, will be utilized to model the random fluctuations of the transmission rate in the SIRS epidemic model.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing financial or personal interests that could potentially influence the findings or interpretations presented in this paper.

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