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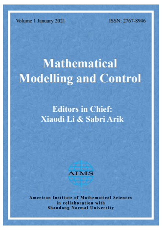

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Research article

Stability analysis of Cohen-Grossberg quaternion-valued inertial neural networks with unbounded time-varying delays in Lagrange sense

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Abstract: The article addressed the Lagrange exponential stability of Cohen-Grossberg inertial neural networks (CGINNs) with quaternion-valued state variables and unbounded time-varying delays. The study extended stability analysis to a more complex and realistic model that included the inertial term, representing the system's acceleration-like memory effect, and quaternion-valued variables, representing multidimensional signals such as 3D rotations or color images. Unbounded time-varying delays added complexity while properly replicating real-world neural network (NN) dynamics. Unlike traditional studies that relied on order-reduction transformations, this work introduced a non-reduction order approach. To achieve stability, Lyapunov functional approaches were used with advanced inequality techniques to generate adequate criteria that ensured Lagrange stability under these complex conditions. Finally, Example 1 was presented to validate the efficiency of the proposed method, while Example 2, related to Quaternion-valued NNs (QVNNs), exhibited the efficiency of storing true color image patterns.

Keywords: Cohen-Grossberg neural networks; quaternion-valued inertial neural networks; Lagrange stability; unbounded time-varying delay

1. Introduction

Neural networks (NNs) have gained significant attention in recent years due to their ability to model complex systems and solve various real-world problems [1–3]. They consist of interconnected nodes (neurons) organized in layers that work together to process input data, learn from it, and make predictions or decisions based on the learned patterns. NNs are particularly effective in fields such as image and speech recognition, natural language processing, and autonomous systems [4, 5]. Cohen-Grossberg NNs (CGNNs) have been the subject of much research as a result of their stability and dynamic behavior among different types of NNs, including recurrent, cellular, Hopfield, and associative memory NNs. In particular, each of these NNs can be thought of as a specific instance of a CGNN. CGNNs, introduced by Cohen and Grossberg in the 1980s [6], are particularly known for their ability to model and simulate brain-like activities and cognitive processes. CGNNs are characterized by their use of differential equations to describe the interactions between neurons, which allows them to capture the dynamic nature of neural activities. A key feature of CGNNs is their ability to handle a variety of dynamic behaviors, including oscillations and chaotic dynamics, which are often observed in biological neural systems. This makes them valuable for modeling complex cognitive processes and understanding brain function. In addition, CGNNs have been applied to various practical problems, such as optimization, pattern recognition, and control systems [7, 8].

Quaternion-valued NNs (QVNNs) have emerged as a powerful extension of traditional NNs, allowing for the representation and processing of multidimensional data [9–11]. Quaternions extend complex numbers to four dimensions and provide a robust mathematical framework for handling problems involving rotations and three-dimensional spatial data. This makes QVNNs particularly suitable for image processing, computer graphics, and robotics applications [12–14]. Combining CGNNs with neurons and inertia with quaternion values leads to the development of Cohen-Grossberg inertial NNs (CGINNs) with the quaternion variable. These networks inherit the advantages of CGNNs, QVNNs, and inertia, offering a more versatile and powerful tool for modeling complex systems, an area that has not yet been fully explored. In 1986, Babcock and Westervelt [15] investigated electronic NNs by incorporating inertial inductors into neural circuits, unlike standard NNs that utilize first-order neuronal state variables. The dynamical difficulties for such networks were described by a second-order derivative of neuronal state variables, known as inertial NNs (INNs), expressed by second-order differential equations that differed from first- and fractional-order state variables. The existence of inertial terms in NNs results in more complex behavior, such as chaos and bifurcation, in dynamical systems. INNs include inertial terms, which account for the system's memory effects and enhance dynamic behavior. This allows for more accurate modeling of real-world phenomena in which past states influence future behavior. However, unbounded time-varying delays pose significant challenges for the stability analysis of these networks [16, 17]. In NNs, time delays can significantly affect the network's dynamics and stability. Time delays are inherent in many practical systems due to finite signal transmission and processing [18, 19]. Unbounded time-varying delays, where the delay can change over time without any upper bound, introduce additional complexity to stability analysis. These time-varying delays significantly influence the dynamic behavior and stability of NNs. When such delays become unbounded, the analysis becomes more complex but also more realistic, capturing the true nature of communication and processing delays in biological and artificial systems. Understanding and ensuring the stability of quaternion-valued CGINNs with such delays is crucial for their successful application in real-world scenarios.

NN stability analysis typically involves concepts such as asymptotic stability, exponential stability, and Lyapunov stability [20–22]. It should be emphasized that the majority of the research currently available on NN dynamics focuses primarily on Lyapunov stability, which emphasizes equilibrium point stability. The potential for computing limits in numerous real-world applications, such as multiple decision processing, is an important downside of Lyapunov stability. Thus, a study of Lagrange stability has been suggested. Compared to Lyapunov stability, Lagrange stability focuses on the overall stability of the systems. Lagrange stability focuses on the boundedness of solutions over time, ensuring that the system's state remains within a certain bound regardless of initial conditions [23,24]. This is particularly important for quaternion-valued CGINNs with unbounded time-varying delays, as it guarantees that the network's output will not diverge or exhibit unbounded behavior despite delays. By establishing Lagrange stability, we can ensure the reliable operation of quaternion-valued CGINNs in practical applications. Numerous studies have been conducted on the stability analysis of NNs, particularly focusing on Lyapunov stability and its variants. These studies have provided valuable information on the conditions required to ensure the stability of various types of NNs. However, the existing literature has several limitations, particularly regarding quaternion-valued CGINNs with unbounded time-varying delays.

Motivated by the discussion above, the main contributions of this article are as follows:

(1) The stability of the second-order Cohen-Grossberg QVNN has been explored.
(2) To achieve the desired criteria, a non-reduction order approach is used in contrast to the traditional method, like the variable transformation of reduced-order INNs.
(3) By developing an appropriate Lyapunov functional, Lagrange stability has been achieved.

The article is organized as follows: Section 2 covers the model's description, preliminary, and definitions. Section 3 presents the main result. Numerical validation is discussed in Section 4, followed by the conclusion in Section 5.

Notations: In this article, $\mathbb{R}$ and $\mathbb{Q}$ denote the real and quaternion fields, respectively. Vectors in these fields are represented by $\mathbb{R}^n$ and $\mathbb{Q}^n$. Matrices are similarly denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{Q}^{n\times n}$. The modulus of a quaternion $p = p^R + p^I e_1 + p^J e_2 + p^K e_3$ is given by $|p| = \sqrt{(p^R)^2 + (p^I)^2 + (p^J)^2 + (p^K)^2}$. The norm of a quaternion vector $p = (p_1, p_2, \ldots, p_\xi) \in \mathbb{Q}^n$, $p_\Gamma = p_\Gamma^R + p_\Gamma^I e_1 + p_\Gamma^J e_2 + p_\Gamma^K e_3$, is defined in two ways: $\|p\|_1 = \sum_{\Gamma=1}^{\tilde{q}}(|p_\Gamma^R| + |p_\Gamma^I| + |p_\Gamma^J| + |p_\Gamma^K|)$ and $\|p\|_2 = \sum_{\Gamma=1}^{\tilde{q}} \sqrt{(p_\Gamma^R)^2 + (p_\Gamma^I)^2 + (p_\Gamma^J)^2 + (p_\Gamma^K)^2}$ where $m = 1, 2, \ldots, \tilde{q}$.

2. Preliminaries

Consider the unbounded time-varying delayed quaternion-valued CGINNs model, given by

$$\ddot{p}_\Gamma(\mathbf{t}) = -d_\Gamma \dot{p}_\Gamma(\mathbf{t}) - u_\Gamma(p_\Gamma(\mathbf{t}))\Big[c_\Gamma(p_\Gamma(\mathbf{t})) - \sum_{\xi=1}^{\tilde{q}} w_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t})) - \sum_{\xi=1}^{\tilde{q}} v_{\Gamma\xi} g_\xi(p_\xi(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t}))\Big] + B_\Gamma(\mathbf{t}), \tag{2.1}$$

where $\Gamma \in J = \{1, 2, 3..., \tilde{q}\}$, and where q represents the total number of neurons. The variable $(p_1, p_2, \ldots, p_{\tilde{q}}) \in \mathbb{Q}^{\tilde{q}}$ denotes the state of the q-th neuron at time $(\mathbf{t})$. The term $\ddot{p}_\Gamma(\mathbf{t})$, the second-order derivative of the state variable, corresponds to the inertial component of the system. Here, $d_\Gamma \in \mathbb{Q}$ are the positive constants, $u_\Gamma(.)$ is the quaternion-valued continuous amplification function, and $c_\Gamma(.)$ is the quaternion-valued characteristic function. $W = (w_{\Gamma\xi})_{\tilde{q}\times\tilde{q}} \in \mathbb{Q}^{\tilde{q}\times\tilde{q}}$ and $V = (v_{\Gamma\xi})_{\tilde{q}\times\tilde{q}} \in \mathbb{Q}^{\tilde{q}\times\tilde{q}}$ indicate the connection weight matrices, and $f_\xi(p_\xi(\mathbf{t}))$ and $g_\xi(p_\xi(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t}))$ denote the activation functions without and with unbounded time-varying delays, respectively. $B_\Gamma(\mathbf{t}) \in \mathbb{Q}$ is the external input.

Remark 2.1. *The chosen model incorporates the inertial term, augmenting the system's capacity to depict acceleration-like memory effects, so aligning it more closely with actual neural dynamics. The quaternion-valued state variables allow efficient handling of multidimensional signals such as color images and spatial rotations. Moreover, the model incorporates unbounded time-varying delays, providing a more realistic framework for practical neural systems.*

Assumption 2.1. *For each $m \in J$, $\exists\, \overline{u}_\Gamma \geq 0$ for every $p \in \mathbb{Q}$, such that*

$$u_\Gamma(p(\mathbf{t})) \leq \overline{u}_\Gamma.$$

Assumption 2.2. *For every $\Gamma \in J$ $\exists$ constants $\overline{c}_\Gamma, \overline{f}_\Gamma, \overline{g}_\Gamma > 0$ s.t. the following inequality holds for any $r, \overline{r} \in \mathbb{R}$,*

$$|c_\Gamma(r) - c_\Gamma(\overline{r})| \leq \overline{c}_\Gamma |r - \overline{r}|,$$
$$|f_\Gamma(r) - f_\Gamma(\overline{r})| \leq \overline{f}_\Gamma |r - \overline{r}|,$$
$$|g_\Gamma(r) - g_\Gamma(\overline{r})| \leq \overline{g}_\Gamma |r - \overline{r}|.$$

Assumption 2.3. *The external input $B_\Gamma(\mathbf{t})$ is bounded $\forall\, t \geq 0$, i.e., $\exists$ a constant $\overline{B}_\Gamma > 0$ s.t. $|B_\Gamma(\mathbf{t})| \leq \overline{B}_\Gamma$.*

Assumption 2.4. *Let $\sigma(\mathbf{t}) \geq 0$ be a continuously differentiable time-varying delay and satisfy $\dot{\sigma}(\mathbf{t}) \leq \overline{\sigma} < 1$, where $\overline{\sigma} > 0$.*

The initial conditions for Eq (2.1) are as follows:

$$p_\Gamma(s) = \psi_\Gamma(s),\ \dot{p}_\Gamma(s) = \phi_\Gamma(s),\ -q \leq s \leq 0, \tag{2.2}$$

where $\Gamma = 1, 2, 3..., \tilde{q}$; $p = \max\limits_{1\leq\Gamma,\xi\leq q}\{p_{\Gamma\xi}\}$ and $\psi_\Gamma(s)$, $\phi_\Gamma(s)$ are the bounded and continuous functions.

Definition 2.1. *[25] Network (2.1) is considered uniformly stable in the Lagrange sense if, for any given $B > 0$, $\exists$ constant $A = A(B) > 0$ s.t. $\|p(\mathbf{t})\| < A$, $\forall\, h \in \mathbb{Q}_B = \{h \in \mathbb{Q} : \|h\| \leq B\}$.*

Definition 2.2. *[25] A network described by Eq (2.1) is termed globally Lagrange exponentially stable if $\exists$ constants $l, x > 0$ for any $B > 0$, there exists a corresponding $A = A(B) > 0$ s.t. the inequality $\|p(\mathbf{t})\| < S + De^{-x(\mathbf{t})}$, holds for all $\mathbf{t} \geq 0$, and any initial condition $p_0 \in \mathbf{Q}_B$. Additionally, the network (2.1) is referred to as globally uniform and exponentially convergent with rate x within the ball $S(l) = \{p(\mathbf{t}) \in \mathbf{Q}^k : \|p(\mathbf{t})\| \leq l\}$.*

Lemma 2.1. *[25] Consider a continuous function $L(\mathbf{t})$ defined on $[0, \infty)$. Suppose $\exists$ constants $D > 0$ and $S > 0$ s.t.*

$$D^+ L(\mathbf{t}) \leq -DL(\mathbf{t}) + S.$$

Then, it follows that

$$L(\mathbf{t}) - \frac{S}{D} \leq \Big(L(0) - \frac{S}{D}\Big)e^{-D(\mathbf{t})}.$$

If $L(\mathbf{t}) \geq \frac{S}{D}$, then, as $(\mathbf{t})$ increases, $L(\mathbf{t})$ exponentially approaches $\frac{S}{D}$.

Lemma 2.2. *[26] For $k,\ l \in \mathbb{Q}$, a constant $\mathcal{N} > 0$, and positive definite Hermitian matrix $E \in \mathbb{Q}^{q\times q}$, the following inequality is satisfied.*

$$k^* l + l^* k \le \mathcal{N} k^* E k + \mathcal{N}^{-1} l^* E^{-1} l.$$

Lemma 2.3. *[26] The inequality below is valid for any quaternion values $n,\ m \in \mathbb{Q}$,*

$$(n+m)^*(n+m) \le 2n^* n + 2m^* m.$$

3. Lagrange stability

Theorem 3.1. *Following from Assumptions (2.1)–(2.4), if there exist constants $\eta_\Gamma, \beta_{\Gamma\xi}, \alpha_{\Gamma\xi}, \delta_\Gamma > 0$ such that*

$$D^1_\Gamma = d_\Gamma - \eta_\Gamma^{-1}\overline{u}_\Gamma - \sum_{\xi=1}^{\tilde q}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w^*_{\Gamma\xi} w_{\Gamma\xi} - \sum_{\xi=1}^{\tilde q}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v^*_{\Gamma\xi} v_{\Gamma\xi} - \delta_\Gamma^{-1} - 1 > 0,$$

$$D^2_\Gamma = \eta_\Gamma^{-1}\overline{u}_\Gamma - \sum_{\xi=1}^{\tilde q}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w^*_{\Gamma\xi} w_{\Gamma\xi} - \sum_{\xi=1}^{\tilde q}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v^*_{\Gamma\xi} v_{\Gamma\xi} - \delta_\Gamma^{-1} > 0,$$

$$D^3_\Gamma = \eta_\Gamma^{-1}\overline{u}_\Gamma - 2\eta_\Gamma \overline{u}_\Gamma \overline{c}_\Gamma - \sum_{\xi=1}^{\tilde q}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w^*_{\Gamma\xi} w_{\Gamma\xi} - \sum_{\xi=1}^{\tilde q}\beta_{\xi\Gamma}\overline{u}_\xi \overline{f}^2 p^*_\Gamma(\mathbf{t})$$
$$- \sum_{\xi=1}^{\tilde q}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v^*_{\Gamma\xi} v_{\Gamma\xi}\delta_\Gamma^{-1} - \sum_{\xi=1}^{\tilde q}\frac{2\alpha_{\xi\Gamma}\overline{u}_\xi \overline{g}_\Gamma^2}{1-\bar\sigma} > 0,$$

then QVNN (2.1) is globally Lagrange exponentially stable (LES). Moreover, QVNN (2.1) is globally exponential and converges to the ball

$$\mathbf{P} = \left\{ p(\mathbf{t}) \in \mathbb{Q}^{\tilde q} : \|p(\mathbf{t})\|_2 = \sqrt{\sum_{\Gamma=1}^{\tilde q} p^*_\Gamma(\mathbf{t}) p_\Gamma(\mathbf{t})} \le \sqrt{\frac{S}{D}} \right\}, \tag{3.1}$$

where

$$\begin{cases} D = \min\limits_{1\le\Gamma\le q}\left\{D^1_\Gamma, D^2_\Gamma, D^3_\Gamma\right\}, \\ S = 2\sum_{\Gamma=1}^{\tilde q}\eta_\Gamma \overline{u}_\Gamma \overline{c}^2 c^*_\Gamma(0) c_\Gamma(0) \\ \quad +2\sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}\beta_{\Gamma\xi}\overline{u}_\Gamma \overline{f}^2 f^*_\xi(0) f_\xi(0) \\ \quad +2\sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}\alpha_{\Gamma\xi}\overline{u}_\Gamma g^*_\xi(0) g_\xi(0) + \sum_{\Gamma=1}^{\tilde q}\delta_\Gamma \overline{B}_\Gamma^2. \end{cases}$$

Proof. Define a Lyapunov function as follows:

$$L(\mathbf{t}) = \frac{1}{2}\sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^*(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))$$
$$+ \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma \overline{g}_\xi^2}{1-\bar\sigma}\int_{\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})}^{t} p^*_\Gamma(s) p_\Gamma(s)\,ds. \tag{3.2}$$

The derivative of the Lypunov function is given by

$$\dot L(\mathbf{t}) = \sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^*(\ddot p_\Gamma(\mathbf{t}) + \dot p_\Gamma(\mathbf{t}))$$
$$+ \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma \overline{g}_\xi^2}{1-\bar\sigma} p^*_\Gamma(\mathbf{t}) p_\Gamma(\mathbf{t})$$
$$- \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q} 2\alpha_{\Gamma\xi}\overline{u}_\Gamma \overline{g}_\xi^2 p^*_\Gamma(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})) p_\Gamma(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})).$$

Using Eq (2.1), we can obtain

$$\dot L(\mathbf{t}) = \sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^*\Big(\Big[- d_\Gamma \dot p_\Gamma(\mathbf{t}) - u_\Gamma(p_\Gamma(\mathbf{t}))\Big(c_\Gamma(p_\Gamma(\mathbf{t}))$$
$$- \sum_{\xi=1}^{\tilde q} w_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t})) - \sum_{\xi=1}^{\tilde q} v_{\Gamma\xi} g_\xi(p_\xi(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})))\Big) + B_\Gamma(\mathbf{t})\Big]$$
$$+ \dot p_\Gamma(\mathbf{t})\Big) + \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma \overline{g}_\xi^2}{1-\bar\sigma} p^*_\Gamma(\mathbf{t}) p_\Gamma(\mathbf{t})$$
$$- \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q} 2\alpha_{\Gamma\xi}\overline{u}_\Gamma \overline{g}_\xi^2 p^*_\Gamma(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})) \times p_\Gamma(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t}))$$
$$= - \sum_{\Gamma=1}^{\tilde q} d_\Gamma(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* \dot p_\Gamma(\mathbf{t})$$
$$+ \sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* u_\Gamma(p_\Gamma(\mathbf{t})) \times c_\Gamma(p_\Gamma(\mathbf{t}))$$
$$+ \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* u_\Gamma(p_\Gamma(\mathbf{t})) w_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t}))$$
$$+ \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* u_\Gamma(p_\Gamma(\mathbf{t})) v_{\Gamma\xi} g_\xi(p_\xi(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})))$$
$$+ \sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* B_\Gamma(\mathbf{t}) + \sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* \dot p_\Gamma(\mathbf{t})$$
$$+ \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma \overline{g}_\xi^2}{1-\bar\sigma} p^*_\Gamma(\mathbf{t}) p_\Gamma(\mathbf{t})$$
$$- \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q} 2\alpha_{\Gamma\xi}\overline{g}_\xi^2 p^*_\Gamma(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})) \times p_\Gamma(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t}))$$
$$\le - \sum_{\Gamma=1}^{\tilde q} d_\Gamma(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* \dot p_\Gamma(\mathbf{t}) + \sum_{\Gamma=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* \overline{u}_\Gamma$$
$$\times c_\Gamma(p_\Gamma(\mathbf{t})) + \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* \overline{u}_\Gamma w_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t}))$$
$$+ \sum_{\Gamma=1}^{\tilde q}\sum_{\xi=1}^{\tilde q}(\dot p_\Gamma(\mathbf{t}) + p_\Gamma(\mathbf{t}))^* \overline{u}_\Gamma v_{\Gamma\xi} g_\xi(p_\xi(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})))$$

$$+\sum_{\Gamma=1}^{\tilde{q}}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^* B_\Gamma(\mathbf{t})+\sum_{\Gamma=1}^{\tilde{q}}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\dot{p}_\Gamma(\mathbf{t})$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma\overline{g}_\xi^2}{1-\bar{\sigma}}p_\Gamma^*(\mathbf{t})p_\Gamma(\mathbf{t})-\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}2\alpha_{\Gamma\xi}\overline{g}_\xi^2$$
$$\times p_\Gamma^*(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t}))p_\Gamma(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t}))$$
$$\le -\sum_{\Gamma=1}^{\tilde{q}} d_\Gamma(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\dot{p}_\Gamma(\mathbf{t})$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\eta_\Gamma^{-1}\overline{u}_\Gamma(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\times(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\eta_\Gamma\overline{u}_\Gamma c_\Gamma^*(p_\Gamma(\mathbf{t}))c_\Gamma(p_\Gamma(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w_{\Gamma\xi}^* w_{\Gamma\xi}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}\overline{u}_\Gamma f_\xi^*(p_\xi(\mathbf{t}))f_\xi(p_\xi(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v_{\Gamma\xi}^* v_{\Gamma\xi}\times(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}\overline{u}_\Gamma g_\xi^*(p_\xi(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})))\times g_\xi(p_\xi(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\delta_\Gamma^{-1}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\delta_\Gamma B_\Gamma^*(\mathbf{t})B_\Gamma(\mathbf{t})+\sum_{\Gamma=1}^{\tilde{q}}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\dot{p}_\Gamma(\mathbf{t})$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma\overline{g}_\xi^2}{1-\bar{\sigma}}p_\Gamma^*(\mathbf{t})p_\Gamma(\mathbf{t})$$
$$-\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}2\alpha_{\Gamma\xi}\overline{u}_\Gamma\overline{g}_\xi^2 p_\Gamma^*(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t}))\times p_\Gamma(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})). \tag{3.3}$$

Using Lemma 2.3, we derive the following inequality:

$$c_\Gamma^*(p_\Gamma(\mathbf{t}))c_\Gamma(p_\Gamma(\mathbf{t}))\le 2\overline{c}_\Gamma p_\Gamma^*(\mathbf{t})p_\Gamma(\mathbf{t})+2\overline{c}^2 c_\Gamma^*(0)c_\Gamma(0),$$
$$f_\xi^*(p_\xi(\mathbf{t}))f_\xi(p_\xi(\mathbf{t}))\le 2\overline{f}^2 p_\xi^*(\mathbf{t})p_\xi(\mathbf{t})+2\overline{f}^2 f_\xi^*(0)f_\xi(0),$$
$$g_\xi^*(p_\xi(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})))g_\xi(p_\xi(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})))$$
$$\le 2\overline{g}^2 p_\xi^*(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})))p_\xi(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t})))+2g_\xi^*(0)g_\xi(0).$$

It can be inferred that

$$\dot{L}(\mathbf{t})\le -\sum_{\Gamma=1}^{\tilde{q}} d_\Gamma(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\dot{p}_\Gamma(\mathbf{t})$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\eta_\Gamma^{-1}\overline{u}_\Gamma(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\times(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+2\sum_{\Gamma=1}^{\tilde{q}}\eta_\Gamma\overline{u}_\Gamma\overline{c}_\Gamma p_\Gamma^*(\mathbf{t})p_\Gamma(\mathbf{t})+2\sum_{\Gamma=1}^{\tilde{q}}\eta_\Gamma\overline{u}_\Gamma\overline{c}^2\times c_\Gamma^*(0)c_\Gamma(0)$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w_{\Gamma\xi}^* w_{\Gamma\xi}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+2\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}\overline{u}_\Gamma\overline{f}^2 p_\xi^*(\mathbf{t})p_\xi(\mathbf{t})$$
$$+2\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}\overline{u}_\Gamma\overline{f}^2 f_\xi^*(0)f_\xi(0)$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v_{\Gamma\xi}^* v_{\Gamma\xi}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+2\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}\overline{u}_\Gamma g_\xi^*(0)g_\xi(0)$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\delta_\Gamma^{-1}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\delta_\Gamma\overline{B}_\Gamma^2+\sum_{\Gamma=1}^{\tilde{q}}(\dot{p}_\Gamma(\mathbf{t})+p_\Gamma(\mathbf{t}))^*\dot{p}_\Gamma(\mathbf{t})$$
$$+\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\frac{2\alpha_{\Gamma\xi}\overline{u}_\Gamma\overline{g}_\xi^2}{1-\bar{\sigma}}p_\Gamma^*(\mathbf{t})p_\Gamma(\mathbf{t})$$
$$\le -\sum_{\Gamma=1}^{\tilde{q}}\Big(d_\Gamma-\eta_\Gamma^{-1}\overline{u}_\Gamma-\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w_{\Gamma\xi}^* w_{\Gamma\xi}$$
$$-\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v_{\Gamma\xi}^* v_{\Gamma\xi}-\delta_\Gamma^{-1}-1\Big)\dot{p}_\Gamma^*(\mathbf{t})\dot{p}_\Gamma(\mathbf{t})$$
$$-\sum_{\Gamma=1}^{\tilde{q}}\Big(\eta_\Gamma^{-1}\overline{u}_\Gamma-\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w_{\Gamma\xi}^* w_{\Gamma\xi}$$
$$-\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v_{\Gamma\xi}^* v_{\Gamma\xi}-\delta_\Gamma^{-1}\Big)\dot{p}_\Gamma^*(\mathbf{t})p_\Gamma(\mathbf{t})$$
$$-\sum_{\Gamma=1}^{\tilde{q}}\Big(-d_\Gamma-\eta_\Gamma^{-1}\overline{u}_\Gamma-\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w_{\Gamma\xi}^* w_{\Gamma\xi}$$
$$-\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v_{\Gamma\xi}^* v_{\Gamma\xi}-\delta_\Gamma^{-1}-1\Big)p_\Gamma^*(\mathbf{t})\dot{p}_\Gamma(\mathbf{t})$$

$$-\sum_{\Gamma=1}^{\tilde{q}}\Big(\eta_\Gamma^{-1}\overline{u}_\Gamma - 2\eta_\Gamma\overline{u}_\Gamma\overline{c}_\Gamma - \sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}^{-1}\overline{u}_\Gamma w_{\Gamma\xi}^{*}w_{\Gamma\xi}$$
$$-\sum_{\xi=1}^{\tilde{q}}\beta_{\xi\Gamma}\overline{u}_\xi\overline{f}^2 - \sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}^{-1}\overline{u}_\Gamma v_{\Gamma\xi}^{*}v_{\Gamma\xi}\delta_\Gamma^{-1} - \sum_{\xi=1}^{\tilde{q}}\frac{2\alpha_{\xi\Gamma}\overline{u}_\xi\overline{g}_\Gamma^2}{1-\bar{\sigma}}\Big)$$
$$\times p_\Gamma^{*}(\mathbf{t})p_\Gamma(\mathbf{t}) + 2\sum_{\Gamma=1}^{\tilde{q}}\eta_\Gamma\overline{u}_\Gamma\overline{c}^2c_\Gamma^{*}(0)c_\Gamma(0)$$
$$+2\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\beta_{\Gamma\xi}\overline{u}_\Gamma\overline{f}^2\times f_\xi^{*}(0)f_\xi(0)$$
$$+2\sum_{\Gamma=1}^{\tilde{q}}\sum_{\xi=1}^{\tilde{q}}\alpha_{\Gamma\xi}\overline{u}_\Gamma g_\xi^{*}(0)g_\xi(0) + \sum_{\Gamma=1}^{\tilde{q}}\delta_\Gamma\overline{B}_\Gamma^2$$
$$\leq -DL(\mathbf{t}) + S. \tag{3.4}$$

From Lemma 2.1, when $L(\mathbf{t}) \geq \frac{S}{D}$, then we deduce

$$L(\mathbf{t}) - \frac{S}{D} \leq \Big(V(0) - \frac{S}{D}\Big)e^{-Dt}, \quad \mathbf{t} \geq 0, \tag{3.5}$$

$$\text{i.e., } \|p(\mathbf{t})\| \leq \frac{S}{D} + V(0)e^{-Dt}. \tag{3.6}$$

By definition (2.2), the system (2.1) is globally LES. Also, the system (2.1) is globally exponential convergent to the ball $\mathbf{P}$ with a rate D. □

Remark 3.1. *The Lyapunov functional method is a well-known approach for stabilizing NNs. In most applications of Lyapunov functionals, the state variables of the model under consideration are accessible. Here, Eq* (2.1) *is considered a second-order differential system to investigate its stabilization. A newly developed Lyapunov functional* (3.2) *has been introduced, which includes both the state variables* $p_\Gamma(\mathbf{t})$ *and their derivatives* $\dot{p}_\Gamma(\mathbf{t})$ *of the addressed model* (2.1). *This approach differs from traditional Lyapunov functionals.*

Remark 3.2. *When* $u_\Gamma(p_\Gamma(\mathbf{t})) = 1$ *and* $c_\Gamma(p_\Gamma(\mathbf{t})) = c_\Gamma p_\Gamma(\mathbf{t})$ *are used as amplification functions, and* $c_\Gamma > 0$ *for all* $\Gamma \in J$, *the mentioned CGINN model* (2.1) *transforms into Hopfield or cellular type inertial NNs.*

$$\ddot{p}_\Gamma(\mathbf{t}) = -d_\Gamma\dot{p}_\Gamma(\mathbf{t}) - c_\Gamma p_\Gamma(\mathbf{t}) - \sum_{\xi=1}^{\tilde{q}}w_{\Gamma\xi}f_\xi(p_\xi(\mathbf{t}))$$
$$-\sum_{\xi=1}^{\tilde{q}}v_{\Gamma\xi}f_\xi(p_\xi(\mathbf{t}-\sigma_{\Gamma\xi}(\mathbf{t}))) + B_\Gamma(\mathbf{t}),$$

where $\Gamma \in J$. *A reduced-order method is applied to examine this equation [27–29]. However, finding suitable control gains for stability results in practical applications is challenging due to the complex conditions. This article introduces a more general type of INNs, accompanied by a simpler inequality presented in Theorem 3.1, which can streamline the computational process in comparison to existing results [27, 28, 30].*

Remark 3.3. *Researchers commonly analyze the stability of CGINNs with time delay by employing a variable substitution method to transform second-order CGINNs into first-order differential equations [29–32]. They then apply the Lyapunov functional approach to establish relevant stability criteria. This reduced-order method, however, effectively doubles the system's dimensions, increasing the complexity of the resulting theoretical analysis. In contrast, this article presents exponential stability criteria for CGINNs using a newly developed Lyapunov functional approach, which does not require order reduction and results in simpler inequalities. Consequently, the findings in this article are more straightforward and reasonable.*

Remark 3.4. *In [33], the finite-time and fixed-time stability of CGINNs is investigated using the reduction-order method and bounded time-varying delay. However, in this article, the exponential stability of CGINNs with unbounded time-varying delays is analyzed using a non-reduced order approach. This approach employs a newly developed Lyapunov functional alongside a control scheme to avoid the reduction-order method and reduce the complexity of the theoretical work. Unbounded time-varying delays represent more realistic and challenging scenarios where delays can grow without bounds, making the stability analysis more complex and crucial for practical applications. The newly developed Lyapunov functional and control scheme addresses these challenges, providing a more comprehensive and less conservative stability criterion by removing the need for differentiability and monotonicity of the behaved functions, which were constraints in previous methods [34].*

4. Numerical simulation

The validation of the theoretical findings discussed earlier, assessing both their effectiveness and efficiency, is demonstrated through Example 4.1. Example 4.2 illustrates the application of the QVNN model by retrieving a color image.

Example 4.1. *Let us consider the quaternion-valued CGINNs with time-varying delays as*

$$\ddot{p}_\Gamma(\mathbf{t}) = -d_\Gamma \dot{p}_\Gamma(\mathbf{t}) - u_\Gamma(p_\Gamma(\mathbf{t}))\Big[c_\Gamma(p_\Gamma(\mathbf{t})) - \sum_{\xi=1}^{2} w_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t})) - \sum_{\xi=1}^{2} v_{\Gamma\xi} \times f_\xi(p_\xi(\mathbf{t} - \sigma_{\Gamma\xi}(\mathbf{t})))\Big] + B_\Gamma(\mathbf{t}), \quad \Gamma = 1, 2, \tag{4.1}$$

with the parameters as

$$d_1 = 10, d_2 = 20,$$
$$u_1 = 1.3 - \frac{0.2}{p^2+1}, u_2 = 1.2 - \frac{0.2}{p^2+1}, c_\Gamma(p_\Gamma(\mathbf{t})) = tanh(p_\Gamma(\mathbf{t})),$$
$$f_\xi(p_\xi(\mathbf{t})) = tanh(p_\xi(\mathbf{t})),\ g_\xi(p_\xi(\mathbf{t} - \sigma(t))) = tanh(p_\xi(\mathbf{t} - \sigma(t)))$$

for

$$\xi = 1, 2, \sigma_{11}(\mathbf{t}) = \sigma_{12}(\mathbf{t}) = \sigma_{21}(\mathbf{t}) = \sigma_{22}(\mathbf{t}) = \frac{t}{4},$$
$$W = [w_{\Gamma\xi}]_{2\times2}$$
$$= \begin{bmatrix} 0.2 - 0.3e_1 + 0.1e_2 - 0.4e_3 & -0.8 + 0.3e_1 - 0.5e_2 + 0.4e_3 \\ 0.8 + e_1 + 0.4e_2 + 0.3e_3 & 0.7 - 0.5e_1 + 0.9e_2 - 0.1e_3 \end{bmatrix},$$
$$V = [v_{\Gamma\xi}]_{2\times2}$$
$$= \begin{bmatrix} 0.8 - 0.4e_1 + 0.7e_2 - 0.1e_3 & -0.5 + 0.4e_1 - 0.7e_2 + 0.3e_3 \\ 0.4 + e_1 + 0.7e_2 + 1.5e_3 & 0.2 - 0.1e_1 + 0.3e_2 - 0.8e_3 \end{bmatrix}.$$
$$B_1(\mathbf{t}) = 0.2cos(\mathbf{t}) + 0.4sin(\mathbf{t})e_1 + 0.2cos(\mathbf{t})e_2 + 0.3sin(\mathbf{t})e_3,$$
$$B_2(\mathbf{t}) = 0.4cos(\mathbf{t}) + 0.1sin(\mathbf{t})e_1 + 0.3cos(\mathbf{t})e_2 + 0.2sin(\mathbf{t})e_3.$$

The initial values are given as

$$p_1^R(0) = -0.2, p_1^I(0) = 0.5, p_1^J(0) = -0.8,$$
$$p_1^K(0) = -0.3, p_2^R(0) = 0.4, p_2^I(0) = -0.1,$$
$$p_2^J(0) = 0.3, p_2^K(0) = -0.4, \dot{p}_1^R(0) = 0.3,$$
$$\dot{p}_1^I(0) = 0.6, \dot{p}_1^J(0) = 0.2, \dot{p}_1^K(0) = 0.4,$$
$$\dot{p}_2^R(0) = 0.3, p_2^I(0) = 0.6, \dot{p}_2^J(0) = -0.5, \dot{p}_2^K(0) = -0.1.$$

Through straightforward calculations, we get $\overline{u}_1 = 1.3, \overline{u}_2 = 1.2$*, therefore the Lipschitz constants are* $\overline{c}_\Gamma = \overline{f}_\Gamma = \overline{g}_\Gamma = 1, \Gamma = 1, 2$*; thus, the assumptions 2.1–2.4 are satisfied. Now, choosing* $\eta_1 = 1/6, \eta_2 = 1/7, \beta_{\Gamma\xi} = \alpha_{\Gamma\xi} = \delta_\Gamma = 1$*, we get* $D_1^1 = 0.8089, D_2^1 = 5.747, D_1^2 = 2.4089, D_2^2 = 1.3466, D_1^3 = 1.432, D_2^3 = 2.314,\ D = 1.34 > 0,\ \overline{B}_1 = 0.5745, \overline{B}_2 = 0.5477, S = 0.09$*, and* $\mathbf{P} = \sqrt{\frac{S}{D}} = 0.24$*, which satisfy all the conditions of Theorem 3.1.*

The graph of stability of quaternion-valued CGINNs in the Lagrange sense is shown through Figure 1. If $\overline{B}_\Gamma = \overline{c}_\Gamma = 0, f_\xi = g_\xi = 0, \Gamma, \xi = 1, 2$*, implies* $S = 0$*, in this case, the Lagrange stability will be converted to Lyapunov stability, and we will achieve complete stability, which is depicted in Figure 2.*

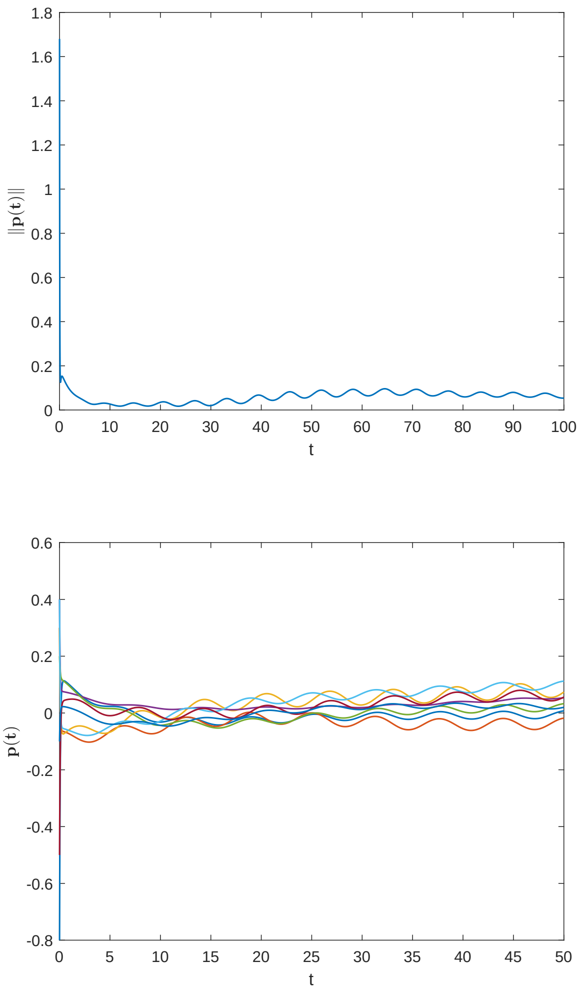

Figure 1. Plots of stability of quaternion-valued CGINNs in Lagrange sense.

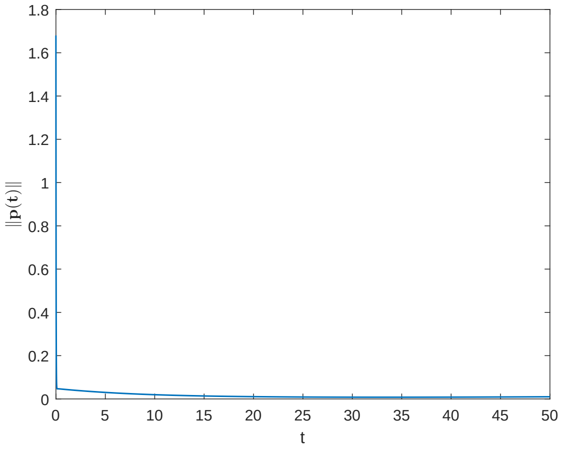

Figure 2. Plot of stability of quaternion-valued CGINN in the Lyapunov sense.

Remark 4.1. *In [27], the stability of neutral-type delayed INNs is analyzed via a linear matrix inequality (LMI) approach, where the examples employ small constant delays (e.g., 0.2 s) and constant connection weights, requiring the delay bounds to be known and strictly limited. Similarly, [28] considers the synchronization of INNs with mixed delays using fixed gain matrices tuned through iterative LMI feasibility tests. In contrast, Example 4.1 in this article uses unbounded time-varying delays* $\sigma_{\Gamma\xi}(t) = t/4$ *and quaternion-valued weight matrices with nontrivial imaginary components, which cannot be handled by the LMI-based examples in [27, 28] without significant conservatism. Our non-reduction order approach avoids transforming the system to first-order form, preserves the original dynamics, and yields stability conditions in the form of direct algebraic inequalities (Theorem 3.1), which are easier to verify for large-scale or high-dimensional quaternion-valued systems.*

Example 4.2. *To demonstrate the use of traditional QVNNs, let's examine a* 12×12 *pixel image pattern, referred to as "H". Figure 3 depicts this pattern and feature-dimensional equilibrium. In this scenario, a set of QVNNs configured as per Eq* (4.2) *is employed, consisting of* 144 *neurons. These QVNNs feature* 144*-dimensional equilibrium points, enabling the storage of the colored "H" patterns. Consequently, we will now focus on traditional QVNNs to continue our discussion.*

$$\dot{p}_\Gamma(\mathbf{t}) = -c_\Gamma p_\Gamma(\mathbf{t}) - \sum_{\xi=1}^{\tilde{q}} w_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t})) - \sum_{\xi=1}^{p} v_{\Gamma\xi} f_\xi(p_\xi(\mathbf{t} - \sigma_\xi(\mathbf{t}))) + B_\Gamma(\mathbf{t}). \tag{4.2}$$

Set the parameters of (4.2) *as follows:*

$$c_\Gamma = 1.0, \tag{4.3}$$

$$w_{\Gamma\xi} = \begin{cases} 4.0 + 0.40e_1 - 0.30e_2 + 0.50e_3, m = n \\ 0.40 - 0.50e_1 + 0.50e_2 - 0.30e_3, m \neq n, \end{cases} \tag{4.4}$$

$$v_{\Gamma\xi} = \begin{cases} -0.20 + 0.20e_1 - 0.50e_2 + 0.40e_3, m < n \\ 0.20 + 0.30e_1 - 0.20e_2 - 0.30e_3, m = n \\ -0.10 + 0.20e_1 + 0.30e_2 - 0.50e_3, m > n, \end{cases} \tag{4.5}$$

$$f^{(c)}(p(\mathbf{t})) = g^{(c)}(p(\mathbf{t})) = tanh(p(\mathbf{t})), \sigma_\xi = 1 \ \forall \ \Gamma, \xi, \tag{4.6}$$

where $m, n = 1, 2, \ldots, 144$ *and* $c = 1, 2, 3$.

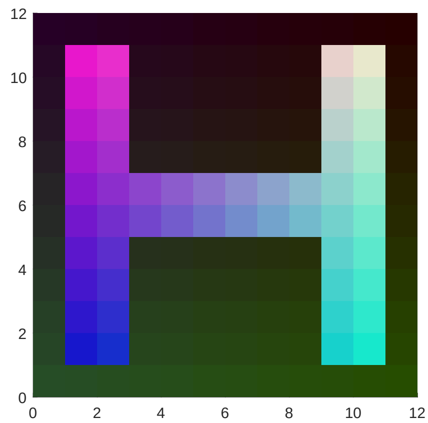

Figure 3. Original color image of pattern "*H*".

The equilibrium point of the proposed QVNNs must be $p = (p_1, p_2, \ldots, p_{144})^T \in \mathbb{Q}^{144}$ *in order to recall the color image pattern "H", where* $p_1 = 0 + 0.15e_1 + 0.30e_2 + 0.1500e_3$, $p_2 = 0 + 0.1500e_1 + 0.30e_2 + 0.1400e_3, \ldots, p_{144} = 0 + 0.150e_1 + 0e_2 + 0e_3$, *which corresponds to the colors* $(0.1500, 0.30, 0.250), (0.150, 0.30, 0.140), \ldots, (0.150, 0, 0)$ *of the pixels in the image pattern "H". The external input B can be determined using the equilibrium point p as follows:*

$$B = (B_1, B_2, \ldots, B_{144})^T \in \mathbb{Q}^{144}, \tag{4.7}$$

where $B_1 = -62.80 + 22.70e_1 - 90.50e_2 - 7.250e_3, B_2 = -630 + 20.950e_1 - 90.50e_2 - 5.660e_3, \ldots, B_{144} = -91.40 - 234.60e_1 - 90.80e_2 + 221.40e_3$. *Due to space considerations, we can only list three components of B and p here. The QVNNs developed with parameters* (4.3)–(4.6) *are capable of recalling the pattern "H", as demonstrated by a simulation with random initial values shown in Figure 4.*

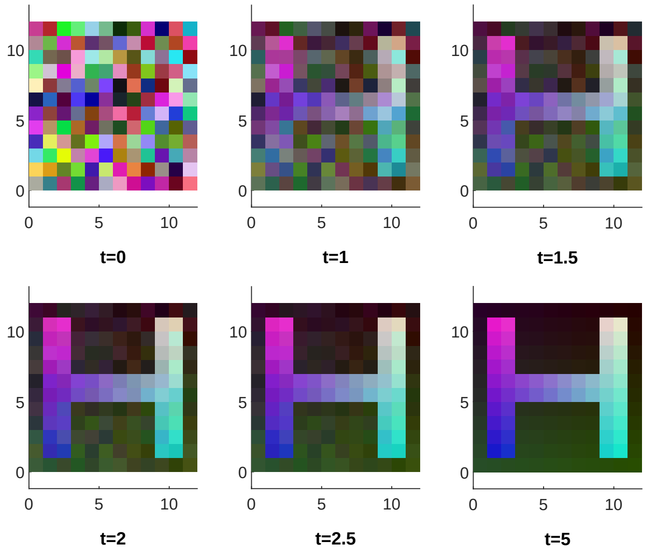

Figure 4. Simulations of retrieving image "*H*" with random initial values of time t.

Remark 4.2. *Example 4.2 illustrates that the proposed system* (4.2)*, defined by Eqs* (4.3)–(4.6) *for its parametric values, requires 144 neurons to store a* 12 × 12 *pixel image pattern. In contrast, findings in [35] indicate that a comparable image stored in a CVNN would need* 432 *neurons—a substantially larger amount. This difference underscores the superior storage efficiency of QVNNs compared to CVNNs. The article specifically focuses on QVNNs utilizing tanh activation functions. Notably, the reconstruction of the image "H" can be approximated to occur within a period of* $t = 2.5$.

5. Conclusions

This article investigates the Lagrange exponential stability of quaternion-valued CGINN with unbounded time-varying delays by developing a Lyapunov functional. The study employs a non-reduction order approach to solve the stability problem of inertial NNs directly, simplifying the analysis process and providing simpler inequality conditions with multiple parameters to ensure model stability. This approach contrasts with the traditional reduced-order method. The efficiency and effectiveness of the proposed theoretical work are validated through a numerical example. Additionally, the potential of QVNNs in true color image retrieval is demonstrated, showcasing the versatility and practical utility of neural networks. For future research, the synchronization of fractional-order octonion-valued CGINNs, incorporating an adaptive control strategy, could be explored by applying the research concepts and the non-reduction order approach utilized in this study. Extending quaternion to octonion-valued networks can improve the expressive power and compactness of high-dimensional signals. Integrating fractional-order dynamics refines memory and damping control, enhancing stability and flexibility. The study of synchronization with time-varying delays contributes to robust coordination in multi-agent and distributed sensing systems.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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